Problem 1

Part A

States of nature $\Theta = \{\theta_0, \theta_1\}$ where θ_0 denotes that there is no snow tomorrow, and θ_1 denotes that there is snow tomorrow.

Action space $\mathcal{A} = \{a_0, a_1\}$ where a_0 denotes the action of opening the school tomorrow, and a_1 denotes the action of closing the school tomorrow.

Appropriate loss function $L(\theta, a)$ for this problem:

$$\begin{array}{ll} L(\theta_0, a_0) = 0 & L(\theta_0, a_1) = 1 \\ L(\theta_1, a_0) = 2 & L(\theta_1, a_1) = 0 \end{array}$$

Part B

Expert $i = 1, 2$ forecasting snow:

$Y_i \sim^{iid} Y$ for $i = 1, 2$ with $S_Y = \{0, 1\}$
where 0 denotes a no-snow forecast, and 1 denotes a snow forecast.

$$\begin{array}{ll} P(Y = 0 \mid \theta_0) = 0.50 & P(Y = 1 \mid \theta_0) = 0.50 \\ P(Y = 0 \mid \theta_1) = 0.25 & P(Y = 1 \mid \theta_1) = 0.75 \end{array}$$

Number of experts forecasting snow:

$X = Y_1 + Y_2$ with $S_X = \{0, 1, 2\}$

Distribution of the number of experts forecasting snow conditioned on the states of nature:

$$p_{X|\Theta}(x \mid \theta) = \sum_{y_2 \in S_{Y_2}} p_{Y_1}(x - y_2 \mid \theta)\, p_{Y_2}(y_2 \mid \theta)$$

$$p_{X|\Theta}(x \mid \theta_0) = \begin{cases} P(Y_1 = 0 \mid \theta_0)\, P(Y_2 = 0 \mid \theta_0) + P(Y_1 = -1 \mid \theta_0)\, P(Y_2 = 1 \mid \theta_0) = 0.25 & x = 0 \\ P(Y_1 = 1 \mid \theta_0)\, P(Y_2 = 0 \mid \theta_0) + P(Y_1 = 0 \mid \theta_0)\, P(Y_2 = 1 \mid \theta_0) = 0.50 & x = 1 \\ P(Y_1 = 2 \mid \theta_0)\, P(Y_2 = 0 \mid \theta_0) + P(Y_1 = 1 \mid \theta_0)\, P(Y_2 = 1 \mid \theta_0) = 0.25 & x = 2 \\ 0 & \text{ow} \end{cases}$$

$$p_{X|\Theta}(x \mid \theta_1) = \begin{cases} P(Y_1 = 0 \mid \theta_1)\, P(Y_2 = 0 \mid \theta_1) + P(Y_1 = -1 \mid \theta_1)\, P(Y_2 = 1 \mid \theta_1) = 0.0625 & x = 0 \\ P(Y_1 = 1 \mid \theta_1)\, P(Y_2 = 0 \mid \theta_1) + P(Y_1 = 0 \mid \theta_1)\, P(Y_2 = 1 \mid \theta_1) = 0.3750 & x = 1 \\ P(Y_1 = 2 \mid \theta_1)\, P(Y_2 = 0 \mid \theta_1) + P(Y_1 = 1 \mid \theta_1)\, P(Y_2 = 1 \mid \theta_1) = 0.5625 & x = 2 \\ 0 & \text{ow} \end{cases}$$

Part C

Given decision rules:

x	$\mathbf{d_1}$	$\mathbf{d_2}$	$\mathbf{d_3}$	$\mathbf{d_4}$	$\mathbf{d_5}$	$\mathbf{d_6}$	$\mathbf{d_7}$	$\mathbf{d_8}$
0	a_0	a_1	a_0	a_1	a_0	a_1	a_0	a_1
1	a_0	a_0	a_1	a_1	a_0	a_0	a_1	a_1
2	a_0	a_0	a_0	a_0	a_1	a_1	a_1	a_1

We calculate corresponding risk points:

$$\begin{aligned} R(\theta_0, d_1) &= L(\theta_0, a_0)\ p_{X|\Theta}(0 \mid \theta_0) + L(\theta_0, a_0)\ p_{X|\Theta}(1 \mid \theta_0) + L(\theta_0, a_0)\ p_{X|\Theta}(2 \mid \theta_0) \\ &= 0 * 0.25 + 0 * 0.50 + 0 * 0.25 = 0.000 \end{aligned}$$

$$\begin{aligned} R(\theta_0, d_2) &= L(\theta_0, a_1)\ p_{X|\Theta}(0 \mid \theta_0) + L(\theta_0, a_0)\ p_{X|\Theta}(1 \mid \theta_0) + L(\theta_0, a_0)\ p_{X|\Theta}(2 \mid \theta_0) \\ &= 1 * 0.25 + 0 * 0.50 + 0 * 0.25 = 0.250 \end{aligned}$$

$$\begin{aligned} R(\theta_0, d_3) &= L(\theta_0, a_0)\ p_{X|\Theta}(0 \mid \theta_0) + L(\theta_0, a_1)\ p_{X|\Theta}(1 \mid \theta_0) + L(\theta_0, a_0)\ p_{X|\Theta}(2 \mid \theta_0) \\ &= 0 * 0.25 + 1 * 0.50 + 0 * 0.25 = 0.500 \end{aligned}$$

$$\begin{aligned} R(\theta_0, d_4) &= L(\theta_0, a_1)\ p_{X|\Theta}(0 \mid \theta_0) + L(\theta_0, a_1)\ p_{X|\Theta}(1 \mid \theta_0) + L(\theta_0, a_0)\ p_{X|\Theta}(2 \mid \theta_0) \\ &= 1 * 0.25 + 1 * 0.50 + 0 * 0.25 = 0.750 \end{aligned}$$

