Linear Algebra & its Applications

Unit 4 - Orthogonalization, Eigen Values & Eigen Vectors

Lectures 1&2

Topic 1 - Orthogonalization - Orthogonal matrices, Properties, Rectangular matrices with orthonormal columns

→ In an orthogonal basis, every vector is perpendicular to every other vector. The coordinate axes are mutually orthogonal.

→ Mutually perpendicular unit vectors are called Orthonormal vectors.

→ For the vector space $\mathbb{R}^2$,

1. The set (2,0), (0,2) is an orthogonal basis.
2. The set (1,-2), (2,1) is an orthogonal basis
3. The set (1,0), (0,1) is an orthonormal basis.

→ A matrix with orthonormal columns will be called Q.

→ A square matrix with orthonormal columns is called an Orthogonal matrix denoted by Q.

→ Example : Rotation matrix, Any Permutation matrix.

Properties of Q :-

→ If Q (square or rectangular) has orthonormal columns, then $Q^T \cdot Q = I$.

→ An orthogonal matrix is a square matrix with orthonormal columns. Then $Q^T = Q^{-1}$

→ If Q is rectangular, then Q^T is left inverse of Q.

→ Multiplication by any Q preserves length. The norms of x & Qx are equal.

→ Also Q preserves inner products and angles, since $(Qx)^T(Qy) = x^T \cdot Q^T \cdot Q \cdot y = x^T y$

→ If $q_1, q_2 \dots q_n$ are orthonormal basis of $\mathbb{R}^n$ then any vector b from $\mathbb{R}^n$ can be expressed as

$$b = x_1 \cdot q_1 + x_2 q_2 + \dots + x_n q_n \quad - \text{(1)}$$

Multiplying both sides by q_1^T

$$q_1^T \cdot b = x_1 \cdot q_1^T \cdot q_1 + x_2 \cdot \underbrace{q_2 q_1^T}_{0} + \dots + x_n \cdot \underbrace{q_n \cdot q_1^T}_{0}$$

$$\therefore q_1^T \cdot b = x_1$$

Similarly $x_2 = q_2^T \cdot b \quad \dots \quad x_n = q_n^T \cdot b$

Hence $b = (q_1^T \cdot b) q_1 + (q_2^T \cdot b) q_2 + \dots + (q_n^T \cdot b) q_n$

$=$ sum of one dimensional projections onto q_i's.

→ The matrix form of (1) is $\boxed{Qx = b}$ and the solution of this system of equations is

$$x = Q^{-1} \cdot b$$

$$\boxed{x = Q^T \cdot b}$$

→ The rows of a square matrix are orthonormal whenever the columns are :-

Orthonormal columns

orthonormal rows

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

Rectangular matrices with orthonormal columns

→ If Q has orthonormal columns, the least squares problem becomes easy.

→ $Q^TQx = Q^Tb$ are the normal equations for the best solution in which $Q^T.Q = I$.

→ $x = Q^Tb$

→ $p = Qx$, the projection of b is $(a_1^T.b)a_1 + \dots + (a_n^T b)a_n$

→ $p = Q.Q^Tb$, the projection matrix is $P = Q.Q^T$

Special cases

1) If Q is a square matrix, $\hat{x} = Q^Tb$

$\hat{x} = Q^{-1}.b$ $\left[\because Q^T = Q^{-1}\right]$

2) If Q is a square matrix, $P = Q\hat{x}$

$P = Q.Q^{-1}.b$

$\boxed{P = b}$

Example Project $b = (0,3,0)$ onto each of the orthonormal vectors $a_1 = \left(2/3, 2/3, -1/3\right)$; $a_2 = \left(-1/3, 2/3, 2/3\right)$ & then find its projection its projection P onto the plane of a_1 & a_2.

④

Solution The projection of b onto a_1 is given by

$$P_1 = \left(\frac{a_1^T \cdot b}{a_1^T a_1}\right) a_1 \quad \text{but} \quad a_1^T \cdot a_1 = \|a_1\| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = 1$$

$$P_1 = \begin{bmatrix} 2/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} * \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix} = 2 \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 4/3 \\ 4/3 \\ -2/3 \end{bmatrix}$$

Similarly, $P_2 = \dfrac{(a_2^T b) a_2}{a_2^T a_2}$; $a_2^T a = \|a_2\| = \sqrt{1/9 + 4/9 + 4/9}$
$= 1$

$$P_2 = \begin{bmatrix} -1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 4/3 \\ 4/3 \end{bmatrix}$$

∴ P is the projection onto the plane containing a_1 & a_2 ie $P = P_1 + P_2 = \begin{bmatrix} 4/3 \\ 4/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} -2/3 \\ 4/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix}$

Example Find a third column so that the matrix is orthogonal. Verify that the rows automatically become orthonormal at the same time.

