LINEAR ALGEBRA AND ITS APPLICATIONS

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CLASS-8

POWERS AND PRODUCTS OF MATRICES

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Computation of Powers of a square matrix:

Diagonization of a square matrix A also helps to find the Powers of $A, A^2, \ldots$ etc.

We have $D = S^{-1} A S$

$$D^2 = (S^{-1} A S)(S^{-1} A S)$$

$$D^2 = S^{-1} A^2 S$$

Pre-multiplying by S and Post-multiplying by S^{-1}

$$S D^2 S^{-1} = S S^{-1} A^2 S S^{-1} = I A^2 I = A^2$$

$$A^2 = S D^2 S^{-1}$$

Powers and Products of matrices

In general, $A^n = S D^n S^{-1}$

where $D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 & 0 & \cdots & \\ 0 & \lambda_2^n & 0 & 0 & \cdots & \\ 0 & 0 & \lambda_3^n & 0 & \cdots & \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

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Problems:-

1. Diagonalize ~~the~~ matrix $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ and find one of its square roots. How many square roots will be there?

Ans:

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)^2 - 16 = 0$

$\Rightarrow \lambda = 1, 9$

consider $[A - \lambda I]X = 0$

$\Rightarrow (5-\lambda)x + 4y = 0$

$4x + (5-\lambda)y = 0$

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Case-1:- When $\lambda = 1$

$4x + 4y = 0$ $\therefore X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$4x + 4y = 0$

Case 2:- When $\lambda = 9$

$-4x + 4y = 0$ $\therefore X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$4x - 4y = 0$

$\therefore S = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}, S^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

Powers and Products of matrices

We know that $A = S\Lambda S^{-1}$

$$A^{1/2} = S\Lambda^{1/2}S^{-1}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}^{1/2} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^{1/2} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

We have 4 square roots $\because \Lambda^{1/2} = \sqrt{1} = \pm 1$ & also $\sqrt{9} = \pm 3$.

We get different values when we consider $\pm 1, \pm 3$.

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(2): Diagonalize $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ & hence find A^{100}.

Show that $A^{100} = A$.

Ans: $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - \lambda \end{vmatrix} \Rightarrow \lambda^2 - \lambda = 0$

$\lambda = 0, \lambda = 1$

Consider $[A - \lambda I]x = 0 \Rightarrow (\frac{1}{2} - \lambda)x + \frac{1}{2}y = 0$

$\frac{1}{2}x + (\frac{1}{2} - \lambda)y = 0$

Case 1:- When $\lambda = 0 \Rightarrow \frac{x}{2} + \frac{y}{2} = 0 \Rightarrow x = -y \quad x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\frac{x}{2} + \frac{y}{2} = 0$

Case 2: When $\lambda = 1$

$$-\frac{x}{2} + \frac{y}{2} = 0 \quad \Rightarrow x = y \qquad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{x}{2} - \frac{y}{2} = 0$$

$$A = S \Lambda S^{-1}$$

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^{100} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = A$$

THANK YOU

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