## University of New South Wales School of Mathematics and Statistics

#### MATH5905 Statistical Inference Term One 2021

#### Assignment Two

Given: Wednesday 7 April 2021 Due date: Wednesday 21 March 2021

Instructions: This assignment is to be completed **collaboratively** by a group of **at most** 3 students. The same mark will be awarded to each student within the group, unless I have good reasons to believe that a group member did not contribute appropriately. This assignment must be submitted no later than 11:59 pm on Wednesday, 14 April 2021. The first page of the submitted PDF should be **this page**. Only one of the group members should submit the PDF file on Moodle, with the names of the other students in the group clearly indicated in the document.

I/We declare that this assessment item is my/our own work, except where acknowledged, and has not been submitted for academic credit elsewhere. I/We acknowledge that the assessor of this item may, for the purpose of assessing this item reproduce this assessment item and provide a copy to another member of the University; and/or communicate a copy of this assessment item to a plagiarism checking service (which may then retain a copy of the assessment item on its database for the purpose of future plagiarism checking). I/We certify that I/We have read and understood the University Rules in respect of Student Academic Misconduct.

Name	Student No.	Signature	Date
Yaroslav Akimov	z3152502		21/04/2021
Mohit Khanna	z5266543		21/04/2021
Emma Oliver	z5344057		21/04/2021

# **Assignment 2**

## **Problem 1**

$$f(x, heta) = \left\{ egin{array}{ll} rac{\sqrt{ heta}}{x\sqrt{2\pi}} e^{-rac{ heta}{2}log^2x} & ext{if } x>0 \ 0 & ext{otherwise} \end{array} 
ight., \qquad heta>0$$

### Part A

Let  $Y_i = log X_i = g(X_i)$  that is a one to one mapping.

Then 
$$X_i=g^{-1}(Y_i)=e^{Y_i}$$
 with  $rac{dX_i}{dY_i}=e^{Y_i}.$ 

Using transformation density formula:

$$egin{aligned} f_{Y_i}(y) &= f_{X_i}(g^{-1}(y)) \left| rac{dX_i}{dY_i} 
ight| = rac{\sqrt{ heta}}{e^y\sqrt{2\pi}} exp(-rac{ heta}{2}log^2 e^y) \left| e^y 
ight| \ &= rac{\sqrt{ heta}}{\sqrt{2\pi}} exp(rac{- heta}{2}y^2) = rac{1}{\sqrt{2\pi}\sqrt{ heta}^{-1}} exp(rac{-(y-0)^2}{2 heta^{-1}}) \end{aligned}$$

Therefore: 
$$Y_i \sim N(0, heta^{-1}), \quad \mathbb{E}[Y_i] = 0, \quad \mathbb{V}\mathrm{ar}(Y_i) = rac{1}{ heta}$$

Finally 
$$\mathbb{E}[log^2X_i]=\mathbb{E}[Y_i^2]=\mathbb{V}\mathrm{ar}(Y_i)+\mathbb{E}[Y_i]^2=rac{1}{ heta}+0=rac{1}{ heta}$$

### Part B

$$f(x, heta) = rac{\sqrt{ heta}}{x\sqrt{2\pi}} exp(-rac{ heta}{2}log^2x)$$

$$log f(x, heta) = rac{1}{2} log heta - log x - rac{1}{2} log 2\pi - rac{ heta}{2} log^2 x$$

$$rac{\partial}{\partial heta} log f(x, heta) = rac{1}{2 heta} - rac{1}{2} log^2 x$$

$$rac{\partial^2}{\partial heta^2} log f(x, heta) = -rac{1}{ heta^2}$$

For a single observation 
$$I_{X_1}( heta) = -\mathbb{E}[rac{\partial^2}{\partial heta^2}logf(x, heta)] = \mathbb{E}[rac{1}{ heta^2}] = rac{1}{ heta^2}$$

For a sample of 
$$n$$
 i.i.d. r.v.  $I_X( heta) = nI_{X_1} = rac{n}{ heta^2}$ 

#### Part C

Let 
$$h( heta) = rac{1}{ heta}$$

$$L(x, heta)=\prod_{i=1}^nrac{\sqrt{ heta}}{x\sqrt{2\pi}}exp(-rac{ heta}{2}log^2x)=rac{1}{\prod_{i=1}^nx_i}ig(rac{ heta}{2\pi}ig)^rac{ heta}{2}exp(-rac{ heta}{2}\sum_{i=1}^nlog^2x_i)$$

$$logL(x, heta) = -\sum_{i=1}^n logx_i + rac{n}{2}log heta - rac{n}{2}log2\pi - rac{ heta}{2}\sum_{i=1}^n log^2x_i$$

$$rac{\partial}{\partial heta} log L(x, heta) = V(x, heta) = rac{n}{2 heta} - rac{1}{2} \sum_{i=1}^n log^2 x_i$$

