ASSIGNMENT 1

PROBLEM 1

a) List all possible functions f : {a, b, c} → {0, 1}

There are 8 possible functions namely :-

1. f(a) = 0, f(b) = 0, f(c) = 0
2. f(a) = 0, f(b) = 0, f(c) = 1
3. f(a) = 0, f(b) = 1, f(c) = 0
4. f(a) = 0, f(b) = 1, f(c) = 1
5. f(a) = 1, f(b) = 0, f(c) = 0
6. f(a) = 1, f(b) = 0, f(c) = 1
7. f(a) = 1, f(b) = 1, f(c) = 0
8. f(a) = 1, f(b) = 1, f(c) = 1

b) Describe a connection between your answer for (a) and Pow({a, b, c})

If S = Pow({a, b, c })
Then subset of set S are namely:-

1. {}
2. Pow{a}
3. Pow{b}
4. Pow{c}
5. Pow{a, b}
6. Pow{a, c}
7. Pow{b, c}
8. Pow{a, b, c}

We can clearly see that (a) and (b) both have same number of subsets/functions

c) In general, if card(A) = m and card(B) = n, how many

i. functions are there from A to B?

Let A have a elements and B have b elements.
b^a number of functions.

ii. **relations are there between A and B?**

The total number of relations that can be formed between two sets is the number of subsets of their Cartesian product.

n(A) = p
n(B) = q
n(AXB) = pq
Number of relations between A and B is 2^{pq}

iii. **symmetric relations are there on A?**

Let A contains a elements
To be symmetric, whenever it includes a pair (x,y) (x,y), it must include the pair (y,x)
So A has a elements with (a/2) subsets of size 2.
Hence it has $2^{a/2}$ symmetric relations.

PROBLEM 2

For $x, y \in Z$ we define the set: $S_{x,y} = \{mx + ny : m, n \in Z\}$.

a) **Give five elements of $S_{2,-3}$.**

For $S_{2,-3}$
Set $\{2m - 3n : m, n \in Z\}$
Let $m = 0, n = 0 \Rightarrow 0$
Let $m = 1, n = 0 \Rightarrow 2$
Let $m = 0, n = 1 \Rightarrow -3$
Let $m = 1, n = 1 \Rightarrow -1$
Let $m = -1, n = -1 \Rightarrow 1$
Hence five elements are:-
$\{0,2,-3,-1,1\}$

b) **Give five elements of $S_{12,16}$.**

For $S_{12,16}$
Set $\{12m + 16n : m, n \in Z\}$
Let $m = 0, n = 0 \Rightarrow 0$
Let $m = 1, n = 0 \Rightarrow 12$
Let $m = 0, n = 1 \Rightarrow 16$
Let $m = 1, n = 1 \Rightarrow 28$
Let $m = -1, n = 1 \Rightarrow 4$
Hence five elements are:-
$\{0,12,16,28,4\}$

For the following questions, let $d = \gcd(x, y)$ and z be the smallest positive number in $S_{x,y}$.

c) **Show that $S_{x,y} \subseteq \{n : n \in Z \text{ and } d|n\}$.**

$x = k_1 . d$
$y = k_2 . d$ (for some k_1 and $k_2 \in Z$)
We have
$S_{x,y} = \{mx + ny : m, n \in Z\}$
So $mx + ny = m(k_1 . d) + n(k_2 . d)$
$d(m . k_1 + n . k_2)$
Thus $d \mid mx + ny$ which further implies, $d \mid n$
therefore $S_{x,y} \subseteq \{n : n \in Z \text{ and } d|n\}$

d) Show that $\{n : n \in Z \text{ and } z|n\} \subseteq Sx,y$.

$n = k \,.\, z$ (for some $k \in Z$)
$k \,.\, z \in S_{x,y}$
$k \,.z = k\,(mx + ny)$
$\Rightarrow \quad kmx + kny$
$\Rightarrow \quad (km)x + (kn)y = n \in Z$
therefore $\{n : n \in Z \text{ and } z|n\} \subseteq S_{x,y}$

e) Show that $d \leq z$. (Hint: use (c))

$z = mx + ny$
$\Rightarrow \quad z = m\,(k_1 \,.\, d) + n\,(k_2 \,.\, d)$ (where $x = k_1 \,.\, d$ and $y = k_2 \,.\, d$)
$\Rightarrow \quad z = (m \,.\, k_1 + n \,.\, k_2)\, d$
$\Rightarrow \quad d = z \,/\, (\, m \,.\, k_1 + n \,.\, k_2\,)$
Hence $d \leq z$

f) Show that $z \leq d$. (Hint: use (d))

$d \in S_{x,y}$
$d = mx + ny$
$\Rightarrow \quad d = m\,(k_1 \,.\, d) + n\,(k_2 \,.\, d)$
$\Rightarrow \quad d = (m \,.\, k_1 + n \,.\, k_2)\, d$
$\Rightarrow \quad (m \,.\, k_1 + n \,.\, k_2) = 1$ and $\in Z$

$K_1x + k_2y = 1$ (where $x,y \in Z$)
Hence $z \leq d$

PROBLEM 3

We define the operation $*$ on subsets of a universal set U as follows. For any two sets.

