

## **LINEAR ALGEBRA**

# **UE19MA251**

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# Agenda



- Singular value and singular vector
- Positivity of a Singular value
- Number of Singular values
- Singular value decomposition(SVD)
- Proofs of SVD
- Matrices and SVD
- Singular vectors and fundamental subspaces

# Singular Value and Singular Vector

A **singular value** of a real matrix A is the square root of a non-zero eigenvalue of  $(A^TA)$ .

It means to find the singular values of A, one needs to find the non-zero eigenvalues of  $(A^TA)$ .

**Singular vector.** If  $\sigma$  is a singular value of A, then there exists  $v \neq 0$  such that

$$\left(oldsymbol{A}^Toldsymbol{A}
ight)oldsymbol{v}=\sigma^2oldsymbol{v}$$

Such a v is called a right singular vector of A with singular value  $\sigma$ . It is an eigenvector of  $\begin{pmatrix} A^TA \end{pmatrix}$  with eigenvalue  $\sigma^2$ .



# Positivity of a Singular Value



A singular value is always positive.

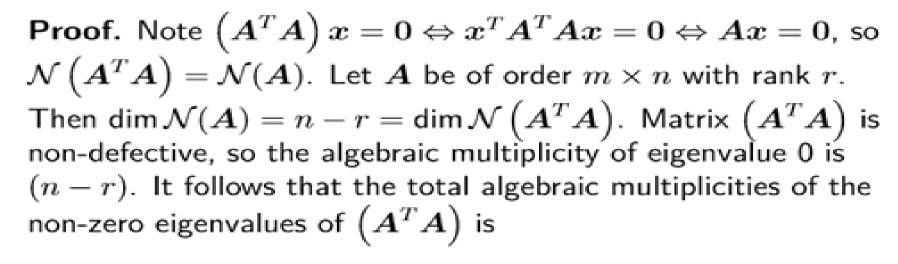
The matrix  $(A^TA)$  is positive semi-definite:

$$x^{T}(A^{T}A)x = (Ax)^{T}(Ax) = ||Ax||^{2} \ge 0$$

so the eigenvalues of  $(A^TA)$  must be non-negative, and the non-zero eigenvalues must be positive. Hence a singular value is positive.

# Number of Singular Values

A matrix of rank r has exactly r singular values.



$$n - (n - r) = r$$

**Notation.** Singular values are denoted by  $\sigma_1, \ldots, \sigma_r$ .



# Singular Value Decomposition (SVD)

A real matrix can be decomposed by its singular values and singular vectors. This is called singular value decomposition.

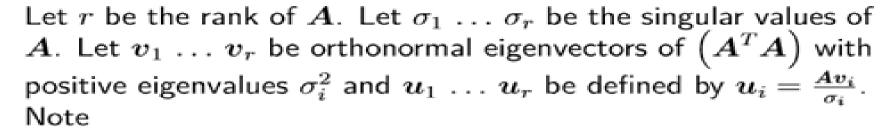


$$A = U\Sigma V^T$$

where  $\Sigma$  is an  $m \times n$  "diagonal" matrix with the singular values of  $\boldsymbol{A}$  as the leading diagonal elements,  $\boldsymbol{U}$  is an  $m \times m$  orthogonal matrix with the eigenvectors of  $\left(\boldsymbol{A}\boldsymbol{A}^T\right)$  as columns, and  $\boldsymbol{V}$  is an  $n \times n$  orthogonal matrix with the eigenvectors of  $\left(\boldsymbol{A}^T\boldsymbol{A}\right)$  as columns.



#### Proof of SVD



$$\left(\mathbf{A}\mathbf{A}^T\right)\mathbf{u}_i = \frac{\mathbf{A}\mathbf{A}^T\mathbf{A}\mathbf{v}_i}{\sigma_i} = \frac{\mathbf{A}\sigma_i^2\mathbf{v}_i}{\sigma_i} = \sigma_i^2\mathbf{u}_i, \ i = 1, \dots, r$$

So  $u_i$  is an eigenvector of  $\left(AA^T\right)$  with the same eigenvalue  $\sigma_i^2$ . Let  $v_{r+1}\ldots v_n$  be orthonormal eigenvectors of  $\left(A^TA\right)$  with eigenvalue 0, and  $u_{r+1}\ldots u_m$  be eigenvectors of  $\left(AA^T\right)$  with eigenvalue 0. Construct matrices U and V by

$$oldsymbol{U} = egin{bmatrix} oldsymbol{u}_1 & \dots & oldsymbol{u}_m \end{bmatrix}, \; oldsymbol{V} = egin{bmatrix} oldsymbol{v}_1 & \dots & oldsymbol{v}_n \end{bmatrix}$$





## **THANK YOU**

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