Final Exam MATH5905

I declare that this test is my own work.

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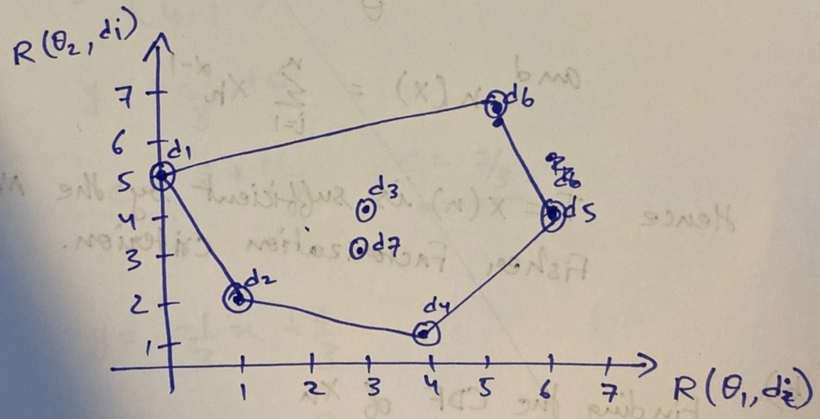
Q1.

a)

i	1	2	3	4	5	6	7
$R(\theta, d_i)$	0	1	3	4	6	5	3
$R(\theta, d_i)$	5	2	4	1	4	6	3
Max	5	2	4	4	6	6	3

Since d_2 is the minimum of maximum values, d_2 is the minimax rule with a minimax risk of 2.

b)



The risk set of the randomized decision rules D. Generated by the non-randomized decision rules D is convex risk set above.

(1)

c) The minimax rule is a set D of randomized decision rules is attained by finding the intersection of the $y=x$ with the most south-west part convex risk set.

$\overline{d_2 d_4}$ intersection by line $x=y$.

$(1,2)\ d_2$
$(4,1)\ d_4$

$$y = mx + c$$

$$y = \left(\frac{1-2}{4-1}\right) x + c$$

$$y = \frac{-1}{3} + c$$

$$c = 2 + 1/3 \quad ; \quad \underline{c = 7/3}$$

Now

$$y = \frac{-1}{3}x + \frac{7}{3}$$

$$x + \frac{1}{3}x = \frac{7}{3}$$

$$\therefore x = 7/4$$

$$\boxed{x = y = \frac{7}{4}}$$

$\therefore$ The risk point is $\left(\frac{7}{4}, \frac{7}{4}\right)$

2

The minimax risk for non-randomized decision rule D is 2. This is higher than 7/4 for randomized D.

d) δ^* is a randomized of rules d_2 & d_4.

Finding $\alpha \in [0,1]$ such that δ^* chooses d_2 with prob α & d_4 with $(1-\alpha)$.

$$\cancel{x_0\alpha + x_1(1-\alpha)}$$

$$R(\theta_0, d_2)\alpha + R(\theta_0, d_4)(1-\alpha) = \sup_{\theta \in \Theta} R(\theta, \delta^*)$$

$$\alpha + 4(1-\alpha) = 7/4$$

$$-3\alpha = \frac{7}{4} - 4$$

$$\alpha = 3/4$$

δ^* chooses d_2 with probability 3/4 & d_4 with probability 1/4.

e) The prior is $(p, 1-p)$

slop of line $\overline{d_2 d_4}$ is $-1/3$

$$\frac{-p}{1-p} = \frac{-1}{3} = 3p = 1-p$$

$$p = 1/4.$$

$\therefore$ The prior is $\left(\frac{1}{4}, \frac{3}{4}\right)$.

3

Q2.

a) $f(x,\theta) = \theta^{-1} e^{-x/\theta}$, $x > 0$

We can find it belongs to exponential family.

$a(\theta) = 1/\theta$ $b(x) = 1$, $c(\theta) = -1/\theta$, $d(x) = x$.

Therefore $T = \sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i$ is complete and sufficient statistics. for θ.

