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Agenda – Applications of SVD

- Covariance of a matrix (2 x 2)
- Example on constructing covariance matrix

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Applications : Covariance of a 2x2 matrix

we can represent the covariance matrix by its eigenvectors and eigenvalues:

$$\Sigma\vec{v} = \lambda\vec{v}$$

where $\vec{v}$ is an eigenvector of Σ, and λ is the corresponding eigenvalue.

Equation above

holds for each eigenvector-eigenvalue pair of matrix Σ. In the 2D case, we obtain

two eigenvectors and two eigenvalues. The system of two equations defined by equation be represented efficiently using matrix notation:

$$\Sigma V = V L$$

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Applications : Covariance of a 2x2 matrix

This means that we can represent the covariance matrix as a function of its eigenvectors and eigenvalues:

$$\Sigma = V\,L\,V^{-1}$$

is called the eigendecomposition of the covariance matrix and can be obtained

using a Singular Value Decomposition algorithm. Whereas the eigenvectors represent the directions of the largest variance of the data, the eigenvalues represent the magnitude of this variance in those directions. In other words, V represents a rotation matrix, while $\sqrt{L}$ represents a scaling matrix. The covariance matrix can thus be decomposed further as:

$$\Sigma = R\,S\,S\,R^{-1}$$

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Applications : Covariance of a 2x2 matrix

where $R = V$ is a rotation matrix and $S = \sqrt{L}$ is a scaling matrix.

linear transformation $T = R\,S$. Since S is a diagonal scaling matrix, $S = S^\intercal$. Furthermore, since R is an orthogonal matrix, $R^{-1} = R^\intercal$. Therefore, $T^\intercal = (R\,S)^\intercal = S^\intercal\,R^\intercal = S\,R^{-1}$. The covariance matrix can thus be written as:

$$\Sigma = R\,S\,S\,R^{-1} = T\,T^\intercal,$$

In other words, if we apply the linear transformation defined by $T = R\,S$ to the original data D shown by figure we obtain the rotated and scaled data D' with covariance matri: $T\,T^\intercal = \Sigma' = R\,S\,S\,R^{-1}$. This is illustrated by figure below

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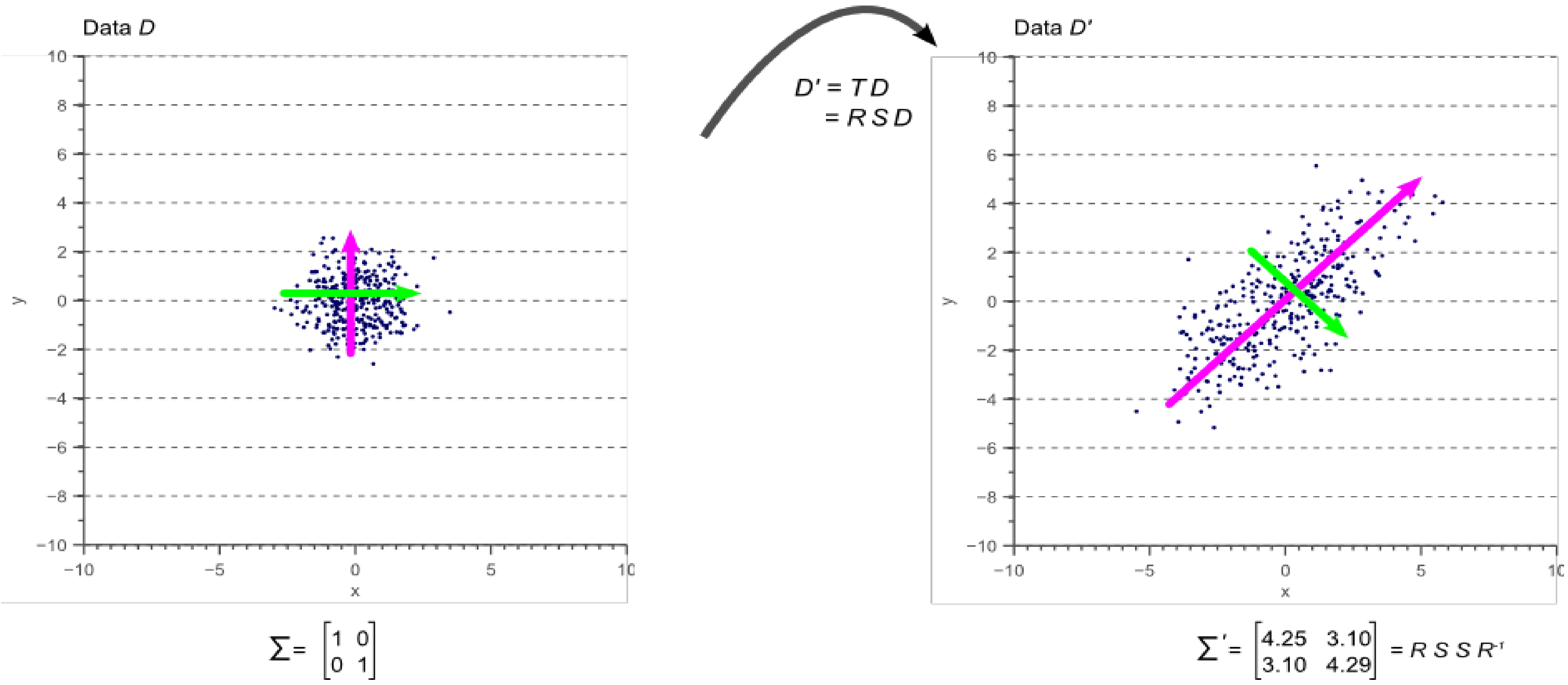
Applications : Covariance of a 2x2 matrix



