THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Term One 2020

MATH5905
Statistical Inference

(1) TIME ALLOWED – THREE (3) HOURS

(2) TOTAL NUMBER OF QUESTIONS – 5

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE **NOT** OF EQUAL VALUE

(5) START EACH QUESTION ON A NEW PAGE.

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

YOU ARE TO COMPLETE THE TEST UNDER STANDARD EXAM CONDITIONS, WITH HANDWRITTEN SOLUTIONS.

YOU WILL THEN SUBMIT ONE OR MORE FILES CONTAINING YOUR SOLUTIONS. **MAKE SURE YOU SUBMIT ALL YOUR ANSWERS.**

ONE OF THE SUBMITTED FILES MUST INCLUDE A PHOTOGRAPH OF YOUR **STUDENT ID CARD** WITH THE **SIGNED**, HANDWRITTEN STATEMENT:
"**I declare that this submission is entirely my own original work.**"

YOU CAN DELETE AND/OR RELOAD FILES UNTIL THE DEADLINE.

Start a new page clearly marked Question 1

1. [**12 marks**] Consider a decision problem with parameter space $\Theta = \{\theta_1, \theta_2\}$ and a set of non-randomized decisions $D = \{d_i, 1 \le i \le 6\}$ with risk points $\{R(\theta_1, d_i), R(\theta_2, d_i)\}$ as follows:

i	1	2	3	4	5	6
$R(\theta_1, d_i)$	0	1	3	5	7	6
$R(\theta_2, d_i)$	7	4	6	2	5	7

a) Find the minimax rule(s) amongst the **non-randomized** rules in D.

b) Sketch the risk set of all **randomized** rules $\mathcal{D}$ generated by the set of rules in D.

c) Find the risk point of the minimax rule in $\mathcal{D}$ and determine its minimax risk.

d) Define the minimax rule in the set $\mathcal{D}$ in terms of rules in D.

e) Find the prior on $\{\theta_1, \theta_2\}$ such that the minimax rule in the set $\mathcal{D}$ is also a Bayes rule.

Start a new page clearly marked Question 2

2. [11 marks]

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random sample from $N(0, \theta)$ where $\theta > 0$ is the variance and the parameter of interest.

a) Find a complete and minimal sufficient statistic for θ.

b) Find an unbiased estimator of θ.

c) Hence or otherwise determine the UMVUE of θ.

d) Find the Cramer-Rao lower bound for the variance of the unbiased estimator of θ which you found in part c).

e) Find the variance of the UMVUE from part c).
Hint: The variance of a χ^2_p random variable might provide useful.

f) Compare the variance of the estimator found in part d) and part e).

Start a new page clearly marked Question 3

3. [**14 marks**] Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be independent and identically distributed random variables from a population with density

$$f(x, \theta, \phi) = \frac{1}{\theta\phi}\left(\frac{x}{\theta}\right)^{\frac{1-\phi}{\phi}}, \quad 0 \le x \le \theta, \ \ \phi < 1,$$

where θ and ϕ are unknown parameters.

a) Show that a two-dimensional sufficient statistic for (θ, ϕ) is

$$T = (T_1, T_2) = \left(\prod_{i=1}^{n} X_i, X_{(n)}\right).$$

b) Show that the sufficient statistic T in part (a) is also minimal sufficient statistic for (θ, ϕ).

c) Show that the density of the statistic $T = X_{(n)}$ is given by

$$f_{X_{(n)}}(x) = \frac{n}{\theta\phi}\left(\frac{x}{\theta}\right)^{n/\phi - 1} \qquad \text{for} \qquad 0 \le x \le \theta$$

otherwise zero.
Hint: You might consider using the following

$$P(X_{(n)} < x) = P(X_1 < x, X_2 < x, \ldots, X_n < x).$$

d) For the remainder of this question assume that the parameter ϕ is known. Find the maximum likelihood estimator of θ and provide justification.

e) Show that the MLE is a biased estimator.

f) Show that $T = X_{(n)}$ is complete and hence find the UMVUE of θ.

Start a new page clearly marked Question 4

4. **[11 marks]** Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a random sample from a normal distribution $\mathrm{N}(\theta, 4)$.

a) Write down the likelihood function for this sample.

b) Find a complete and minimal sufficient statistic for this family.

c) Show that this family has a monotone likelihood ratio in its complete and sufficient statistic.

d) Write down in words the definition of a uniformly most powerful α-test and then give the structure of the UMP α-test for testing H_0: $\theta = 5$ versus $H_1 : \theta > 5$.

e) Find the sample size n with power function $K(\theta)$ so that approximately $K(5) = 0.90$ and $K(3) = 0.1$.
Hint: The distribution of $\sum_{i=1}^{n} X_i$ is normal with mean θn and variance $4n$ and the upper 0.10 percentile of a standard normal is 1.28.