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & - \\ 1/\sqrt{3} & 2/\sqrt{14} & - \\ 1/\sqrt{3} & -3/\sqrt{14} & - \end{bmatrix}$$

Solution By the definition of orthogonal matrix, first and second columns are orthogonal to the required third column. Let the third column be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & x \\ 1/\sqrt{3} & 2/\sqrt{14} & y \\ 1/\sqrt{3} & -3/\sqrt{14} & z \end{bmatrix} \Rightarrow a^T c = 0$$

$$\begin{array}{ccc} a & b & c \end{array}$$

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = 0$$

$$\Rightarrow x + y + z = 0 \quad - (1)$$

Similarly, $b^T c = 0 \Rightarrow \underbrace{\begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix}}_{b} \Rightarrow \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$$\frac{x}{\sqrt{14}} + \frac{2y}{\sqrt{14}} - \frac{3z}{\sqrt{14}} = 0 \Rightarrow x + 2y - 3z = 0$$

$$x + y + z = 0$$
$$x + 2y - 3z = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -4 \end{bmatrix}$$

$$x + y + z = 0$$
$$y - 4z = 0$$
$$y = 4z$$
$$x + 4z + z = 0$$
$$x = -5z$$
$$y = 4z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5z \\ 4z \\ z \end{pmatrix} = z \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}$$

(6)

→ For rows to be orthonormal, the normal must be earual to 1.

$$\left(\frac{1}{\sqrt{3}}\right)^2 + \left[\frac{1}{\sqrt{14}}\right]^2 + (-5z)^2 = 1$$

$$\frac{1}{3} + \frac{1}{14} + 25z^2 = 1$$

$$25z^2 = \frac{1}{1} - \frac{1}{3} - \frac{1}{14}$$

$$25z^2 = \frac{42-14-3}{42}$$

$$25z^2 = \frac{25}{42}$$

$$z^2 = 1/42 \Rightarrow z = \frac{1}{\sqrt{42}}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}$$

$$\therefore Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & \frac{-5}{\sqrt{42}} \\ 1/\sqrt{3} & \cancel{2}/\sqrt{14} & 4/\sqrt{42} \\ 1/\sqrt{3} & -3/\sqrt{14} & 1/\sqrt{42} \end{bmatrix}$$

Lectures 3&4

TOPIC 2 The Gram-Schmidt orthogonalization, $A = QR$ factorization :-

The gram-schmidt process

→ It is a process of converting linearly independent vectors into orthonormal vectors.

→ consider any 3 independent vectors a, b, c. Then the first orthonormal vector $q_1 = \frac{a}{\text{norm}\{a\}}$

→ If 'b' is perpendicular to the vector 'a' then

$q_2 = \frac{b}{\text{norm}(b)}$ otherwise $B = b - (q_1^T b) q_1$ and

$$q_2 = \frac{B}{\text{norm}(B)}$$

→ If 'c' is perpendicular to the plane spanned by the vectors 'a' and 'b' then $q_3 = \frac{c}{\text{norm}(c)}$ otherwise

$C = c - (q_1^T c) q_1 - (q_1^T \cdot c) q_2$ and $q_3 = \frac{C}{\text{norm}(C)}$

→ This is the idea of the whole Gram-Schmidt process, to subtract from every new vector its components in the directions that are already settled. That idea is used over & over again. When there is a fourth vector, we subtract away its components in the directions of q_1, q_2, q_3.

Example From the vectors a, b, c find orthonormal vectors q_1, q_2 and q_3 given $a = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $c = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

solution :- From Gram-Schmidt Process :-

$$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

8

$$a_2 = \frac{e_2}{\|e_2\|}$$

$$e_2 = b - (a_1^T b)\, a_1$$

$$e_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 1/\sqrt{2} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/\sqrt{2} * \frac{1}{\sqrt{2}} \\ 1/2 \\ 0 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$a_2 = \frac{e_2}{\|e_2\|} = \frac{1}{\sqrt{1/4 + 1/4 + \frac{1}{1}}} \cdot \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{\frac{1+1+4}{4}}} \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$a_2 = \frac{2}{\sqrt{6}} \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$a_3 = \frac{e_3}{\|e_3\|} \Rightarrow e_3 \Rightarrow c - (a_1^T c) a_1 - (a_2^T \cdot c) a_2$$

$$e_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$- \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$a_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} - \frac{1}{\sqrt{6}} \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = e_3$$

$$a_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/6 \\ -1/6 \\ 2/6 \end{pmatrix} = e_3$$

$$e_3 = \begin{pmatrix} -1/2 - 1/6 \\ \frac{1}{1} - 1/2 + 1/6 \\ \frac{1}{1} - 2/6 \end{pmatrix} = \begin{pmatrix} \frac{-3-1}{6} \\ \frac{6-3+1}{6} \\ \frac{6-2}{6} \end{pmatrix} = \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$a_3 = \frac{e_3}{\|e_3\|} = \frac{1}{\sqrt{4/9 + 4/9 + 4/9}} \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$= \frac{\sqrt{3}}{\sqrt{4}} \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$\therefore \ a_3 = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$A = QR$ factorization :-

→ Let A be a matrix whose columns are a, b, c

→ Let Q be the matrix whose columns are q_1, q_2 and q_3 which are determined using Gram Schmidt process.

→ Then to find the third matrix which connects A & Q, express a, b, c as a linear combination of q_1, q_2, q_3.

