ASSIGNMENT 2

PROBLEM 1

Recall the relation composition operator ; defined as:

$$R_1; R_2 = \{(a, c) : \text{there is a b with } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

For any set S, and any binary relations $R_1, R_2, R_3 \subseteq S \times S$, prove or give a counter-example to disprove the following:

a) $(R_1; R_2); R_3 = R_1;(R_2; R_3)$

$(R_1; R_2)$ $= \{(a, c) : \text{there is a b with } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$
$(R_1; R_2); R_3$ $= \{(a, d) : \text{there is a c with } (a,c) \in (R_1; R_2) \text{ and } (c, d) \text{ with } R_3\}$
$= \{(a, d) : \text{there is a b, c with } (a,b) \in R_1, (b,c) \in R_2 \text{ and } (c, d) \in R_3\}$

$(R_2; R_3)$ $= \{(b, d) : \text{there is a c with } (b, c) \in R_2 \text{ and } (c, d) \in R_3\}$
$(R_1; R_2); R_3$ $= \{(a, d) : \text{there is a b with } (a,b) \in R_1 \text{ and } (b, d) \in (R_2; R_3)\}$
$= \{(a, d) : \text{there is a b, c with } (a,b) \in R_1, (b,c) \in R_2 \text{ and } (c, d) \in R_3\}$

Hence Proved, $(R_1; R_2); R_3 = R_1;(R_2; R_3)$

b) $I; R_1 = R_1; I = R_1$ where $I = \{(x, x) : x \in S\}$

$(I:R_1) = \{(a, c) : \text{there is a b with } (a, b) \in I \text{ and } (b, c) \in R_1\}$
Given $I = \{(x, x) : x \in S\}$ and $(a,b) \in I$
then $a=b$
Therefore, $(I:R_1)$ $= \{(b, c) : \text{there is a b with } (b, b) \in I \text{ and } (b, c) \in R_1\}$
$= \{(b, c) : (b, c) \in R_1\}$
$= R_1$
$(R_1:I)$ $= \{(a, c) : \text{there is a b with } (a, b) \in R_1 \text{ and } (b, c) \in I\}$
Given $I = \{(x, x) : x \in S\}$ and $(b,c) \in I$
then $b=c$
Therefore, $(I:R_1)$ $= \{(a, b) : \text{there is a b with } (a, b) \in R_1 \text{ and } (b, b) \in I\}$
$= \{(a,b) : (a,b) \in R_1\}$
$= R_1$

Hence Proved, $I; R_1 = R_1; I = R_1$ where $I = \{(x, x) : x \in S\}$

c) $(R_1; R_2)^{\leftarrow} = R_1^{\leftarrow}; R_2^{\leftarrow}$

$R_1^{\leftarrow}; R_2^{\leftarrow}$ = {(a, c) : there is a b with $(a, b) \in R_1^{\leftarrow}$ and $(b, c) \in R_2^{\leftarrow}$}
= {(a, c) : there is a b with $(b, a) \in R_1^{\leftarrow}$ and $(c, b) \in R_2$}
$(R_1; R_2)$ = {(c, a) : there is a b with $(c, b) \in R_1$ and $(b, a) \in R_2$}
$(R_1; R_2)^{\leftarrow}$ = {(a, c) : there is a b with $(c, b) \in R_1$ and $(b, a) \in R_2$}

If R_1 = {(a, b),(c, d)} and R_2 = {(b, a),(d, c)},

then counter example of $(R_1; R_2)^{\leftarrow} = R_1^{\leftarrow}; R_2^{\leftarrow}$ is
$(R_1; R_2)^{\leftarrow}$ = {(a, a),(c, c)} and $R_1^{\leftarrow}; R_2^{\leftarrow}$ = {(b, b),(d, d)}

Therefore, $(R_1; R_2)^{\leftarrow} \neq R_1^{\leftarrow}; R_2^{\leftarrow}$

d) $(R_1 \cup R_2); R_3 = (R_1; R_3) \cup (R_2; R_3)$

$(R_1 \cup R_2); R_3$ = {(a, c) : there is a b with $(a, b) \in (R_1 \cup R_2)$ and $(b, c) \in R_3$}

$(R_1;R_3)$ = {(a, c) : there is a b_1 with $(a, b_1) \in R_1$ and $(b_1, c) \in R_3$}
$(R_2;R_3)$ = {(a, c) : there is a b_2 with $(a, b_2) \in R_2$ and $(b_2, c) \in R_3$}
$(R_1;R_3) \cup (R_2;R_3)$ = {(a, c) : $(a, c) \in (R_1; R_3)$ or $(a, c) \in (R_2; R_3)$}
= {(a, c) : there is a b_1,b_2 with $(a, b_1) \in R_1$ and $(b_1, c) \in R_3$ or $(a, b_2) \in R_2$ and $(b_2, c) \in R_3$}
= {(a, c) : there is a b with $(a, b) \in R_1$ or R_2 and $(b, c) \in R_3$}
= {(a, c) : there is a b with $(a, b) \in (R_1 \cup R_2)$ and $(b, c) \in R_3$}

