Using Limits to Compare Order of Growth

$$\lim_{n\to\infty}\frac{t(n)}{g(n)} = \begin{cases} 0 & \text{implies that } t(n) \text{ has a smaller order of growth than } g(n), \\ c & \text{implies that } t(n) \text{ has the same order of growth as } g(n), \\ \infty & \text{implies that } t(n) \text{ has a larger order of growth than } g(n). \end{cases}$$

Case1: $t(n) \in O(g(n)$

Case2: $t(n) \in \Theta(g(n))$

Case3: $g(n) \in O(t(n))$

t'(n) and g'(n) are first-order derivatives of t(n) and g(n)

L'Hopital's Rule $$\lim_{n\to\infty}\frac{t(n)}{g(n)} = \lim_{n\to\infty}\frac{t'(n)}{g'(n)}$$

Stirling's Formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ for large values of n

Using Limits to Compare Order of Growth: Example 1

Compare the order of growth of f(n) and g(n) using method of limits

$t(n) = 5n3 + 6n + 2 ,$ $\qquad g(n) = n4$

$$\lim_{n\to\infty} \frac{\mathrm{t}(n)}{g(n)} = \lim_{n\to\infty} \frac{5n^3 + 6n + 2}{n^4} = \lim_{n\to\infty} \left(\frac{5}{n} + \frac{6}{n^3} + \frac{2}{n^4} \right) = 0$$

As per case1

t(n) = O(g(n))

$5n^3 + 6n + 2 = O(n^4)$

Using Limits to Compare Order of Growth: Example 2

$$t\,(n) = \sqrt{5n^2 + 4n + 2}$$

using the Limits approach determine g(n) such that f(n) = Θ(g(n))

Leading term in square root n2

$$g(n) = \sqrt{n^2} = n$$

$$\lim_{n\to\infty} \frac{t(n)}{g(n)} = \lim_{n\to\infty} \frac{\sqrt{5n^2 + 4n + 2}}{\sqrt{n^2}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{5n^2 + 4n + 2}{n^2}} = \lim_{n\to\infty} \sqrt{5 + \frac{4}{n} + \frac{2}{n^2}} = \sqrt{5}$$

non-zero constant

Hence, t(n) = Θ(g(n)) = Θ(n)

Using Limits to Compare Order of Growth

$$lim_{n \to \infty} t(n)/g(n) \neq 0, \infty \Rightarrow t(n) \in \Theta(g(n))$$

$$lim_{n \to \infty} t(n)/g(n) \neq \infty \quad \Rightarrow t(n) \in O(g(n))$$

$$lim_{n \to \infty} t(n)/g(n) \neq 0 \quad \Rightarrow t(n) \in \Omega(g(n))$$

$$lim_{n \to \infty} t(n)/g(n) = 0 \quad \Rightarrow t(n) \in o(g(n))$$

$$lim_{n \to \infty} t(n)/g(n) = \infty \quad \Rightarrow t(n) \in \omega(g(n))$$

Using Limits to Compare Order of Growth: Example 3

Compare the order of growth of t(n) and g(n) using method of limits

$$t(n) = \log_2 n,\ g(n) = \sqrt{n}$$

$$\lim_{n\to\infty} \frac{\log_2 n}{\sqrt{n}} = \lim_{n\to\infty} \frac{(\log_2 n)'}{(\sqrt{n})'} = \lim_{n\to\infty} \frac{(\log_2 e)\,\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = 2\log_2 e \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$$\log_2 n \in o(\sqrt{n})$$