$$\begin{aligned} R(\theta_0, d_5) &= L(\theta_0, a_0)\ p_{X|\Theta}(0 \mid \theta_0) + L(\theta_0, a_0)\ p_{X|\Theta}(1 \mid \theta_0) + L(\theta_0, a_1)\ p_{X|\Theta}(2 \mid \theta_0) \\ &= 0 * 0.25 + 0 * 0.50 + 1 * 0.25 = 0.250 \end{aligned}$$

$$\begin{aligned} R(\theta_0, d_6) &= L(\theta_0, a_1)\ p_{X|\Theta}(0 \mid \theta_0) + L(\theta_0, a_0)\ p_{X|\Theta}(1 \mid \theta_0) + L(\theta_0, a_1)\ p_{X|\Theta}(2 \mid \theta_0) \\ &= 1 * 0.25 + 0 * 0.50 + 1 * 0.25 = 0.500 \end{aligned}$$

$$\begin{aligned} R(\theta_0, d_7) &= L(\theta_0, a_0)\ p_{X|\Theta}(0 \mid \theta_0) + L(\theta_0, a_1)\ p_{X|\Theta}(1 \mid \theta_0) + L(\theta_0, a_1)\ p_{X|\Theta}(2 \mid \theta_0) \\ &= 0 * 0.25 + 1 * 0.50 + 1 * 0.25 = 0.750 \end{aligned}$$

$$\begin{aligned} R(\theta_0, d_8) &= L(\theta_0, a_1)\ p_{X|\Theta}(0 \mid \theta_0) + L(\theta_0, a_1)\ p_{X|\Theta}(1 \mid \theta_0) + L(\theta_0, a_1)\ p_{X|\Theta}(2 \mid \theta_0) \\ &= 1 * 0.25 + 1 * 0.50 + 1 * 0.25 = 1.000 \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_1) &= L(\theta_1, a_0)\ p_{X|\Theta}(0 \mid \theta_1) + L(\theta_1, a_0)\ p_{X|\Theta}(1 \mid \theta_1) + L(\theta_1, a_0)\ p_{X|\Theta}(2 \mid \theta_1) \\ &= 2 * 0.0625 + 2 * 0.3750 + 2 * 0.5625 = 2.000 \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_2) &= L(\theta_1, a_1)\ p_{X|\Theta}(0 \mid \theta_1) + L(\theta_1, a_0)\ p_{X|\Theta}(1 \mid \theta_1) + L(\theta_1, a_0)\ p_{X|\Theta}(2 \mid \theta_1) \\ &= 0 * 0.0625 + 2 * 0.3750 + 2 * 0.5625 = 1.875 \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_3) &= L(\theta_1, a_0)\ p_{X|\Theta}(0 \mid \theta_1) + L(\theta_1, a_1)\ p_{X|\Theta}(1 \mid \theta_1) + L(\theta_1, a_0)\ p_{X|\Theta}(2 \mid \theta_1) \\ &= 2 * 0.0625 + 0 * 0.3750 + 2 * 0.5625 = 1.250 \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_4) &= L(\theta_1, a_1)\ p_{X|\Theta}(0 \mid \theta_1) + L(\theta_1, a_1)\ p_{X|\Theta}(1 \mid \theta_1) + L(\theta_1, a_0)\ p_{X|\Theta}(2 \mid \theta_1) \\ &= 0 * 0.0625 + 0 * 0.3750 + 2 * 0.5625 = 1.125 \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_5) &= L(\theta_1, a_0)\, p_{X|\Theta}(0 \mid \theta_1) + L(\theta_1, a_0)\, p_{X|\Theta}(1 \mid \theta_1) + L(\theta_1, a_1)\, p_{X|\Theta}(2 \mid \theta_1) \\ &= 2 * 0.0625 + 2 * 0.3750 + 0 * 0.5625 = 0.875 \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_6) &= L(\theta_1, a_1)\, p_{X|\Theta}(0 \mid \theta_1) + L(\theta_1, a_0)\, p_{X|\Theta}(1 \mid \theta_1) + L(\theta_1, a_1)\, p_{X|\Theta}(2 \mid \theta_1) \\ &= 0 * 0.0625 + 2 * 0.3750 + 0 * 0.5625 = 0.750 \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_7) &= L(\theta_1, a_0)\, p_{X|\Theta}(0 \mid \theta_1) + L(\theta_1, a_1)\, p_{X|\Theta}(1 \mid \theta_1) + L(\theta_1, a_1)\, p_{X|\Theta}(2 \mid \theta_1) \\ &= 2 * 0.0625 + 0 * 0.3750 + 0 * 0.5625 = 0.125 \end{aligned}$$

$$\begin{aligned} R(\theta_1, d_8) &= L(\theta_1, a_1)\, p_{X|\Theta}(0 \mid \theta_1) + L(\theta_1, a_1)\, p_{X|\Theta}(1 \mid \theta_1) + L(\theta_1, a_1)\, p_{X|\Theta}(2 \mid \theta_1) \\ &= 0 * 0.0625 + 0 * 0.3750 + 0 * 0.5625 = 0.000 \end{aligned}$$

	$\mathbf{d_1}$	$\mathbf{d_2}$	$\mathbf{d_3}$	$\mathbf{d_4}$	$\mathbf{d_5}$	$\mathbf{d_6}$	$\mathbf{d_7}$	$\mathbf{d_8}$
$\mathbf{R(\theta_0, d_i)}$	0.000	0.250	0.500	0.750	0.250	0.500	0.750	1.000
$\mathbf{R(\theta_1, d_i)}$	2.000	1.875	1.250	1.125	0.875	0.750	0.125	0.000