$$\mathbf{A \text{ and } B: A * B := A^c \cup B^c\ .}$$

Answer the following questions using the Laws of Set Operations (and any derived results given in lectures) to justify your answer:

a) **What is $(A * B) * (A * B)$?**

$(A^c \cup B^c) * (A^c \cup B^c)$	[using given $A * B := A^c \cup B^c$]
$(A^c \cup B^c)^c \cup (A^c \cup B^c)^c$	[using given $A * B := A^c \cup B^c$]
$((A^c)^c \cap (B^c)^c) \cup ((A^c)^c \cap (B^c)^c)$	[using de Morgan's Law]
$(A \cap B) \cup (A \cap B)$	[using Double complementation Law]
$A \cap B$	[using Idempotence Law]

b) **Express A^c using only A, $*$ and parentheses (if necessary).**

$A^c = A * A$

Proof :

$A * A = A^c \cup A^c$	[using given $A * B := A^c \cup B^c$]
$(A - U) \cup (A - U)$	[as we know $A^c = A - U$]
$A - U$	[using Idempotence Law]
A^c	[as we know $A^c = A - U$]

c) **Express $\emptyset$ using only A, $*$ and parentheses (if necessary).**

$\emptyset = ((A * A) * A) * ((A * A) * A)$

Proof:

$\emptyset = \emptyset \cup \emptyset$	[using Idempotence Law]
$(A \cap A^c) \cup (A \cap A^c)$	[using Complementation Law]
$(A \cup A^c)^c \cup (A \cup A^c)^c$	[using de Morgan's Law]
$((A^c)^c \cup A^c)^c \cup ((A^c)^c \cup A^c)^c$	[using Double complementation Law]
$(A^c * A)^c \cup (A^c * A)^c$	[using given $A * B := A^c \cup B^c$]
$((A * A) * A)^c \cup ((A * A) * A)^c$	[using $A^c = A * A$, from 2b]
$((A * A) * A) * ((A * A) * A)$	[using given $A * B := A^c \cup B^c$]

d) Express A \ B using only A, B, ∗ and parentheses (if necessary).

$A \setminus B = (A * (B * B)) * (A * (B * B))$

Proof:

$A \setminus B = A \cap B^c$

$(A \cap B^c) \cup (A \cap B^c)$	[using Idempotence Law]
$(A^c \cup B)^c \cup (A^c \cup B)^c$	[using de Morgan's Law]
$(A * B^c)^c \cup (A * B^c)^c$	[using given $A * B := A^c \cup B^c$]
$(A * (B * B))^c \cup (A * (B * B))^c$	[using $A^c = A * A$, from 2b]
$(A * (B * B)) * (A * (B * B))$	[using given $A * B := A^c \cup B^c$]

PROBLEM 4

Let $\Sigma = \{a, b\}$. Define $R \subseteq \Sigma^* \times \Sigma^*$ as follows: $(w, v) \in R$ if there exists $z \in \Sigma^*$ such that $v = wz$.

a) Give two words $w, v \in \Sigma^*$ such that $(w, v) \notin R$ and $(v, w) \notin R$.

1)
$w = a$
$v = b$
here $w, v \in \Sigma^*$
but $(w, v) \notin R$ and $(v, w) \notin R$
as $v \neq wz$ where $z \in \Sigma^*$

2)
$w = b$
$v = a$
here $w, v \in \Sigma^*$
but $(w, v) \notin R$ and $(v, w) \notin R$
as $v \neq wz$ where $z \in \Sigma^*$

b) What is $R^{\leftarrow}(\{aba\})$?

$f(w) = wz$
$v = wz$
$w = vz$
$w/z = vz/z$
$v = w/z$
$f^{\leftarrow}(w) = w/z$ (where $z \in \Sigma^*$)
now solve for 'w' we get
$R = \{ (ab,aba) , (a,aba) , (\lambda,aba) , (aba,aba) \}$
therefore
$R^{\leftarrow}(\{aba\}) \quad \Rightarrow \quad \{ aba, ab, a, \lambda \}$

c) Show that R is a partial order

For partial order a relation must be Reflexive, Anti-Symmetric and Transitive

1. **Reflexive**

   if $z = \lambda$
   so $v = w$
   $(w, w) \in R$ and $(v, v) \in R$
   Hence the given relation is reflexive.

2. **Anti-Symmetric**

   Proof:
   $(w, v) \in R$ and $(v, w) \in R$
   $(w, v) \quad \Rightarrow \quad v = wz_1$
   $(v, w) \quad \Rightarrow \quad w = vz_2$
   $\qquad\quad \Rightarrow \quad w = wz_1z_2$
   $\qquad\quad \Rightarrow \quad w = w \qquad (z_1 = z_2 = \lambda)$
   Hence the given relation is anti-symmetric

3. **Transitive**

   Proof:
   $(w, v) \in R$ and $(v, q) \in R$

   $(w, v) \qquad \Rightarrow v = wz_1$
   $(v, q) \qquad \Rightarrow q = vz_2$
   $\qquad\qquad \Rightarrow q = wz_1z_2$
   $\qquad\qquad \Rightarrow q = wz \quad (z_1 = z_2 = z)$
   Therefore $(w,q) \in R$
   Hence the given relation is transitive

   From the above results we can conclude that the given relation is a Partial Order.

PROBLEM 5

Show that for all $x, y, z \in Z$:

If $x|yz$ and $\gcd(x, y) = 1$ then $x|z$.

$yz = kx$ (where $k \in Z$) and $\gcd(x, y) = 1$

for $y = 0$

$z = kx$

$\Rightarrow \quad x \mid z$

for $y \neq 0$

$z = (kx) / y$

$z = k_y x \qquad$ (where $k_y = k / y$ and $k_y \in Z$)

$\Rightarrow \quad x \mid z$

Hence $x|yz$ and $\gcd(x, y) = 1$ then $x|z$.