→ The whole factorization is :-

$$A = \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{bmatrix}$$

$$A = Q \cdot R$$

<u>Note</u> :- consider the normal equation for $Ax = b$

$$A^T \cdot A \cdot \hat{x} = A^T b \quad \text{But} \quad A = QR$$

$$(QR)^T (QR) \hat{x} = (QR)^T b$$

$$R^T \cdot \underbrace{Q^T \cdot Q}_{I} \cdot R \cdot \hat{x} = R^T \cdot Q^T \cdot b$$

$$R^T \cdot R \cdot \hat{x} = R^T \cdot Q^T \cdot b$$

$$R \hat{x} = Q^T b$$

⇒ When $Ax = b$ is not solvable, we consider $R\hat{x} = Q^T b$ and solve.

→ If A is a square matrix of order 'n' then Q & R are also square matrix of order 'n'.

But if A is a matrix of order '$m \times n$' then Q is also a matrix of order '$m \times n$' but R is a square matrix of order 'n'.

$$\begin{bmatrix} \uparrow & \uparrow \\ a & b \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow \\ q_1 & q_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b \\ 0 & q_2^T b \end{bmatrix}$$

Example Factorize $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix}$ into QR

recognizing that first column is already a unit vector.

Solution:- $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix}$ $\quad Q = [q_1 \; q_2]$

$$\begin{array}{cc} \downarrow & \downarrow \\ a & b \end{array}$$

$$q_1 = \frac{a}{\|a\|} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$q_2 = \frac{e_2}{\|e_2\|} \; ; \quad e_2 = b - (q_1^T b) q_1$$

$$e_2 = \begin{pmatrix} \sin\theta \\ 0 \end{pmatrix} - (\cos\theta \;\; \sin\theta) \begin{pmatrix} \sin\theta \\ 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$e_2 = \begin{pmatrix} \sin\theta \\ 0 \end{pmatrix} - (\cos\theta \cdot \sin\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$e_2 = \begin{pmatrix} \sin\theta \\ 0 \end{pmatrix} - \begin{pmatrix} \cos^2\theta \cdot \sin\theta \\ \cos\theta \cdot \sin^2\theta \end{pmatrix}$$

$$e_2 = \begin{pmatrix} \sin\theta (1 - \cos^2\theta) \\ -\cos\theta \cdot \sin^2\theta \end{pmatrix}$$

$$\|e_2\| = \sqrt{\{\sin\theta - \sin\theta \cdot \cos^2\theta\}^2 + \cos^2\theta \cdot \sin^4\theta}$$

$$= \sqrt{\sin^2\theta + \sin^2\theta \cos^4\theta - 2\sin^2\theta \cdot \cos^2\theta + \cos^2\theta \cdot \sin^4\theta}$$

12

$$\|e_2\| = \sqrt{\sin^2\theta + \sin^2\theta \cdot \cos^4\theta - 2\sin^2\theta \cdot \cos^2\theta + \cos^2\theta \cdot \sin^4\theta}$$

$$\|e_2\| = \sqrt{\sin^2\theta + \sin^2\theta \cdot \cos^2\theta(\cos^2\theta + \sin^2\theta) - 2\sin^2\theta \cdot \cos^2\theta}$$

$$\|e_2\| = \sqrt{\sin^2\theta + \sin^2\theta \cdot \cos^2\theta - 2\sin^2\theta\cos^2\theta}$$

$$\|e_2\| = \sqrt{\sin^2\theta - \sin^2\theta \cdot \cos^2\theta}$$

$$\|e_2\| = \sqrt{\sin^2\theta\,(1 - \cos^2\theta)}$$

$$\|e_2\| = \sqrt{\sin^2\theta \cdot \sin^2\theta}$$

$$\|e_2\| = \sin^2\theta$$

$$\therefore\ q_2 = \frac{e_2}{\|e_2\|} = \frac{1}{\sin^2\theta}\begin{bmatrix} \sin\theta\,(1-\cos^2\theta) \\ -\cos\theta \cdot \sin^2\theta \end{bmatrix}$$

$$\therefore\ q_2 = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$$

$$R = \begin{bmatrix} q_1^T a & q_1^T b \\ 0 & q_2^T b \end{bmatrix} ;\ q_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad q_2 = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$$

$$q_1^T a = [\cos\theta \quad \sin\theta] \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = 1$$

$$q_1^T b = [\cos\theta \quad \sin\theta] \begin{bmatrix} \sin\theta \\ 0 \end{bmatrix} = \cos\theta \cdot \sin\theta$$

$$q_2^T b = [\sin\theta \quad -\cos\theta] \begin{bmatrix} \sin\theta \\ 0 \end{bmatrix} = \sin^2\theta$$

$$\therefore\ R = \begin{bmatrix} 1 & \cos\theta \cdot \sin\theta \\ 0 & \sin^2\theta \end{bmatrix}$$

$$\therefore\ A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} 1 & \cos\theta \cdot \sin\theta \\ 0 & \sin^2\theta \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos^2\theta\cdot\sin\theta+\sin^3\theta \\ \sin\theta & \cos\theta\sin^2\theta-\cos\theta\sin^2\theta \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\theta & \sin\theta(\cos^2\theta+\sin^2\theta) \\ \sin\theta & 0 \end{bmatrix}$$

$$\therefore \quad A = QR = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix}$$

<u>Example</u> Find an orthonormal set q_1, q_2, q_3 for which q_1, q_2 span the column space of $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$. What fundamental subspace contains q_3. What is the least squares solution of $Ax=b$ if $b = [1\ 2\ 7]^T$?

