

LINEAR ALGEBRA

UE19MA251

APARNA B S

Department of Science and Humanities

Agenda



- Least square problem Normal equations.
- Positive Definiteness and the Least squares
- A simple example

The Normal Equations



Theorem: Any solution $oldsymbol{x}$ of the least squares problem is a solution of the linear system

$$A^T A x = A^T b.$$

The system is nonsingular if and only if $m{A}$ has linearly independent columns.

Proof. Since
$$m{b}-m{A}m{x}\in\ker(m{A}^T)$$
, we have $m{A}^T(m{b}-m{A}m{x})=m{0}$ or $m{A}^Tm{A}m{x}=m{A}^Tm{b}$.

 $m{A}^Tm{A}$ is nonsingular. Suppose $m{A}^Tm{A}m{x}=m{0}$ for some $m{x}\in\mathbb{R}^n$. Then $0=m{x}^Tm{A}^Tm{A}m{x}=(m{A}m{x})^Tm{A}m{x}=\|m{A}m{x}\|_2^2$. Hence $m{A}m{x}=m{0}$ which implies that $m{x}=m{0}$ if and only if $m{A}$ has linearly independent columns.

The linear system $A^TAx = A^Tb$ is called the normal equations.

Positive Definite Matrices and Least Squares



We have learned that **least square** comes from **projection**:

$$b-p=e\Rightarrow, A^T(b-A\hat{x})=0\Rightarrow, A^TA\hat{x}=A^Tb$$

Consequently, only if A^TA is invertible, then we can use linear regression to find approximate solutions $\hat{x}=(A^TA)^{-1}A^Tb$ to unsolvable systems of linear equations.

Positive Definite Matrices and Least Squares



According to the reasoning before, we know as long as all columns of $A_{m\times,n}$ are mutual independent, then A^TA is invertible. At the same time we ought to notice that the columns of A are guaranteed to be independent if they are orthogonal and even orthonormal.

In another prospective, if A^TA is positive definite, then $A_{m\times,n}$ has rank n (independent columns) and thus A^TA is invertible.

Positive Definite Matrices and Least Squares



Overall, if A^TA is positive definite or invertible, then we can find approximate solutions of least square.

Find the function of the form

$$F(x) = c_1 + c_2 x \lg x + c_3 e^x$$

that is the best least-squares fit to the data points

Positive Definite Matrices and Least Squares



First we form the A matrix

$$A = egin{pmatrix} 1 & 0 & e \ 1 & 2 & e^2 \ 1 & 3\lg 3 & e^3 \ 1 & 8 & e^4 \end{pmatrix}$$

We compute the pseudoinverse, then multiply it by y, to obtain the coefficient vector

$$c = \begin{pmatrix} 0.411741 \\ -0.20487 \\ 0.16546 \end{pmatrix}.$$

Positive Definite Matrices and Least Squares



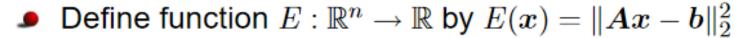
- lacksquare Given $A^{m,n}$ and $b\in\mathbb{R}^m$.
- **•** The system Ax = b is over-determined if m > n.
- **▶** This system has a solution if $b \in \text{span}(A)$, the column space of A, but normally this is not the case and we can only find an approximate solution.
- A general approach is to choose a vector norm ||·|| and find x which minimizes ||Ax b||.
- We will only consider the Euclidian norm here.

Positive Definite Matrices and Least Squares



- Given $A^{m,n}$ and $b \in \mathbb{R}^m$ with $m \ge n \ge 1$. The problem to find $x \in \mathbb{R}^n$ that minimizes $||Ax b||_2$ is called the least squares problem.
- lacktriangle A minimizing vector $m{x}$ is called a least squares solution of $m{A}m{x}=m{b}$.
- Several ways to analyze:
- Quadratic minimization
- Orthogonal Projections
- SVD

Positive Definite Matrices and Least Squares



•
$$E(x) = (Ax - b)^T (Ax - b) = x^T Bx - 2c^T x + \alpha$$
, where

- $m{B} := m{A}^T m{A}, \ m{c} := m{A}^T m{b} \ \text{and} \ \alpha := m{b}^T m{b}.$
- B is positive semidefinite and positive definite if A has rank n.
- Since the Hessian $HE(x) := \left(\frac{\partial^2 E(x)}{\partial x_i \partial x_j}\right) = 2B$ we can find minimum by setting partial derivatives equal zero.

• Normal equations $A^T A x = A^T b$.



A simple example



$$egin{aligned} x_1 &= 1 \ x_1 &= 1, \quad oldsymbol{A} &= egin{bmatrix} 1 \ 1 \ 1 \end{bmatrix}, \quad oldsymbol{x} &= [x_1], \quad oldsymbol{b} &= egin{bmatrix} 1 \ 1 \ 2 \end{bmatrix}, \ x_1 &= 2 \end{aligned}$$

Quadratic minimization problem:

$$\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_{2}^{2} = (x_{1} - 1)^{2} + (x_{1} - 1)^{2} + (x_{1} - 2)^{2}.$$

- Setting the first derivative with respect to x_1 equal to zero we obtain $2(x_1-1)+2(x_1-1)+2(x_1-2)=0$ or $6x_1-8=0$ or $x_1=4/3$
- The second derivative is positive (it is equal to 6) and x = 4/3 is a global minimum.



THANK YOU

Aparna B. S

Department of Science & Humanities

aparnabs@pes.edu