BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE, PILANI

First Semester, 2020-21

MATH F111 : Mathematics-I

Comprehensive Examination(Open Book)

Max. Time : 120 Minutes **Date : 19/02/2021** **Max. Marks: 120**

Notes:

- There are two parts of question paper. Each part is of 60 marks and 60 minutes.
- Write your name and ID no. at the top of each sheet.
- You have to upload Part A by 11:10 AM and Part B by 12:20 PM in google classroom of your tutorial section.
- If there is any difficulty in uploading, email the file immediately to your tutorial instructor.

Max Time: 60 Minutes **Part-A** **Max. Marks: 60**

1. (a) Find the angle between the curves $r = \dfrac{a}{1+\cos\theta}$ and $r = \dfrac{b}{1-\cos\theta}$, with $a > 0, b > 0$ at the points of intersection. **[4]**

   (b) Find the area of the region shared by the curves $r = 1,\ r = 2\cos\theta$, and $r = 2\sin\theta$ with out using double integral. **[8]**

2. Consider the curve $\mathbf{r} = a\cos(b\theta)\mathbf{i} + a\sin(b\theta)\mathbf{j} + (ab\cot\beta)\theta\mathbf{k}$ for $a > 0,\ b > 0,\ \beta \in (\frac{\pi}{2}, \pi), \theta \geq 0$.

   (a) Find the arc length parameter of the curve. **[5]**

   (b) Find the equation of the osculating circle for the given curve at $\theta = 2\pi/b$. **[7]**

3. Set up and evaluate the triple integral to find the outward flux of the field

$$\mathbf{F} = (x+y)\mathbf{i} + (x+z)\mathbf{j} + (zx^2 + zy^2)\mathbf{k}$$

   through the closed surface S, which is surface boundary of the solid region enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 12$. **[12]**

4. Calculate the surface area of

$$S : z = \frac{h}{b^2}\left(x^2 + y^2\right)$$

   where h and b are the height and base radius of the revolving paraboloid respectively. **[12]**

5. On 16th January 2021, India started its national vaccination programme against COVID-19 pandemic. Suppose, in the first year (i.e., in 2021) we successfully vaccinate 5 crore Indians. If successful vaccination is expected to rise by 15% each year, then after how many years (or after the end of which year) we can expect to vaccinate a minimum 100 crore Indians to aquire herd immunity against COVID-19? **[6]**

6. Determine the radius of convergence of the series

$$\frac{(x-1)}{3} - \frac{(x-1)^2}{2.3^2} + \frac{(x-1)^3}{3.3^3} - \frac{(x-1)^4}{4.3^4} + ... + (-1)^{n-1}\frac{(x-1)^n}{n.3^n} + \cdots .$$

   What is the interval of convergence of this series (discuss at the end points also)? **[6]**

*****END of PART A*****

Ans 10 For the curve $r = \frac{a}{1+\cos\theta} = f(\theta)$

the slope m_1 of the curve is given by

$$m_1 = \frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$= \frac{\frac{a\sin^2\theta}{(1+\cos\theta)^2} + \frac{a\cos\theta}{1+\cos\theta}}{\frac{a\sin\theta\cos\theta}{(1+\cos\theta)^2} - \frac{a\sin\theta}{1+\cos\theta}}$$

$$= -\frac{a(1+\cos\theta)}{a\sin\theta}$$

$$= -\frac{1+\cos\theta}{\sin\theta} = -\cot\frac{\theta}{2}$$ [2]

For the curve $r = \frac{b}{1-\cos\theta} = f(\theta)$.

the slope m_2 of the curve is given by

$$m_2 = \frac{dy}{dx} = \frac{-\frac{b\sin^2\theta}{(1-\cos\theta)^2} + \frac{b\cos\theta}{(1-\cos\theta)}}{-\frac{b\sin\theta\cos\theta}{(1-\cos\theta)^2} - \frac{b\sin\theta}{1-\cos\theta}}$$

$$\Rightarrow m_2 = \frac{b(\cos\theta - 1)}{b\sin\theta} = \tan\frac{\theta}{2}$$