<u>solution</u> $a = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ $\Rightarrow q_1 = \dfrac{a}{\|a\|} = \dfrac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$$\therefore \quad q_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

$$q_2 = \frac{e_2}{\|e_2\|} \;;\; e_2 = b - (q_1^T b) q_1$$

$$e_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1/3 & 2/3 & -2/3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}\begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} - \left(\frac{1}{3} - \frac{2}{3} - \frac{8}{3}\right)\begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

(14)

$$e_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} - (-3)\begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ +2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$a_2 = \frac{e_2}{\|e_2\|} = \frac{1}{\sqrt{9}}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\therefore \quad a_2 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

Let $e_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. To find a_3, let the vector be assumed as orthogonal to the plane spanned by a_1 and a_2 such that $a_1^T \cdot e_3 = 0$ & $a_2^T \cdot e_3 = 0$

$$\begin{bmatrix} 1/3 & 2/3 & -2/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad ; \quad \begin{bmatrix} 2/3 & 1/3 & 2/3 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$x + 2y - 2z = 0 \qquad\qquad 2x + y + 2z = 0$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 6 \end{bmatrix}$$

$$-3y + 6z = 0$$
$$x + 2y - 2z = 0$$
$$-3y = -6z$$
$$\boxed{y = 2z}$$
$$x + 2(2z) - 2z = 0$$
$$x + 4z - 2z = 0$$
$$x + 2z = 0$$
$$\boxed{x = -2z}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2z \\ 2z \\ z \end{pmatrix} = z\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1/3 & 2/3 & -2z \\ 2/3 & 1/3 & 2z \\ -2/3 & 2/3 & z \end{pmatrix}$$

rows should be orthonormal

$\therefore \quad \sqrt{1/9 + 4/9 + 4z^2} = 1$

$$1/9 + 4/9 + 4z^2 = 1$$

$$4z^2 = 1 - \frac{5}{9}$$

$$4z^2 = 4/9$$

$$z^2 = 1/9$$

$$\boxed{z = 1/3}$$

$$\therefore \quad Q = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

consider $Ax = b \Rightarrow A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

We can verify that the above equation is not solvable & hence we consider $R\hat{x} = Q^T b$ where

$$R = \begin{bmatrix} q_1^T a & q_1^T b \\ 0 & q_2^T b \end{bmatrix}$$

$$q_1^T a = \begin{bmatrix} 1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = 1/3 + 4/3 + 4/3 = 3$$

$$q_1^T b = \begin{bmatrix} 1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = 1/3 - 2/3 - 8/3 = -3$$

(16) $a_2^T b = \begin{bmatrix} 2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \frac{2}{3} - \frac{1}{3} + \frac{8}{3} = 3$

$\therefore \quad R = \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}$

$\therefore \quad R \cdot \hat{x} = Q^T b$

$\begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} \cdot \hat{x} = Q^T \cdot b$

$Q = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \\ -2/3 & 2/3 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$

$Q^T b = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 1/3 + 4/3 - \frac{14}{3} \\ 2/3 + 2/3 + \frac{14}{3} \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

$\therefore \quad \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} \hat{x} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

$\hat{x} = \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

$\hat{x} = \frac{1}{9} \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

$= \frac{1}{9} \begin{bmatrix} -9 + 18 \\ 18 \end{bmatrix}$

$\therefore \quad \boxed{\hat{x} = \begin{matrix} 1 \\ 2 \end{matrix}}$

Lectures 5 to 7 - Introduction to Eigen values & Eigen vectors, Properties of Eigen values and Eigen vectors, Spectral theorem, Symmetric matrices, Cayley-Hamilton theorem statement only

Let A be a square matrix of order n. If there exists a real or complex number λ and a non zero vector x such that $Ax = \lambda x$, then x is called the eigen vector of A and λ is its corresponding eigen value.

→ The vector x is in null space of $A - \lambda I$.

→ The number λ is chosen so that $A - \lambda I$ has a null space.

→ $[A - \lambda I]$ must be singular

→ $Det\{A - \lambda I\} = 0$ is called the characteristic equation of A and roots of this equation are called characteristic roots or eigen values or latent roots.

→ corresponding to 'n' distinct eigen values we get 'n' independent eigen vectors. But when 2 or more eigen vectors are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to repeated roots.

Procedure to find eigen values & eigen vectors :-

→ compute the determinant of $A - \lambda I$. With a λ subtracted along the diagonal, this determinant is a polynomial of degree 'n'. It starts with $(-\lambda)^n$.

→ Find the roots of this polynomial. The n roots are the eigen values of A.