Setting the score to zero to find the MLE:

$$V(x, \theta) = \frac{n}{2\theta} - \frac{1}{2} \sum_{i=1}^{n} log^{2} x_{i} = 0$$

$$\frac{1}{\theta} = \frac{1}{n} \sum_{i=1}^{n} log^2 x_i$$

$$\hat{ heta}_{MLE} = rac{n}{\sum_{i=1}^{n} log^2 x_i}$$

As 
$$h( heta)=rac{1}{ heta}$$
 the MLE of  $h( heta)$  is  $\widehat{h( heta)}_{MLE}=rac{1}{\hat{ heta}_{MLE}}=rac{\sum_{i=1}^n log^2 x_i}{n}$ 

Now to prove that this is unbiased

$$\mathbb{E}[h(\hat{ heta}_{MLE})] = \mathbb{E}[rac{\sum_{i=1}^n log^2 x_i}{n}] = rac{1}{n} n \mathbb{E}[log^2 x_i] = \mathbb{E}[log^2 x_i]$$

From Part A we know that  $\mathbb{E}[log^2x_i]=rac{1}{ heta}$ 

Therefore  $\mathbb{E}[h(\hat{ heta}_{MLE})] = rac{1}{ heta}$  that is unbiased for  $h( heta) = rac{1}{ heta}$ 

#### Part D

Does the variance of the MLE for  $h(\theta)$  attain the Cramer Rao bound?

$$\hat{ heta}_{MLE} = rac{n}{\sum_{i=1}^n log^2 X_i}$$

Therefore 
$$h( heta) = rac{1}{\hat{ heta}_{\scriptscriptstyle MLE}} = rac{1}{n} \sum_{i=1}^{n} log^2 X_i$$

$$Var(\widehat{h( heta)}_{MLE}) = Var(rac{1}{n}\sum_{i=1}^{n}log^{2}X_{i}) = rac{1}{n^{2}}nVar(log^{2}X_{i}) = rac{1}{n}Var(log^{2}X_{i})$$

As 
$$Y_i = log X_i hicksim N(0, rac{1}{ heta})$$

We standardise rv and get  $\sqrt{ heta}Y_i hicksim N(0,1)$ 

Therefore: 
$$heta Y_i = \chi_1^2$$

So: 
$$Var(\chi_1^2) = Var(\theta Y_i^2) = heta^2 Var(Y_i^2) = 2$$

From this 
$$Var(Y_i^2)=2 heta^2$$

Therefore: 
$$Var(\widehat{h(\theta)}_{MLE}) = \frac{2\theta^2}{n}$$

As: 
$$rac{\partial}{\partial heta} log L(x, heta) = rac{n}{2 heta} - rac{1}{2} \sum_{i=1}^n log^2 x_i$$

Then: 
$$rac{\partial^2}{\partial heta^2} log L(x, heta) = -rac{n}{2 heta^2}$$

$$-\mathbb{E}[-rac{n}{2 heta^2}] = rac{n}{2 heta^2}$$

Therefore: 
$$I_{X(n)}^{-1}=rac{2 heta^2}{n}$$

The CRLB is attained by the variance of the MLE which makes it also the UMVUE.

#### Part E

From the Delta Method:

$$egin{aligned} \sqrt{n}(h(\hat{ heta}_{MLE})-h( heta)->N(0,[rac{\partial h}{\partial heta}( heta_0)]^2I^{-1}( heta_0)\ \sqrt{n}(\widehat{h( heta)}-h( heta))->N(0,rac{(h'( heta))^2}{I_{X_1}}) \end{aligned}$$

$$h(\theta) = \frac{1}{\theta}$$

$$egin{aligned} h( heta) &= rac{1}{ heta} \ h^{'}( heta) &= -rac{1}{ heta^2} \ h^{'}( heta)^2 &= rac{1}{ heta^4} \end{aligned}$$

$$h^{'}( heta)^2 = rac{1}{ heta^4}$$

$$CRLB = rac{1}{ heta^4} * rac{2 heta^2}{n} = rac{2}{ heta^2 n}$$

$$\sqrt{n}(\widehat{h( heta)}-h( heta))\sim N(0,rac{2}{ heta^2})$$

For 
$$au( heta) = e^{- heta}$$
:

$$au^{'}( heta)=-e^{- heta}$$

$$au^{'}( heta) = -e^{- heta} \ ( au^{'}( heta))^2 = e^{-2 heta}$$

$$\sqrt{n}(\widehat{ au( heta)}- au( heta))\sim N(0,2e^{-2 heta} heta^2)$$

$$f(x, heta) = \left\{ egin{array}{ll} rac{ au x^{ au-1}}{ heta^ au} & ext{if } 0 < x < heta \ 0 & ext{if otherwise} \end{array} 
ight.$$

