Design and Analysis of Algorithms

Decision Trees

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- Dynamic Programming
  - Computing a Binomial Coefficient
  - The Knapsack Problem
  - Memory Functions
  - Warshall's and Floyd's Algorithms
  - Optimal Binary Search Trees
- Limitations of Algorithmic Power
  - Lower-Bound Arguments
  - **Decision Trees**
  - P, NP, and NP-Complete, NP-Hard Problems
- Coping with the Limitations
  - Backtracking
  - Branch-and-Bound

Concepts covered

- Decision Trees
  - Smallest of three numbers
  - Sorting
  - Searching

Problem Types: Optimization and Decision

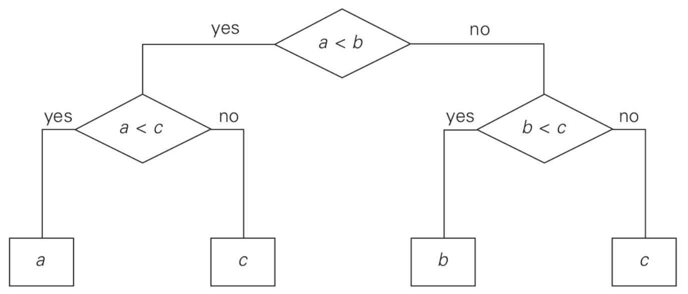
- Optimization problem: find a solution that maximizes or minimizes some objective function
- Decision problem: answer yes/no to a question
- Many problems have decision and optimization versions
  - Ex: traveling salesman problem
  - optimization: find Hamiltonian cycle of minimum length
  - decision: find Hamiltonian cycle of length m
- Decision problems are more convenient for formal investigation of their complexity

Introduction

- Many important algorithms, especially those for sorting and searching, work by comparing items of their inputs
- We can study the performance of such algorithms with a device called the decision tree

Example: Decision tree for minimum of three numbers

Decision tree for a determining the minimum of three numbers



Central Idea

- The central idea behind this model lies in the observation that a tree with a given number of leaves, which is dictated by the number of possible outcomes, has to be tall enough to have that many leaves
- Specifically, it is not difficult to prove that for any binary tree with leaves and height h

$$h \geq \lceil \log_2 l \rceil$$

- A binary tree of height h with the largest number of leaves has all its leaves on the last level
- Hence, the largest number of leaves in such a tree is 2^h
- In other words, $2^h \geq l$ which implies $h \geq \lceil \log_2 l \rceil$

Decision Trees for Sorting Algorithms

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- Most sorting algorithms are comparison-based, i.e., they work by comparing elements in a list to be sorted
- By studying properties of binary decision trees, for comparison-based sorting algorithms, we can derive important lower bounds on time efficiencies of such algorithms
- We can interpret an outcome of a sorting algorithm as finding a permutation of the element indices of an input list that puts the list's elements in ascending order
- For example, for the outcome $a < c < b$ obtained by sorting a list a, b, c
- The number of possible outcomes for sorting an arbitrary n-element list is equal to $n!$

Decision Trees for Sorting Algorithms

- The height of a binary decision tree for any comparison-based sorting algorithm and hence the worst -case number of comparisons made by such an algorithm cannot be less than

$$C_{worst}(n) \geq \lceil \log_2 n! \rceil$$

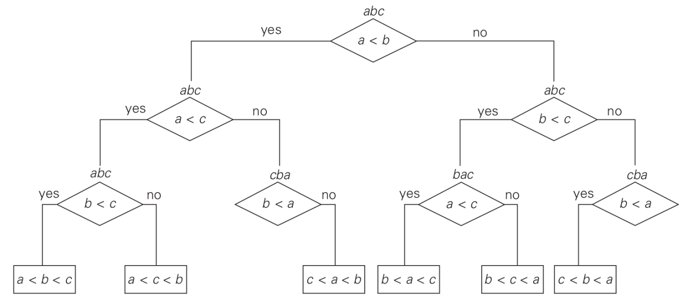
- Using Stirling's formula:

$$\lceil \log_2 n! \rceil \approx \log_2 \sqrt{2\pi n}\left(\frac{n}{e}\right)^n = n\log_2 n - n\log_2 e + \frac{\log_2 n}{2} + \frac{\log_2 \pi}{2} \approx n\log_2 n$$