→ For each eigen value λ, solve the equation $(A-\lambda I)x=0$. Since the determinant $|A-\lambda I|$ is zero, there are solutions other than $x=0$. Those are the eigen vectors.

Example Find eigen values & Eigen vectors :-

$$A=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$$

Solution characteristic equation :- $|A-\lambda I|=0$

$$|A-\lambda I| = \begin{vmatrix}-\lambda & 1\\ -2 & -3-\lambda\end{vmatrix}=0$$

$$(-\lambda)(-3-\lambda)+2=0$$
$$+3\lambda+\lambda^2+2=0$$
$$\lambda^2+2\lambda+\lambda+2=0$$
$$\lambda(\lambda+2)+1(\lambda+2)=0$$
$$\lambda=-2\ ;\ \lambda=-1$$

Let's find the eigen vector v_1 associated with the eigen value $\lambda_1=-1$.

$$A\cdot v_1=\lambda_1\cdot v_1$$
$$(A-\lambda_1 I)v_1=0$$

$$\begin{bmatrix}-\lambda_1 & 1\\ -2 & -3-\lambda_1\end{bmatrix}v_1=0$$

$\boxed{\lambda_1=-1}$ $$\begin{bmatrix}1 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}v_{1,1}\\ v_{1,2}\end{bmatrix}=0$$

$$v_{1,1}+v_{1,2}=0 \qquad -2v_{1,1}-2\cdot v_{1,2}=0$$
$$-2v_{1,1}=2v_{1,2}$$
$$v_{1,1}=-v_{1,2}$$

Scanned with CamScanner

$$\begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = \begin{bmatrix} v_{1,1} \\ -v_{1,1} \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

→ Going through the same procedure for the second eigen value :-

$$A \cdot v_2 = \lambda_2 \cdot v_2$$

$$(A - \lambda I) v_2 = 0$$

$$\begin{bmatrix} -\lambda_2 & 1 \\ -2 & -3-\lambda_2 \end{bmatrix} v_2 = 0$$

for $\lambda_2 = -2 \Rightarrow \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} = 0$

$$2 v_{2,1} + v_{2,2} = 0$$

$$-2 v_{2,1} - v_{2,2} = 0$$

$$2(-v_{2,1}) = v_{2,2}$$

$$v_{2,2} = -2 v_{2,1}$$

$$\therefore \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} = \begin{bmatrix} v_{2,1} \\ -2v_{2,1} \end{bmatrix} = K \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Properties of Eigen Values & Eigen Vectors & Cayley Hamilton theorem :-

Properties of Eigen Values & Eigen vectors :-

→ If λ is an eigen value of A with x as the corresponding eigen vector then λ^2 is an eigen value of A^2 with the same eigen vector x.

→ For a given Eigen vector x, there corresponds only one eigen value λ.

→ For a given Eigen value, there corresponds infinitely many Eigen vectors.

→ $\lambda = 0$ is an eigen value of A, if and only if A is singular i.e $\det\{A\} = 0$.

→ If λ is an eigen value with x as the eigen vector then $\frac{1}{\lambda}$ is an eigen value of A^{-1} provided A^{-1} exists.

→ A and A^T have the same eigen values.

→ The Eigen values of a diagonal matrix are just the diagonal elements of the matrix.

→ The eigen values of an idempotent matrix are either zero or unity.

→ The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.

→ The product of the eigen values of a matrix A is equal to its determinant.

Cayley-Hamilton theorem - statement

→ "Every square matrix satisfies its own characteristic equation".

Example :- Find the eigen values & eigen vectors for the given matrix
$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution characteristic eaquation is $|A-\lambda I|=0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)[(3-\lambda)^2-1]+2[-2(3-\lambda)+2]+2[2-2(3-\lambda)]=0$$

$$(6-\lambda)[\lambda^2+9-6\lambda-1]+2[-6\lambda+2\lambda+2]+2[2-6+2\lambda]=0$$

$$(6-\lambda)[\lambda^2-6\lambda+8]+2[-4\lambda+2]+2[-4+2\lambda]=0$$

$$6\lambda^2-36\lambda+48-\lambda^3+6\lambda^2-8\lambda-8\lambda+4-8+4\lambda=0$$

$$-\lambda^3+12\lambda^2-48\lambda+44=0$$

$$\lambda^3-12\lambda^2+48\lambda-44=0$$

$$(6-\lambda)[9+\lambda^2-6\lambda-1]+2[-6+2\lambda+2]+2[2-6+2\lambda]=0$$

$$(6-\lambda)[\lambda^2-6\lambda+8]+2[-4+2\lambda]+2[-4+2\lambda]=0$$

$$6\lambda^2-36\lambda+48-\lambda^3+6\lambda^2-8\lambda-8+4\lambda-8+4\lambda=0$$