### Part A

Using 
$$P(X_{(n)} \leq x) = P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n)$$
  
So  $P(X_{(n)} \leq x) = P(X_1 \leq x)^n$ 

Computing the CDF firstly to find the density function:

$$F(x) = \int_0^1 au heta^{- au} x^{ au-1} dx \ = au heta^{- au} \int_0^1 x^{ au-1} dx \ = au heta^{- au} \int_t^1 x^{ au-1} dx \ = rac{ au}{ heta^{- au}} [rac{x^ au}{x}]_{x=t}^{x= heta} \ = \left(rac{t}{ heta}
ight)^ au$$

From above  $P(X_{(n)} \leq x) = \left( rac{t}{ heta} 
ight)^{ au n}$ 

Differentiate this to find the density function  $\frac{\partial}{\partial \theta}(\frac{t}{\theta})^{\tau n}=\frac{\tau n t^{\tau n-1}}{\theta^{\tau n}}$ 

Hence

$$f_T(t) = egin{cases} rac{ au n t^{ au n - 1}}{ heta^{ au n}} & ext{if } 0 < x < heta \ 0 & ext{if otherwise} \end{cases}$$

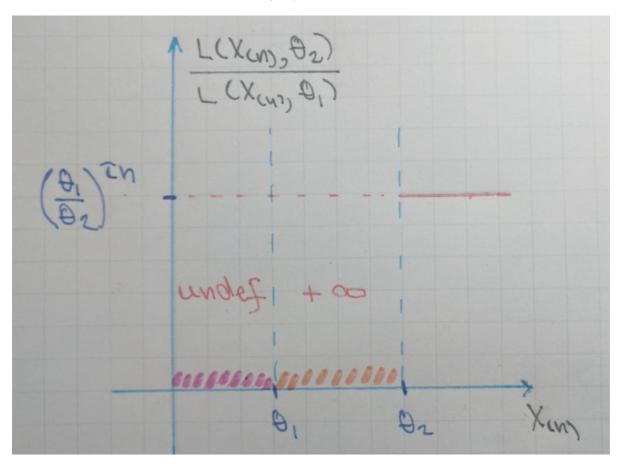
 $\{L(X, heta), heta > 0\}$  has a monotone likelihood ratio in the statistic  $T = X_{(n)}$ 

Let  $heta_1 < heta_2$ , therefore we need to look at the ratio  $rac{L(X, heta_2)}{L(X, heta_1)}$ 

$$L(x, heta) = \prod_{i=1}^n rac{ au x^{ au-1}}{ heta^ au} = rac{ au^n}{ heta^{ au n}} \prod_{i=1}^n X_i^{ au-1} I_{(X_{(n)},\infty)}( heta)$$

$$\frac{L(X,\theta_2)}{L(X,\theta_1)} = \frac{\frac{\tau^n}{\theta_2^{\tau n}} \prod_{i=1}^n X_i^{\tau-1} I_{(X_{(n)},\infty)}(\theta_2)}{\frac{\tau^n}{\theta_1^{\tau n}} \prod_{i=1}^n X_i^{\tau-1} I_{(X_{(n)},\infty)}(\theta_1)} = \left(\frac{\theta_1}{\theta_2}\right)^{\tau n} \frac{I_{(X_{(n)},\infty)}(\theta_2)}{I_{(X_{(n)},\infty)}(\theta_1)}$$

$$rac{L(X, heta_2)}{L(X, heta_1)} = egin{cases} ext{undefined} & ext{if } X_{(n)} < heta_1 \ \infty & ext{if } heta_1 < X_{(n)} < heta_2 \ \left(rac{ heta_1}{ heta_2}
ight)^{ au n} & ext{if } X_{(n)} > heta_2 \end{cases}$$



### Part C

Finding the Uniformly most powerful lpha-size test  $\phi^*$  for  $H_0: heta \leq au$  vs  $H_1: heta > au$ 