- About $nlog_2 n$ comparisons are necessary to sort an arbitrary n-element list by any comparison-based sorting algorithm

Decision Trees for Sorting Algorithms

Decision Tree for Three Element Selection Sort



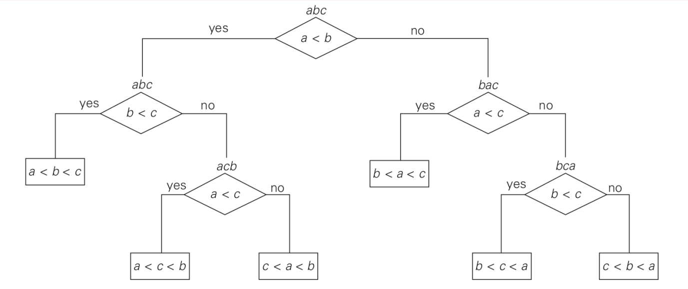
Decision Trees for Sorting Algorithms

- We can also use decision trees for analyzing the average-case behavior of a comparison based sorting algorithm
- We can compute the average number of comparisons for a particular algorithm as the average depth of its decision tree's leaves, i.e., as the average path length from the root to the leaves
- For example, for the three-element insertion sort this number is:

$$\frac{2+3+3+2+3+3}{6} = 2\frac{2}{3}$$

Decision Trees for Sorting Algorithms

Decision Tree for Three Element Insertion Sort



Decision Trees for Sorting Algorithms

- Under the standard assumption that all n! outcomes of sorting are equally likely, the following lower bound on the average number of comparisons Cavg made by any comparison-based algorithm in sorting an n-element list has been proved

$$C_{avg}(n) \geq \log_2 n!$$

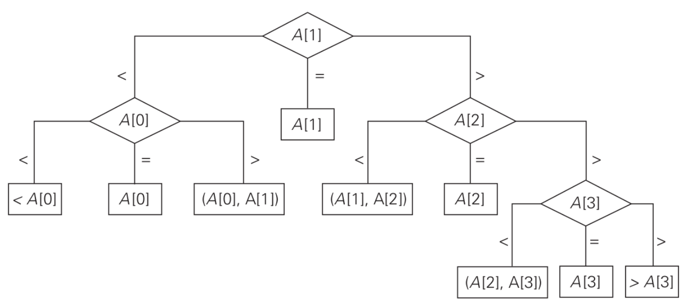
Decision Trees for Searching Algorithms

- Decision trees can be used for establishing lower bounds on the number of key comparisons in searching a sorted array of n keys: $A[O] < A[1] < \ldots < A[n-1]$
- The number of comparisons made by binary search in the worst case:

$$C_{worst}^{bs}(n) = \lfloor \log_2 n \rfloor + 1 = \lceil \log_2(n+1) \rceil$$

Decision Trees for Searching Algorithms

Ternary Decision Tree for Four Element Array Binary Search



Decision Trees for Searching Algorithms

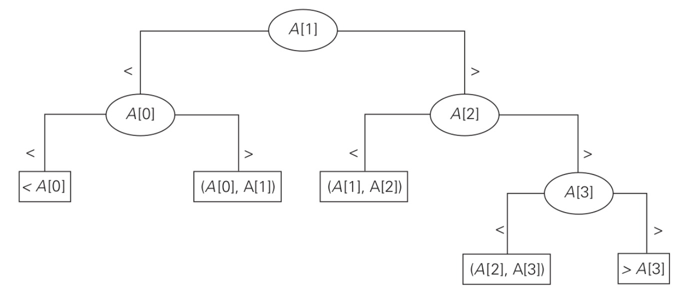
- For an array of n elements, all such decision trees will have 2n +1 leaves (n for successful searches and n + 1 for unsuccessful ones)
- Since the minimum height h of a ternary tree with l leaves is floor(log3l), we get the following lower bound on the number of worst-case comparisons:

$$C_{worst}(n) \geq \lceil \log_3(2n+1) \rceil$$

- This lower bound is smaller than $\lceil \log_2(n+1) \rceil$, the number of worst-case comparisons for binary search
- Can we prove a better lower bound, or is binary search far from being optimal?

Decision Trees for Searching Algorithms

Binary Decision Tree for Four Element Array Binary Search



- The binary decision tree is simply the ternary decision tree with all the middle subtrees eliminated

$$C_{worst}(n) \geq \lceil \log_2(n+1) \rceil$$