Design and Analysis of Algorithms

The Knapsack Problem

Reetinder Sidhu

Department of Computer Science and Engineering

Design and Analysis of Algorithms

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- Dynamic Programming
  - Computing a Binomial Coefficient
  - **The Knapsack Problem**
  - Memory Functions
  - Warshall's and Floyd's Algorithms
- Limitations of Algorithmic Power
  - Lower-Bound Arguments
  - Decision Trees
  - P, NP, and NP-Complete, NP-Hard Problems
- Coping with the Limitations
  - Backtracking
  - Branch-and-Bound. Architecture (microprocessor instruction set)

Concepts covered

- The Knapsack Problem
  - Introduction
  - Recurrence
  - Example

Problem Definition

- Given
  - n items of
    - integer weights : $w_1 \quad w_2 \quad \ldots \quad w_n$
    - values : $v_1 \quad v_2 \quad \ldots \quad v_n$
  - knapsack of capacity W (integer $W > 0$)
- Find the most valuable subset of items such that sum of their weights does not exceed W

The Knapsack Problem
Kanpsack Recurrence

- To design a dynamic programming algorithm, we need to derive a recurrence relation that expresses a solution to an instance of the knapsack problem in terms of solutions to its smaller subinstances
- Consider the smaller knapsack problem where number of items is i ($i \leq n$) and the knapsack capacity id j ($j \leq W$)
- Then

$$F(i,j) = \begin{cases} \max(F(i-1,j),\ \ v_i + F(i-1,j-w_i)) & \text{if } j - w_i \geq 0 \\ F(i-1,j) & \text{if } j - w_i < 0 \end{cases}$$

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Dynamic Programming Example

item i	weight w_i	value v_i
1	2	12
2	1	10
3	3	20
4	2	15

What is the maximum value that can be stored in a knapsack of capacity 6?

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item i	weight w_i	value v_i
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	capacity j				
i	1	2	3	4	5
1					
2					
3					
4					

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3	10	12	22	30	
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2	10	12	22	22	22
3	10	12	22	30	32
4	10	15	25		

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3	10	12	22	30	32
4	10	15	25	30	37

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4	10	15	25	30	37

Given above 6 items, maximum value that can be stored in a knapsack of capacity 5 is **37**

Complexity

- Space complexity: $\Theta(nW)$
- Time complexity: $\Theta(nW)$
- Time to compose optimal solution: $O(n)$

Think About It

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- Write pseudocode of the bottom-up dynamic programming algorithm for the knapsack problem

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- Write pseudocode of the bottom-up dynamic programming algorithm for the knapsack problem
- True or False:
  1. A sequence of values in a row of the dynamic programming table for the knapsack problem is always nondecreasing?

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Think About It

- Write pseudocode of the bottom-up dynamic programming algorithm for the knapsack problem
- True or False:
  1. A sequence of values in a row of the dynamic programming table for the knapsack problem is always nondecreasing?
  2. A sequence of values in a column of the dynamic programming table for the knapsack problem is always nondecreasing?