From part B this has a monotone likelihood ration in the statistic  $T=X_{(n)}$ . Therefore:

$$\phi^* = egin{cases} 1 & ext{if } X_{(n)} < k \ 0 & ext{if } X_{(n)} \geq k \end{cases}$$

To find k we must "exhaust the lpha-level":  $lpha = P(X_{(n)} < k | heta = au)$ 

From Part A:

$$lpha = (rac{k}{ au})^{ au n} \ (rac{k}{ au})^{ au n} = 1 - lpha \ rac{k}{ au} = lpha^{rac{1}{ au n}} \ k = au lpha^{rac{1}{ au n}}$$

Hence

$$\phi^*(X) = egin{cases} 1 & ext{if } X_{(n)} < au lpha^{rac{1}{ au n}} \ 0 & ext{if } X_{(n)} \geq au lpha^{rac{1}{ au n}} \end{cases}$$

### Part D

$$egin{aligned} Power( heta) &= \mathbb{E}_{ heta}[\phi^*(X)] \ &= P(X_{(n)} < au lpha^{rac{1}{ au n}}) \ &= \left(rac{ heta}{ au lpha^{rac{1}{ au n}}}
ight)^{ au n} \ &= \left(rac{ heta}{ au}
ight)^{ au n} rac{1}{lpha} \end{aligned}$$

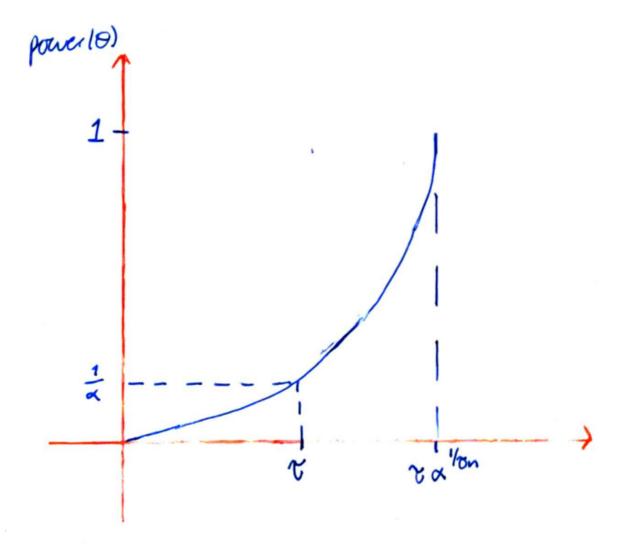
# Part E

 $Powerig( aulpha^{rac{1}{ au n}}ig)=lpharac{1}{lpha}=1$ 

 $Power(\tau) = (1)^{\tau n} \frac{1}{\alpha} = \frac{1}{\alpha}$ 

Power(0) = 0

 $lim_{ heta o \infty} \ Power( heta) = 1$  since power is bound by 1.



#### Part A

Let  $X=(X_1,\ldots,X_n)$  be a sample of n i.i.d. observations from this distribution.

We can show that family  $\{L(X,\theta)\}$  has a monotone likelihood ratio in the statistic  $T=X_{(1)}$ .

i)

The density can be directly computed from the formula noting that the distribution function for each  $X_i$  is  $f(x,\theta) = e^{-(x-\theta)}$  for  $\theta < x < \infty$ 

$$egin{align} F(X) &= \int_{ heta}^{x} f(x, heta) dx \ &= \int_{ heta}^{x} e^{-(x- heta)} dx \ &= [-e^{-(x- heta)}]_{ heta}^{x} \ &= 1 - e^{-(x- heta)} ext{ for } heta < x < \infty \ &= F_{X_{(n)}}(x) = n[1-F(x)]^{n-1} f(x) \ &= n[1-1+e^{-(x- heta)}]^{n-1} e^{-(x- heta)} \ &= ne^{-n(x- heta)} \end{aligned}$$

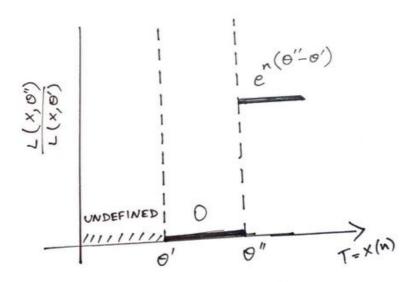
Therefore,

$$egin{aligned} F_{X_{(n)}}(x) &= \int_{ heta}^{x} n e^{-n(x- heta)} dx \ &= 1 - e^{-n(x- heta)} ext{ for } heta < x < \infty \end{aligned}$$

ii)