Part D

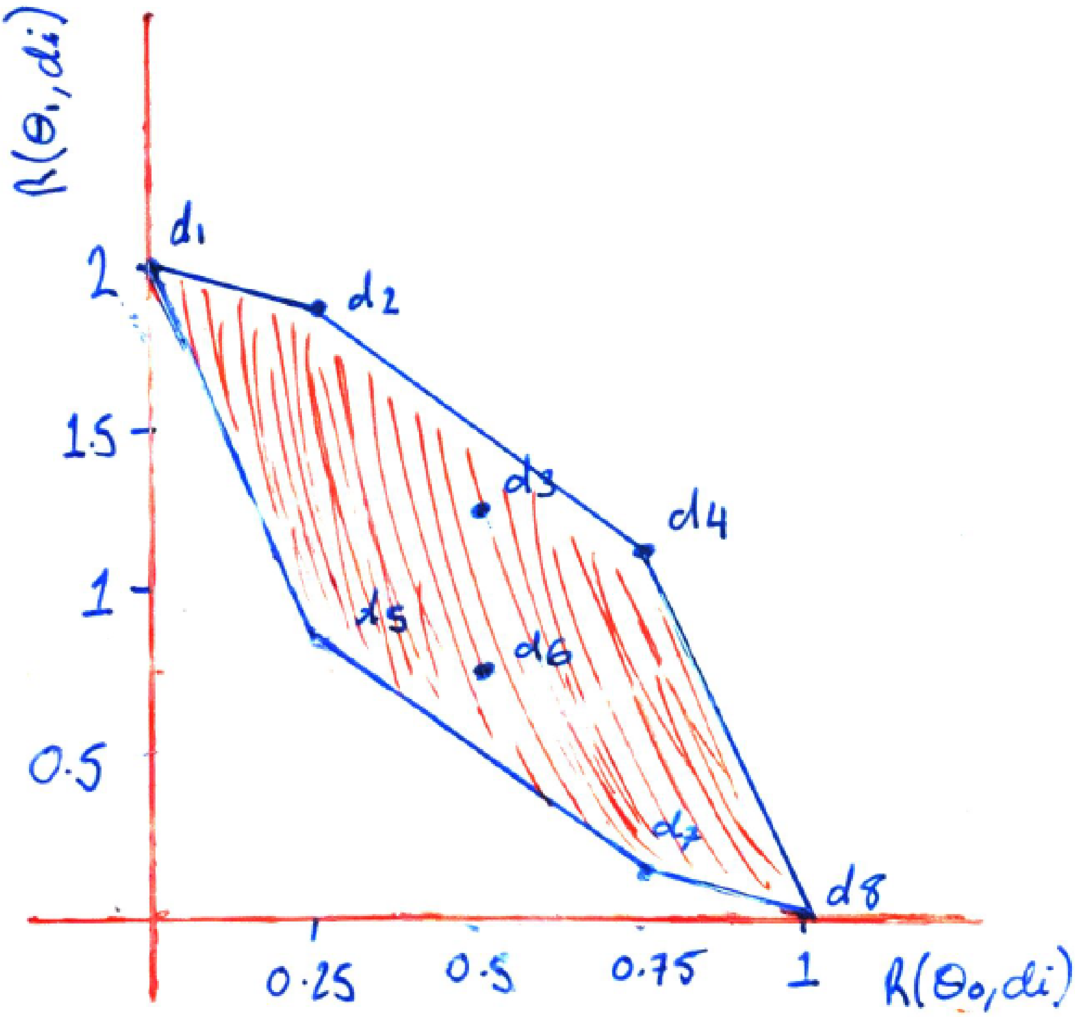
	$\mathbf{d_1}$	$\mathbf{d_2}$	$\mathbf{d_3}$	$\mathbf{d_4}$	$\mathbf{d_5}$	$\mathbf{d_6}$	$\mathbf{d_7}$	$\mathbf{d_8}$
$\mathbf{R(\theta_0, d_i)}$	0.000	0.250	0.500	0.750	0.250	0.500	0.750	1.000
$\mathbf{R(\theta_1, d_i)}$	2.000	1.875	1.250	1.125	0.875	0.750	0.125	0.000
$\sup_{\theta \in \boldsymbol{\Theta}} \mathbf{R(\theta, d_i)}$	2.000	1.875	1.250	1.125	0.875	0.750	0.750	1.000

Hence minimax rules among non-randomised rules in **D** are d_6 and d_7 with a risk value of:

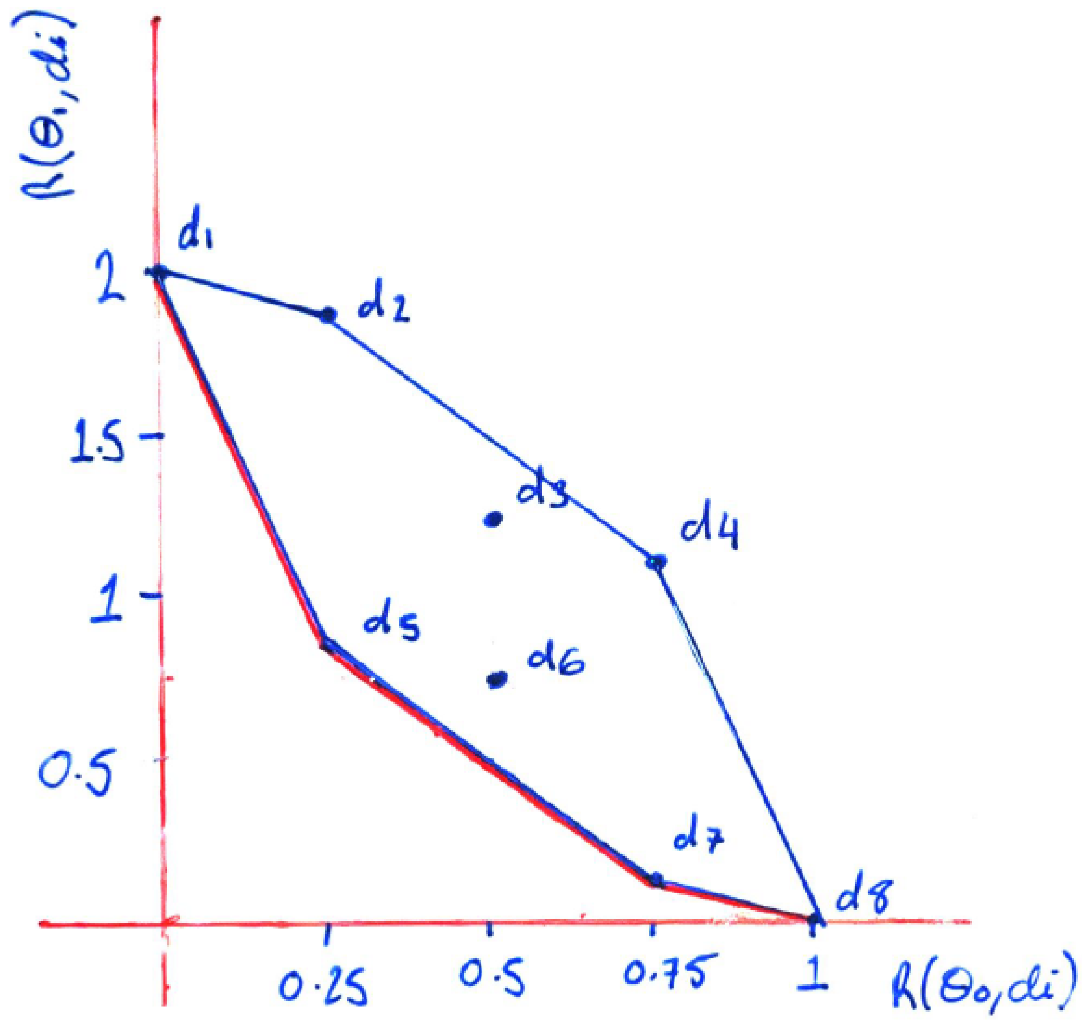
$$\inf_{d \in D} \sup_{\theta \in \Theta} R(\theta, d) = 0.75$$

Part E

Risk set of all randomized rules $\mathcal{D}$ generated by the set of rules in **D**:

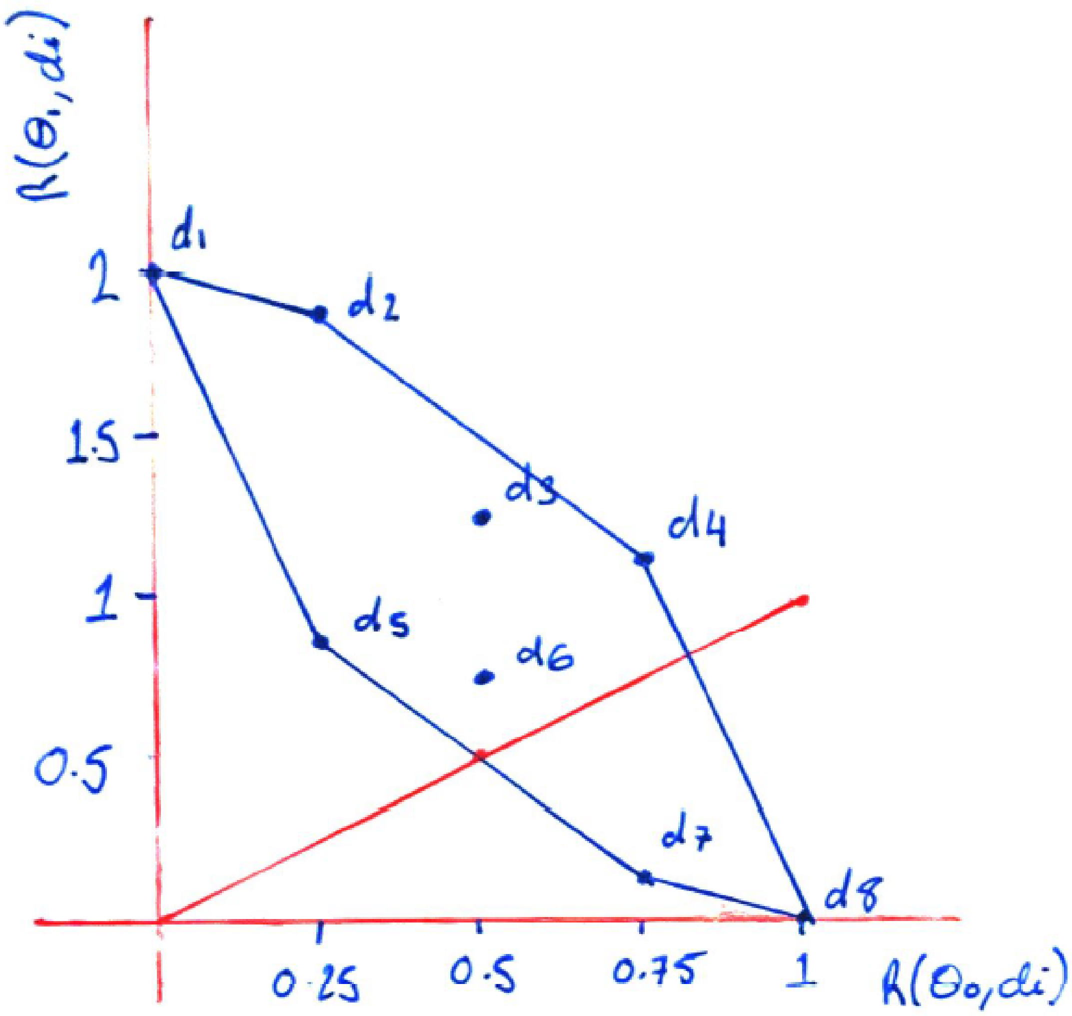


Part F



Admissible decision rules are decision rules that are not strcitkly dominated by any other rules. For a risk plot, these are risk points that have either lower x or y coordinate than all other risk points, i.e. lower left boundary of the convex hull, marked by a red line on our graph above.