[1]

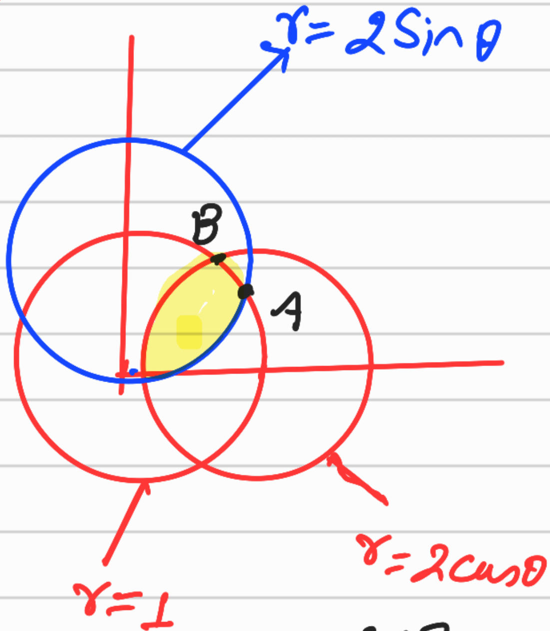
At the point of intersection product of slopes will be

$m_1 m_2 = -1$

Hence the curves are perpendicular to each other at the point of intersection. [1]

Ans 1b The shared region is shaded region

The Point A is point of intersection of the curves $r=1$ and $r=2\sin\theta$.



[1]

Hence $2\sin\theta = 1$

$\Rightarrow \theta = \pi/6$

The coordinates of point A is

$\left(1, \frac{\pi}{6}\right)$ [1]

The point B is point of intersection of $r=1$ & $r = 2\cos\theta$.

Hence $2\cos\theta = 1 \Rightarrow \theta = \frac{\pi}{3}$

the coordinates of point B is

$\left(1, \frac{\pi}{3}\right)$. [1]

Therefore the area of shaded region will be given by

$$\frac{1}{2}\int_0^{\pi/6} (2\sin\theta)^2 d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/3} 1^2 d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (2\cos\theta)^2 d\theta$$ [3]

$$= \int_0^{\pi/6} 2\sin^2\theta + \frac{1}{2}\cdot\frac{\pi}{6} + \int_{\pi/3}^{\pi/2} 2\cos^2\theta \, d\theta$$

$$= \int_0^{\pi/6} (1-\cos 2\theta)\, d\theta + \frac{\pi}{12} + \int_{\pi/3}^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \frac{\pi}{6} - \frac{1}{2}\frac{\sqrt{3}}{2} + \frac{\pi}{12} + \frac{\pi}{6} + \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)$$

$$= \frac{5\pi}{12} - \frac{\sqrt{3}}{2}$$

$$= \frac{5\pi - 6\sqrt{3}}{12} \qquad [2]$$

Ans 29

$$s(\theta) = \int_0^{\theta} |v(\theta)|\, d\theta$$

$$\bar{r} = a\cos(b\theta)\,\hat{i} + a\sin(b\theta)\,\hat{j} + ab\theta\cot\beta\,\hat{k}$$

$$\frac{d\bar{r}}{d\theta} = -ab\sin(b\theta)\,i + ab\cos(b\theta)\,j + ab\cot\beta\,k \qquad [1]$$

$$|v(\theta)| = \left|\frac{dr}{d\theta}\right| = \sqrt{a^2b^2 + a^2b^2\cot^2\beta}$$

$$= ab\,\operatorname{cosec}\beta \quad \text{as } \beta \in \left(\frac{\pi}{2}, \pi\right)$$

$$\neq 0 \qquad [2]$$

Thus $s(\theta) = \int_0^{\theta} ab\,\operatorname{cosec}\beta\, d\theta$

$$= (ab\,\operatorname{cosec}\beta)\,\theta \qquad [2]$$

Ans 2b

$$\bar{T} = \frac{1}{|v|}\frac{dr}{d\theta}$$

$$= \frac{1}{\operatorname{cosec}\beta}\left[-\sin b\theta\, i + \cos b\theta\, j + \cot\beta\,\hat{k}\right] \quad [1]$$

