LINEAR ALGEBRA

UE19MA251

APARNA B S
Department of Science and Humanities

LINEAR ALGEBRA – UE19MA251

Agenda

- Singular value and singular vector
- Positivity of a Singular value
- Number of Singular values
- Singular value decomposition(SVD)
- Proofs of SVD
- Matrices and SVD
- Singular vectors and fundamental subspaces

Singular Value and Singular Vector

> A **singular value** of a real matrix A is the square root of a non-zero eigenvalue of $\left(A^T A\right)$.

It means to find the singular values of A, one needs to find the non-zero eigenvalues of $\left(A^T A\right)$.

Singular vector. If σ is a singular value of A, then there exists $v \neq 0$ such that

$$\left(A^T A\right) v = \sigma^2 v$$

Such a v is called a right singular vector of A with singular value σ. It is an eigenvector of $\left(A^T A\right)$ with eigenvalue σ^2.

Positivity of a Singular Value

A singular value is always positive.

The matrix $\left(\boldsymbol{A}^T\boldsymbol{A}\right)$ is positive semi-definite:

$$\boldsymbol{x}^T\left(\boldsymbol{A}^T\boldsymbol{A}\right)\boldsymbol{x} = (\boldsymbol{A}\boldsymbol{x})^T(\boldsymbol{A}\boldsymbol{x}) = \|\boldsymbol{A}\boldsymbol{x}\|^2 \geq 0$$

so the eigenvalues of $\left(\boldsymbol{A}^T\boldsymbol{A}\right)$ must be non-negative, and the non-zero eigenvalues must be positive. Hence a singular value is positive.

Number of Singular Values

PES UNIVERSITY ONLINE

> A matrix of **rank** r has exactly r singular values.

Proof. Note $\left(\boldsymbol{A}^T\boldsymbol{A}\right)\boldsymbol{x} = \boldsymbol{0} \Leftrightarrow \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0 \Leftrightarrow \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, so $\mathcal{N}\left(\boldsymbol{A}^T\boldsymbol{A}\right) = \mathcal{N}(\boldsymbol{A})$. Let $\boldsymbol{A}$ be of order $m \times n$ with rank r. Then $\dim \mathcal{N}(\boldsymbol{A}) = n - r = \dim \mathcal{N}\left(\boldsymbol{A}^T\boldsymbol{A}\right)$. Matrix $\left(\boldsymbol{A}^T\boldsymbol{A}\right)$ is non-defective, so the algebraic multiplicity of eigenvalue 0 is $(n - r)$. It follows that the total algebraic multiplicities of the non-zero eigenvalues of $\left(\boldsymbol{A}^T\boldsymbol{A}\right)$ is

$$n - (n - r) = r$$

Notation. Singular values are denoted by $\sigma_1, \ldots, \sigma_r$.

Singular Value Decomposition (SVD)

> A real matrix can be decomposed by its singular values and singular vectors. This is called singular value decomposition.

A matrix of order $m \times n$ has SVD

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$$

where $\boldsymbol{\Sigma}$ is an $m \times n$ "diagonal" matrix with the singular values of $\boldsymbol{A}$ as the leading diagonal elements, $\boldsymbol{U}$ is an $m \times m$ orthogonal matrix with the eigenvectors of $(\boldsymbol{A}\boldsymbol{A}^T)$ as columns, and $\boldsymbol{V}$ is an $n \times n$ orthogonal matrix with the eigenvectors of $(\boldsymbol{A}^T\boldsymbol{A})$ as columns.

LINEAR ALGEBRA – UE19MA251

Proof of SVD

Let r be the rank of A. Let $\sigma_1 \ldots \sigma_r$ be the singular values of A. Let $v_1 \ldots v_r$ be orthonormal eigenvectors of $\left(A^T A\right)$ with positive eigenvalues σ_i^2 and $u_1 \ldots u_r$ be defined by $u_i = \frac{Av_i}{\sigma_i}$.
Note

$$\left(AA^T\right) u_i = \frac{AA^T Av_i}{\sigma_i} = \frac{A\sigma_i^2 v_i}{\sigma_i} = \sigma_i^2 u_i, \quad i = 1, \ldots, r$$

So u_i is an eigenvector of $\left(AA^T\right)$ with the same eigenvalue σ_i^2. Let $v_{r+1} \ldots v_n$ be orthonormal eigenvectors of $\left(A^T A\right)$ with eigenvalue 0, and $u_{r+1} \ldots u_m$ be eigenvectors of $\left(AA^T\right)$ with eigenvalue 0. Construct matrices U and V by

$$U = \begin{bmatrix} u_1 & \ldots & u_m \end{bmatrix}, \; V = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$$

THANK YOU

Aparna B. S
Department of Science & Humanities
aparnabs@pes.edu