So $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is complete & sufficient for θ.

b) $E(X_i) = \theta$, $E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n} n\theta$

$= \theta.$

So $\bar{X}$ is unbiased estimator for θ.

UMVUE : $\hat{\tau}(T) = E_\theta(\bar{X} \mid x) = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} X_i$

c) ~~Likehoot~~ Likelihood function $L(x,\theta) = \theta^{-n} e^{-\frac{\sum_{i=1}^{n} X_i}{\theta}}$

So now $\log L(x,\theta) = -n\log\theta - \sum_{i=1}^{n} X_i/\theta$

Derivative, . $= \frac{-n}{\theta} + \sum_{i=1}^{n} X_i/\theta$

setting this equation equation to 0.

1

So, $\hat{\theta}_{mle} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$

and $\frac{\partial^2 \log L(x,\theta)}{\partial^2 \theta^2} \Big|_{\hat{\theta}_{mle}} < 0$

d) We have $\tau = 1/\theta > 0$, $\hat{\theta}_{mle} = \bar{x}$

So $\hat{\tau}_{mle} = 1/\hat{\theta}_{mle} = 1/\bar{x}$

2

Q3.

a) $f(x,\theta,\phi) = \frac{\phi}{\theta}\left(\frac{\theta}{x}\right)^{\phi+1}$

$x \geq \theta,\ \theta > 0,\ \phi > 0$

So, $L(x,\theta,\phi) = \frac{\phi^n}{\theta^n}\left(\frac{\theta^n}{\prod_{i=1}^{n} x_i}\right)^{\phi+1} I_{(-\infty,x)}\theta$

$= \left(\frac{\phi}{\theta}\right)^n \theta^{n\phi+n} \prod_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\phi+1} I_{(+\infty,x_{(1)})}\theta$

Hence $T = \begin{pmatrix} \prod_{i=1}^{n} x_i \\ X_{(1)} \end{pmatrix}$

b) $\frac{L(Y,\theta,\phi)}{L(X,\theta,\phi)} = \frac{\left(\frac{\phi}{\theta}\right)^n \theta^{n\phi+n} \prod_{i=1}^{n}\left(\frac{1}{Y_i}\right)^{\phi+1} I_{(-\infty,Y_{(1)})}\theta}{\left(\frac{\phi}{\theta}\right)^n \theta^{n\phi+n} \prod_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\phi+1} I_{(-\infty,x_{(1)})}\theta}$

$= \left(\frac{\prod x_i}{\prod Y_i}\right)^{\phi+1} \frac{I_{(-\infty,Y_{(1)})}\theta}{I_{(-\infty,x_{(1)})}\theta}$

$\therefore$ only if $\prod x_i = \prod Y_i$ $\quad Y_{(1)} = X_{(1)}$

$\frac{L(Y,\theta,\phi)}{L(X,\theta,\phi)}$ does not depend on θ and ϕ

Hence T is the part a) is also minimal sufficient for $f(\theta,\phi)$

c) $F_X(x) = \int_0^x \frac{\phi}{\theta}\left(\frac{\theta}{y}\right)^{\phi+1} dy = 1-\left(\frac{x}{\theta}\right)^{-\phi}$

$$F_{X_{(1)}}(x) = n\left[1 - F_X(x)\right]^{n-1} \cancel{f(x)(x)}\, f_X(x)$$

$$= n\left(1-\left(1-\frac{\theta}{x}\right)^{\phi}\right)\frac{\phi}{\theta}\left(\frac{\theta}{x}\right)^{\phi+1}$$

$$= \frac{n\phi}{\theta}\left(\frac{\theta}{x}\right)^{n\phi+1}$$

Hence $f_X(x) = \begin{cases} \frac{n\phi}{\theta}\left(\frac{\theta}{x}\right)^{n\phi+1} & x \in \theta \\ 0 & \text{otherwise} \end{cases}$

d) $L(x,\theta,\phi) = \left(\frac{\phi}{\theta}\right)^n \theta^{n\phi+n} \prod_{i=1}^{n} (x_i)^{-\phi-1} I_{(-\infty,\, x_{(1)})}\theta$

This is a increasing function.

