Mid Sem Exam MATH5905

1)

a) Using the property of the one-parameter exponential family, we observe that

$$
\begin{aligned}
f(x,\theta) &= \theta(1-\theta)^x \\
&= \exp\left(\log(\theta(1-\theta)^x\right) \\
&= \exp\left(\log\theta + \log(1-\theta)^x\right) \\
&= \exp\left(\log\theta + x\log(1-\theta)^x\right) \\
&= \theta + e^{x\log(1-\theta)}
\end{aligned}
$$

and we know

$a(\theta) = \theta$, $b(x) = 1$, $c(\theta) = \log(1-\theta)$, $d(x) = x$

Thus, the Geometric distribution belongs to the ~~exponential~~ one-parameter exponential family.

$\therefore$ this implies that

$$T(X) = \sum_{i=1}^{n} X_i$$

This is minimal & sufficient and complete for θ.

b)

i) T is sufficient and compelete for θ.

ii) Let $W = I_{\{X_1 = 1, \dots X_k = 1\}}(x)$ and then using the fact that $P(X_i = 0) = \theta$

we see that

$$E(W) = P(X_1 = 0, \dots X_k = 0) = P(X_1 = 0)$$

$$= \theta,$$

which is unbiased for $h(\theta)$. Now, we apply the Theoram of Lehmann - Scheffe and obtain the following

$$\hat{\tau}(T) = E(W \mid T = t) = E\left(I_{\{X_1 = 0 \dots X_k = 0\}} \,\Big|\, \sum_{i=1}^{n} X_i = t\right)$$

$$= P\left(X_1 = 0, \dots X_k = 0 \,\Big|\, \sum_{i=1}^{n} X_i = t\right)$$

$$= \frac{P(X_1 = 0, \dots X_k = 0 \mid \Sigma_{i=1}^{n} X_i = t)}{P(\Sigma_{i=1}^{n} X_i = t)}$$

Now we observe that the events in the numerator must be satisfied simultaneously to have a non-zero probability and hence this reduces to.

$$\hat{\tau}(T) = \frac{P\left(X_1 = 0, \dots, X_k = 0, \sum_{i=k+1}^{n} X_i = t\right)}{P\left(\sum_{i=1}^{n} X_i = t\right)}$$

$$= \frac{\theta \, P\left(\sum_{i=k+1}^{n} X_i = t\right)}{P\left(\sum_{i=1}^{n} X_i = t\right)}$$

$$= \frac{\cancel{\theta} \binom{n+t-2}{t} \theta^{\cancel{n-1}} (1-\cancel{\theta})^t}{\binom{n+t-1}{t} \cancel{\theta^n} (1-\cancel{\theta})^t}$$

$$= \frac{\binom{n+t-2}{t}}{\binom{n+t-1}{t}} = \frac{\dfrac{(n+t-2)!}{\cancel{t!}\,(n+t-2-t)!}}{\dfrac{(n+t-1)!}{(n+t-1-t)!\,\cancel{t!}}}$$

$$= \frac{\dfrac{(n+t-2)!}{(n-2)!}}{\dfrac{(n+t-1)!}{(n-1)(n-2)!}} = \frac{n-1}{n+t-1}$$

$$= \frac{n-1}{n+n\bar{X}-1}$$

So, UMVUE of $h(\theta) = \theta$ is $\underline{\underline{\dfrac{n-1}{n+n\bar{X}-1}}}$.

c) First we calculate the fisher information as follows. We have that

$$\log f(x) = \log\left[\theta(1-\theta)^x\right]$$
$$= \log\theta + x\log(1-\theta)$$

$$\frac{\partial}{\partial\theta}\log f(x) = \frac{1}{\theta} + \frac{x}{1-\theta}(-1)$$
$$= \cancel{\frac{1}{\theta} +} \frac{1}{\theta} - \frac{x}{1-\theta}$$

$$\frac{\partial^2}{\partial\theta^2}\log f(x) = -\frac{1}{\theta^2} + \frac{x(-1)}{(1-\theta)^2}$$
$$= -\frac{1}{\theta^2} - \frac{x}{(1-\theta)^2}$$

and so the fisher information in a single sample is

$$I_{X_1}(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\log(x)\right]$$
$$= -E\left[-\frac{1}{\theta^2} - \frac{x}{(1-\theta)^2}\right]$$
$$= \frac{1}{\theta^2} + \frac{E(x)}{(1-\theta)^2}$$
$$= \frac{1}{\theta^2} + \frac{\frac{1-\theta}{\theta}}{(1-\theta)^2}$$
$$= \frac{1}{\theta^2} + \frac{1-\theta}{\theta(1-\theta)^2}$$

