University of New South Wales
School of Mathematics and Statistics

MATH5905 Statistical Inference
Term One 2021

Assignment Two

Given: Wednesday 7 April 2021 **Due date: Wednesday 21 March 2021**

Instructions: This assignment is to be completed **collaboratively** by a group of **at most** 3 students. The same mark will be awarded to each student within the group, unless I have good reasons to believe that a group member did not contribute appropriately. This assignment must be submitted no later than 11:59 pm on Wednesday, 14 April 2021. The first page of the submitted PDF should be **this page**. Only one of the group members should submit the PDF file on Moodle, with the names of the other students in the group clearly indicated in the document.

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Assignment 2

Problem 1

$$f(x,\theta)=\begin{cases}\frac{\sqrt{\theta}}{x\sqrt{2\pi}}e^{-\frac{\theta}{2}log^2x} & \text{if } x>0\\ 0 & \text{otherwise}\end{cases}\quad ,\qquad \theta>0$$

Part A

Let $Y_i = logX_i = g(X_i)$ that is a one to one mapping.

Then $X_i = g^{-1}(Y_i) = e^{Y_i}$ with $\frac{dX_i}{dY_i} = e^{Y_i}$.

Using transformation density formula:

$$\begin{aligned} f_{Y_i}(y) &= f_{X_i}(g^{-1}(y))\left|\frac{dX_i}{dY_i}\right| = \frac{\sqrt{\theta}}{e^y\sqrt{2\pi}}exp(-\frac{\theta}{2}log^2e^y)\left|e^y\right| \\ &= \frac{\sqrt{\theta}}{\sqrt{2\pi}}exp(\frac{-\theta}{2}y^2) = \frac{1}{\sqrt{2\pi}\sqrt{\theta}^{-1}}exp(\frac{-(y-0)^2}{2\theta^{-1}}) \end{aligned}$$

Therefore: $\quad Y_i \sim N(0,\theta^{-1}), \quad \mathbb{E}[Y_i]=0, \quad \mathbb{V}\text{ar}(Y_i)=\frac{1}{\theta}$

Finally $\mathbb{E}[log^2X_i] = \mathbb{E}[Y_i^2] = \mathbb{V}\text{ar}(Y_i) + \mathbb{E}[Y_i]^2 = \frac{1}{\theta} + 0 = \frac{1}{\theta}$

Part B

$f(x,\theta) = \frac{\sqrt{\theta}}{x\sqrt{2\pi}}exp(-\frac{\theta}{2}log^2x)$

$logf(x,\theta) = \frac{1}{2}log\theta - logx - \frac{1}{2}log2\pi - \frac{\theta}{2}log^2x$

$\frac{\partial}{\partial\theta}logf(x,\theta) = \frac{1}{2\theta} - \frac{1}{2}log^2x$

$\frac{\partial^2}{\partial\theta^2}logf(x,\theta) = -\frac{1}{\theta^2}$

For a single observation $I_{X_1}(\theta) = -\mathbb{E}[\frac{\partial^2}{\partial\theta^2}logf(x,\theta)] = \mathbb{E}[\frac{1}{\theta^2}] = \frac{1}{\theta^2}$

For a sample of n i.i.d. r.v. $I_X(\theta) = nI_{X_1} = \frac{n}{\theta^2}$

Part C

Let $h(\theta) = \frac{1}{\theta}$

$L(x, \theta) = \prod_{i=1}^{n} \frac{\sqrt{\theta}}{x\sqrt{2\pi}} exp(-\frac{\theta}{2} log^2 x) = \frac{1}{\prod_{i=1}^{n} x_i} \left(\frac{\theta}{2\pi}\right)^{\frac{n}{2}} exp(-\frac{\theta}{2} \sum_{i=1}^{n} log^2 x_i)$

$logL(x, \theta) = -\sum_{i=1}^{n} logx_i + \frac{n}{2} log\theta - \frac{n}{2} log2\pi - \frac{\theta}{2} \sum_{i=1}^{n} log^2 x_i$

$\frac{\partial}{\partial\theta} logL(x, \theta) = V(x, \theta) = \frac{n}{2\theta} - \frac{1}{2} \sum_{i=1}^{n} log^2 x_i$

Setting the score to zero to find the MLE:

$V(x, \theta) = \frac{n}{2\theta} - \frac{1}{2} \sum_{i=1}^{n} log^2 x_i = 0$

$\frac{1}{\theta} = \frac{1}{n} \sum_{i=1}^{n} log^2 x_i$

$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} log^2 x_i}$

As $h(\theta) = \frac{1}{\theta}$ the MLE of $h(\theta)$ is $\widehat{h(\theta)}_{MLE} = \frac{1}{\hat{\theta}_{MLE}} = \frac{\sum_{i=1}^{n} log^2 x_i}{n}$

Now to prove that this is unbiased
$\mathbb{E}[h(\hat{\theta}_{MLE})] = \mathbb{E}[\frac{\sum_{i=1}^{n} log^2 x_i}{n}] = \frac{1}{n} n\mathbb{E}[log^2 x_i] = \mathbb{E}[log^2 x_i]$

From Part A we know that $\mathbb{E}[log^2 x_i] = \frac{1}{\theta}$

Therefore $\mathbb{E}[h(\hat{\theta}_{MLE})] = \frac{1}{\theta}$ that is unbiased for $h(\theta) = \frac{1}{\theta}$

Part D

Does the variance of the MLE for $h(\theta)$ attain the Cramer Rao bound?