$$\frac{d\bar{T}}{d\theta} = \frac{b}{\operatorname{cosec}\beta}\left[-\cos b\theta\, i - \sin b\theta\, j\right]$$

$$k = \frac{1}{|v|}\left|\frac{dT}{d\theta}\right| \quad [1]$$

$$= \frac{1}{a\, b \operatorname{cosec}\beta}\,\frac{b}{\operatorname{cosec}\beta} = \frac{1}{a \operatorname{cosec}^2\beta}$$

Hence radius of curvature

$$\rho = \frac{1}{k} = a \operatorname{cosec}^2\beta. \quad [1]$$

$$\bar{N} = \frac{d\bar{T}}{d\theta} \Big/ \left|\frac{d\bar{T}}{d\theta}\right|$$

$$= -\cos(b\theta)\, i - \sin(b\theta)\, j$$

at $\theta = 2\pi/b$,

$$\bar{N} = -i \quad [1]$$

and $\bar{r}\left(\frac{2\pi}{b}\right) = ai + 2\pi a\cot\beta k$

Therefore center of the osculating circle will be

$$(a - \rho, 0, 2\pi a \cot\beta)$$

$$= (a(1 - \operatorname{cosec}^2\beta), 0, 2\pi a \cot\beta)$$

$$= (a\cot^2\beta, 0, 2\pi a\cot\beta) \quad [1]$$

So the eqn of osculating circle is

$$(x - a\cot^2\beta)^2 + y^2 + (z - 2\pi a\cot\beta)^2$$

$$= a^2 \operatorname{cosec}^4\beta \quad [2]$$

<u>Ans 3</u> Using divergence theorem

$$\iint_S \bar{F}\cdot\bar{n}\, d\sigma = \iiint_D \nabla\cdot\bar{F}\, dV \quad [1]$$

Here $\vec{F} = (x+y)i + (x+z)k + (zx^2+zy^2)k$

$\nabla\cdot F = 1 + 0 + x^2+y^2$

$= 1 + x^2+y^2$ [2]

So $\iint\limits_S F\cdot n\, d\sigma = \iiint\limits_D (2+x^2+y^2)\,dV$

$$= \int_0^{2\pi}\int_0^{\sqrt{1/2}}\int_{r^2}^{1/2} (1+r^2)\, r\, dz\, dr\, d\theta$$

[4]

$$= \int_0^{2\pi}\int_0^{\sqrt{12}} r(1+r^2)(12-r^2)\,dr\,d\theta \quad [1]$$

$$= 2\pi\left(6\times 12 + \frac{11}{4}12^2 - \frac{1}{6}12^3\right) \quad [2]$$

$$= 2\pi\left(\frac{13\times 12^2}{4} - 2\ 12^2\right)$$

$$= 2\pi \times \frac{5}{4} \times 144$$

$$= 360\pi. \quad [2]$$

Ans 4 Let $v = z$, $0 \le z \le h$

So $v = \frac{h}{b^2}(x^2+y^2)$

$\Rightarrow \frac{h}{vb^2}x^2 + \frac{h}{vb^2}y^2 = 1$

Let $x = b\sqrt{\frac{v}{h}}\cos u$, $y = b\sqrt{\frac{v}{h}}\sin u$

$0 \le u \le 2\pi$

So the parametrization of the surface is

$$\vec{g}(u,v) = x(u,v)i + y(u,v)j + z(u,v)$$

$$= b\sqrt{\frac{v}{h}}\cos u\, i + b\sqrt{\frac{v}{h}}\sin u\, j + v k$$

[4]