Start a new page clearly marked Question 5

5. [**12 marks**] Consider n independent identically distribution (iid) observations from Gamma(α, β) distribution.

a) Determine the cumulant generating function for a single observation from a Gamma(α, β).

b) Compute the first and second derivative of the cumulant generating function.

c) Find the saddlepoint $\hat{t}$.

d) Evaluate $K_X(\hat{t})$ and $K_X''(\hat{t})$ and simplify as much as possible.

e) Show that the first order saddlepoint approximation for the density of $\bar{X}$ is

$$\hat{f}(\bar{x}) \approx \sqrt{\frac{n\alpha}{2\pi}}(n\alpha\beta)^{-n\alpha}e^{n\alpha}(n\bar{x})^{n\alpha-1}e^{-n\frac{\bar{x}}{\beta}}n.$$

Hint: The first order saddlepoint approximation is

$$\hat{f}(\bar{x}) \approx \sqrt{\frac{n}{2\pi K_X''(\hat{t})}}e^{\{nK_X(\hat{t})-n\hat{t}\bar{x}\}}.$$

f) Hence, determine the first order saddlepoint approximation for the density of the sum of n iid observations $Y = \sum_{i=1}^{n} X_i$ from this distribution.
Hint: Consider using the density transformation formula

$$f_Y(y) = f_X(x(y))\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right|.$$

g) What is the approximate distribution of this saddlepoint approximation? Keep in mind the famous *Stirling approximation* of the Gamma function that states

$$\sqrt{\frac{2\pi}{n\alpha}}(n\alpha)^{n\alpha}e^{-n\alpha} \approx \Gamma(n\alpha).$$

Table of Common Distributions

Discrete Distributions

Bernoulli(p)

pmf $P(X = x|p) = p^x(1-p)^{1-x}; \quad x = 0, 1; \quad 0 \le p \le 1$

mean and variance $\mathbb{E}X = p, \quad Var(X) = p(1-p)$

mgf $M_X(t) = (1-p) + pe^t$

Binomial(n, p)

pmf $P(X = x|n, p) = \binom{n}{x} p^x(1-p)^{n-x}; \quad x = 0, 1, \ldots n; \quad 0 \le p \le 1$

mean and variance $\mathbb{E}X = np, \quad Var(X) = np(1-p)$

mgf $M_X(t) = [(1-p) + pe^t]^n$

Poisson(λ)

pmf $P(X = x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}; \quad x = 0, 1, \ldots; \quad 0 \le \lambda < \infty$

mean and variance $\mathbb{E}X = \lambda, \quad Var(X) = \lambda$

mgf $M_X(t) = e^{\lambda(e^t - 1)}$

Continuous Distributions

Beta(α, β)

pdf $f(x|\alpha, \beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}; \quad 0 \le x \le 1; \quad \alpha, \beta > 0, \ B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and $\Gamma(n) = (n-1)!$

mean and variance $\mathbb{E}X = \frac{\alpha}{\alpha+\beta}, \quad Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

mgf $M_X(t) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r} \right) \frac{t^k}{k!}$

Cauchy (θ, σ)

pdf $f(x|\theta, \sigma) = \frac{1}{\pi\sigma} \frac{1}{1+\left(\frac{x-\theta}{\sigma}\right)^2}, \quad -\infty < x < \infty, -\infty < \theta < \infty, \sigma > 0$

mean and variance do not exist

mgf does not exist

Chi squared(p)

pdf $f(x|p) = \frac{1}{\Gamma(p/2)\, 2^{p/2}} x^{(p/2)-1} e^{-x/2}$**,** $\quad 0 \le x < \infty$**,** $\quad p = 1, 2, 3, \ldots$

mean and variance $\mathbb{E}X = p$**,** $\quad Var(X) = 2p$

mgf $M_X(t) = \left(\frac{1}{1-2t}\right)^{p/2}$**,** $t < \frac{1}{2}$

Exponential(β)

pdf $f(x|\beta) = \frac{1}{\beta} e^{\frac{-x}{\beta}}$**;** $0 \leq x \leq \infty$**;** $\beta > 0$

mean and variance $\mathbb{E}X = \beta$**,** $Var(X) = \beta^2$

mgf $M_X(t) = \frac{1}{1-\beta\, t}$**,** $t < \frac{1}{\beta}$

Gamma(α, β)

pdf $f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\; x^{\alpha-1}\; e^{\frac{-x}{\beta}}$**;** $0 \leq x \leq \infty$**;** $\alpha, \beta > 0$

mean and variance $\mathbb{E}X = \alpha\,\beta$**,** $Var(X) = \alpha\,\beta^2$

mgf $M_X(t) = \left(\frac{1}{1-\beta\, t}\right)^\alpha$**,** $t < \frac{1}{\beta}$

Normal(μ, σ^2)

pdf $f(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma}\; e^{\frac{-(x-\mu)^2}{2\sigma^2}}$**;** $-\infty \leq x \leq \infty$**;** $-\infty < \mu < \infty,\, \sigma > 0$

mean and variance $\mathbb{E}X = \mu$**,** $Var(X) = \sigma^2$

mgf $M_X(t) = \exp\{\mu\, t + \frac{\sigma^2 t^2}{2}\}$

Uniform (a, b)

pdf $f(x|a, b) = \frac{1}{b-a}$**;** $a \leq x \leq b$

mean and variance $\mathbb{E}X = \frac{a+b}{2}$**,** $Var(X) = \frac{(b-a)^2}{12}$

mgf $M_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$

END OF EXAMINATION