Design and Analysis of Algorithms

All Pairs Shortest Path (Floyd's Algorithm

Reetinder Sidhu

Department of Computer Science and Engineering

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- Dynamic Programming
  - Computing a Binomial Coefficient
  - The Knapsack Problem
  - Memory Functions
  - **Warshall's and Floyd's Algorithms**
- Limitations of Algorithmic Power
  - Lower-Bound Arguments
  - Decision Trees
  - P, NP, and NP-Complete, NP-Hard Problems
- Coping with the Limitations
  - Backtracking
  - Branch-and-Bound. Architecture (microprocessor instruction set)

Concepts covered

- All Pairs Shortes Path (Floyd's Algorithm)
  - Definition
  - Algorithm
  - Example

Problem Definition

- Given a undirected or directed graph, with weighted edges, find the shortest path between every pair of vertices
  - Dijkstra's algorithm found shortest paths from given vertex to remaining $n-1$ vertices ($\Theta(n)$ paths)
  - Current problem is to find the shortest path between every pair of vertices ($\Theta(n^2)$ paths)
- Solution approach is similar to the transitive closure approach: Compute transitive closure via sequence of $n \times n$ matrices $R^{(0)}, \ldots, R^{(k)}, \ldots, R^{(n)}$ where $R^{(k)}[i,j] = 1$ iff there is nontrivial path from i to j with only first k vertices allowed as intermediate vertices
- Compute all pairs shortest paths via sequence of $n \times n$ matrices $D^{(0)}, \ldots, D^{(k)}, \ldots, D^{(n)}$ where $D^{(k)}[i,j]$ is the shortest path from i to j with only first k vertices allowed as intermediate vertices

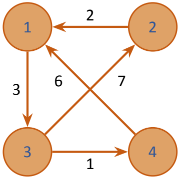
Example

- Example of all pairs shortest paths:

All Pairs Shortest Path (Floyd's Algorithm)

Example

- Example of all pairs shortest paths:



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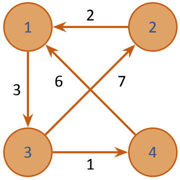

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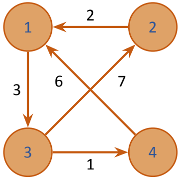

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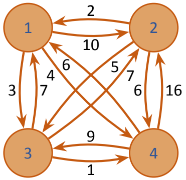

All Pairs Shortest Path (Floyd's Algorithm)

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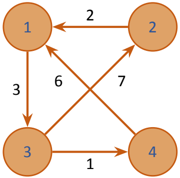

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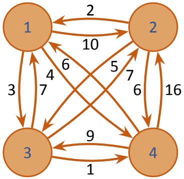

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All Pairs Shortest Path (Floyd's Algorithm)

Algorithm

Transitive Closure (Floyd's Algorithm)

procedure FLOYD(()$A[1 \dots n, 1 \dots n]$)
    ▷ **Input:** Weight matrix A of a graph with no negative length cycles
    ▷ **Output:** Distance matrix of shortest paths
    $D \leftarrow W$
    for $k \leftarrow 1$ **to** n **do**
        for $i \leftarrow 1$ **to** n **do**
            for $j \leftarrow 1$ **to** n **do**
                $D[i,j] \leftarrow \min(D[i,j], D[i,k] + D[k,j])$
    return D

All Pairs Shortest Path (Floyd's Algorithm)

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Transitive Closure (Floyd's Algorithm)

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        for i ← 1 to n do
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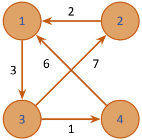
                D[i, j] ← min(D[i, j], D[i, k] + D[k, j])
    return D
```

- Complexity: $\Theta(n^3)$

All Pairs Shortest Path (Floyd's Algorithm)

Example



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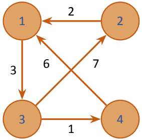

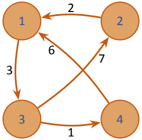

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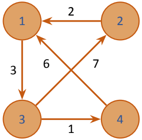
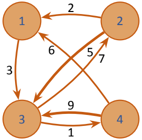
ALL PAIRS SHORTEST PATH (FLOYD'S ALGORITHM)

Example





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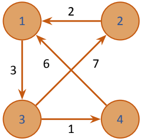

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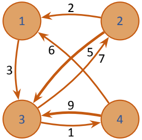

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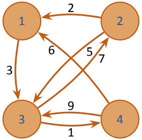

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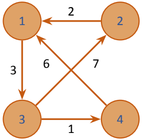

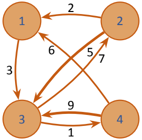

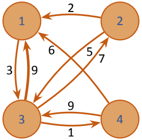

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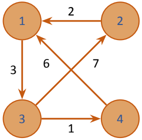

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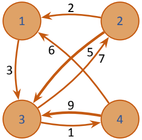

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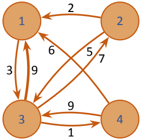

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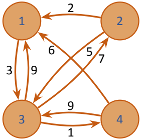

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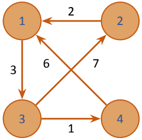

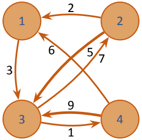

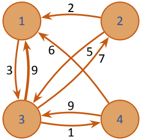

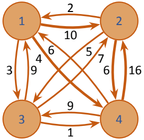

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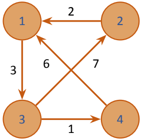

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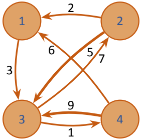

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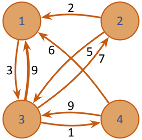

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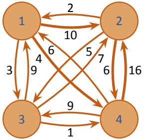

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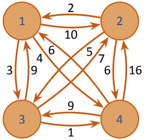

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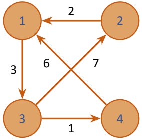

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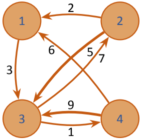

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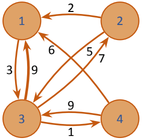

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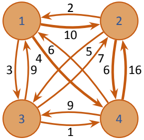

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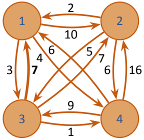

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Think About It

- Give an example of a graph with negative weights for which Floyd's algorithm does not yield the correct result
- Enhance Floyd's algorithm so that shortest paths themselves, not just their lengths, can be found