$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} log^2 X_i}$

Therefore $h(\theta) = \frac{1}{\hat{\theta}_{MLE}} = \frac{1}{n} \sum_{i=1}^{n} log^2 X_i$

$Var(\widehat{h(\theta)}_{MLE}) = Var(\frac{1}{n} \sum_{i=1}^{n} log^2 X_i) = \frac{1}{n^2} nVar(log^2 X_i) = \frac{1}{n} Var(log^2 X_i)$

As $Y_i = logX_i$ ~ $N(0, \frac{1}{\theta})$

We standardise rv and get $\sqrt{\theta} Y_i$ ~ $N(0, 1)$

Therefore: $\theta Y_i = \chi_1^2$

So: $Var(\chi_1^2) = Var(\theta Y_i^2) = \theta^2 Var(Y_i^2) = 2$

From this $Var(Y_i^2) = 2\theta^2$

Therefore: $Var(\widehat{h(\theta)}_{MLE}) = \frac{2\theta^2}{n}$

As: $\frac{\partial}{\partial\theta} logL(x, \theta) = \frac{n}{2\theta} - \frac{1}{2} \sum_{i=1}^{n} log^2 x_i$

Then: $\frac{\partial^2}{\partial\theta^2}logL(x,\theta) = -\frac{n}{2\theta^2}$

$-\mathbb{E}[-\frac{n}{2\theta^2}] = \frac{n}{2\theta^2}$

Therefore: $I_{X(n)}^{-1} = \frac{2\theta^2}{n}$

The CRLB is attained by the variance of the MLE which makes it also the UMVUE.

Part E

From the Delta Method:

$$\sqrt{n}(h(\hat{\theta}_{MLE}) - h(\theta)- > N(0, [\frac{\partial h}{\partial\theta}(\theta_0)]^2 I^{-1}(\theta_0)$$
$$\sqrt{n}(\widehat{h(\theta)} - h(\theta))- > N(0, \frac{(h^{'}(\theta))^2}{I_{X_1}})$$

$$h(\theta) = \frac{1}{\theta}$$
$$h^{'}(\theta) = -\frac{1}{\theta^2}$$
$$h^{'}(\theta)^2 = \frac{1}{\theta^4}$$

$$CRLB = \frac{1}{\theta^4} * \frac{2\theta^2}{n} = \frac{2}{\theta^2 n}$$

$$\sqrt{n}(\widehat{h(\theta)} - h(\theta)) \sim N(0, \frac{2}{\theta^2})$$

For $\tau(\theta) = e^{-\theta}$:

$$\tau^{'}(\theta) = -e^{-\theta}$$
$$(\tau^{'}(\theta))^2 = e^{-2\theta}$$

$$\sqrt{n}(\widehat{\tau(\theta)} - \tau(\theta)) \sim N(0, 2e^{-2\theta}\theta^2)$$

Problem 2

$$f(x,\theta) = \begin{cases} \frac{\tau x^{\tau-1}}{\theta^\tau} & \text{if } 0 < x < \theta \\ 0 & \text{if otherwise} \end{cases}$$

Part A

Using $P(X_{(n)} \leq x) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n)$
So $P(X_{(n)} \leq x) = P(X_1 \leq x)^n$

Computing the CDF firstly to find the density function:

$$\begin{aligned} F(x) &= \int_0^1 \tau\theta^{-\tau} x^{\tau-1} dx \\ &= \tau\theta^{-\tau} \int_0^1 x^{\tau-1} dx \\ &= \tau\theta^{-\tau} \int_t^\theta x^{\tau-1} dx \\ &= \frac{\tau}{\theta^{-\tau}} [\frac{x^\tau}{\tau}]_{x=t}^{x=\theta} \\ &= \left(\frac{t}{\theta}\right)^\tau \end{aligned}$$

From above $P(X_{(n)} \leq x) = \left(\frac{t}{\theta}\right)^{\tau n}$

Differentiate this to find the density function $\frac{\partial}{\partial\theta}(\frac{t}{\theta})^{\tau n} = \frac{\tau n t^{\tau n-1}}{\theta^{\tau n}}$

Hence

$$f_T(t) = \begin{cases} \frac{\tau n t^{\tau n-1}}{\theta^{\tau n}} & \text{if } 0 < x < \theta \\ 0 & \text{if otherwise} \end{cases}$$