Hence Proved, $(R_1 \cup R_2); R_3 = (R_1; R_3) \cup (R_2; R_3)$

e) $R_1;(R_2 \cap R_3) = (R_1; R_2) \cap (R_1; R_3)$

$R_1;(R_2 \cap R_3)$ = {(a, c) : there is a b with $(a, b) \in R_1$, $(b, c) \in (R_2 \cap R_3)$}
$R_1;(R_2 \cap R_3)$ = {(a, c) : there is a b with $(a, b) \in R_1$, $(b, c) \in R_2$ and $(b, c) \in R_3$)}

$(R_1; R_2)$ = {(a, c) : there is a b_1 with $(a, b_1) \in R_1$ and $(b_1, c) \in R_2$}
$(R_1; R_3)$ = {(a, c) : there is a b_2 with $(a, b_2) \in R_1$ and $(b_2, c) \in R_3$}
$(R_1; R_2) \cap (R_1; R_3)$ = {(a, c) : $(a, c) \in (R_1; R_2)$ and $(a, c) \in (R_1; R_3)$}
= {(a, c) : there is a b_1, b_2 with (a, b_1),$(a, b_2) \in R_1$ and $(b_1, c) \in R_2$ and $(b_2, c) \in R_3$}

If R_1 = {(a, b),(a, c),(a, e)}, R_2 = {(b, d),(e, f)} and R_3 = {(c, d),(e, f)} $\forall$ a, b, c, d, e, f $\in$ S
then counter example of $R_1;(R_2 \cap R_3) = (R_1; R_2) \cap (R_1; R_3)$
where $R_1;(R_2 \cap R_3)$ = {(a, f) and $(R_1; R_2) \cap (R_1; R_3)$ = {(a, d),(a, f)}}

Therefore, $R_1;(R_2 \cap R_3) \neq (R_1; R_2) \cap (R_1; R_3)$

PROBLEM 2

Let $R \subseteq S \times S$ be any binary relation on a set S. Consider the sequence of relations R^0, R^1, R^2, . . ., defined as follows:

$R^0 := I = \{(x, x) : x \in S\}$, and

$R^{i+1} := R^i \cup (R; R^i)$ for $i \geq 0$

a) **Prove that if there is an i such that $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$.**

For i = 0

$R^1 := R^0 \cup (R; R^0)$

Given, $R^0 := I = \{(x, x) : x \in S\}$

Therefore

$R^1 := I \cup (R; I)$

$R^1 := I \cup R$ [using $I; R_1 = R_1; I = R_1$ (Problem 1.b))]

$R^1 := R$ [using $I \cup R = R$]

Therefore $R^0 = R^0$, $R^0 = R^1$, $R^1 = R^2$ and so on

Hence Proved there is an i such that $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$.

b) **Prove that if there is an i such that $R^i = R^{i+1}$, then $R^k \subseteq R^i$ for all $k \geq 0$.**

For i = 0

$R^1 := R^0 \cup (R; R^0)$

Given, $R^0 := I = \{(x, x) : x \in S\}$

Therefore

$R^1 := I \cup (R; I)$

$R^1 := I \cup R$ [using $I; R_1 = R_1; I = R_1$ (Problem 1.b))]

$R^1 := R$ [using $I \cup R = R$]

Therefore $R^0 \subseteq R^0$, $R^1 \subseteq R^0$, $R^2 \subseteq R^0$ and so on

Hence Proved there is an i such that $R^i = R^{i+1}$, then $R^k \subseteq R^i$ for all $k \geq 0$

c) Let P(n) be the proposition that for all m ∈ N: R^n ; $R^m = R^{n+m}$. Prove that P(n) holds for all n ∈ N.

Goal: Show P(n) holds for all n ∈ N

Base Case[B]: P(0) holds

Proof
⇒ R^0 ; $R^m = R^{0+m}$
⇒ R^0 ; $R^m = R^m$
⇒ I; $R^m = R^m$
using I; $R_1 = R_1$; I = R_1 (Problem 1.b))
Therefore P(0) holds.

