Design and Analysis of Algorithms

P, NP, and NP-Complete Problems

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- Dynamic Programming
  - Computing a Binomial Coefficient
  - The Knapsack Problem
  - Memory Functions
  - Warshall's and Floyd's Algorithms
  - Optimal Binary Search Trees
- Limitations of Algorithmic Power
  - Lower-Bound Arguments
  - Decision Trees
  - **P, NP, and NP-Complete, NP-Hard Problems**
- Coping with the Limitations
  - Backtracking
  - Branch-and-Bound

Concepts covered

- Decision Trees
  - Smallest of three numbers
  - Sorting
  - Searching

Classifying Problem Complexity

- Is the problem tractable, i.e., is there a polynomial-time ($O(p(n))$) algorithm that solves it?
- Possible answers:
  - yes
  - no
    - because it's been proved that no algorithm exists at all (e.g., Turing's halting problem)
    - because it's been be proved that any algorithm takes exponential time
- unknown

Problem Types: Optimization and Decision

- Optimization problem: find a solution that maximizes or minimizes some objective function
- Decision problem: answer yes/no to a question

- Many problems have decision and optimization versions
- Example: traveling salesman problem
  - optimization: find Hamiltonian cycle of minimum length
  - decision: find Hamiltonian cycle of length $\leq m$
- Decision problems are more convenient for formal investigation of their complexity

Class P

Class P (Polynomial

The class of decision problems that are solvable in $O(p(n))$ time, where $p(n)$ is a polynomial of problem's input size n

- searching
- element uniqueness
- graph connectivity
- graph acyclicity
- primality testing

Class NP

Class NP (Nondeterministic Polynomial)

class of decision problems whose proposed solutions can be verified in polynomial time = solvable by a nondeterministic polynomial algorithm

- A nondeterministic polynomial algorithm is an abstract two-stage procedure that:
  - generates a random string purported to solve the problem checks
  - whether this solution is correct in polynomial time
- By definition, it solves the problem if it's capable of generating and verifying a solution on one of its tries
- Why this definition?
  - led to development of the rich theory called "computational complexity"

Example: CNF satisfiability

Boolean Satisfiability (CNF)

Is a Boolean expression in its conjunctive normal form (CNF) satisfiable, i.e., are there values of its variables that makes it true?

- This problem is in NP. Nondeterministic algorithm:
  - Guess truth assignment
  - Substitute the values into the CNF formula to see if it evaluates to true
- Example: Consider the Boolean expression in CNF form:

$$(a + \overline{b} + \overline{c})(\overline{a} + b)(\overline{a} + \overline{b} + \overline{c})$$

- Can values *false* and *true* (or 0 and 1) be assigned to a, b and c such that above expression evaluates to 1?
- $a = 1, b = 1, c = 0$
- Checking phase: $\Theta(n)$

What problems are in NP?

- Hamiltonian circuit existence
- Partition problem: Is it possible to partition a set of n integers into two disjoint subsets with the same sum?
- Decision versions of TSP, knapsack problem, graph coloring, and many other combinatorial optimization problems. (Few exceptions include: MST, shortest paths)
- All the problems in P can also be solved in this manner (but no guessing is necessary), so we have:

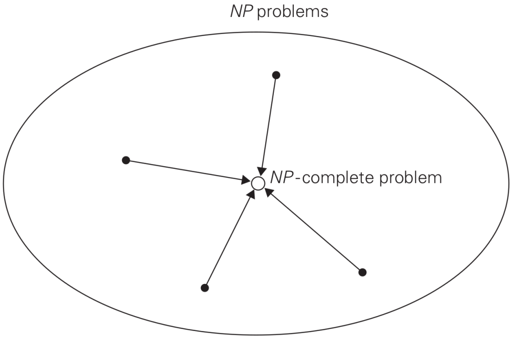
$$P \subseteq NP$$

- Big question:

$$P = NP \ ?$$

NP-Complete Problems

- A decision problem D is NP-complete if it's as hard as any problem in NP, i.e.,
  - D is in NP
  - every problem in NP is polynomial-time reducible to D



  - Cook's theorem (1971): CNF-sat is NP-complete

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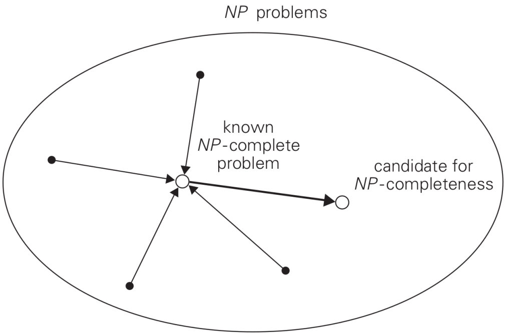
NP-Complete Problems

- Other NP-complete problems obtained through polynomial- time reductions from a known NP-complete problem

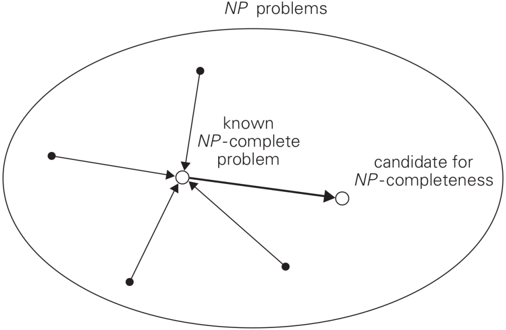


- Examples: TSP, knapsack, partition, graph-coloring and hundreds of other problems of combinatorial nature

P, NP, and NP-Complete Problems

P = NP ? Dilemma Revisited

- $P = NP$ would imply that every problem in NP, including all NP-complete problems, could be solved in polynomial time
- If a polynomial-time algorithm for just one NP-complete problem is discovered, then every problem in NP can be solved in polynomial time, i.e., $P = NP$



- Most but not all researchers believe that $P \neq NP$