Part B

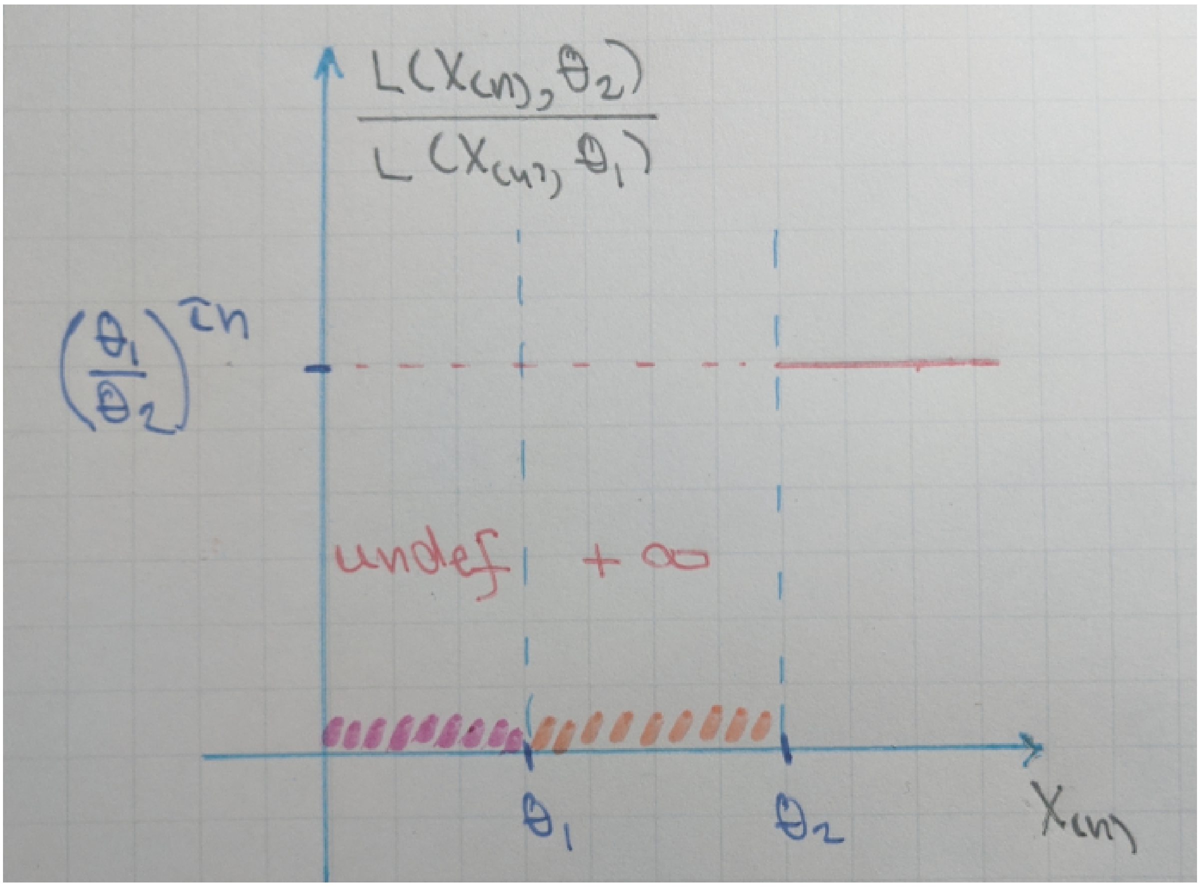
$\{L(X,\theta), \theta > 0\}$ has a monotone likelihood ratio in the statistic $T = X_{(n)}$

Let $\theta_1 < \theta_2$, therefore we need to look at the ratio $\frac{L(X,\theta_2)}{L(X,\theta_1)}$

$L(x,\theta) = \prod_{i=1}^{n} \frac{\tau x^{\tau-1}}{\theta^{\tau}} = \frac{\tau^n}{\theta^{\tau n}} \prod_{i=1}^{n} X_i^{\tau-1} I_{(X_{(n)},\infty)}(\theta)$

$\frac{L(X,\theta_2)}{L(X,\theta_1)} = \frac{\frac{\tau^n}{\theta_2^{\tau n}} \prod_{i=1}^{n} X_i^{\tau-1} I_{(X_{(n)},\infty)}(\theta_2)}{\frac{\tau^n}{\theta_1^{\tau n}} \prod_{i=1}^{n} X_i^{\tau-1} I_{(X_{(n)},\infty)}(\theta_1)} = (\frac{\theta_1}{\theta_2})^{\tau n} \frac{I_{(X_{(n)},\infty)}(\theta_2)}{I_{(X_{(n)},\infty)}(\theta_1)}$

$$\frac{L(X,\theta_2)}{L(X,\theta_1)} = \begin{cases} \text{undefined} & \text{if } X_{(n)} < \theta_1 \\ \infty & \text{if } \theta_1 < X_{(n)} < \theta_2 \\ (\frac{\theta_1}{\theta_2})^{\tau n} & \text{if } X_{(n)} > \theta_2 \end{cases}$$



Part C

Finding the Uniformly most powerful α-size test ϕ^* for $H_0 : \theta \leq \tau$ vs $H_1 : \theta > \ \tau$

From part B this has a monotone likelihood ration in the statistic $T = X_{(n)}$. Therefore:

$$\phi^* = \begin{cases} 1 & \text{if } X_{(n)} < k \\ 0 & \text{if } X_{(n)} \geq k \end{cases}$$

To find k we must "exhaust the α-level": $\quad \alpha = P(X_{(n)} < k | \theta = \tau)$

From Part A:

$\alpha = (\frac{k}{\tau})^{\tau n}$
$(\frac{k}{\tau})^{\tau n} = 1 - \alpha$
$\frac{k}{\tau} = \alpha^{\frac{1}{\tau n}}$
$k = \tau \alpha^{\frac{1}{\tau n}}$

Hence

$$\phi^*(X) = \begin{cases} 1 & \text{if } X_{(n)} < \tau \alpha^{\frac{1}{\tau n}} \\ 0 & \text{if } X_{(n)} \geq \tau \alpha^{\frac{1}{\tau n}} \end{cases}$$

Part D

$Power(\theta) = \mathbb{E}_\theta[\phi^*(X)]$
$= P(X_{(n)} < \tau \alpha^{\frac{1}{\tau n}})$
$= \left(\frac{\theta}{\tau \alpha^{\frac{1}{\tau n}}}\right)^{\tau n}$
$= \left(\frac{\theta}{\tau}\right)^{\tau n} \frac{1}{\alpha}$