Hence to maximize $L(x,\theta,\phi)$ we should take $x_{(1)} = \theta$

Hence $\hat{\theta}_{mle} = X_{(1)}$

e) $E(X_{(1)}) = \int_0^\infty x \cdot \frac{n\phi}{\theta}\left(\frac{\theta}{x}\right)^{n\phi+1} dx$

$$= \frac{n\phi}{\theta}\,\theta^{n\phi+1}\int_0^\infty x \cdot x^{-n\phi-1}\, dx$$

$$= \frac{n\phi\,\theta\,\phi}{n\phi-1}$$

2

f) $E_\theta\left(g(t)\right) = 0$

$$\Rightarrow \int_0^\infty g(\theta)\, \frac{n\phi}{\theta} \left(\frac{\theta}{x}\right)^{n\phi+1} dt = 0$$

$$g(\theta) = \frac{n\phi}{\theta} \left(\frac{\theta}{\theta}\right)^{n\phi+1} = 0$$

$$\frac{n\phi}{\theta} \neq 0$$

Hence $g(\theta) \equiv 0$.

$E_\theta\left(g(\theta)\right) = 1$.

Consider $W = \frac{n\phi \cdot 1}{n\phi}\left(X_{(1)}\right)$

$$E(W|T) = \frac{n\phi - 1}{n\phi} \left(X_{(1)}\right)$$

$\therefore$ UMVUE $= \frac{n\phi-1}{n\phi} X_{(1)}$

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Q4. $N(\theta, 8)$

a) $f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/\sigma^2}$

$E(X) = \mu$, $Var(X) = \sigma^2$

$f(x,\theta) = \frac{1}{\sqrt{16\pi}} \exp\left[\frac{1}{-16}(x-\theta)^2\right]$

$L(x,\theta) = \prod_{i=1}^{n}\left[\frac{1}{\sqrt{16\pi}} \exp\left[\frac{-1}{16}(x-\theta)^2\right]\right]$

$= \left(\frac{1}{\sqrt{16\pi}}\right)^n \exp\left[\frac{-1}{16}\sum_{i=1}^{n}(x_i-\theta)^2\right]$

b) $\left(\frac{1}{\sqrt{16\pi}}\right)\exp\left[\frac{-1}{16}(x^2 - 2x\theta + \theta^2)\right] = f(x,\theta)$

$= \frac{1}{\sqrt{16\pi}} \exp\left\{\frac{-x^2}{16} + \frac{2x\theta}{16} - \frac{\theta^2}{16}\right]$

$\therefore$ $a(x) = \frac{1}{\sqrt{16\pi}} \exp\left(\frac{-x^2}{16}\right)$

$b(\theta) = \cancel{\exp}\left\{\frac{-\theta^2}{16}\right\}$

$c(\theta) = \theta/8$

$d(x) = x$

Hence, it is sufficient and complete statistic is $\sum_{i=1}^{n} d(x_i)$, $T(x) = \sum_{i=1}^{n} X_i$ is min

1

c) θ_0, θ_1 and $0 < \theta_0 < \theta_1$

$$\frac{L(X,\theta_1)}{L(X,\theta_0)} = \frac{\exp\left\{\frac{-1}{16}\sum_{i=1}^{n}(x_i-\theta_1)^2\right\}}{\exp\left\{\frac{-1}{16}\sum_{i=1}^{n}(x_i-\theta_2)^2\right\}}$$

$$= \frac{\exp\left\{\frac{-1}{16}\left(\sum_{i=1}^{n}x_i^2 - 2\sum_{i=1}^{n}x_i\theta_1 + n\theta^2\right)\right\}}{\exp\left\{\frac{-1}{16}\left(\sum_{i=1}^{n}x_i^2 - 2\sum_{i=1}^{n}x_i\theta_0 + n\theta_0^2\right)\right\}}$$