$$= \frac{1}{\theta^2} + \frac{1}{\theta(1-\theta)}$$

$$= \frac{\cancel{\theta}(1-\theta) + \theta}{\theta^2(1-\theta)}$$

$$= \frac{1-\theta+\theta}{\theta^2(1-\theta)} = \frac{1}{\theta^2(1-\theta)}.$$

Hence the Fisher Information for the whole sample

$$I_X = n I_{X_1}(\theta) = \frac{n}{\theta^2(1-\theta)}$$

Then notice that $\frac{\partial}{\partial\theta} h(\theta) = 1$

Hence the CRLB

$$\frac{\left(\frac{\partial}{\partial\theta} h(\theta)\right)^2}{I_X(\theta)} = \frac{1^2}{\frac{n}{\theta^2(1-\theta)}}$$

$$= \frac{\theta^2(1-\theta)}{n}.$$

d) To show that the Cramer-Rao bound is attainable we will look at the score function.

$$V(X,\theta) = \frac{\partial}{\partial\theta}\left[\log\left(L(x,\theta)\right)\right]$$

$$= \frac{\partial}{\partial\theta}\left[\log\left(\prod_{i=1}^{n}\left(\theta(1-\theta)^{x}\right)\right)\right]$$

$$= \frac{\partial}{\partial\theta}\left[\log \theta^{n}(1-\theta)^{\sum_{i=1}^{n} x}\right]$$

$$= \frac{\partial}{\partial\theta}\left[n\log\theta + n\bar{X}\log(1-\theta)\right]$$

$$= \frac{n}{\theta} + \frac{n\bar{X}}{1-\theta}$$

$$\cancel{= \quad \theta}$$

$$= \frac{-n}{\theta^2}\left(\frac{\bar{X}\theta^2}{1-\theta} - \theta\right)$$

$\frac{\bar{x}\,\theta^2}{1-\theta}$ is not a statistic and it depends on θ. Then UMVUE of $h(\theta)$ does not attain CRLB

~~e) The MLE for θ is simply $\hat{\theta} = \bar{X}$ and~~

~~Hence the~~

$$\cancel{\widehat{h(\theta)}_{mle} \neq h\left(\hat{\theta}_{mle}\right) = \bar{X}}$$

e) MLE of $h(\theta) = \theta$ is

$$L(x,\theta) = \theta^n (1-\theta)^{\sum_{i=1}^{n} x_i}$$

$$\log L(x,\theta) = n \log \theta + \sum_{i=1}^{n} x_i \log(1-\theta)$$

$$\frac{\partial}{\partial \theta} \log L(x,\theta) = \frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{1-\theta}$$

Putting it as equal to 0

$$\hat{\theta} = \frac{1}{\bar{x}+1}$$

Therefore MLE is $\underline{\underline{\dfrac{1}{\bar{x}+1}}}$.

f) The UMVUE is

$$\widehat{h(\theta)}_{UMVUE} = \frac{6-1}{6+6\times 3-1} = \frac{5}{23}.$$

~~The MLE is~~

$$\cancel{\widehat{h(\theta)}_{mle} = \frac{10}{5} = 2 \left[\because \bar{X} = \frac{T}{n}\right]}$$

The MLE is

$$\widehat{h(\theta)}_{mle} = \frac{1}{3+1} = \frac{1}{4}.$$

As $n \to \infty$ the UMVUE would converge to the MLE while for finite sample size these values are slightly different.

g)

$$\varphi^* = \begin{cases} \text{Reject } H_0 & \text{if } P(\theta \in \Theta_0 | x) < 1/2 \\ \text{Accept } H_0 & \text{if } P(\theta \in \Theta_0 | x) > 1/2 \end{cases}$$

$$P(\theta \in \Theta_0(x)) = \int_{\Theta_0} h(\theta|x)\, d\theta$$

$$h(\theta|x) \propto \tau(\theta) \cdot L(x|\theta)$$

$$\propto \theta^2 (1-\theta)^2 \cdot \theta^n \cdot (1-\theta)^{\sum_{i=1}^{n} x_i}$$

$$\propto \theta^5 (1-\theta)^3$$

So $h(\theta|x) \sim \text{Beta}(6,4)$

$$\therefore \quad P(\theta \in \Theta|x) = \int_{0.7}^{1} \frac{1}{B(6,4)} x^5 (1-x)^3 dx$$

$$= \frac{\Gamma(6+4)}{\Gamma(6)\Gamma(4)} \int_{0.7}^{1} x^5 (1-x)^3 dx$$

$$= \frac{\Gamma(10)}{\Gamma(6)\Gamma(4)} \times 0.000536$$

$$= \frac{9 \times \overset{2}{\cancel{8}} \times 7 \times \cancel{6} \times \cancel{5!}}{\cancel{5!} \times \cancel{4} \times \cancel{3} \times \cancel{2}} \times 0.000536$$

$$= 0.270144 < 1/2$$

Hence, we reject H_0

2)

a) $L(X,\theta) = \prod_{i=1}^{n} \frac{\alpha}{\theta^{\alpha}} x^{\alpha-1} I(x_i, \infty)\theta$