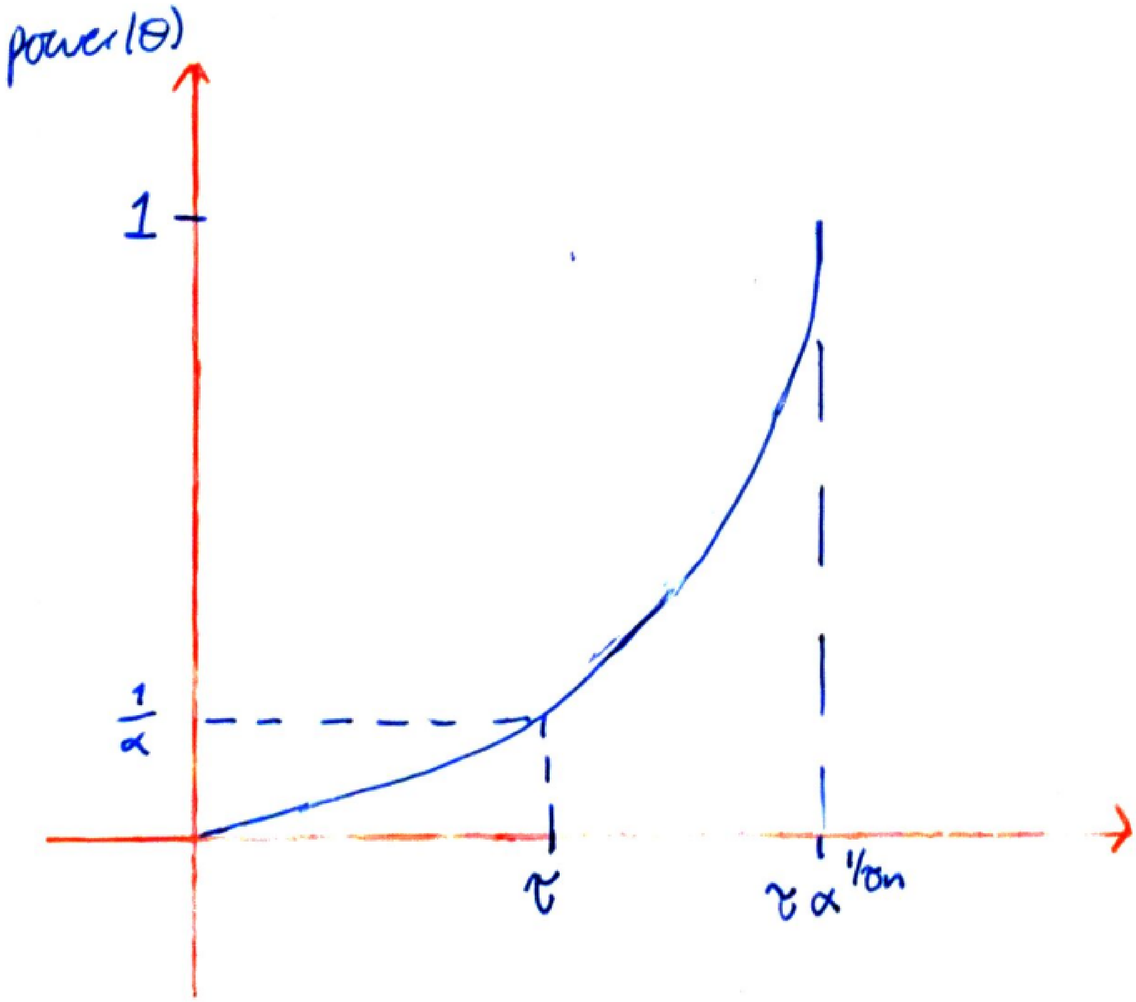
Part E

$Power\left(\tau\alpha^{\frac{1}{\tau n}}\right) = \alpha\frac{1}{\alpha} = 1$

$Power(\tau) = \left(1\right)^{\tau n}\frac{1}{\alpha} = \frac{1}{\alpha}$

$Power(0) = 0$

$lim_{\theta\to\infty}\ Power(\theta) = 1$ since power is bound by 1.



Problem 3

Part A

Let $X = (X_1, \dots, X_n)$ be a sample of n i.i.d. observations from this distribution.

We can show that family $\{L(X, \theta)\}$ has a monotone likelihood ratio in the statistic $T = X_{(1)}$.

i)

The density can be directly computed from the formula noting that the distribution function for each X_i is $f(x, \theta) = e^{-(x-\theta)}$ for $\theta < x < \infty$

$$F(X) = \int_\theta^x f(x, \theta) dx$$

$$= \int_\theta^x e^{-(x-\theta)} dx$$

$$= [-e^{-(x-\theta)}]_\theta^x$$

$$= 1 - e^{-(x-\theta)} \text{ for } \theta < x < \infty$$

$$F_{X_{(n)}}(x) = n[1 - F(x)]^{n-1} f(x)$$

$$= n[1 - 1 + e^{-(x-\theta)}]^{n-1} e^{-(x-\theta)}$$

$$= ne^{-n(x-\theta)}$$

Therefore,

$$F_{X_{(n)}}(x) = \int_\theta^x ne^{-n(x-\theta)} dx$$

$$= 1 - e^{-n(x-\theta)} \text{ for } \theta < x < \infty$$

ii)