Now

$$\frac{\partial \bar{g}}{\partial u} = -b\sqrt{\frac{v}{h}}\sin u\, i + b\sqrt{\frac{v}{h}}\cos u\, j$$

and $$\frac{\partial \bar{g}}{\partial v} = \frac{b}{2\sqrt{vh}}\left(\cos u\, i + \sin u\, j + \frac{2\sqrt{vh}}{b}\hat{k}\right)$$

$$\frac{\partial \bar{g}}{\partial u} \times \frac{\partial \bar{g}}{\partial v} = b\sqrt{\frac{v}{h}}\cos u\,\hat{i} + b\sqrt{\frac{v}{h}}\sin u\,\hat{j} - \frac{b^2}{2h}\hat{k})$$

So $$\left|\frac{\partial \bar{g}}{\partial u} \times \frac{\partial \bar{g}}{\partial v}\right| = \frac{b}{2h}\sqrt{4vh + b^2}$$

[4]

Therefore surface area

$$A = \int_0^h \int_0^{2\pi} \frac{b}{2h}\sqrt{4vh + b^2}\, du\, dv$$

$$= 2\pi \int_0^h \frac{b}{2h} \sqrt{4vh + b^2}\, dv \quad [2]$$

$$= \pi \frac{b}{h} \frac{2}{3} \left[(4h^2 + b^2)^{3/2} - b^3 \right] \cdot \frac{1}{4h}$$

$$= \frac{\pi}{6} \frac{b}{h^2} \left[(4h^2 + b^2)^{3/2} - b^3 \right] \quad [2]$$

Ans5

Here $a_1 = 5$

$a_2 = 1.15 a_1 + a_1$

$a_3 = 1.15 a_2 + a_1$

$= \left((1.15)^2 + 1.15 + 1 \right) a_1$

$\vdots$

$a_n = \left[(1.15)^{n-1} + \dots 1.15 + 1 \right] a_1$

$= 5 \left[(1.15)^n - 1 \right] / (1.15 - 1) \quad [3]$

Now our target is to vaccinate max 100crore.

ie $b_n = \frac{(1.15)^n - 1}{.15} \times 5 \geq 100$

or $(1.15)^n \geq 4$

$\Rightarrow \quad n \geq \log 4 / \log(1.15)$

$\Rightarrow n \geq 9.91869$

Therefore it will take 10 years (ie by 2030) to vaccinate 100 crore Indians.

[3]

Ans 6 Here $a_n = \dfrac{(-1)^{n+1}}{n\,3^n}$

$$R = \frac{1}{\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{\lim\limits_{n\to\infty}\frac{n}{3(n+1)}}$$

$= 3$ [3]

So the given power series is convergent $\forall x$ st

$|x-1| < 3$ or $-2 < x < 4$

For the boundary point $x=-2$ the power series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} = \sum_{n=1}^{\infty} -\frac{1}{n}$$

which is divergent series.

Now at $x=4$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

which is convergent. So the interval of convergence is

$$[-2, 4] \qquad [3]$$

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Max Time: 60 Minutes **Part-B** **Max. Marks: 60**

1. Given that $z = f(x, y)$ has continuous second order partial derivatives and that $x = r^2 + s^2$, $y = 2rs$, find z_{rr} in terms of partial derivates of z with respect to x and y. **[6]**

2. Find the absolute maxima and minima of $f(x, y) = -x^3 + 4xy - 2y^2 + 1$ over the triangular region whose vertices are $(0, 0)$, $(1, 0)$ and $(0, 1)$. **[9]**

3. Find the equation of tangent plane to the paraboloid $z = 1 - \frac{1}{10}(x^2 + 4y^2)$ at the point $\left(1, 1, \frac{1}{2}\right)$. **[9]**

4. Using spherical coordinates, find the volume of the solid bounded below by $z = \sqrt{x^2 + y^2}$ and bounded above by $z^2 + x^2 + y^2 = z$. **[12]**