For 
$$\theta'' > \theta'$$
,

$$egin{aligned} L(X, heta) &= \prod_{i=1}^n e^{-(x_i- heta)} \mathrm{I}_{(-\infty,x_i)}( heta) \ &= e^{\sum X_i}.\,e^{n heta} \mathrm{I}_{(-\infty,X_i)}( heta) \ &rac{L(X, heta'')}{L(X, heta')} &= rac{e^{n heta''}}{e^{n heta'}}.\,rac{\mathrm{I}_{(-\infty,X)}( heta'')}{\mathrm{I}_{(-\infty,X)}( heta')} \ &= egin{cases} undefined & ext{if } X_1 < heta'' \ 0 & ext{if } heta' < X_1 < heta'' \ e^{n( heta''- heta')} & ext{if } X_1 > heta'' \end{cases} \end{aligned}$$



From the graph, we can see it is a non-decreasing function.

 $T=X_{\left( n
ight) }$  has the MLR property.

iii)

Using Blackwell and Girshick theoram the UMP lpha-test is given by

$$arphi^* = egin{cases} 1 & ext{if } X_n < \mathsf{k} \ 0 & ext{if } X_n \geq k \end{cases}$$

To find k we must exhaust the  $\alpha$ -test

$$lpha=E_{ heta_0}arphi^*=P_{ heta_0}(X_n< k)=1-P heta_0(X_n> k)$$
  $lpha=1-[P heta_0(X_1> k)]^n$   $lpha=1-e^{-n(k- heta_0)}$ 

Hence, Rearanging to solve for k

 $k= heta_0-rac{1}{n}ln(1-lpha)$  and  $arphi^*$  is completely determine.

$$arphi^* = egin{cases} 1 & ext{if } X_n < \mathrm{k} = heta_0 - rac{1}{n} ln(1-lpha) \ 0 & ext{if } X_n \geq k = heta_0 - rac{1}{n} ln(1-lpha) \end{cases}$$

iv)

Given,

$$X$$
 = (1, 2, 1.01, 3, 1:45),  $lpha=0.10$  and  $heta_0=1$ 

$$k = \text{1 -} \tfrac{1}{5}ln(1-0.10) = 1.02107$$

We observe that  $X_{\left(1\right)} < k$  and we should reject  $H_0$ 

Let,

$$egin{align} Z_n &= n(X_{(1)} - heta) \ F_{Z_n}(z) &= P(Z_n < Z) = P(nX_{(1)} - n heta < z) \ &= P(X_{(n)} < rac{z + n heta}{n}) = 1 - P(X_{(n)} > rac{z + n heta}{n}) \ &= 1 - P(X_{(1)} > rac{z + n heta}{n})]^n \ &= 1 - e^{-n}(rac{z + n heta}{n} - heta) \ &= 1 - e^{-z} \qquad z > 0 \ \end{array}$$

we see that,

$$\lim_{n o\infty}F_{Z_n}(z)=F_Z(z)=\left\{egin{array}{ll} 0 & ext{if } ext{z}<0 \ 1-e^{-z} & ext{if } ext{z}>0 \end{array}
ight.$$

Here,  $F_Z(z)$  is the distribution function of an exponential random variable with mean one. Therefore,  $Z_n$  converges in distribution to an exponential random variable with mean one as  $n\to\infty$ .

#### vi)

We consider evaluating  $P(|X_{(n)} - \theta| < \epsilon)$  directly by noting that X(n) cannot possibly be greater than  $\theta$ . Hence,

$$P(|X_{(n)} - \theta| < \epsilon) = P(X_{(n)} > \theta - \epsilon) = 1 - P(X_{(n)} \le \theta - \epsilon)$$

Now, the maximum  $X_{(n)}$  is less than some constant if and only if each of the random variables  $X_1, \ldots, X_n$  is less than that constant. Therefore, since the  $X_i$  are i.i.d.,

$$P(X_{(n)} \leq heta - \epsilon) = [P(X_1 \leq heta - \epsilon)]^n = \left\{egin{array}{ll} (1 - rac{\epsilon}{ heta})^n & ext{if } 0 \leq \epsilon < heta \ 0 & ext{if } \epsilon \geq heta \end{array}
ight.$$

since  $1-rac{\epsilon}{ heta}$  is strictly less than one, we conclude no matter what positive value  $\epsilon$  takes,

$$P(X_{(n)} \le \theta - \epsilon) o 0$$

as desired.