$$= \exp\left\{\frac{1}{8}\theta_1\sum_{i=1}^{n}x_i - \frac{1}{8}\sum_{i=1}^{n}x_i\theta_0\right] \frac{e^{n/16\,\theta_0^2}}{e^{n/16\,\theta_1^2}}$$

$$= \frac{e^{\frac{n}{16}\theta_0^2}}{e^{\frac{n}{16}\theta_1^2}} e\left[\left(\frac{1}{8}\theta_1 - \frac{1}{8}\theta_0\right)\sum_{i=1}^{n}x_i\right]$$

$\therefore$ This family has MLR in $T(X) = \sum_{i=1}^{n} X_i$

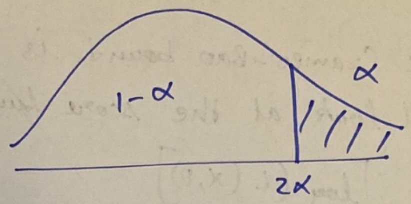
d) ~~H_0~~: We must exhaust the given level α which means

$$E_{\theta_0}(\delta^*) = 1.\ P(2X_i < C \mid \theta - 10) = \alpha \text{ must hold.}$$

$$\sum X_i = \frac{\sum X_i - n\mu}{\sigma\sqrt{n}}, \quad Z \sim N(0,1)$$

Hence $$E\theta_0(\delta^*) = P\left(\frac{\sum x_1 - n\theta}{\sqrt{8n}} < \frac{C - n\theta}{\sqrt{8n}}\right)$$

2



$$= P\left(Z < \frac{c - 10n}{\sqrt{8n}}\right)$$

$$= P(Z < Z_\alpha) = \alpha$$

where $Z \sim N(0,1)$

Hence $\frac{c - n(10)}{\sqrt{8n}} = Z_\alpha$ must hold and

$$c = Z_\alpha \sqrt{8n} + 10n$$

$$= 2\sqrt{2} Z_\alpha + 10n$$

Then, we get the rejection region S

$$S = \{ X_1 : \Sigma X_i < 2\sqrt{2n} Z_\alpha + 10n \}$$

Hence the UMP α-test of H_0 v H_1 is.

$$\delta^* = \begin{cases} 1 & \text{if } \Sigma X_1 < 2\sqrt{2n} Z_\alpha + 10n \\ 0 & \text{if } \Sigma X_i > 2\sqrt{2n} Z_\alpha + 10n \end{cases}$$

e) $\gamma(8) = 0.95$, $\gamma(10) = 0.05$

$\mu = \theta n$, Var $= 8n = \sigma^2$

$$= \sqrt{8n} = \underline{\underline{2\sqrt{2n}}}.$$

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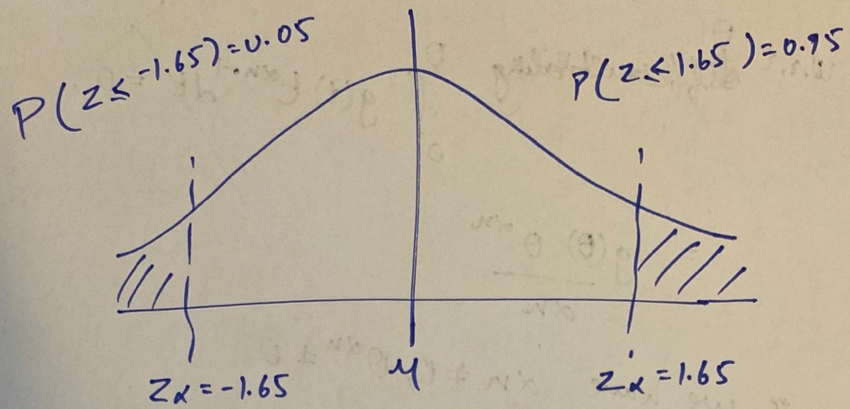