Part G



To find the risk point of the minimax rule for randommized decision rules $\mathcal{D}$ we need to find the intersection of the lines $y = x$ and $\overline{d_5 d_7}$:

$$\begin{cases} y = x \\ \frac{y-0.875}{x-0.25} = \frac{0.125-0.875}{0.75-0.25} \end{cases} \qquad \begin{cases} y = x \\ y = -1.5x + 1.25 \end{cases} \qquad \begin{cases} x = 0.5 \\ y = 0.5 \end{cases}$$

Hence $(0.5, 0.5)$ is the risk point of the minimax decision rule δ^* in set $\mathcal{D}$ of randomized decision rules generated by **D**.

Thus minimax risk is equal to $\sup_{\theta\in\Theta} R(\theta, \delta^*) = \sup\{0.5, 0.5\} = 0.5$

Therefore minimax risk of 0.5 from the minimax decision rule in set $\mathcal{D}$ of randomized decision rules is lower than the correponding minimax risk of 0.75 from the minimax decision rule in set **D** of non-randomized decision rules.

Part H

We express the rule δ^* in the terms of d_5 and d_7 by finding a value $\alpha \in [0, 1]$ such that we choose d_5 with probability α, and d_7 with probability $(1 - \alpha)$.

$$R(\theta_0, d_5)\ \alpha + R(\theta_0, d_7)\ (1 - \alpha) = \sup_{\theta \in \Theta} R(\theta, \delta^*)$$
$$0.25\ \alpha + 0.75\ (1 - \alpha) = 0.5$$
$$\alpha = 0.5$$

The randomised decision rule δ^* is to choose d_5 with probabiltiy 0.5 and to choose d_7 with probabiltiy 0.5.

Part I

Let a prior be $(p, 1 - p)$. It should be perpencidular to $\overline{d_5 d_7}$.
Hence a slope of it's normal should be equal to the slope of $\overline{d_5 d_7}$:

$$\frac{-p}{1-p} = \frac{0.125-0.875}{0.75-0.25} = -1.5$$
$$p = 0.6$$
$$1 - p = 0.4$$

Thus the least favouriable prior with respect to δ^* is $\tau^* = (0.6, 0.4)$

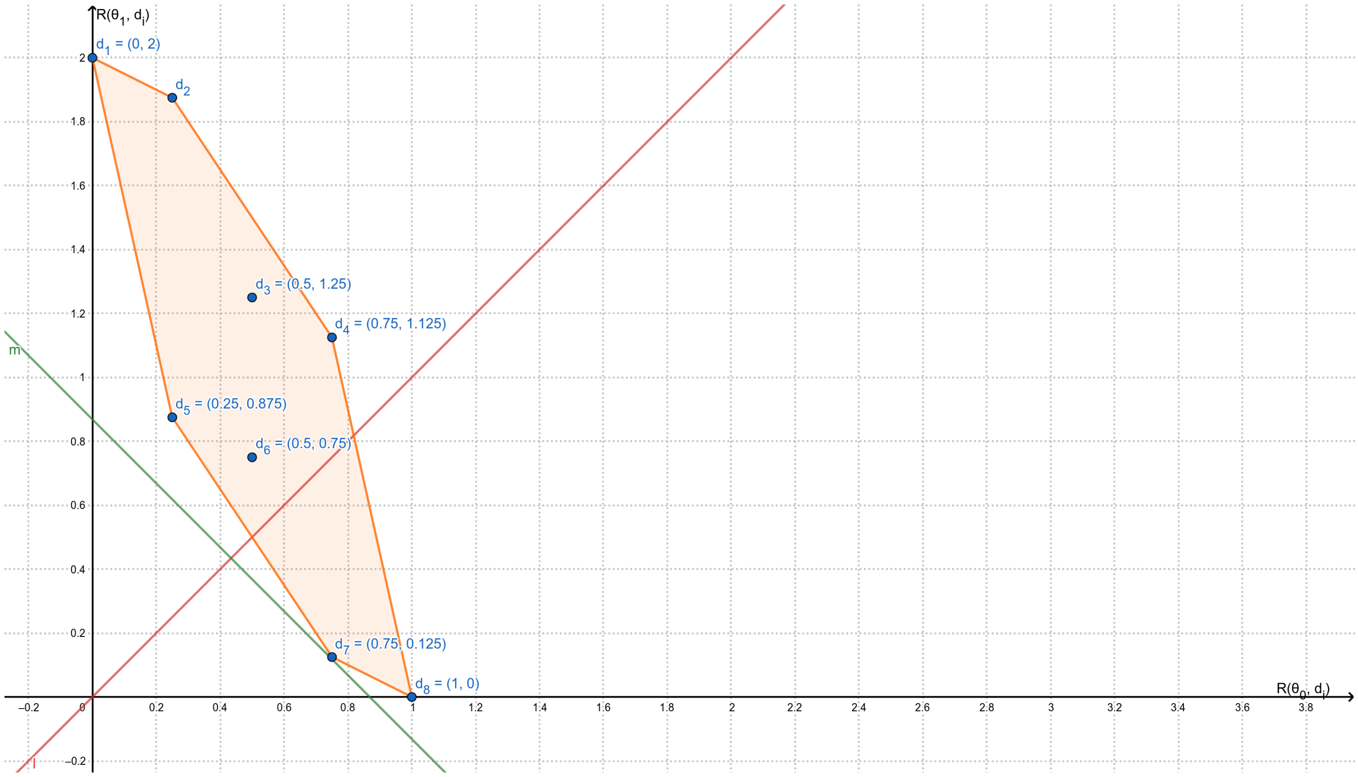
Part J

The principle believes that there is snow with a probabiltiy of 0.5, therefore he also believes that there is no snow with a probability of 0.5.