Inductive Case[I]:

Assume P(n) holds for n ≥ 0.
⇒ P(n) := R^n ; $R^m = R^{n+m}$ [Inductive step]

If P(n) holds then P(n + 1) holds
We will show P(n+1) holds

Proof:

⇒ P(n+1) := R^{n+1} ; $R^m = R^{(n+1)+m}$
⇒ := R + R^n ; R^m
⇒ := R^{n+1} ; R^m
⇒ := $R^{(n+1)+m}$
Therefore P(n+1) holds

So, P(n) implies P(n+1)
Therefore by principle of induction P(n) holds for all n ∈ N.

d) If |S| = k, explain why $R^k = R^{k+1}$. (Hint: Show that if (a, b) ∈ R^{k+1} then (a, b) ∈ R^i for some i < k + 1.)

Suppose (a, b) ∈ R^{k+1}
We have $R^k \subseteq R^i$ for all k ≥ 0 [using part b)]
$R^{k+1} \subseteq R^i$
Therefore (a, b) ∈ R^i

Hence proved, $R^k = R^{k+1}$

e) If |S| = k, show that R^k is transitive.

Suppose $(a, b) \in R^k$ and $(b, c) \in R^k$
Then there exists (a, c) such that $\in R^k$

Hence R^k is transitive

f) If |S| = k, show that $(R \cup R^{\leftarrow})^k$ is an equivalence relation.

For equivalence relation a relation must be Reflexive, Symmetric, Transitive

1) Reflexive

for all $x \in S$
$(x, x) \in R$, and $(x, x) \in R^{\leftarrow}$

$\Rightarrow \quad (R \cup R^{\leftarrow}) = \{(x, x)\}$

Hence $(R \cup R^{\leftarrow})$ is a Reflexive relation

2) Symmetric

for all $x, y \in S$
$(x, y) \in R$, and $(y, x) \in R^{\leftarrow}$

$\Rightarrow \quad (R \cup R^{\leftarrow}) = \{(x, y),(y, x)\}$

Hence $(R \cup R^{\leftarrow})$ is Symmetric relation

3) Transitive

For all $x, y, z \in S$
(x, y) and $(y, z) \in R$
(y, x) and $(z, y) \in R^{\leftarrow}$

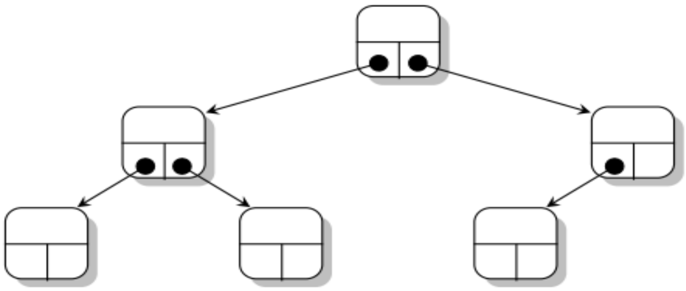
$\Rightarrow \quad (R \cup R^{\leftarrow}) = \{(x, y),(y, z),(x, z),(y, x),(z, y),(z, x)\}$

Hence $(R \cup R^{\leftarrow})$ is Transitive relation

From the above results we can conclude that the given relation is a equivalence relation.

PROBLEM 3

A binary tree is a data structure where each node is linked to at most two successor nodes:



If we allow empty binary trees (trees with no nodes), then we can simplify the description by saying a node has exactly two children which are binary trees.

a) **Give a recursive definition of the binary tree data structure. Hint: review the recursive definition of a Linked List**

Recursive Definition of a binary tree data structure is :-

(B) an empty tree, or
(R) an node pointing to two binary trees, one its left child and other one its right child

A leaf in a binary tree is a node that has no successors (i.e. it has two empty trees as children). A fully internal node in a binary tree is a node that has two successors. The example above has 3 leaves and 2 fully-internal nodes.

b) **Based on your recursive definition above, define the function count(T) that counts the number of nodes in a binary tree T.**

```
        count(T):
(B)             if(T.isEmpty()):
                        return -1
(R)             if(T.isNotEmpty()):
                        if(T.left.isNotEmpty()):
                                return 1 + count(T.left)
                        if(T.right.isNotEmpty()):
                                return 1 + count(T.right)
```

c) Based on your recursive definition above, define the function leaves(T) that counts the number of leaves in a binary tree T.

```
        leaves(T):
(B)             if(T.isEmpty()):
                        return 0
                if(T.left.isEmpty() && T.right.isEmpty()):
                        return 1
(R)             else:
                        return leaves(T.left) + leaves(T.right)
```

d) Based on your recursive definition above, define the function internal(T) that counts the number of fully-internal nodes in a binary tree T. Hint: it is acceptable to define an empty tree as having −1 fully internal nodes.

```
        internal(T):
(B)             if(T.isEmpty()):
                        return -1
                if(T.left.isEmpty() && T.right.isEmpty()):
                        return 0
(R)             else:
                        return 1 + internal(T.left) + internal(T.right)
```

e) If T is a binary tree, let P(T) be the proposition that leaves(T) = 1 + internal(T). Prove that P(T) holds for all binary trees T.

