LINEAR ALGEBRA

UE19MA251

APARNA B S
Department of Science and Humanities

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Agenda

- Problems on SVD
- SVD and Rank one matrices
- SVD and Pseudoinverse

Find SVD of

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

The eigenvalues of

$$A^T A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

are $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0$. Hence the singular values of A are

$$\sigma_1 = \sqrt{3}, \; \sigma_2 = 1$$

Orthonormal eigenvectors of $\left(A^T A\right)$ are

$$v_1 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix},\ v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix},\ v_3 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

The corresponding left singular vectors of A are

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix},\ u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

So

$$A = U\Sigma V^T = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\sqrt{3} & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}$$

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Rank One matrices and SVD

Every real matrix of rank r is the sum of r real matrices of rank 1 based on singular values and singular vectors.

By SVD

$$\begin{aligned} \boldsymbol{A} = \boldsymbol{U}\Sigma\boldsymbol{V}^T &= \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \cdots + \sigma_r \boldsymbol{u}_r \boldsymbol{v}_r^T \\ &= \boldsymbol{A}_1 + \cdots + \boldsymbol{A}_r \end{aligned}$$

Image approximation. For an image of size 1000×1000, a compression rate of 90% is achieved if 50 terms are used.

Let $A = U\Sigma V^T$ be an SVD of A. For a rectangular system of linear equations $Ax = b$, the least-squares solution with the minimum length is $x^+ = V\Sigma^+U^Tb$.

Pseudo-inverse. The minimum-length least-squares solution can be written as $x^+ = A^+b$, where $A^+ = V\Sigma^+U^T$. A^+ is called the **pseudo-inverse** of A.

The Pseudoinverse of a matrix generalizes the notion of the inverse in a similar manner that SVD generalized the diagonalization of a matrix.

Not every matrix has an inverse, but every matrix has a psuedoinverse.

Computing the pseudoinverse from SVD is simple.

If $A = U\Sigma V^*$ then $A^+ = V\Sigma^+ U^*$

where Σ^+ is formed from Σ by taking the reciprocal of all the non-zero elements, leaving all the zeros alone, and making the matrix the right shape: if Σ is an *m* by *n* matrix, then Σ^+ must be an *n* by *m* matrix.

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Consider the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & -2 \end{bmatrix}$

The Singular value decomposition of A is

$$\begin{bmatrix} \frac{1}{\sqrt{26}} & -\frac{5}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{bmatrix} \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{11}{\sqrt{195}} & \frac{7}{\sqrt{195}} & -\sqrt{\frac{5}{39}} \\ -\frac{3}{\sqrt{26}} & 2\sqrt{\frac{2}{13}} & -\frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{30}} & \sqrt{\frac{2}{15}} & \sqrt{\frac{5}{6}} \end{bmatrix}$$

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SVD and Pseudoinverse

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The Pseudoinverse may be computed from SVD using the definition explained earlier.

More on Pseudoinverse $A^{\dagger}$.

- If A is square and nonsingular then $A^{\dagger} = A^{-1}$.
- $A^{\dagger}$ is always defined.
- Thus $A^{\dagger}$ is a generalization of usual inverse.
- If $B \in \mathbb{R}^{n,m}$ satisfies
  1. $ABA = A$
  2. $BAB = B$
  3. $(BA)^T = BA$
  4. $(AB)^T = AB$

  then $B = A^{\dagger}$.
- Thus $A^{\dagger}$ is uniquely defined by these axioms.

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Example on Pseudoinverse $\boldsymbol{A}^{\dagger}$.

Show that the pseudoinverse of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$ is $B = \frac{1}{4}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

We have $BA = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $AB = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus

1. $ABA = A$
2. $BAB = B$
3. $(BA)^T = BA$
4. $(AB)^T = AB$

and hence $A^{\dagger} = B$.

THANK YOU

Aparna B. S

Department of Science & Humanities

aparnabs@pes.edu