$$-\lambda^3+12\lambda^2-36\lambda+32=0$$

$$\lambda^3-12\lambda^2+36\lambda-32=0$$

$$8^3-12(64)+36(8)-32=0$$

$\lambda=8$ is one of the roots

8	1	-12	36	-32
	0	8	-32	32
	1	-4	4	$\boxed{0}$

$$\lambda^2-4\lambda+4=0$$

$$\lambda^2-2\lambda-2\lambda+4=0$$

$$\lambda(\lambda-2)-2(\lambda-2)=0$$

$$(\lambda-2)(\lambda-2)=0$$

$$\therefore \lambda=2,2$$

$\therefore$ The three eigen values are $\lambda_1=8, \lambda_2=2, \lambda_3=2$

Consider $|(A-\lambda I)|x=0$

$$(6-\lambda)x-2y+2z=0$$

$$-2x+(3-\lambda)y-z=0$$

$$2x-y+(3-\lambda)z=0$$

when $\lambda_1=8$

$$\left.\begin{aligned} -2x-2y+2z&=0 \\ -2x-5y-z&=0 \\ 2x-y-5z&=0 \end{aligned}\right\}$$

(22)

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

x, y = pivot variable

z = free variable

$-3y - 3z = 0$ $-2x - 2y + 2z = 0$

$-3y = 3z$ $-2x - 2(-z) + 2z = 0$

$\boxed{y = -z}$ $-2x + 2z + 2z = 0$

$-2x = -4z$

$\boxed{x = 2z}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

let $z = 1$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

when $\lambda_2 = 2$

$\Rightarrow$ $4x - 2y + 2z = 0$

$-2x + y - z = 0$

$2x - y + z = 0$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \xrightarrow[2R_3-R_1]{2R_2+R_1} \begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$4x - 2y + 2z = 0$

x = pivot variable; y & z = free variables

let $y = 1$ & $z = 0$ let $y = 0$ & $z = 1$

$4x - 2 = 0$ $4x + 2 = 0$

$4x = 2$ $4x = -2$

$x = 1/2$ $\boxed{x = -1/2}$

$4x - 2y + 2z = 0$

let $z = k_2$ & $y = k_1$

$$4x - 2k_1 + 2k_2 = 0$$

$$4x = 2k_1 - 2k_2$$

$$x = k_1/2 - k_2/2$$

$$\therefore \quad X_2 = \begin{pmatrix} k_1/2 - k_2/2 \\ k_1 \\ k_2 \end{pmatrix} = X_3$$

$\therefore$ Eigen vector $X = [X_1 \; X_2 \; X_3]$

where k_1 & k_2 are arbitrary and can take any value.

<u>Example</u> Find the eigen values and eigen vectors for the matrix :-

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

<u>Solution</u> :- $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & -\lambda & 0 \\ -1 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-\lambda)(-1-\lambda) + 1(1(-1-\lambda)) + 1(1-\lambda) = 0$$

$$(1-\lambda)\{\lambda + \lambda^2\} + 1\{-1-\lambda\} + 1 - \lambda = 0$$

$$\cancel{\lambda} + \cancel{\lambda^2} - \cancel{\lambda^2} - \lambda^3 - \cancel{1} - \cancel{\lambda} + \cancel{1} - \lambda = 0$$

$$-\lambda^3 - \lambda = 0$$

$$\Rightarrow \quad \lambda^3 + \lambda = 0$$

$\lambda(\lambda^2 + 1) = 0 \Rightarrow$ roots are $\lambda = 0$ & $\lambda = \pm i$

(24) Now consider $|A-\lambda I|x = 0$

$$(1-\lambda)x - y + z = 0$$
$$x - \lambda \cdot y + 0 \cdot z = 0$$
$$-x + y + (-1-\lambda)z = 0$$

Case i) When $\lambda = 0$ $\quad x - y + z = 0$

$$x = 0$$
$$-x + y - z = 0$$

$-y + z = 0 \qquad y = z$

$-y = -z \qquad$ let $z = k_2$ & $y = k_1$

$\boxed{y = z}$

$X_1 = \begin{bmatrix} 0 \\ k_1 \\ k_2 \end{bmatrix}$; let $k_1 = k_2 = 1$

$\therefore \quad X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Case ii) When $\lambda = i$; $\quad (1-i)x - y + z = 0$

$$-x + y + (-1-i)z = 0$$

$x = y(1-i) \qquad x - iy = 0$

$x = y + (-1-i)z \qquad x = iy$

$\cancel{y(1-i) = y + (-1-i)z} \qquad iy = y + (-1-i)z$

$\cancel{y(1-i) - y = (-1-i)z} \qquad y(i-1) = (-1-i)z$

$y(i-1) = (-1-i)z$

$z = \dfrac{y(i-1)}{-1-i} \times \dfrac{-1+i}{-1+i}$

$z = \dfrac{y(-i+i^2+1-i)}{(-1)^2-(i^2)}$

$z = \dfrac{y(-i-1+1-i)}{2}$

$\boxed{z = -i \cdot y}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} iy \\ y \\ -iy \end{pmatrix} = y\begin{pmatrix} i \\ 1 \\ -i \end{pmatrix}$