$$\gamma(\theta) = P\left(Z < \frac{c - \theta n}{2\sqrt{2n}}\right) \qquad \left[\text{We have } Z = \frac{X-\mu}{\sigma}\right]$$

$$\gamma(8) = 0.95$$
$$= P\left(Z < \frac{c-8n}{2\sqrt{2n}}\right)$$
$$c = (1.65)\, 2\sqrt{2n} + 8n$$

$$\gamma(10) = 0.05$$
$$= P\left(Z < \frac{c-10n}{2\sqrt{2n}}\right)$$
$$c = (-1.65)\, 2\sqrt{2n} + 10n$$

Now, $1.65(2\sqrt{2n}) + 8n = -1.65(2\sqrt{2n}) + 10n$

$$10.89 \cdot 2n = n^2$$
$$n = 21.78$$
$$\approx 22$$

Q5.

a) Taking the value of mgf from Table of Distribution for chi-squared (χ^2).

mgf: $$M_X(t) = \left(\frac{1}{1-2t}\right)^{P/2}$$

The cgf for a single observation is
$$K_X(t) = \log M_X(t) = \log\left(\frac{1}{1-2t}\right)^{P/2}$$

$$\therefore\ K_X(t) = \frac{-P}{2}\log(1-2t)$$

b) First derivative of cgf

$$K'_X(t) = -P/2\,\frac{\partial}{\partial t}\left(\log(1-2t)\right)$$
$$= \frac{+P}{1-2t}$$

$$K''_X(t) = P\cdot\frac{2}{(1-2t)^2}$$
$$= \frac{2P}{(1-2t)^2}$$

(1)

c) The saddlepoint equation $K_X'(\hat{t}) = \bar{X}$ gives the saddlepoint as

$$\frac{p}{1-2\hat{t}} = \bar{X}$$

$$\frac{p}{\bar{X}} = 1-2\hat{t}$$

$$2\hat{t} = 1-\frac{p}{\bar{X}}$$

$$\therefore \hat{t} = \frac{\bar{X}-p}{2\bar{X}}$$

d)

$$K_X(t) = \frac{-p}{2}\log(1-2\hat{t})$$

$$= \frac{-p}{2}\log\left(1-2\left(\frac{\bar{X}-p}{2\bar{X}}\right)\right)$$

$$= \frac{-p}{2}\log\left(1-\left(\frac{\bar{X}-p}{\bar{X}}\right)\right)$$

$$= \frac{-p}{2}\log\left(\frac{\bar{X}-\bar{X}+p}{\bar{X}}\right)$$

$$\therefore \quad K_X(t) = \frac{-p}{2}\log\left(\frac{p}{\bar{X}}\right)$$

3

$$K''_x(\hat{t}) = \frac{2p}{(1-2\hat{t})^2}$$

$$= \frac{2p}{\left(1 - 2\left[\frac{\bar{x} - 2p}{2\bar{x}}\right]\right)^2}$$

$$= \frac{2p}{\left(1 - \frac{\bar{x} - 2p}{\bar{x}}\right)^2} = \frac{2p}{\left(p/\bar{x}^2\right)^2}$$

$$\cancel{\frac{4p}{\left(\frac{2p}{\bar{x}}\right)^2} = \frac{4p}{4p^2} \times \bar{x}^2}$$

$$\boxed{K''_x(\hat{t}) = \frac{2\bar{x}^2}{p}}$$

$$\cancel{\therefore K''_x(\hat{t}) = \frac{\bar{x}}{p}}$$

e) First order saddlepoint approximation is

$$\hat{f}(x) = \sqrt{\frac{n}{2\pi K''_x(\hat{t})}} \exp\left\{n K_x(\hat{t}) - n \hat{t} \bar{x}\right\}$$

$$= \sqrt{\frac{n}{2\pi(\bar{x}/p)}} \exp\left\{n\left(-\frac{p}{2}\log(1-2\hat{t})\right) - n\left(\frac{\bar{x}-2p}{2\bar{x}}\right)\bar{x}\right\}$$

