UNSW Sydney
Department of Statistics
Term 1, 2020
MATH5905 - Statistical Inference
Assignment 2

Instructions: This assignment must be submitted no later than **6 pm on Thursday, 23rd April 2020 (week 10)**. Please, declare on the first page that the assignment is your own work, except where acknowledged. State also that you have read and understood the University Rules in respect to Student Academic Misconduct. You need to submit the PDF file on **Moodle** using the assignment submission link by 6 pm on Thursday 23rd April 2020 (week 10).

Problem 1
Let $X = (X_1, X_2, \ldots, X_n)$ be sample of n i.i.d. random variables, each with a density

$$f(x, \theta) = \frac{1}{x\sqrt{2\pi\theta}} \exp\left(-\frac{1}{2\theta} \log^2(x) \right)$$

when $x > 0$ otherwise zero and where $\theta > 0$ is a parameter.

a) Find the distribution of $Y_i = \log X_i$ and hence or otherwise compute $E(\log^2 X_i)$.

b) Find the Fisher information about θ in one observation and in the sample of n observations.

c) Find the Maximum Likelihood Estimator (MLE) of θ and show that it is unbiased.

d) Does the variance of the MLE attain the Cramer Rao bound?
Note: The χ^2_k distribution has mean k and variance $2k$.

e) Determine the asymptotic distribution of the MLE of θ and also the asymptotic distribution of $h(\theta) = e^\theta$.

Problem 2

Suppose $X = X_1, X_2, \ldots, X_n$ is a sample of n i.i.d. random variables from a population with a density

$$f(x; \theta) = \begin{cases} \tau\theta^\tau x^{-(\tau+1)} & \text{if} \quad x > \theta \\ 0 & \text{if} \quad \text{otherwise} \end{cases}.$$

where $\tau > 0$ is a known constant and $\theta > 0$ is an unknown parameter.

a) Show that the family $\{L(X, \theta\}, \theta > 0$ has a monotone likelihood ratio in the statistic $T = X_{(1)}$.

b) Show that the density of $T = X_{(1)}$ is

$$f_{X_{(1)}}(x) = \begin{cases} \tau n\theta^{\tau n} x^{-(\tau n+1)} & \text{if} \quad x > \theta \\ 0 & \text{if} \quad \text{otherwise} \end{cases}.$$

Hint: It might provide useful to consider the following

$$P(X_{(1)} \leq x) = 1 - P(X_1 \geq x, X_2 \geq x, \ldots, X_n \geq x).$$

c) Find the uniformly most powerful α-size test φ^* of

$$H_0 : \theta \geq \tau \qquad \text{versus} \qquad H_1 : \theta < \tau.$$

d) Calculate the power function of φ^*.

e) Compute the value of the power function at zero, τ and at the threshold constant of the test. Then, sketch a graph of the power function as precisely as possible.

Problem 3

Suppose that X is uniform on $[0, \theta]$ with density $f(x, \theta) = 1/\theta,\ 0 < x < \theta$, zero elsewhere.

a) Let $X = (X_1, \dots, X_n)$ be a sample of n i.i.d. observations from this distribution. Similar to the first question it can be shown that the family $\{L(X, \theta)\}$ has a monotone likelihood ratio in the statistic $T = X_{(n)}$.

  i) Compute the density and distribution function for $T = X_{(n)}$.

  ii) Determine the uniformly most powerful α-size test of

$$H_0 : \theta \geq \frac{1}{2} \qquad \text{versus} \qquad H_1 : \theta < \frac{1}{2}.$$

  iii) Show that the random variable

$$W_n = n\left(1 - \frac{X_{(n)}}{\theta}\right)$$

  converges in distribution to the exponential distribution with mean one as $n \to \infty$.

  iv) Hence or otherwise justify that $X_{(n)}$ is a consistent estimator of θ.

b) Now suppose that $X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}, X_{(5)}$ are the order statistics of a random sample of size five from this distribution. Let the observed value of $X_{(5)}$ be $x_{(5)}$. The test rejects $H_0 : \theta = 2$ and accepts $H_1 : \theta \neq 2$ when either $x_{(5)} \leq 1$ or $x_{(5)} > 2$.

  i) Find the power function $\gamma(\theta)$ for $0 < \theta$ of this particular test.

  ii) Plot the power function $\gamma(\theta)$ for all values $\theta > 0$.

Problem 4

Suppose $X_1, \dots, X_n$ are an i.i.d. sample with density

$$f(x, , \theta) = \theta(1 - x)^{\theta - 1} \qquad 0 < x < 1,$$

zero elsewhere and $\theta > 0$.

a) Find the form (you do not have to find the threshold constant) of the uniformly most powerful test of

$$H_0 : \theta = 1 \qquad \text{against} \qquad H_1 : \theta > 1.$$

b) Find the likelihood ratio for testing

$$H_0 : \theta = 1 \qquad \text{against} \qquad H_1 : \theta \neq 1.$$

Problem 5
Suppose $X_{(1)} < X_{(2)} < X_{(3)} < X_{(4)} < X_{(5)}$ are the order statistics based on a random sample of size five from the standard exponential density $f(x) = e^{-x}, x > 0$.

a) Find $E(X_{(4)})$.
Hint: Use a computer package to calculate the integral.

b) Find the density of the midrange $A = \frac{1}{2}(X_{(1)} + X_{(5)})$.
Hint: Use the following expansion

$$(e^{-v} - e^{-2u+v})^3 = -3e^{-2u-v} + 3e^{v-4u} - e^{3v-6u} + e^{-3v}.$$

c) Using this result (or otherwise), find $P(A > 2)$.
Hint: Use a computer package to calculate the integral.