let $y = k_1 = 1$

$\therefore X_2 = \begin{bmatrix} i \\ 1 \\ -i \end{bmatrix}$

Case iii) when $\lambda = -i$:-

$$(1+i)x - y + z = 0$$

$$x + iy = 0 \Rightarrow \boxed{x = -iy}$$

$$-x + y + (-1+i)z = 0$$

$$x = y + (-1+i)z$$

$$-iy = y + (-1+i)z$$

$$(-1+i)z = -iy - y$$

$$(-1+i)z = y(-i-1)$$

$$z = y \, \frac{-i-1}{-1+i} \times \frac{-1-i}{-1-i}$$

$$z = \frac{y\{i + i^2 + 1 + i\}}{(-1)^2 - i^2}$$

$$z = \frac{y\{2i + 1 - 1\}}{2}$$

$$\boxed{z = iy}$$

$\therefore$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -iy \\ y \\ iy \end{pmatrix} = y \begin{pmatrix} -i \\ 1 \\ i \end{pmatrix}$ let $y = k_1 = 1$

$\therefore$ $X_3 = \begin{pmatrix} -i \\ 1 \\ i \end{pmatrix}$

$\therefore$ Eigen vectors corresponding to $\lambda = 0, i, -i$:-

$$X = [X_1 \; X_2 \; X_3] = \begin{bmatrix} 0 & i & -i \\ 1 & 1 & 1 \\ 1 & -i & i \end{bmatrix}$$

Example If $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ Evaluate A^{-1} & A^{-2} using Cayley Hamilton theorem

Solution :- $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\{(3-\lambda)(-3-\lambda)+8\} - 6\{1(-3-\lambda)+2\} + 6\{-4+(3-\lambda)\} = 0$$

$$(4-\lambda)\{-9-3\lambda+3\lambda+\lambda^2+8\} - 6\{-3-\lambda+2\} + 6\{-4+3-\lambda\} = 0$$

$$(4-\lambda)\{\lambda^2-1\} - 6\{-\lambda-1\} + 6\{-\lambda-1\} = 0$$

$$4\lambda^2 - 4 - \lambda^3 + \lambda + \cancel{6\lambda} + \cancel{6} - \cancel{6\lambda} - \cancel{6} = 0$$

$$-\lambda^3 + 4\lambda^2 + \lambda - 4 = 0$$

$$\boxed{\lambda^3 - 4\lambda^2 - \lambda + 4 = 0}$$

According to CH theorem :- λ to be replaced by A.

$$\therefore \quad A^3 - 4A^2 - A + 4I = 0 \quad - (1)$$

To find A^{-1}, multiply equation (1) by A^{-1}

$$A^{-1}\{A^3 - 4A^2 - A + 4I\} = 0$$

$$A^2 - 4A - I + 4A^{-1} = 0$$

$$4A^{-1} = -A^2 + 4A + I$$

$$A^{-1} = \frac{-A^2 + 4A + I}{4}$$

$$A^{-1} = \frac{1}{4}\left[-\underbrace{\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}}_{} + 4\underbrace{\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}}_{} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{} \right]$$

$$\therefore \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}$$

To find A^{-2}, multiply ① by A^{-2}

eqn ①

$$A^3 - 4A^2 - A + 4I = 0$$

$$A^{-2}\{A^3 - 4A^2 - A + 4I\} = 0$$

$$A - 4I - A^{-1} + 4A^{-2} = 0$$

$$4A^{-2} = -A + A^{-1} + 4I$$

$$A^{-2} = \frac{1}{4}\left[-A + A^{-1} + 4I\right]$$

$$A^{-2} = \frac{1}{4}\left[\underbrace{-\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}} + \underbrace{\frac{1}{4}\begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}} + \underbrace{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}} \right]$$

we can verify that :-

$$A^{-2} = \frac{1}{16}\begin{bmatrix} 1 & -18 & -18 \\ -5 & 10 & -6 \\ 5 & 6 & 22 \end{bmatrix}$$

Lectures 8 & 9 - Diagonalization of a matrix, Powers & Products of Matrices

Diagonalization of a Matrix :-

Statement :- If A is a square matrix of order 'n' has 'n' linearly independent vectors, then a matrix 'S' can be found such that $S^{-1}AS$ is a diagonal matrix.

Proof :- Let A be a square matrix of order 3.

Let λ_1, λ_2 and λ_3 be its eigen values and $X_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, $X_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ and $X_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$ be the corresponding eigen vectors.

Let S be a square matrix $S = [X_1 \; X_2 \; X_3]$

i.e $S = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$

multiplying by A, $AS = A[X_1, X_2, X_3]$

$$= [AX_1, AX_2, AX_3]$$

$$\Rightarrow AS = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$$

$$AS = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}_{D}$$

$$\boxed{\therefore AS = SD}$$

where D is the diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$$AS = SD$$

Multiplying both sides by S^{-1}

$$S^{-1}AS = SDS^{-1}$$

$$\boxed{S^{-1}AS = D} \quad \text{or} \quad S^{-1}AS = \Lambda$$

$$\boxed{A = SDS^{-1}}$$

S is invertible beceause its columns (the eigen vectors) were assumed to be independent.

Note :

1) Any matrix with distinct eigen values can be diagonalizable.