Goal: Show P(T) holds for all binary trees T

Base Case [B]:

[B1]: T = 0
Proof:

$\Rightarrow$ P(0) = 1 + 0
= 1

A binary tree with only 1 node has 0 full internal nodes and 1 leaf (the node itself is the leaf), so P(0) is true.
Therefore P(0) holds.

[B2]: T = 1
Proof:

$\Rightarrow$ P(1) = 1 + 1
= 2

A binary tree with 1 full internal node has 2 leaves, so P(1) is true.
Therefore P(1) holds.

Inductive Case [I]:

Assume P(T) holds.
$\Rightarrow$ P(T) := leaves(T) = 1 + internal(T) [Inductive step]

If P(T) holds then P(T + 1) holds
We will show P(T+1) holds

Proof:

Enlarging a binary tree by adding a node to an existing node that has 1 child, makes an initial binary tree with n full nodes now have n+1 full nodes. Since, the parent node becomes a full node, then the number of leaves in the binary tree increases by 1. So, the number of leaves in the binary tree becomes
$\Rightarrow$ (n+1) + 1 = n + 2

Therefore the number of leaves minus the number of full nodes is given by
$\Rightarrow$ (n+2) – (n+1) = 1, as desired

So, P(T) implies P(T+1).

Therefore, by principle of induction, P(T) holds for all binary Tree T

PROBLEM 4

Four WIFI networks, Alpha, Bravo, Charlie and Delta, all exist within close proximity to one another as shown below.

Networks connected with an edge in the diagram above can interfere with each other. To avoid interference networks can operate on one of two channels, hi and lo. Networks operating on different channels will not interfere; and neither will networks that are not connected with an edge.

Our goal is to determine (algorithmically) whether there is an assignment of channels to networks so that there is no interference. To do this we will transform the problem into a problem of determining if a propositional formula can be satisfied.

a) **Carefully defining the propositional variables you are using, write propositional formulas for each of the following requirements:**

Propositional Variables :

A_{hi} = Alpha uses hi channel
A_{lo} = Alpha uses lo channel
B_{hi} = Bravo uses hi channel
B_{lo} = Bravo uses lo channel
C_{hi} = Charlie uses hi channel
C_{lo} = Charlie uses lo channel
D_{hi} = Delta uses hi channel
D_{lo} = Delta uses lo channel

i. **φ_1: Alpha uses channel hi or channel lo; and so does Bravo, Charlie and Delta.**

$\varphi_1 = (A_{hi} \vee A_{lo}) \wedge (B_{hi} \vee B_{lo}) \wedge (C_{hi} \vee C_{lo}) \wedge (D_{hi} \vee D_{lo})$

ii. **φ_2: Alpha does not use both channel hi and lo; and the same for Bravo, Charlie and Delta.**

$\varphi_2 = (A_{hi} \rightarrow \neg A_{lo}) \wedge (B_{hi} \rightarrow \neg B_{lo}) \wedge (C_{hi} \rightarrow \neg C_{lo}) \wedge (D_{hi} \rightarrow \neg D_{lo})$

iii. φ_3: Alpha and Bravo do not use the same channel; and the same applies for all other pairs of networks connected with an edge.

$\varphi_3 = (A_{hi} \rightarrow \neg B_{hi}) \wedge (A_{lo} \rightarrow \neg B_{lo}) \wedge (B_{hi} \rightarrow \neg C_{hi}) \wedge (B_{lo} \rightarrow \neg C_{lo}) \wedge (C_{hi} \rightarrow \neg D_{hi}) \wedge (C_{lo} \rightarrow \neg D_{lo})$

b)

i. Show that $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ is satisfiable; so the requirements can all be met. Note that it is sufficient to give a satisfying truth assignment, you do not have to list all possible combinations

A_{hi}	A_{lo}	B_{hi}	B_{lo}	C_{hi}	C_{lo}	D_{hi}	D_{lo}	$\varphi_1 \wedge \varphi_2 \wedge \varphi_3$
T	F	F	T	T	F	F	T	T

OR

A_{hi}	A_{lo}	B_{hi}	B_{lo}	C_{hi}	C_{lo}	D_{hi}	D_{lo}	$\varphi_1 \wedge \varphi_2 \wedge \varphi_3$
F	T	T	F	F	T	T	F	T

ii. Based on your answer to the previous question, which channels should each network use in order to avoid interference?

Alpha uses hi channel, Bravo uses lo channel, Charlie uses hi channel and Delta uses lo channel

OR

Alpha uses lo channel, Bravo uses hi channel, Charlie uses lo channel and Delta uses hi channel