$= \frac{\alpha^n}{\theta^{n\alpha}} \prod_{i=1}^{n} x_i^{\alpha-1} I(X_i, \infty)\theta$

$\cancel{= \frac{\alpha^n}{\theta^{n\alpha}} \prod_{i=1}^{n} (x_i \ldots}$

$\therefore \quad L(T,\theta) = \frac{\alpha^n}{\theta^{n\alpha}} I(T, \infty)\theta$

and $\quad h(x) = \sum_{i=1}^{n} x_h^{\alpha-1}$

Hence $T = X(n)$ is sufficient by the Neyman Fisher Factorization criterion.

b) Finding the CDF of x_n

$= \int_{-\infty}^{x} \frac{\alpha y^{\alpha-1}}{\theta^{\alpha}}\, dy = \frac{y^{\alpha}}{\theta^{\alpha}}\Big|_{-\infty}^{x}$

Hence $F_T(x,\theta) = P(T \le t)$

$= P(X_n < t, \ldots X_n < t)$

$= \left(\frac{x}{\theta}\right)^{\alpha n}$

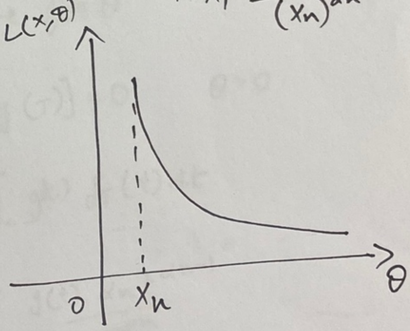
~~b)~~ Then $F_T(x,\theta) = \begin{cases} 0 & , \ t<0 \\ (t/\theta)^{\alpha n} & , \ 0\le t\le\theta \\ 1 & , \ t>0 \end{cases}$

Differentiating w.r.t to t.

$$f_T(t,\theta) = \begin{cases} \dfrac{\alpha n t^{\alpha n-1}}{\theta^{\alpha n}} & 0< t<\theta \\ 0 & \text{otherwise} \end{cases}$$

c) $L(X,\theta) = \dfrac{\alpha^n}{\theta^{\alpha n}} \prod_{i=1}^{n} x_i \, I_{(x_n,\infty)}\theta$

$\prod x_i \dfrac{\alpha^n x_n}{(x_n)^{\alpha n}}$



The graph starts with the highest value of the likelihood $L(X,\theta)$ ~~and~~ the i.e X_n.

Hence MLE is X_n

d) $E(X_n) = \int_0^\theta \frac{t \cdot \alpha n \, t^{\alpha n+1}}{\theta^{\alpha n}} \, dt$

$= \frac{\alpha n}{\theta^{\alpha n}} \int_0^\theta t^{\alpha n} \, dt$

$= \frac{\alpha n}{\theta^{\alpha n}} \left[\frac{t^{\alpha n+1}}{\alpha n+1} \Big|_0^\theta \right]$

$= \frac{\alpha n \, \theta}{\alpha n+1}$

$\neq \theta$

Hence MLE is a biased estimation.

e) $T = X_n$ is complete for θ

Lets assume $E_\theta [g(T)] = 0 \quad \theta > 0$

$E_\theta [g(T)] = \int_0^\theta g(t) \, f_T(t) \, dt$

$= \int_0^\theta g(t) \frac{\alpha n \, t^{\alpha n-1}}{\theta^{\alpha n}}$

$\because \alpha n \neq 0 \, , \; \theta^{\alpha n} \neq 0$

then

diff w.r.t θ ~~differentiating~~ $\int_0^\theta g(t)\, t^{\alpha n-1}\, dt = 0$

$$= \frac{g(\theta)\, \theta^{\alpha n}}{\alpha n}$$

we have $\alpha n \neq 0$, $\theta^{\alpha n} \neq 0$

then $g(\theta) = 0$

Hence $T = X_n$ is complete for θ.

f) We have from d)

$$W = \frac{X_n(\alpha n + n)}{\cancel{\theta}\,\alpha n}$$

By Lehmann Scheffe theory

$$\theta^n = E(W \mid X_n) = \frac{X_n(\alpha n + 1)}{\alpha^n}$$

is unique UMVUE.