5. An engine of a jetty is set to have the force $\Phi(y)\mathbf{i}$ at the location (x, y) in the still water of a circular reservoir such that $\Phi(0) = -\frac{1}{2}$. In fact, the flow of water in the reservoir induces an additional force
$$\mathbf{F}(x, y) = 2x\mathbf{i} + \left[2y + xye^{-y^2}\right]\mathbf{j}$$
at the location (x, y) in the reservoir and all the round trips of the jetty in the reservoir are performed with no work done. Find the work done to go from the point $P(0, 0)$ to a point $Q(4, 5)$ in the reservoir. **[12]**

6. Use Green's theorem to find the counterclockwise circulation of the field
$$\mathbf{F}(x, y) = \left(y^3 + e^{\sin x}\right)\mathbf{i} + \left(x^3 + \sin y\right)\mathbf{j}$$
on the boundary of the region in the first quadrant bounded by the curves
$$y = x,\ \ y = \sqrt{3}x,\ \ x^2 + y^2 = 1 \text{ and } x^2 + y^2 = 4.$$ **[12]**

*****END*****

Ans1

$x = r^2 + s^2$

$y = 2rs$

$z = f(x, y)$

using chain rule

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial z}{\partial x} \cdot 2r + \frac{\partial z}{\partial y} \cdot 2s \quad [2]$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)$$

$$= 2\frac{\partial}{\partial r}\left[r\frac{\partial z}{\partial x} + s\frac{\partial z}{\partial y}\right] \checkmark$$

$$= 2\left[\frac{\partial z}{\partial x} + r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\right]$$

$$= 2\left[\frac{\partial z}{\partial x} + r\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + r\frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial r}\right.$$

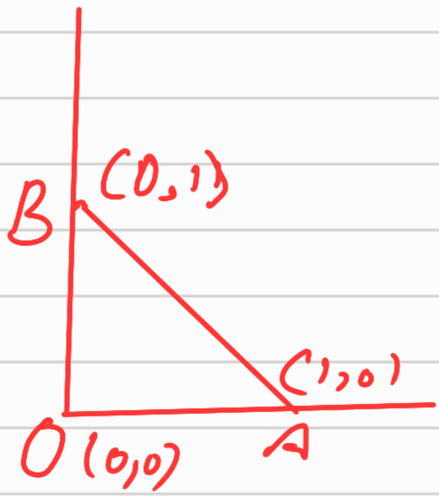
$$\left. + s\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r} + s\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right]$$

$$= 2\left[z_x + 2r^2 z_{xx} + 4sr\, z_{xy} + 2s^2 z_{yy}\right] \rightarrow [4]$$

Ans2

$f_x = -3x^2 + 4y$

$f_y = 4x - 4y$



for Critical point

$\left.\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 3x^2 = 4y \\ x = y \end{array}\right\} \Rightarrow x = 0 \text{ or } \frac{4}{3}$

[1]

The point $\left(\frac{4}{3}, \frac{4}{3}\right)$ does not belong to the triangular region.

$f(0,0) = \boxed{1}$ [1]

Along the boundary OA, $y=0$

$g(x): f(x,0) = -x^3 + 1$

$g'(x) = 0 \Rightarrow -3x^2 = 0 \Rightarrow x = 0$

which gives the point (0,0).

$f(1,0) = \boxed{0}$

[1]

Along AB, $x+y=1$

or $y=1-x$

$$g(x)=f(x,1-x)$$

$$=-x^3+4x(1-x)-2(1-x)^2+1$$

$$=-x^3+8x-6x^2-1$$

$$g'(x)=-3x^2+8-12x$$

$$g'(x)=0 \Rightarrow -3x^2-12x+8=0$$

$$\Rightarrow 3x^2+12x-8=0$$

$$x=\frac{-12\pm\sqrt{144+4\times3\times8}}{2\times3}$$

$$=\frac{-12\pm\sqrt{240}}{6}$$

$$=-2\pm\frac{2}{3}\sqrt{15}$$

$$x=-2-\frac{2}{3}\sqrt{15}<-2$$

Since $0<-2+2\sqrt{\frac{5}{3}}<1$, So

$$f\left(-2+2\sqrt{\frac{5}{3}},\ 3-2\sqrt{\frac{5}{3}}\right)=1.4265$$

and $f(0,1) = -1$ [3]