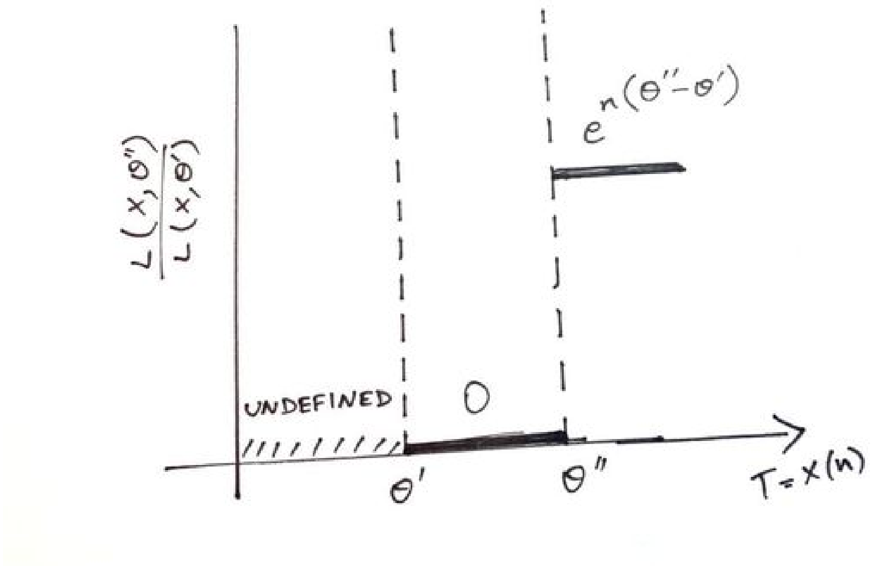
For $\theta^{''} > \theta^{'}$,

$$L(X, \theta) = \prod_{i=1}^n e^{-(x_i-\theta)} \mathrm{I}_{(-\infty, x_i)}(\theta)$$

$$= e^{\sum X_i} . e^{n\theta} \mathrm{I}_{(-\infty, X_i)}(\theta)$$

$$\frac{L(X,\theta^{''})}{L(X,\theta^{'})} = \frac{e^{n\theta^{''}}}{e^{n\theta^{'}}} . \frac{\mathrm{I}_{(-\infty,X)}(\theta^{''})}{\mathrm{I}_{(-\infty,X)}(\theta^{'})}$$

$$= \begin{cases} undefined & \text{if } X_1 < \theta^{''} \\ 0 & \text{if } \theta^{'} < X_1 < \theta^{''} \\ e^{n(\theta^{''} - \theta^{'})} & \text{if } X_1 > \theta^{''} \end{cases}$$



From the graph, we can see it is a non-decreasing function.

$T = X_{(n)}$ has the MLR property.

iii)

Using Blackwell and Girshick theoram the UMP α-test is given by

$$\varphi^* = \begin{cases} 1 & \text{if } X_n < \text{k} \\ 0 & \text{if } X_n \geq k \end{cases}$$

To find k we must exhaust the α-test

$$\alpha = E_{\theta_0}\varphi^* = P_{\theta_0}(X_n < k) = 1 - P\theta_0(X_n > k)$$

$$\alpha = 1 - [P\theta_0(X_1 > k)]^n$$

$$\alpha = 1 - e^{-n(k-\theta_0)}$$

Hence, Rearanging to solve for k

$k = \theta_0 - \frac{1}{n}ln(1-\alpha)$ and φ^* is completely determine.

$$\varphi^* = \begin{cases} 1 & \text{if } X_n < \text{k} = \theta_0 - \frac{1}{n}ln(1-\alpha) \\ 0 & \text{if } X_n \geq k = \theta_0 - \frac{1}{n}ln(1-\alpha) \end{cases}$$

iv)

Given,
X = (1, 2, 1.01, 3, 1:45), $\alpha = 0.10$ and $\theta_0 = 1$

k = 1 - $\frac{1}{5}ln(1-0.10) = 1.02107$

We observe that $X_{(1)} < k$ and we should reject H_0

v)

Let,

$$Z_n = n(X_{(1)} - \theta)$$

$$F_{Z_n}(z) = P(Z_n < Z) = P(nX_{(1)} - n\theta < z)$$

$$= P(X_{(n)} < \tfrac{z+n\theta}{n}) = 1 - P(X_{(n)} > \tfrac{z+n\theta}{n})$$

$$= 1 - P(X_{(1)} > \tfrac{z+n\theta}{n})]^n$$

$$= 1 - e^{-n}(\tfrac{z+n\theta}{n} - \theta)$$

$$= 1 - e^{-z} \qquad z \geq 0$$

we see that,

$$\lim_{n\to\infty} F_{Z_n}(z) = F_Z(z) = \begin{cases} 0 & \text{if z} < 0 \\ 1 - e^{-z} & \text{if z} > 0 \end{cases}$$

Here, $F_Z(z)$ is the distribution function of an exponential random variable with mean one. Therefore, Z_n converges in distribution to an exponential random variable with mean one as $n \to \infty$.

vi)

We consider evaluating $P(|X_{(n)} - \theta| < \epsilon)$ directly by noting that X(n) cannot possibly be greater than θ. Hence,

$$P(|X_{(n)} - \theta| < \epsilon) = P(X_{(n)} > \theta - \epsilon) = 1 - P(X_{(n)} \leq \theta - \epsilon)$$

Now, the maximum $X_{(n)}$ is less than some constant if and only if each of the random variables $X_1, \ldots.., X_n$ is less than that constant. Therefore, since the X_i are i.i.d.,

$$P(X_{(n)} \leq \theta - \epsilon) = [P(X_1 \leq \theta - \epsilon)]^n = \begin{cases} (1 - \frac{\epsilon}{\theta})^n & \text{if } 0 \leq \epsilon < \theta \\ 0 & \text{if } \epsilon \geq \theta \end{cases}$$

since $1 - \frac{\epsilon}{\theta}$ is strictly less than one, we conclude no matter what positive value ϵ takes,

$$P(X_{(n)} \leq \theta - \epsilon) \to 0$$

as desired.