https://www.visiondummy.com/wp-content/uploads/2014/04/lineartrans.png

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Applications : Covariance of a 2x2 matrix

The covariance matrix represents a linear transformation of the original data.

The largest eigenvector, i.e. the

eigenvector with the largest corresponding eigenvalue, always points in the direction of the largest variance of the data and thereby defines its orientation. Subsequent eigenvectors are always orthogonal to the largest eigenvector due to the orthogonality of rotation matrices.

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Example: Covariance of a matrix

1) Find the covariance matrix given that :

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 4 \\ 2 \\ 13 \end{bmatrix} \quad X_3 = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix}$$

and $$X_4 = \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix}$$

Example: Covariance of a matrix

step 1. Find the mean vector.

$$\text{Mean} = \frac{1}{n}\left[X_1 + X_2 + X_3 + X_4\right] = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix} = M \text{ (say)}$$

step 2: The mean deviation of the vectors X_i, $i = 1, 2, 3, 4$

are $\hat{X}_i = X_i - M$, $i = 1, 2, 3, 4$.

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Example: Covariance of a matrix

Therefore $\hat{X}_1 = \begin{bmatrix} -4 \\ -2 \\ -4 \end{bmatrix}$, $\hat{X}_2 = \begin{bmatrix} -1 \\ -2 \\ 8 \end{bmatrix}$ $\hat{X}_3 = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}$ and

$$\hat{X}_4 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

Let the matrix $B = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 & \hat{X}_3 & \hat{X}_4 \end{bmatrix}$

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Example: Covariance of a matrix

Then, $B = \begin{bmatrix} -4 & -1 & 2 & 3 \\ -2 & -2 & 4 & 0 \\ -4 & 8 & -4 & 0 \end{bmatrix}$

The covariance matrix is $S = \dfrac{1}{n-1}\left[B B^T\right]$

$$S = \frac{1}{3}\begin{bmatrix} 30 & 18 & 0 \\ 18 & 24 & -24 \\ 0 & -24 & 96 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 0 \\ 6 & 8 & -8 \\ 0 & -8 & 32 \end{bmatrix}$$

Example: Covariance of a matrix

2) The table below lists the weights and heights of 5 boys. Find the covariance matrix.

Boy:	1	2	3	4	5
Weight:	120	125	125	135	150
Height:	61	60	64	68	72

Example: Covariance of a matrix

Solution: Let $X = \begin{pmatrix} \text{Weight} \\ \text{Height} \end{pmatrix}$

Mean of X is $M = \frac{1}{5}\left[X_1 + X_2 + X_3 + X_4 + X_5\right]$

$\therefore \quad M = \frac{1}{5}\begin{bmatrix} 650 \\ 325 \end{bmatrix} = \begin{bmatrix} 130 \\ 65 \end{bmatrix}$

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Example: Covariance of a matrix

Let $\hat{X}_i = X_i - M$, $i = 1, 2, 3, 4, 5.$

$\therefore \hat{X}_1 = \begin{bmatrix} -10 \\ -4 \end{bmatrix}, \hat{X}_2 = \begin{bmatrix} -5 \\ -3 \end{bmatrix}, \hat{X}_3 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

$\hat{X}_4 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ and $\hat{X}_5 = \begin{bmatrix} 15 \\ 7 \end{bmatrix}$

Example: Covariance of a matrix

let $B = \left[\hat{X}_1 \;\; \hat{X}_2 \;\; \hat{X}_3 \;\; \hat{X}_4 \;\; \hat{X}_5 \right]$

$$= \begin{bmatrix} -10 & -5 & -5 & 5 & 15 \\ -4 & -5 & -1 & 3 & 7 \end{bmatrix}_{2\times 5}$$

$$S = \frac{1}{n-1}\left[BB^T \right] = \begin{bmatrix} 100 & \frac{190}{4} \\ \frac{190}{4} & 25 \end{bmatrix}$$

THANK YOU

Aparna B. S
Department of Science & Humanities

aparnabs@pes.edu