### Part B

i)

We first notice that the distribution function of X is  $F(x,\theta) = e^{-(x-\theta)}$  for  $\theta < x < \infty$  and the density of  $X_1$  is

$$f_{X_{(1)}}(x) = 1 - 5e^{-5(x- heta)} \qquad heta < x < \infty \qquad n = 5$$

When heta < 1

$$egin{aligned} \gamma( heta) &= P(X_{(n)} \geq 2 ext{ or } X_{(n)} < 1) \ &= 1 - P(X_{(n)} < 2) + P(X_{(n)} < 1) \ &= \int_{ heta}^{1} n e^{-n(t- heta)} dt + \int_{2}^{\infty} n e^{-n(t- heta)} dt \ &= -e^{-n(t- heta)}|_{ heta}^{1} + [-e^{-n(t- heta)}|_{2}^{\infty}] \ &= 1 - e^{-n(1- heta)} + e^{-n(2- heta)} \end{aligned}$$

We are given random sample of size five from this distribution.

Hence,

$$= 1 + e^{-5(2-\theta)} - e^{-5(1-\theta)}$$
$$= e^{5\theta}[e^{-10} - e^{-5}] + 1$$

When  $heta \geq 2$ 

$$\gamma(\theta) = 1$$

When  $1 \leq \theta < 2$ 

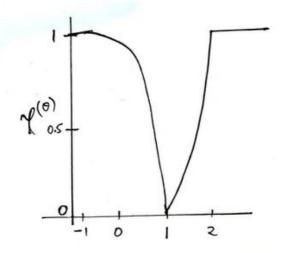
$$egin{split} \gamma( heta) &= P(X_{(n)} \geq 2) + P(X_{(n)} < 1) \ &= 1 - (1 - e^{-n(2 - heta)}) \end{split}$$

We are given random sample of size five from this distribution.

Hence,

$$=e^{-10+5\theta}$$

$$= egin{cases} e^{5 heta}[e^{-10}-e^{-5}]+1 & ext{if } heta < 1 \ e^{-10+5 heta} & ext{if } 1 \leq heta < 2 \ e^{n( heta''- heta')} & ext{if } heta \geq 2 \end{cases}$$



#### Part A

$$\begin{split} f(x;\alpha,\beta) &= \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}} I_{[\beta,\infty)}(x) \\ L(X;\alpha,\beta) &= \prod_{i=1}^{n} \frac{\alpha\beta^{\alpha}}{x_{i}^{\alpha+1}} I_{[\beta,\infty)}(x_{i}) = \alpha^{n}\beta^{n\alpha} \frac{1}{\prod_{i=1}^{n} x_{i}^{\alpha+1}} I_{(0,X_{(1)})}(\beta) \\ logL(X;\alpha,\beta) &= nlog\alpha + n\alpha log\beta + log1 - \sum_{i=1}^{n} logx_{i}^{\alpha+1} \\ logL(X;\alpha,\beta) &= nlog\alpha + n\alpha log\beta - \sum_{i=1}^{n} (\alpha+1) logx_{i} \\ logL(X;\alpha,\beta) &= nlog\alpha + n\alpha log\beta - \alpha \sum_{i=1}^{n} logx_{i} - \sum_{i=1}^{n} logx_{i} \end{split}$$

## i) Finding the MLE of $\boldsymbol{\alpha}$

$$rac{\partial}{\partial lpha} log L(X; lpha) = rac{n}{lpha} + n log eta - \sum_{i=1}^{n} log x_i$$

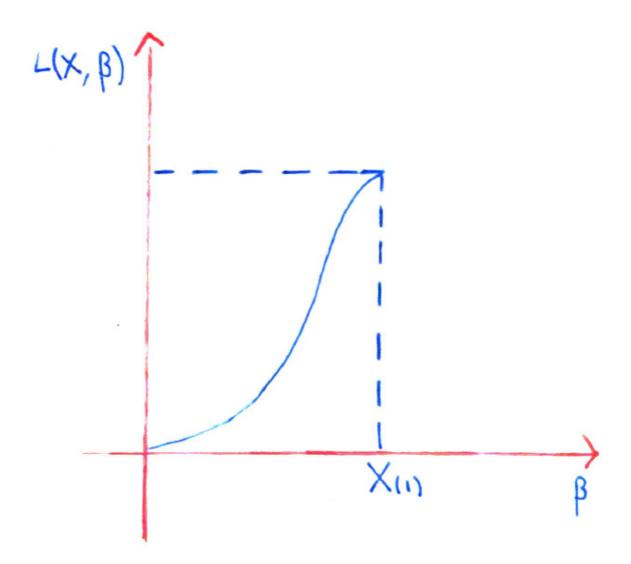
Setting this to zero:

$$rac{n}{lpha} = \sum_{i=1}^n log x_i - nlog eta$$

$$\hat{lpha}_{MLE} = rac{n}{\sum_{i=1}^{n} log x_i - nlog eta}$$

## ii) Finding the MLE of eta

$$L(x,eta)=rac{(lphaeta^lpha)^n}{\prod_{i=1}^nx_i^{lpha+1}}I_{(0,X_{(1)})}(eta)$$



Most value is achieved when  $\beta = X_{(1)}$  Therefore  $\hat{eta}_{MLE} = X_{(1)}$ 

# iii) Finding $\alpha$ in terms of T

$$T=log(rac{\prod_{i=1}^n x_i}{X_{(1)}^n})$$

Subbing 
$$\hat{eta}_{MLE}$$
 into  $\hat{lpha}_{MLE}$ :

Subbing 
$$\hat{eta}_{MLE}$$
 into  $\hat{lpha}_{MLE}$ :  $\hat{lpha}_{MLE} = rac{n}{\sum_{i=1}^{n}logx_{i}-nlog\hat{eta}_{MLE}}$   $\hat{lpha}_{MLE} = rac{n}{log\prod_{i=1}^{n}x_{i}-logX_{(1)}^{n}}$   $\hat{lpha}_{MLE} = rac{n}{log\left(rac{\prod_{i=1}^{n}x_{i}}{logX_{(1)}^{n}}
ight)}$ 

$$\hat{lpha}_{MLE} = rac{n}{log\prod_{i=1}^{n}x_i - logX_{(1)}^n}$$

$$\hat{lpha}_{MLE} = rac{n}{logig(rac{\prod_{i=1}^n x_i}{logX_{(1)}^n}ig)}$$

$$\hat{lpha}_{MLE} = rac{n}{T}$$

### Part B

$$H_0:lpha=1,eta>0$$
 vs  $H_1:lpha
eq 1,eta>0$   $\lambda(X)=rac{L(X,lpha,eta)}{L(X,\hat{lpha}_{MLF},\hat{eta}_{MLF})}I_{(0,X_{(1)})}(eta)$ 

$$L(X,lpha=1,eta)=rac{eta^n}{\prod_{i=1}^n x_i^2}I_{(0,X_{(1)})}(eta)$$

$$L(X,\hat{lpha}_{MLE},\hat{eta}_{MLE}) = rac{(\hat{lpha}_{MLE}\hat{eta}_{MLE}^{\hat{lpha}_{MLE}})^n}{\prod_{i=1}^n x_i^{\hat{lpha}_{MLE}+1}} I_{(0,X_{(1)})}(eta) = rac{\left(rac{n}{T}X_{(1)}^{rac{n}{T}}
ight)^n}{\prod_{i=1}^n x_i^{rac{n}{T}+1}}$$

$$\lambda(X) = rac{eta^n}{\prod_{i=1}^n x_i^2} * rac{\prod_{i=1}^n x_i^{rac{n}{T}+1}}{\left(rac{n}{T} X_{(1)}^{rac{n}{T}}
ight)^n}$$

$$=X_{(1)}^{n-rac{n^2}{T}}ig(rac{T}{n}ig)^n*rac{\prod_{i=1}^n x_i^{rac{T}{T}+1}}{\prod_{i=1}^n x_i^2}$$

$$=X_{(1)}^{n\left(1-rac{n}{T}
ight)}ig(rac{T}{n}ig)^nst\prod_{i=1}^nig(rac{x_i^rac{n}{T}+1}{x_i^2}ig)$$

$$=\left(rac{T}{n}
ight)^n*rac{1}{\left(X_{(1)}^n
ight)^{rac{n}{T}+1}}*\prod_{i=1}^nx_i^{rac{n}{T}-1}$$

$$=\left(rac{T}{n}
ight)^nst\left(rac{\prod_{i=1}^nx_i}{X_{(1)}^n}
ight)^{rac{n}{T}-1}$$

$$=ig(rac{T}{n}ig)^n exp[logig(rac{\prod_{i=1}^n x_i}{X_{(1)}^n}ig)^{rac{n}{T}-1}]$$