Part B

i)

We first notice that the distribution function of X is $F(x, \theta)$ = $e^{-(x-\theta)}$ for $\theta < x < \infty$ and the density of X_1 is

$$f_{X_{(1)}}(x) = 1 - 5e^{-5(x-\theta)} \qquad \theta < x < \infty \qquad n = 5$$

When $\theta < 1$

$$\gamma(\theta) = P(X_{(n)} \geq 2 \text{ or } X_{(n)} < 1)$$

$$= 1 - P(X_{(n)} < 2) + P(X_{(n)} < 1)$$

$$= \int_\theta^1 ne^{-n(t-\theta)}dt + \int_2^\infty ne^{-n(t-\theta)}dt$$

$$= -e^{-n(t-\theta)}|_\theta^1 + [-e^{-n(t-\theta)}|_2^\infty]$$

$$= 1 - e^{-n(1-\theta)} + e^{-n(2-\theta)}$$

We are given random sample of size five from this distribution.

Hence,

$$= 1 + e^{-5(2-\theta)} - e^{-5(1-\theta)}$$

$$= e^{5\theta}[e^{-10} - e^{-5}] + 1$$

When $\theta \geq 2$

$$\gamma(\theta) = 1$$

When $1 \leq \theta < 2$

$$\gamma(\theta) = P(X_{(n)} \geq 2) + P(X_{(n)} < 1)$$

$$= 1 - (1 - e^{-n(2-\theta)})$$

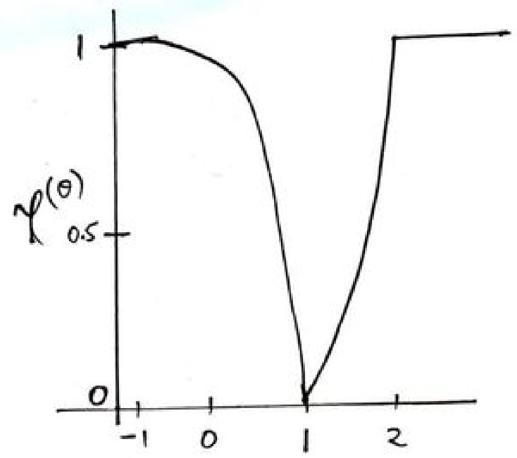
We are given random sample of size five from this distribution.

Hence,

$$= e^{-10+5\theta}$$

$$= \begin{cases} e^{5\theta}[e^{-10} - e^{-5}] + 1 & \text{if } \theta < 1 \\ e^{-10+5\theta} & \text{if } 1 \leq \theta < 2 \\ e^{n(\theta'' - \theta')} & \text{if } \theta \geq 2 \end{cases}$$

ii)



1
$\gamma^{(\theta)}$
0.5
0
-1
0
1
2

Problem 4

Part A

$f(x;\alpha,\beta) = \frac{\alpha\beta^\alpha}{x^{\alpha+1}} I_{[\beta,\infty)}(x)$
$L(X;\alpha,\beta) = \prod_{i=1}^{n} \frac{\alpha\beta^\alpha}{x_i^{\alpha+1}} I_{[\beta,\infty)}(x_i) = \alpha^n \beta^{n\alpha} \frac{1}{\prod_{i=1}^{n} x_i^{\alpha+1}} I_{(0,X_{(1)})}(\beta)$
$logL(X;\alpha,\beta) = nlog\alpha + n\alpha log\beta + log1 - \sum_{i=1}^{n} logx_i^{\alpha+1}$
$logL(X;\alpha,\beta) = nlog\alpha + n\alpha log\beta - \sum_{i=1}^{n}(\alpha+1)logx_i$
$logL(X;\alpha,\beta) = nlog\alpha + n\alpha log\beta - \alpha\sum_{i=1}^{n} logx_i - \sum_{i=1}^{n} logx_i$

i) Finding the MLE of α

$\frac{\partial}{\partial\alpha} logL(X;\alpha) = \frac{n}{\alpha} + nlog\beta - \sum_{i=1}^{n} logx_i$

