THE UNIVERSITY OF NEW SOUTH WALES

DEPARTMENT OF STATISTICS

MID SESSION TEST - 2020 - Thursday, 2nd April (Week 7)

MATH5905

Time allowed: 135 minutes

1. Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ be a sample of i.i.d. Bernoulli(θ) random variables with density function

$$f(x, \theta) = \theta^x (1-\theta)^{1-x}, x = \{0, 1\}; \theta \in (0, 1)).$$

a) The statistic $T(X) = \sum_{i=1}^n X_i$ is complete and sufficient for θ. Provide justification for why this statement is true.

b) Derive the UMVUE of $h(\theta) = \theta^k$ where $k = 1, 2, \dots, n$ is a known integer. You must justify each step in your answer.
Hint: Use the interpretation that $P(X_1 = 1) = \theta$ and therefore we have that $P(X_1 = 1, \dots, X_k = 1) = P(X_1 = 1)^k = \theta^k$.

c) Calculate the Cramer-Rao lower bound for the minimal variance of an unbiased estimator of $h(\theta) = \theta^k$.

d) Find the value of k for which the variance of the UMVUE of $h(\theta)$ attains the Cramer-Rao lower bound found in part (c).

e) Determine the MLE $\hat{h}$ of $h(\theta)$.

f) Suppose that $n = 6$, $T = 3$ and $k = 2$ compute the numerical values of the UMVUE in part (b) and the MLE in part (e). What would happen to these values as $n \to \infty$ but the ratio $T/n = 1/2$ remained the same.

g) Consider testing $H_0 : \theta \leq 0.5$ versus $H_1 : \theta > 0.5$ with a 0-1 loss in Bayesian setting with the prior $\tau(\theta) = 60\theta^3(1-\theta)^2$. What is your decision when $n = 6$ and $T = 3$. You may use:

$$\int_0^{0.5} x^6(1-x)^5 \mathrm{d}x = 0.0000698.$$

Note: The continuous random variable X has a beta density f with parameters $\alpha > 0$ and $\beta > 0$ if

$$f(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, x \in (0, 1)$$

where

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

and

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) = \alpha!$$

2. Let $X_1, X_2, \ldots, X_n$ be independent random variables, with a density

$$f(x;\theta) = \begin{cases} \frac{\theta}{x^2}, & x > \theta, \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$ is an unknown parameter. Let $T = \min\{X_1, \ldots, X_n\} = X_{(1)}$ be the minimal of the n observations.

a) Show that T is a sufficient statistic for the parameter θ.

b) Show that the density of T is

$$f_T(t) = \begin{cases} \frac{n\theta^n}{t^{n+1}}, & t > \theta, \\ 0 & \text{otherwise} \end{cases}$$

Hint: You may compute the CDF first by using

$$P(X_{(1)} < x) = 1 - P(X_1 > x \cap X_2 > x \cdots \cap X_n > x).$$

c) Find the maximum likelihood estimator of θ and provide justification.

d) Show that the MLE is a biased estimator.

e) Show that $T = X_{(1)}$ is complete for θ.

f) Hence determine the UMVUE of θ.