Along the OB, $x=0$

$$h(y) = f(0,y) = -2y^2+1$$

$$h'(y) = 0 \Rightarrow y = 0$$

So No New point is obtained [1]

Thus absolute maximum value is 1.4265 and absolute minimum value is -1. [2]

Ans3 Consider $f(x,y,z) = 1 - \frac{1}{10}(x^2+4y^2) - z = 0$ [1]

$$\nabla f = -\frac{1}{5}x\,i - \frac{4}{5}y\,j - k \quad [3]$$

At $(1,1,\frac{1}{2})$,

$$\nabla f = -\frac{1}{5}i - \frac{4}{5}j - k \quad [1]$$

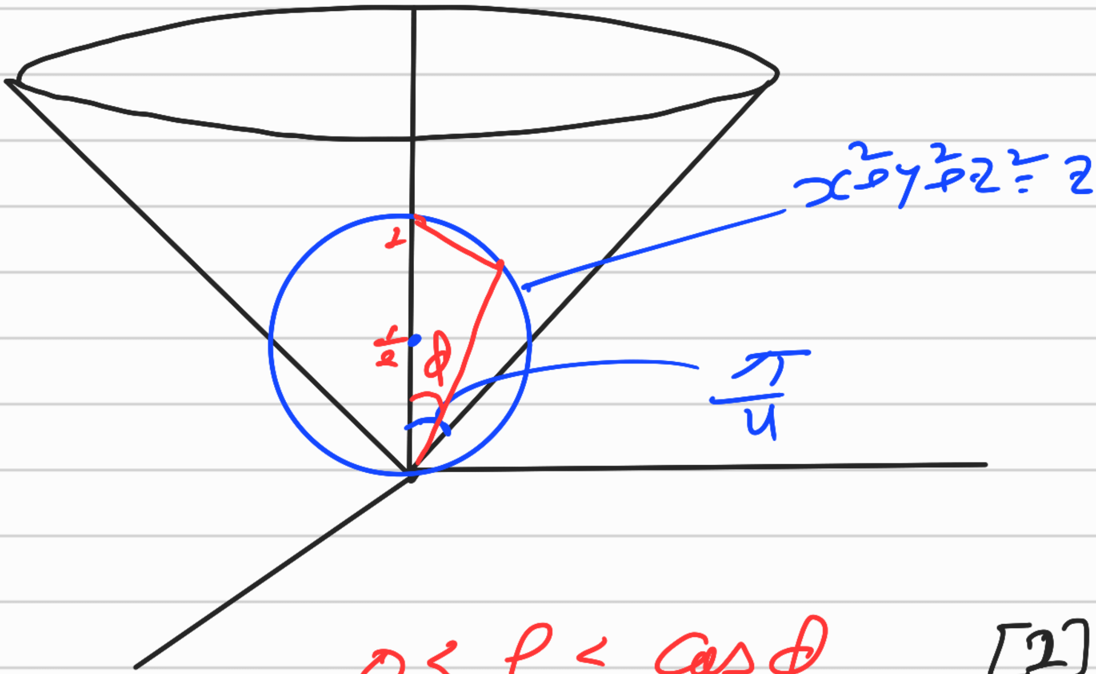
Equation of tangent plane

$$-\frac{1}{5}(x-1) - \frac{4}{5}(y-1) - (z-\tfrac{1}{2}) = 0 \quad [3]$$

$$\Rightarrow 2(x-1) + 8(y-1) + 5(2z-1) = 0$$

$$\Rightarrow \quad 2x + 8y + 10z = 15 \quad [1]$$

Ans 4



$0 \le \rho \le \cos\phi \quad [2]$

$0 \le \phi \le \frac{\pi}{4} \quad [2]$

$0 \le \theta \le 2\pi \quad [1]$

The volume is

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \quad [2]$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{3} \cos^3\phi \sin\phi \, d\phi \, d\theta$$