$$= \sqrt{\frac{n}{2\pi(\bar{x}/p)}} \exp\left\{\frac{-np}{2}\log(1-2\hat{t}) + \left(\frac{-n\bar{x}+2pn}{2}\right)\right\}$$

$$= \sqrt{\frac{np}{2\pi \bar{x}}}\, e^{-np/2} \cdot e^{\log(1-2\hat{t})}$$

3

e) First order saddlepoint approximation is

$$\hat{f}(x) = \sqrt{\frac{n}{2\pi K''_X(\hat{t})}} \exp\left\{ n K_X(\hat{t}) - n\hat{t}\bar{x} \right\}$$

$$= \sqrt{\frac{n}{2\pi\left(2\bar{x}^2/p\right)}} \exp\left\{ n\left(-\tfrac{p}{2}\log\left(\tfrac{2p}{\bar{x}}\right)\right) - n\left(\tfrac{\bar{x}-2p}{2\bar{x}}\right)\bar{x} \right\}$$

$$= \sqrt{\frac{np}{2\pi(2\bar{x}^2)}} \exp\left\{ \frac{-np}{2}\log\left(\frac{2p}{\bar{x}}\right) + \frac{2pn - n\bar{x}}{2} \right\}$$

$$= \sqrt{\frac{np}{4\pi(\bar{x})^2}} \exp\left\{ \frac{-np}{2}\log 2p + \frac{np}{2}\log(\bar{x}) + n\left[\frac{2p-\bar{x}}{2}\right] \right\}$$

$$\cancel{= \sqrt{\frac{np}{2\pi\bar{x}}}\, e^{np/2} \cdot e^{\log 2p}}$$

$$= \sqrt{\frac{np}{4\pi(\bar{x})^2}}\, e^{\frac{-np}{2}\log 2p} \cdot e^{\frac{np}{2}\log(\bar{x})} \cdot e^{\frac{2pn - n\bar{x}}{2}}$$

$$= \sqrt{\frac{np}{4\pi(\bar{x})^2}} \cdot p^{-np/2} \cdot e^{np/2}\, \bar{x}^{\frac{np}{2}-1}\, e^{-n\bar{x}/2} \cdot \bar{x}^{\cancel{1/2}}$$

$$= \sqrt{\frac{np}{4\pi}} \cdot p^{-np/2} \cdot e^{np/2}\, \bar{x}^{\frac{np}{2}-1}\, e^{-n\bar{x}/2}$$

f) $S = \sum_{i=1}^{n} X_i = n\bar{X} = \bar{X} = \frac{S}{n}$

~~Because~~ We have the first order saddlepoint

$$\hat{f}(x) \approx \sqrt{\frac{np}{4\pi}}\, p^{-np/2} \cdot e^{np/2} \cdot \bar{x}^{\frac{np}{2}-1} \cdot e^{-\frac{n\bar{x}}{2}}$$

(4)

$$\hat{f}(s) = \sqrt{\frac{np}{4\pi}} \cdot p^{\frac{-np}{2}} \cdot e^{\frac{-np}{2}} \cdot s^{\frac{np}{2}} \cdot e^{\frac{-ns}{2}} \cdot \left(\frac{1}{n}\right)$$

$$= \sqrt{\frac{np}{4\pi}} \cdot n^{-1} \cdot p^{\frac{-np}{2}} \cdot e^{\frac{-np}{2}} \cdot e^{-\frac{ns}{2}} s^{\frac{np}{2}}$$

g) We are given.

$$\Gamma(np/2) = \sqrt{\frac{4\cancel{p}\pi}{np}} \left(\frac{np}{2}\right)^{np/2} e^{-np/2}$$

$$\therefore \hat{f}(s) = \frac{1}{\Gamma(np/2)} \cdot s^{\frac{np}{2}} \cdot e^{\frac{-ns}{2}}$$

$$\hat{f}(s) \approx \text{Gamma}(np/2, s)$$

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