



DESIGN AND ANALYSIS OF ALGORITHMS

Transitive Closure (Warshall's Algorithm)

Reetinder Sidhu

Department of Computer Science and Engineering

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TRANSITIVE CLOSURE (WARSHALL'S ALGORITHM)

UNIT 5: Limitations of Algorithmic Power and Coping with the Limitations

- Dynamic Programming
 - ▶ Computing a Binomial Coefficient
 - ▶ The Knapsack Problem
 - ▶ Memory Functions
 - ▶ **Warshall's and Floyd's Algorithms**
- Limitations of Algorithmic Power
 - ▶ Lower-Bound Arguments
 - ▶ Decision Trees
 - ▶ P, NP, and NP-Complete, NP-Hard Problems
- Coping with the Limitations
 - ▶ Backtracking
 - ▶ Branch-and-Bound. Architecture (microprocessor instruction set)

Concepts covered

- Transitive Closure (Warshall's Algorithm)
 - ▶ Motivation
 - ▶ Algorithm
 - ▶ Example

TRANSITIVE CLOSURE (WARSHALL'S ALGORITHM)

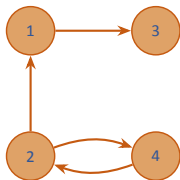
Transitive Closure

- Computes the transitive closure of a relation
- Alternatively: existence of all nontrivial paths in a digraph (directed graph)
- Example of transitive closure:

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Transitive Closure

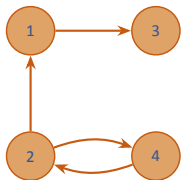
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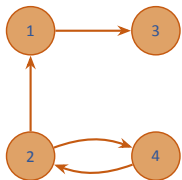


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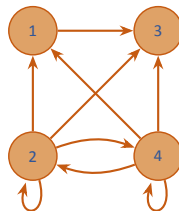
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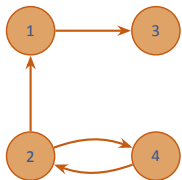
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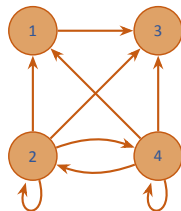
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TRANSITIVE CLOSURE (WARSHALL'S ALGORITHM)

Warshall's Algorithm



- Constructs transitive closure T as the last matrix in the sequence of $n \times n$ matrices $R^{(0)}, \dots, R^{(k)}, \dots, R^{(n)}$ where $R^{(k)}[i, j] = 1$ iff there is nontrivial path from i to j with only first k vertices allowed as intermediate vertices
 - ▶ $R^{(0)} = A$ (adjacency matrix), $R^{(n)} = T$ (transitive closure)
- On the k^{th} iteration, the algorithm computes $R^{(k)}$

$$R^{(k)}[i, j] = \begin{cases} 1 & \text{if path from } i \text{ to } k \text{ and } k \text{ to } j, \text{ i.e., } R^{(k-1)}[i, k] = R^{(k-1)}[k, j] = 1 \\ R^{(k-1)}[i, j] & \text{otherwise} \end{cases}$$

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$$R^{(k)}[i, j] = R^{(k-1)}[i, j] \text{ or } (R^{(k-1)}[i, k] \text{ and } R^{(k-1)}[k, j])$$

TRANSITIVE CLOSURE (WARSHALL'S ALGORITHM)

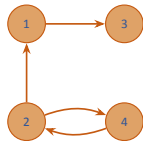
Algorithm

Transitive Closure (Warshall's Algorithm)

```
1: procedure WARSHALL( $A[1 \dots n, 1 \dots n]$ )
2:   ▷ Input: The adjacency matrix  $A$  of a digraph with  $n$  vertices
3:   ▷ Output: The transitive closure of the digraph
4:    $R^{(0)} \leftarrow A$ 
5:   for  $k \leftarrow 1$  to  $n$  do
6:     for  $i \leftarrow 1$  to  $n$  do
7:       for  $j \leftarrow 1$  to  $n$  do
8:          $R^{(k)}[i, j] \leftarrow R^{(k-1)}[i, j] \text{ or } (R^{(k-1)}[i, k] \text{ and } R^{(k-1)}[k, j]);$ 
9:   return  $R$ 
```

TRANSITIVE CLOSURE (WARSHALL'S ALGORITHM)

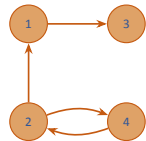
Warshall's Algorithm



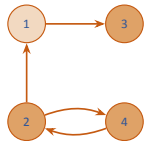
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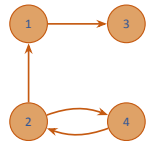
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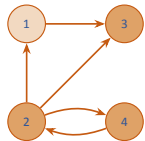
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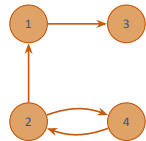
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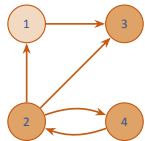
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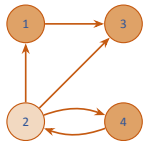
Warshall's Algorithm



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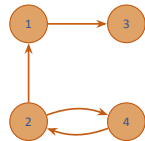
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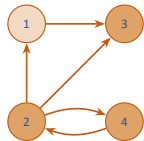
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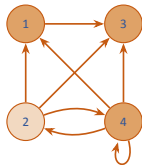
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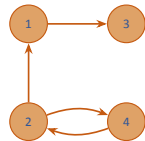
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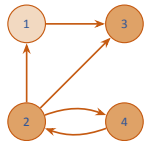
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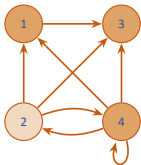
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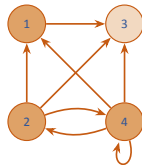
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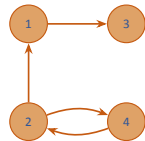
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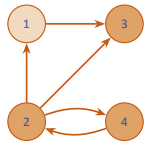
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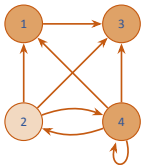
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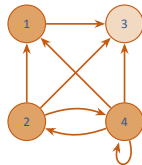
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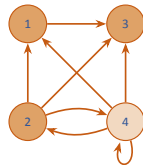
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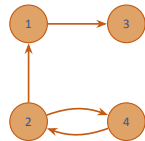
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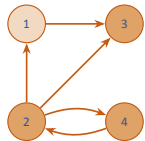
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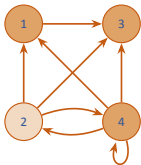
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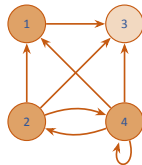
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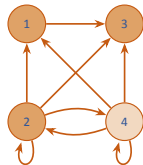
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TRANSITIVE CLOSURE (WARSHALL'S ALGORITHM)

Think About It



- Is Warshall's algorithm efficient for sparse graphs? Why / why not?
- Can Warshall's algorithm be used to determine if a graph is a DAG (Directed Acyclic Graph)? If so, how?