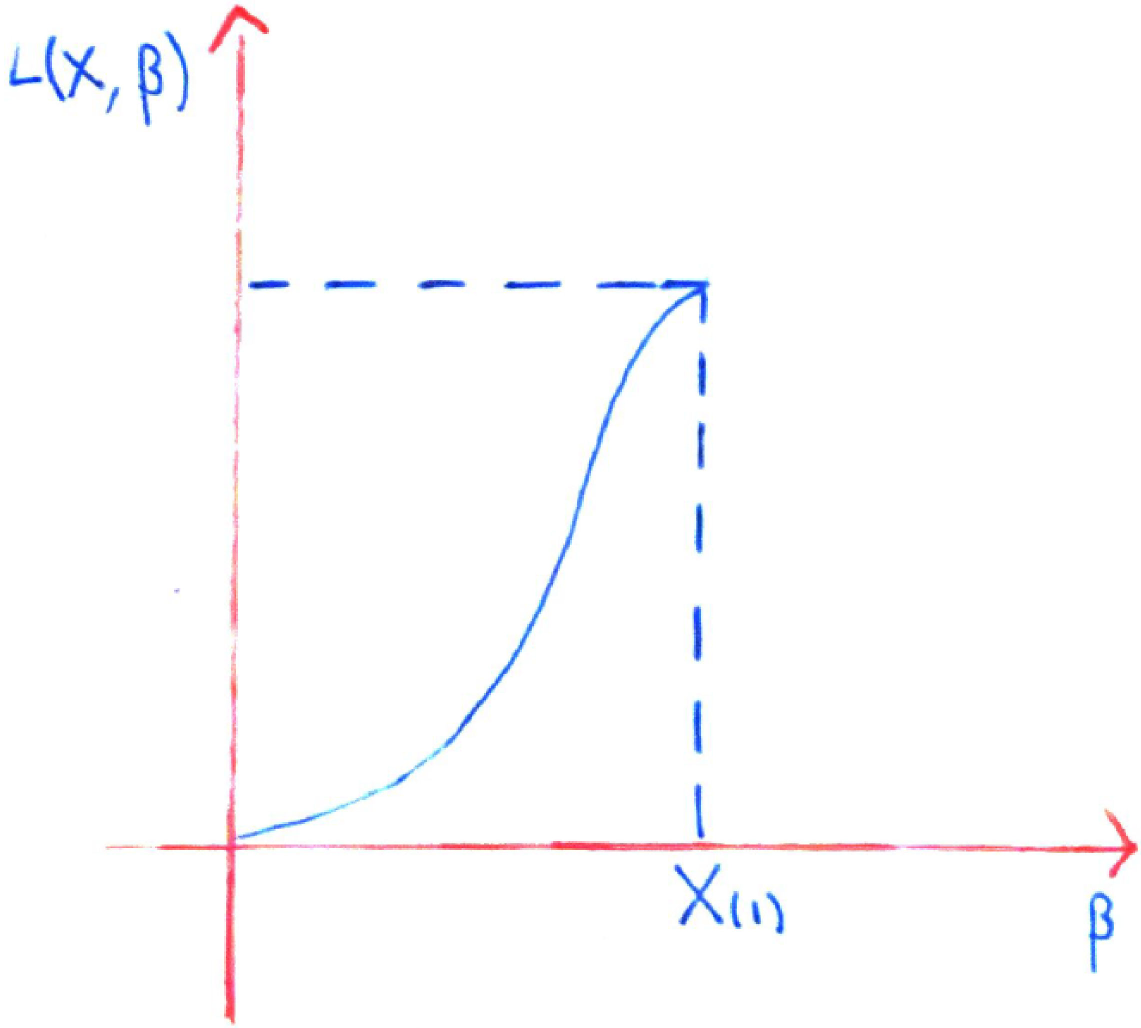
Setting this to zero:

$\frac{n}{\alpha} = \sum_{i=1}^{n} logx_i - nlog\beta$

$\hat{\alpha}_{MLE} = \frac{n}{\sum_{i=1}^{n} logx_i - nlog\beta}$

ii) Finding the MLE of β

$L(x,\beta) = \frac{(\alpha\beta^\alpha)^n}{\prod_{i=1}^{n} x_i^{\alpha+1}} I_{(0,X_{(1)})}(\beta)$



Most value is achieved when $\beta = X_{(1)}$
Therefore $\hat{\beta}_{MLE} = X_{(1)}$

iii) Finding α in terms of T

$T = log(\frac{\prod_{i=1}^{n} x_i}{X_{(1)}^n})$

Subbing $\hat{\beta}_{MLE}$ into $\hat{\alpha}_{MLE}$:

$$\hat{\alpha}_{MLE} = \frac{n}{\sum_{i=1}^{n} logx_i - nlog\hat{\beta}_{MLE}}$$

$$\hat{\alpha}_{MLE} = \frac{n}{log\prod_{i=1}^{n} x_i - logX_{(1)}^n}$$

$$\hat{\alpha}_{MLE} = \frac{n}{log\left(\frac{\prod_{i=1}^{n} x_i}{logX_{(1)}^n}\right)}$$

$$\hat{\alpha}_{MLE} = \frac{n}{T}$$

Part B

$H_0 : \alpha = 1, \beta > 0$ vs $H_1 : \alpha \neq 1, \beta > 0$

$\lambda(X) = \frac{L(X,\alpha,\beta)}{L(X,\hat{\alpha}_{MLE},\hat{\beta}_{MLE})} I_{(0,X_{(1)})}(\beta)$

$L(X, \alpha = 1, \beta) = \frac{\beta^n}{\prod_{i=1}^n x_i^2} I_{(0,X_{(1)})}(\beta)$

$L(X, \hat{\alpha}_{MLE}, \hat{\beta}_{MLE}) = \frac{(\hat{\alpha}_{MLE}\hat{\beta}_{MLE}^{\hat{\alpha}_{MLE}})^n}{\prod_{i=1}^n x_i^{\hat{\alpha}_{MLE}+1}} I_{(0,X_{(1)})}(\beta) = \frac{\left(\frac{n}{T} X_{(1)}^{\frac{n}{T}}\right)^n}{\prod_{i=1}^n x_i^{\frac{n}{T}+1}}$

$\lambda(X) = \frac{\beta^n}{\prod_{i=1}^n x_i^2} * \frac{\prod_{i=1}^n x_i^{\frac{n}{T}+1}}{\left(\frac{n}{T} X_{(1)}^{\frac{n}{T}}\right)^n}$