[2]

$$= \frac{1}{3}\int_0^{2\pi} \frac{1}{4}\left(1 - \frac{1}{4}\right) d\theta \qquad [2]$$

$$= \frac{\pi}{8} \qquad [1]$$

Ans5 Since there is no work done on the round trip. Total force

$$G(x,y) = F(x,y) + \phi(y)i$$

$$= (2x + \phi(y))i + (2y + xye^{-y^2})j$$

this is conservative.

$$\nabla \times G = 0 \qquad [2]$$

$$0i + 0j + k\left(ye^{-y^2} - \phi'(y)\right) = 0$$

$$\Rightarrow \phi'(y) = \frac{1}{2} 2y e^{-y^2}$$

$$\phi(y) = -\frac{1}{2} e^{-y^2} + C$$

$$\phi(0) = -\frac{1}{2} \Rightarrow C = 0$$

$$\Rightarrow \phi(y) = -\frac{1}{2} e^{-y^2}. \qquad [2]$$

For the potential function g of $\bar{G}$

$M = 2x - \frac{1}{2}e^{-y^2}$, $N = 2y + xye^{-y^2}$ [2]

$g_x = M \Rightarrow g = x^2 - \frac{1}{2}xe^{-y^2} + h(y)$ (2)

$g_y = N \Rightarrow xye^{-y^2} + h'(y) = 2y + xye^{-y^2}$

$\Rightarrow h'(y) = 2y$

or $h(y) = y^2 + C$ [2]

So $g(x,y) = x^2 - \frac{1}{2}xe^{-y^2} + y^2 + C$ (1)

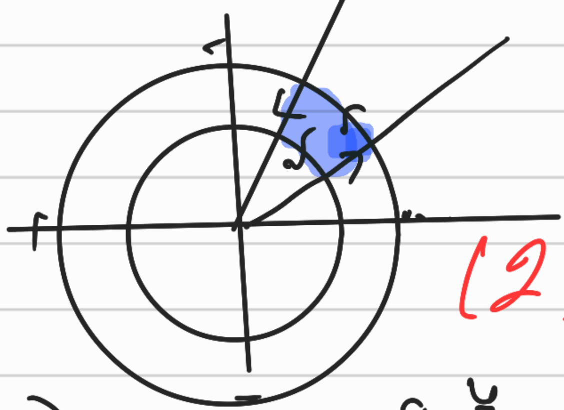
So the work done by $\bar{G}$

$\int_C \bar{G} \cdot d\bar{r} = g(4,5) - g(0,0)$

$= 16 - 2e^{-25} + 25 + \cancel{C} - \cancel{C}$

$= 41 - 2e^{-25}$ (1)

Ans 6



[2]

$$\bar{F} = (y^3 + e^{\sin x})\,i + (x^3 + e^{\sin y})\,j$$

$$M = y^3 + e^{\sin x}$$

$$N = x^3 + \sin y$$ [2]

$$N_x - M_y = 3(x^2 - y^2)$$

Counterwise Circulation $= \iint_R (N_x - M_y)\,dx\,dy$

$$= \iint_R 3(x^2 - y^2)\,dx\,dy$$ [2]

$$= \int_{\pi/4}^{\pi/3}\int_1^2 3r^2\cos 2\theta \; r\,dr\,d\theta$$

$$\int_{\pi/4}^{\pi/3}\int_1^2 3r^3 \cos 2\theta\,dr\,d\theta$$ [4]

$$\frac{3}{4}(16-1)\left.\frac{\sin 2\theta}{2}\right|_{\pi/4}^{\pi/3}$$

$$= \frac{45}{4} \cdot \frac{1}{2} \left(\sin \frac{2\pi}{3} - \sin \frac{\pi}{2} \right)$$

$$= \frac{45}{8} \cdot \left(\frac{\sqrt{3}}{2} - 1 \right)$$

$$= \frac{45}{16} \left(\sqrt{3} - 2 \right) \qquad [2]$$