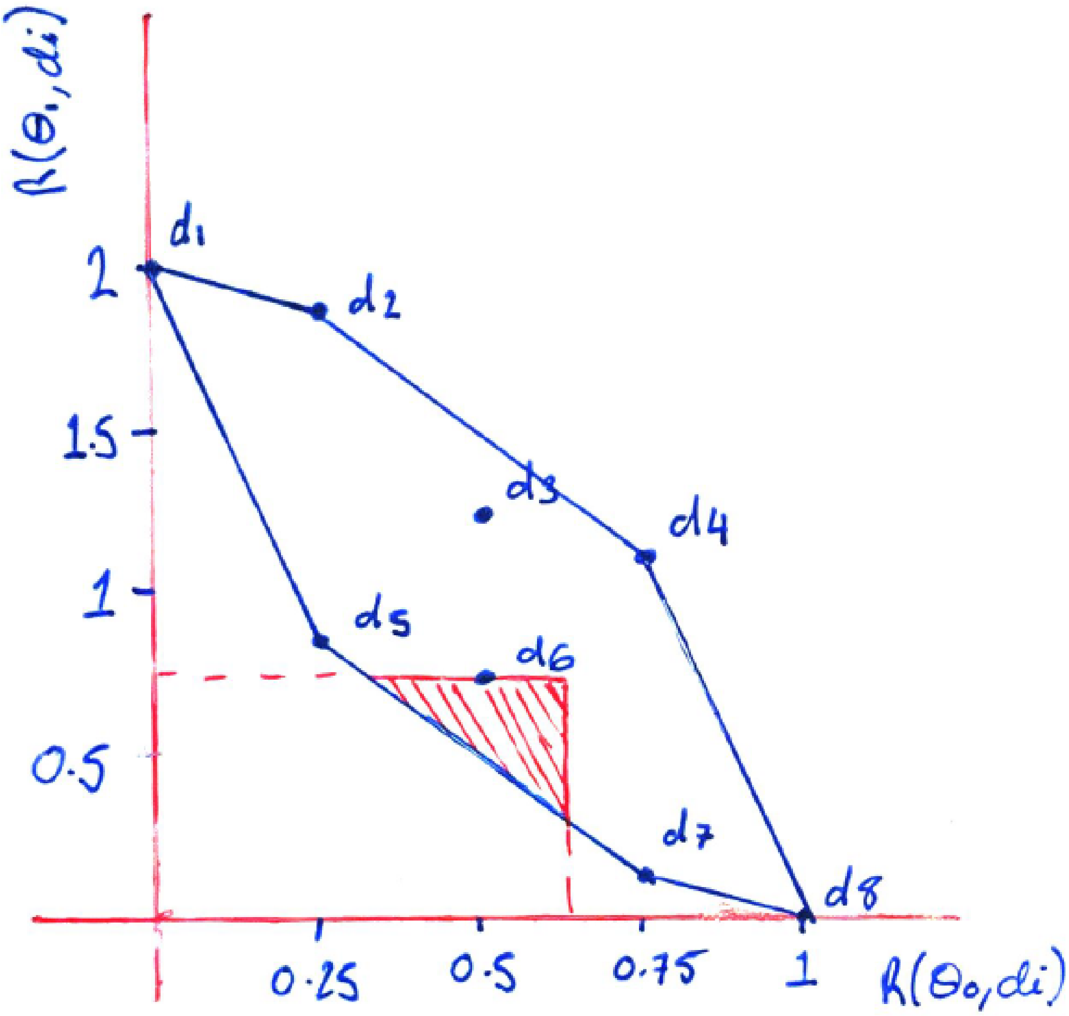
Aim is to determine the Bayes rule and risk of the prior $(0.5, 0.5)$ on $\{\theta_0, \theta_1\}$

Bayes rule chooses d_7 as the first south west point that lies on a normal to principle's prior.

Bayes risk wrt the prior = $(1-p)\ R(\theta_0, d_7) + p\ R(\theta_1, d_7) = 0.5 * 0.750 + 0.5 * 0.125 = 0.4375$



Part K



See the shaded area on the graph, these vertices are $(0.6, 0.35)$, $(0.43, 0.6)$, $(0.6, 0.6)$

Problem 2

$X \sim Bin(n, \theta)$
$\tau(\theta) = 1$ on $[0, 1]$, and 0 otherwise.

