LINEAR ALGEBRA AND ITS APPLICATIONS

Renna Sultana
Department of Science and Humanities

LINEAR ALGEBRA AND ITS APPLICATIONS

MATRICES AND GAUSSIAN ELIMINATION

Renna Sultana

Department of Science and Humanities

PES UNIVERSITY ONLINE

Course Content: Triangular Factors

- Consider the system of equations Ax=b. Using elementary row transformations we reduce A to U an Upper triangular matrix. While doing so Ax=b reduces to Ux=c(i.e [b] reduces to [c]) which can be solved by back substitution to obtain x. The steps involved are $Ax = b \Rightarrow E_{32}E_{31}E_{21}Ax = Ux = c\left(\because E_{32}E_{31}E_{21}A = U\right)$

  To undo the steps of elimination we must trace back the steps instead of subtracting we must add . For this we need inverses of Elementary Matrices i.e $E_{21}^{-1}, E_{31}^{-1}, E_{32}^{-1}$

  E_{21}^{-1} can be obtained by changing –l to l in the transformation. Similarly E_{31}^{-1} and E_{32}^{-1} can also be obtained. But while going from U to A the inverses should be in reverse direction i.e., $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = A$

Triangular Factorization A=LU:

- Any square matrix A can be factored as A=LU where
- L is a Lower Triangular Matrix
- With 1's on the diagonal
- Having multiplier's l_{ij} below the diagonal in their respective positions.
- And
- U is an Upper Triangular Matrix
- Having pivots on the diagonal (u_{11}, u_{22}, u_{33})
- i.e $L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} d_1 & u_{12} & u_{13} \\ 0 & d_2 & u_{23} \\ 0 & 0 & d_3 \end{pmatrix}$

MATRICES AND GAUSSIAN ELIMINATION:

Consider the matrix A= $\begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow[R_3 - 1/3R_1]{R_2 - 2/3R_1} \begin{pmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 5/3 & 1/3 \end{pmatrix}$

$$\xrightarrow{R_3 + (5/11)R_2} \begin{pmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{pmatrix} = U \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -5/11 & 1 \end{pmatrix}$$

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -5/11 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{pmatrix}$$

PES UNIVERSITY ONLINE

Triangular Factorization A=LDU:

- **A=LU** factorization is Unsymmetric as L has 1's on the diagonal whereas U has pivots on the diagonal. In order to make this factorization symmetric we do **A=LDU** factorization
- **D is a Diagonal Matrix with pivots $d_1, d_2 . d_3$ on the diagonal**
- **L is a Lower Triangular Matrix**
- **With 1's on the diagonal**
- **Having multiplier's l_{ij} below the diagonal in their respective positions.**
- **U is an Upper Triangular Matrix with 1's on the diagonal obtained by dividing each row by its pivot.**

i.e

$$L=\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} D=\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} U=\begin{pmatrix} 1 & u_{12}/u_{11} & u_{13}/u_{11} \\ 0 & 1 & u_{23}/u_{22} \\ 0 & 0 & 1 \end{pmatrix}$$

MATRICES AND GAUSSIAN ELIMINATION:

Factorise A=LU and hence A=LDU.

$$A=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \xrightarrow{R_2+(1/2)R_1} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \xrightarrow{R_3+(2/3)R_2} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow{R_4+(3/4)R_3} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{pmatrix} = U \quad A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{pmatrix}$$

$$A = LDU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

MATRICES AND GAUSSIAN ELIMINATION:

PES UNIVERSITY ONLINE

- To solve a system of equations Ax=b, factorize **A=LU, then Ax=Lux where**

  Let Ux=c, then Lc=b=Ax.

  Solve Lc=b, using Forward elimination and then find x using Ux=c by Backward substitution which gives x.

Ex: Solve as triangular systems without multiplying LU to find A.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

Let $Lc = b \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

$$\Rightarrow \begin{cases} c_1 = 2 \\ c_2 = -2 \\ c_3 = 0 \end{cases} \text{then } Ux = c \Rightarrow \begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2x + 4y + 4z = 2 \\ y + 2z = -2 \\ z = 0 \end{cases}$$

$$(x, y, z) = (5, -2, 0)$$

MATRICES AND GAUSSIAN ELIMINATION:

- Find A=LU and LDU factorization given

$$A=\begin{pmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{pmatrix} \xrightarrow[\substack{R_3+2R_1 \\ R_4+R_1}]{R_2-1/2R_1} \begin{pmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 4 & 13 & 0 \\ 0 & -2 & -14 & 11 \end{pmatrix} \xrightarrow[R_4-R_2]{R_3+2R_2} \begin{pmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -10 & 12 \end{pmatrix}$$

$$\xrightarrow{R_4+2R_3} \begin{pmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{pmatrix} = U \quad A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

$$A = LDU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & 5/2 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & -1/3 & -2/3 & 2/3 \\ 0 & 1 & 2 & 1/2 \\ 0 & 0 & 1 & -2/5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

THANK YOU

Renna Sultana

Department of Science and Humanities

rennasultana@pes.edu