$= X_{(1)}^{n-\frac{n^2}{T}} \left(\frac{T}{n}\right)^n * \frac{\prod_{i=1}^n x_i^{\frac{n}{T}+1}}{\prod_{i=1}^n x_i^2}$

$= X_{(1)}^{n\left(1-\frac{n}{T}\right)} \left(\frac{T}{n}\right)^n * \prod_{i=1}^n \left(\frac{x_i^{\frac{n}{T}+1}}{x_i^2}\right)$

$= \left(\frac{T}{n}\right)^n * \frac{1}{\left(X_{(1)}^n\right)^{\frac{n}{T}+1}} * \prod_{i=1}^n x_i^{\frac{n}{T}-1}$

$= \left(\frac{T}{n}\right)^n * \left(\frac{\prod_{i=1}^n x_i}{X_{(1)}^n}\right)^{\frac{n}{T}-1}$

$= \left(\frac{T}{n}\right)^n exp[log\left(\frac{\prod_{i=1}^n x_i}{X_{(1)}^n}\right)^{\frac{n}{T}-1}]$

$= \left(\frac{T}{n}\right)^n exp[(\frac{n}{T} - 1) log(\frac{\prod_{i=1}^n X_i}{X_{(1)}^n})]$

$= \left(\frac{T}{n}\right)^n exp[\left(\frac{n}{T} - 1\right)T]$

$= \left(\frac{T}{n}\right)^n e^{n-T}$

Part C

Rejection region: $\{X \mid -2ln\lambda(X) \geq -2lnC = k\}$

We have $ln\lambda(X) = nln\left(\frac{T}{n}\right) + n - T$
Hence $-2ln\lambda(X) = -2(nln\left(\frac{T}{n}\right) + n - T) \geq k$

$$nln\left(\frac{T}{n}\right) = n - T \leq -\frac{k}{2}$$
$$ln\left(\frac{T}{n}\right) + 1 - \frac{T}{n} \leq -\frac{k}{2n}$$
$$ln\left(\frac{T}{n}\right) - \frac{T}{n} \leq -\frac{k}{2n} - 1$$

Let $X = \frac{T}{n}$, $C = -\frac{k}{2n} - 1$:

$$lnX - X \leq C$$
$$lnX \leq C + X$$

To Satisfy this then $\frac{T}{n} < K_1$ or $\frac{T}{n} > K_2$.

Let $k_1 = nK_1$ and $k_2 = nK_2$. Then:

$T > k_1$ or $T < k_2$ where $0 < k_1 < k_2$

Problem 5

$X_{(1)} < X_{(2)} < X_{(3)} < X_{(4)}$ are order statistics from a sample of size $n = 4$.

Given $f(x) = 2e^{-2x},\ x > 0$ we get $F(x) = 1 - e^{-2x}$.

Part A

Given $n = 4$ and $r = 3$, we calculate the density as follows:

$f_{X_{(3)}} = \frac{4!}{(3-1)!(4-3)!}(1 - e^{-2x})^{3-1}(1 - (1 - e^{-2x}))^{4-3}\ 2e^{-2x} = \frac{4!}{2!1!}(1 - e^{-2x})^2\ e^{-2x}\ 2e^{-2x} = 24(1 - e^{-2x})^2$

Hence $\mathbb{E}[X_{(3)}] = \int_0^\infty 24(1 - e^{-2x})^2\ e^{-4x}\ dx = 1$

Part B

$M = \frac{1}{2}(X_{(2)} + X_{(3)})$

First we calcualte the joint density with $n = 4, i = 2, j = 3$:

$f_{X_{(2)},X_{(3)}}(u, v) = \frac{4!}{(2-1)!(3-1-2)!(4-3)!}\ 2e^{-2u}\ 2e^{-2v}\ (1 - e^{-2u})^{2-1}\ (1 - e^{-2v} - 1 + e^{-2u})^{3-1-2}\ (1 - 1 + e^{-2v})^{4}$

$= 96e^{-2u-2v}\ (1 - e^{-2u})\ e^{-2v} = 96e^{-2u-4v}\ (1 - e^{-2u})$ for $-\infty < u < v < \infty$.

We have $M = \frac{1}{2}(X_{(2)} + X_{(3)})$, let $N = X_{(2)}$. Then $X_{(2)} = N$ and $X_{(3)} = 2M - N$.

Calculating absolute value of a Jacobian:

$$|J(m, n)| = \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} = |-2| = 2$$

Density transformation:

$f_{M,N}(m, n) = f_{X_{(2)},X_{(3)}}(x(m, n), y(m, n))\ \ |J(m, n)| = 2f_{X_{(2)},X_{(3)}}(n, 2m - n))$
$= 192e^{-2n-4(2m-n)}\ (1 - e^{-2n}) = 192e^{2n-8m}\ (1 - e^{-2n})$ for $0 < n < m < \infty$

Integrating out N where $0 < N < M < \infty$:

$f_M(m) = \int_0^m 192\ e^{2n-8m}\ (1 - e^{-2n})\ dn = 96e^{-8m}(e^{2m} - 2m - 1)$ for $m > 0$

Part C

$\mathbb{E}[X] = 0.5$, hence:

$\mathbb{P}(M > \mathbb{E}[X]) = \mathbb{P}(M > 0.5) = \int_{0.5}^\infty 96\ e^{-8m}(e^{2m} - 2m - 1)\ dm = 0.302071$