We are given loss function $L(\theta, d) = \frac{1}{\theta\ (1-\theta)}(\theta - d)^2$

Calculating posterior:

$$h(\theta \mid X) = \frac{f(X|\theta)\ \tau(\theta)}{g(X)} \propto \theta^x(1-\theta)^{n-x}$$

Calculating Bayes rule:

$$Q(X, d) = \int_\Theta L(\theta, d)\ h(\theta \mid X)\ d\theta$$

$$\propto \int_0^1 \frac{(\theta-d)^2}{\theta\ (1-\theta)}\theta^x(1-\theta)^{n-x}\ d\theta = \int_0^1 (\theta - d)^2\ \theta^{x-1}\ (1-\theta)^{n-x-1}\ d\theta$$

Differentiating wrt to d and setting it to 0:

$$\frac{\partial}{\partial d}\Big[\int_0^1 (\theta - d)^2\ \theta^{x-1}\ (1-\theta)^{n-x-1}\ d\theta\Big] = 0$$

$$\int_0^1 (\theta - d)\ \theta^{x-1}\ (1-\theta)^{n-x-1}\ d\theta = 0$$

$$d\int_0^1 \theta^{x-1}\ (1-\theta)^{n-x-1}\ d\theta = \int_0^1 \theta\ \theta^{x-1}\ (1-\theta)^{n-x-1}\ d\theta$$

$$d = \frac{\int_0^1 \theta^{x+1-1}\ (1-\theta)^{n-x-1}\ d\theta}{\int_0^1 \theta^{x-1}\ (1-\theta)^{n-x-1}\ d\theta}$$

Numerator is a Beta function with $\alpha = x + 1$ and $\beta = n - x$, where as denominator is a Beta function with $\alpha = x$ and $\beta = n - x$. Hence:

$$d = \frac{Beta(x+1, n-x)}{Beta(x, n-x)} = \frac{x}{x+n-x} = \frac{x}{n}$$

Hence the Bayes rule:

$$d_\tau(X) = \frac{X}{n}$$

If a Bayes estimator has constant risk, it is minimax. Hence we are calculating the risk:

$$R(\theta, d_\tau) = \mathbb{E}(L(\theta, d_\tau))$$

$$= \mathbb{E}\big(\frac{1}{\theta\ (1-\theta)}(\theta - d_\tau)^2\big)$$

$$= \frac{1}{\theta\ (1-\theta)}\mathbb{E}\big(\big(\theta - \frac{X}{n}\big)^2\big)$$

$$= \frac{1}{\theta\ (1-\theta)}\mathbb{E}\big(\big(\theta - \frac{X}{n}\big)^2\big)$$

$$= \frac{1}{\theta\ (1-\theta)}\big(\mathbb{V}\text{ar}(\frac{X}{n}) + \mathbb{B}\text{ias}^2(\frac{X}{n})\big)$$

$d_\tau = \frac{X}{n}$ is an unbiased estimator, hence:

$$R(\theta, d_\tau) = \frac{1}{\theta\ (1-\theta)} \mathbb{V}\mathrm{ar}(\frac{X}{n}) = \frac{\mathbb{V}\mathrm{ar}(X)}{n^2\ \theta\ (1-\theta)} = \frac{\theta\ (1-\theta)}{n^2\ \theta\ (1-\theta)} = \frac{1}{n}$$

Risk above is a contant. Hence d_τ is a minimax estimator.

Problem 3

$X = (X_1, \ldots, X_n)$
$X_i \sim^{iid} Exp(\theta)$ and $f(x_i;\theta) = \theta e^{-\theta x_i},\ x_i, \theta > 0$
$\theta \sim \Gamma(\alpha, \beta)$
$\tau(\theta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \theta^{\alpha-1} e^{-\theta/\beta},\ \alpha, \beta, \theta > 0$

Part A

Let $s = \sum_{i=1}^n x_i$

Calculating likelihood:

$$f_{X|\Theta}(X \mid \theta) = \prod_{i=1}^n f_{X_i|\Theta}(x_i \mid \theta) = \prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n \theta e^{-\theta x_i} = \theta^n e^{-\theta \sum_{i=1}^n x_i} = \theta^n e^{-\theta s}$$

Calculating posterior:

$$h_{\Theta|X}(\theta \mid X) = \frac{f_{X|\Theta}(X|\theta)\ \tau(\theta)}{\int_0^\infty f_{X|\Theta}(X|\theta)\ \tau(\theta)\ d\theta} = \frac{\theta^n e^{-\theta s}\ \frac{1}{\Gamma(\alpha)\beta^\alpha} \theta^{\,\alpha-1} e^{-\theta/\beta}}{\int_0^\infty \theta^n e^{-\theta s}\ \frac{1}{\Gamma(\alpha)\beta^\alpha} \theta^{\,\alpha-1} e^{-\theta/\beta}\ d\theta} \propto \theta^{\,\alpha+n-1}\, e^{-\theta\,(\frac{1}{\beta}+s)}$$

This can identified as a Gamma ditribution with parameters $\alpha + n$ and $\frac{1}{\beta} + s$.

Thus using a conjugate prior and normalizing appropriately we get:

$$h_{\Theta|X}(\theta \mid X) = \Gamma(\theta; \alpha + n, \tfrac{1}{\beta} + s) = \frac{(\frac{1}{\beta} + s)^{\alpha+n}}{\Gamma(\alpha+n)}\, \theta^{\,a+n-1}\, e^{-\theta(\frac{1}{\beta}+s)}$$

Part B

Bayes estimator wrt quadratic loss $L(a, \theta) = (a - \theta)^2$ is:

$$\begin{aligned} \delta_\tau(X) = \mathbb{E}[\theta \mid X] = \int_\Theta \theta\, h_{\Theta|X}(\theta \mid X)\, d\theta \\ &= \int_0^\infty \theta\, \frac{(\frac{1}{\beta} + s)^{\alpha+n}}{\Gamma(\alpha+n)}\, \theta^{\,a+n-1}\, e^{-\theta(\frac{1}{\beta}+s)}\, d\theta \\ &= \frac{(\frac{1}{\beta} + s)^{\alpha+n}}{\Gamma(\alpha+n)} \int_0^\infty \theta^{\,a+n}\, e^{-\theta(\frac{1}{\beta}+s)}\, d\theta \end{aligned}$$

