

LINEAR ALGEBRA

UE19MA251

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Agenda



- Proofs of SVD (Alternate proof)
- Matrices and SVD
- Singular vectors and fundamental subspaces

Alternate proof of SVD



We show $U^TAV=\Sigma$ which leads to SVD $A=U\Sigma V^T$. For j=1 ... r, we have $u_j=\frac{Av_j}{\sigma_j}$, so $Av_j=\sigma_j u_j$ and

$$(\boldsymbol{U}^T \boldsymbol{A} \boldsymbol{V})_{ij} = \boldsymbol{u}_i^T \boldsymbol{A} \boldsymbol{v}_j = \boldsymbol{u}_i^T (\sigma_j \boldsymbol{u}_j) = \sigma_j \delta_{ij}, \ i = 1, \dots, m$$

For j=r+1 ... n, we have $\left(m{A}^Tm{A}\right)m{v}_j=m{0}$, so $m{A}m{v}_j=m{0}$ and

$$(\boldsymbol{U}^T \boldsymbol{A} \boldsymbol{V})_{ij} = \boldsymbol{u}_i^T \boldsymbol{A} \boldsymbol{v}_j = 0, \ i = 1, \dots, m$$

Combining the results, we get

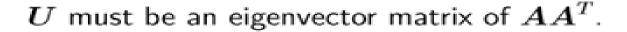
$$U^T A V = \Sigma$$

Hence

$$A = U\Sigma V^T$$

Matrices and SVD

For a matrix of order $m \times n$ with SVD $\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T$, the column vectors of \mathbf{U} (resp. \mathbf{V}) is an orthonormal basis of \mathbb{R}^m (resp. \mathbb{R}^n).



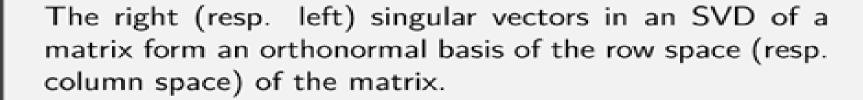
$$oldsymbol{A}oldsymbol{A}^T = \left(oldsymbol{U}oldsymbol{\Sigma}oldsymbol{V}^T
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Similarly, V must be an eigenvector matrix of A^TA .

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Singular Vectors and Fundamental Subspaces



Row space. $\{v_{r+1},\ldots,v_n\}$ contains eigenvectors of (A^TA) with eigenvalue 0, so it is a basis of $\mathcal{N}(A^TA) = \mathcal{N}(A)$. This implies $\{v_1,\ldots,v_r\}$ is a basis of the orthogonal complement of $\mathcal{N}(A)$, i.e. the row space of A.

Column space. We have $AV = U\Sigma$. The first r columns are

$$Av_i = \sigma_i u_i, i = 1, \ldots, r$$

So $\{u_1, \ldots, u_r\}$ is a linearly independent set in the column space of A. Hence, it is a basis of C(A).





THANK YOU

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