$$= (rac{T}{n})^n exp[(rac{n}{T}-1)log(rac{\prod_{i=1}^n X_i}{X_{(1)}^n})]$$

$$=\left(rac{T}{n}
ight)^n exp[\left(rac{n}{T}-1
ight)T]$$

$$= \left(\frac{T}{n}\right)^n e^{n-T}$$

## Part C

Rejection region: 
$$\{X \mid -2ln\lambda(X) \geq -2lnC = k\}$$

We have 
$$ln\lambda(X)=nlnig(rac{T}{n}ig)+n-T$$
 Hence  $-2ln\lambda(X)=-2(nlnig(rac{T}{n}ig)+n-T)\geq k$ 

$$nlnig(rac{T}{n}ig) = n - T \leq -rac{k}{2} \ lnig(rac{T}{n}ig) + 1 - rac{T}{n} \leq -rac{k}{2n} \ lnig(rac{T}{n}ig) - rac{T}{n} \leq -rac{k}{2n} - 1$$

Let 
$$X=rac{T}{n}$$
,  $C=-rac{k}{2n}-1$ :

$$lnX - X \le C \ lnX \le C + X$$

To Satisfy this then 
$$rac{T}{n} < K_1$$
 or  $rac{T}{n} > K_2$ .

Let 
$$k_1=nK_1$$
 and  $k_2=nK_2$ . Then:

$$T > k_1 ext{ or } T < k_2 ext{ \ \ }$$
 where  $0 < k_1 < k_2$ 

 $X_{(1)} < X_{(2)} < X_{(3)} < X_{(4)}$  are order statistics from a sample of size n=4.

Given  $f(x) = 2e^{-2x}, \ x > 0$  we get  $F(x) = 1 - e^{-2x}$ .

### Part A

Given n=4 and r=3, we calculate the density as follows:

$$f_{X_{(3)}} = rac{4!}{(3-1)!(4-3)!} (1-e^{-2x})^{3-1} (1-(1-e^{-2x}))^{4-3} \ 2e^{-2x} = rac{4!}{2!1!} (1-e^{-2x})^2 \ e^{-2x} \ 2e^{-2x} = 24(1-e^{-2x})^2$$

Hence 
$$\mathbb{E}[X_{(3)}] = \int_0^\infty 24(1-e^{-2x})^2 \; e^{-4x} \; dx = 1$$

#### Part B

$$M = \frac{1}{2}(X_{(2)} + X_{(3)})$$

First we calcualte the joint density with n=4, i=2, j=3:

$$f_{X_{(2)},X_{(3)}}(u,v)=rac{4!}{(2-1)!(3-1-2)!(4-3)!}\;2e^{-2u}\;2e^{-2v}\;(1-e^{-2u})^{2-1}\;(1-e^{-2v}-1+e^{-2u})^{3-1-2}\;(1-1+e^{-2v})^{4}$$

$$= 96e^{-2u-2v} \ (1-e^{-2u}) \ e^{-2v} = 96e^{-2u-4v} \ (1-e^{-2u}) \ {
m for} \ -\infty < u < v < \infty.$$

We have 
$$M=rac{1}{2}(X_{(2)}+X_{(3)})$$
, let  $N=X_{(2)}.$  Then  $X_{(2)}=N$  and  $X_{(3)}=2M-N.$ 

Calculating absolute value of a Jacobian:

$$|J(m,n)|=\left|egin{matrix} 0 & 1\ 2 & -1 \end{matrix}
ight|=|-2|=2$$

Density transformation:

$$egin{aligned} f_{M,N}(m,n) &= f_{X_{(2)},X_{(3)}}(x(m,n),y(m,n)) \ |J(m,n)| = 2f_{X_{(2)},X_{(3)}}(n,2m-n)) \ &= 192e^{-2n-4(2m-n)} \ (1-e^{-2n}) = 192e^{2n-8m} \ (1-e^{-2n}) \ ext{for } 0 < n < m < \infty \end{aligned}$$

Integrating out N where  $0 < N < M < \infty$ :

$$f_M(m) = \int_0^m 192 \; e^{2n-8m} \; (1-e^{-2n}) \; dn = 96 e^{-8m} (e^{2m} - 2m - 1) \; {
m for} \; m > 0$$

#### Part C

 $\mathbb{E}[X] = 0.5$ , hence:

$$\mathbb{P}(M>\mathbb{E}[X])=\mathbb{P}(M>0.5)=\int_{0.5}^{\infty}96\;e^{-8m}(e^{2m}-2m-1)\;dm=0.302071$$