Let $t = \theta\left(\frac{1}{\beta} + s\right)$, then $\theta = \frac{t}{\frac{1}{\beta}+s}$ and:

$$\begin{aligned} \delta_\tau(X) &= \frac{(\frac{1}{\beta} + s)^{\alpha+n}}{\Gamma(\alpha+n)} \int_0^\infty \left(\frac{t}{\frac{1}{\beta}+s}\right)^{\,a+n} e^{-t}\, \frac{dt}{\frac{1}{\beta}+s} \\ &= \frac{(\frac{1}{\beta} + s)^{\alpha+n}}{\Gamma(\alpha+n)(\frac{1}{\beta} + s)^{\alpha+n+1}} \int_0^\infty t^{\alpha+n+1-1}\, e^{-t}\, dt \\ &= \frac{1}{\Gamma(\alpha+n)(\frac{1}{\beta}+s)} \Gamma(\alpha+n+1) \end{aligned}$$

Finally $\delta_\tau(X) = \frac{\Gamma(\alpha+n)(a+n)}{\Gamma(\alpha+n)\left(\frac{1}{\beta}+s\right)} = \frac{a+n}{\frac{1}{\beta}+s}$

Part C

Nature states $\Theta = \Theta_0 \cup \Theta_1$ where $\Theta_0 = \{\theta : \theta \leq 2.5\}$ and $\Theta_1 = \{\theta : \theta > 2.5\}$
Hypothesis space $H_0 = \theta \in \Theta_0$ and $H_1 = \theta \in \Theta_1$
Action space $\mathcal{A} = \{a_0, a_1\}$ where a_0 denotes to accept H_0, and a_1 denotes to reject H_0

Losses are given as follows:

$$\begin{aligned} L(\theta, a_0) &= 0 \text{ if } \theta \in \Theta_0 \qquad & L(\theta, a_1) &= 1 \text{ if } \theta \in \Theta_0 \\ L(\theta, a_0) &= 2 \text{ if } \theta \in \Theta_1 \qquad & L(\theta, a_1) &= 0 \text{ if } \theta \in \Theta_1 \end{aligned}$$

Thus 1-0 loss constants are $c_1 = 1, c_2 = 2$

Sample $X = (0.12, 0.28, 0.43, 0.34, 0.47, 0.67, 0.82, 0.12, 0.30, 0.45)$
Hence $n = 10$ and $s = \sum_{i=1}^{10} x_i = 4.00$
Distribution parameters $\alpha = 2, \beta = 1$

Calculating conditional probability:

$$\begin{aligned} P(\theta \in \Theta_0 \mid X) &= \int_{\Theta_0} h(\theta \mid X)\, d\theta \\ &= \int_{\theta \in \Theta_0} \Gamma(\theta; \alpha + n, \tfrac{1}{\beta} + s)\, d\theta \\ &= \int_0^{2.5} \frac{5^{12}}{\Gamma(12)}\, \theta^{11}\, e^{-5\theta}\, d\theta = 0.594239 \end{aligned}$$

Comparing conditional probability with the loss ratio:

$$P(\theta \in \Theta_0 \mid X) = 0.594239 \quad < \quad \frac{c_2}{c_1 + c_2} = \frac{2}{3}$$

Hence choose action a_1 to reject H_0 in favor of H_1

Problem 4

Loss function

$$L(\theta, d) = \begin{cases} \alpha(\theta - d) & \text{if } d \leq \theta \\ \beta(d - \theta) & \text{if } d > \theta \end{cases}$$

Minimizing the following

$$Q(X,d) = \int_\theta L(\theta,d)h(\theta,X)\mathrm{d}\theta = \int_{-\infty}^{d} \beta(d-\theta)f(\theta)\mathrm{d}\theta + \int_{d}^{\infty} \alpha(\theta-d)f(\theta)\mathrm{d}\theta$$

Let the density of Θ be denoted by $f(\theta)$ and the cdf by $F(\theta)$

Taking the derivative with respect to d:

$$\begin{aligned} \frac{\partial}{\partial d}Q(X,d) &= \frac{\partial}{\partial d}\left[\beta \int_{-\infty}^{d}(d-\theta)f(\theta)\mathrm{d}\theta + \alpha\int_{d}^{\infty}(\theta-d)f(\theta)\mathrm{d}\theta\right] \\ &= \frac{\partial}{\partial d}\left[\beta\left[\int_{-\infty}^{d} df(\theta)\mathrm{d}\theta - \int_{-\infty}^{d}\theta f(\theta)\mathrm{d}\theta\right] + \alpha\left[\int_{d}^{\infty}\theta f(\theta)\mathrm{d}\theta - \int_{d}^{\infty} df(\theta)\mathrm{d}\theta\right]\right] \\ &= \frac{\partial}{\partial d}\left[\beta dF(d) - \beta\int_{-\infty}^{d}\theta f(\theta)\mathrm{d}\theta + \alpha\int_{d}^{\infty}\theta f(\theta)\mathrm{d}\theta - \alpha d(1-F(d))\right] \\ &= \beta\left[F(d) + df(d) - df(d) - 0\right] + \alpha\left[0 - df(d) - 1 + F(d) + df(d)\right] \\ &= \beta F(d) - \alpha + \alpha F(d) \end{aligned}$$

Then setting this equal to 0 we get:

$$F(d^*)(\alpha+\beta) - \alpha = 0$$

$$F(d^*) = \frac{\alpha}{\alpha+\beta}$$

The stationary point d^* gives rise to the minimum.

The second derivative delivers the minimum $\frac{\partial^2}{\partial d^2}Q(X,d^*) = (\alpha+\beta)h(d^*|X) > 0$

Since α, β are positive and the posterior evaluated at d^* is also positive.