2) Not all matrices has 'n' linearly independent eigen vectors. Therefore, all the matrices can not be diagonalizable.

Eg $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, here $\lambda_1 = \lambda_2 = 0$. There is only one independent vector. Hence, we cannot construct S.

3) If eigen vectors $x_1, x_2 \dots x_k$ correspond to distinct eigen values $\lambda_1, \lambda_2 \dots \lambda_k$, then those eigen vectors are linearly independent.

4) Eigen vector matrix is not unique since if x is an eigen vector corresponding to λ, then kx is also an eigen vector.

Example Factorize the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ into $S\Lambda S^{-1}$ and also find SAS^{-1}.

Solution consider $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 1 = 0 \qquad 1+\lambda^2-2\lambda-1=0$$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda-2) = 0$$

$$\lambda = 0 \;\&\; \lambda = 2$$

$$\therefore \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

To find eigen ~~value~~ vectors :-

$$[A-\lambda I]x = 0$$

$$(1-\lambda)x + y = 0$$

$$x + (1-\lambda)y = 0$$

case 1 **for $\lambda = 0$**

$$x+y=0$$

$$x+y=0$$

$$x=-y$$

let $y = K$ & $x = -K$

$$X_1 = \begin{bmatrix} -K \\ K \end{bmatrix} = K\begin{bmatrix} -1 \\ 1 \end{bmatrix} ; \text{ If } K=1 \Rightarrow X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

case 2 for $\lambda = 2$

$$-x+y=0$$

$$x-y=0$$

$$x=y$$

$x = y = k$; let $k=1$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ $S = [x_1 \; x_2] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$; $S^{-1} = \frac{1}{-1-1}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

$S^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & +1/2 \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

$A = S \Lambda S^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & +1/2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & +1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\therefore$ $\boxed{A = S \Lambda S^{-1}}$

to find SAS^{-1} :-

$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

$\therefore$ $\boxed{\Lambda = SAS^{-1}}$

Example Find the matrix A whose eigen values are 2, 5 and eigen vectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

solution $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; $S^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

$\therefore$ $A = S \Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

(32)

Example Find the eigen values and Eigen vectors of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and write two different diagonalizing matrices.

Solution: $|A - \lambda I| = 0$ $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)[(1-\lambda)^2 - 1] - 1[(1-\lambda)-1] + 1[1-(1-\lambda)] = 0$$

$$(1-\lambda)[\lambda^2 + \cancel{1} - 2\lambda - \cancel{1}] - 1[\cancel{1} - \lambda - \cancel{1}] + 1[\cancel{1} - \cancel{1} + \lambda] = 0$$

$$(1-\lambda)[\lambda^2 - 2\lambda] - 1[-\lambda] + 1[\lambda] = 0$$

$$\lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2 + \cancel{\lambda} + \cancel{\lambda} = 0$$

$$-\lambda^3 + 3\lambda^2 = 0$$

$$\lambda^3 - 3\lambda^2 = 0$$

$$\lambda^2(\lambda - 3) = 0$$

$$\boxed{\lambda = 0, 0, 3}$$

consider $(A - \lambda I)x = 0$

$$(1-\lambda)x + y + z = 0$$

$$x + (1-\lambda)y + z = 0$$

$$x + y + (1-\lambda)z = 0$$

case 1) $\lambda = 0$

$$x + y + z = 0$$

$$x + y + z = 0$$

$$x + y + z = 0$$

$x = -y - z$

let $y = k_1$; $z = k_2$

$x = -k_1 - k_2$

let $K_1 = K_2 = 1$

$$\therefore X_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

<u>case 2)</u> <u>$\lambda = 0$</u>

let $K_1 = 0$ & $K_2 = 1$ or let $K_1 = 1$ & $K_2 = 0$

$$\therefore X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

<u>case 3)</u> <u>when $\lambda = 3$</u>

$$-2x + y + z = 0$$
$$x - 2y + z = 0$$
$$x + y - 2z = 0$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[2R_3 + R_1]{2R_2 + R_1} \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

x, y = pivot variable & z = free variable

$$-3y + 3z = 0$$
$$-3y = -3z$$
$$\boxed{y = z}$$

let $z = K_1$

$y = K_1$

$$-2x + y + z = 0$$
$$-2x + K_1 + K_1 = 0$$
$$-2x = -2K_1$$
$$x = K_1$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = K_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

let $K_1 = 3$

$$\therefore X_3 = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

(34) $\therefore$ Two diagonalizing matrices are :-

$$S_1 = \begin{bmatrix} -2 & -1 & 3 \\ 1 & 0 & 3 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{or} \quad S_2 = \begin{bmatrix} -2 & -1 & 3 \\ 1 & 1 & 3 \\ 1 & 0 & 3 \end{bmatrix}$$

Powers & Products of Matrices :-

→ computation of Powers of a square matrix :-

→ Diagonalization of a Square matrix A also helps to find the powers of $A, A^2, \ldots$ e.t.c

→ We have $D = S^{-1}AS$

$$D^2 = (S^{-1}AS)(S^{-1}AS)$$

$$\cancel{D^2 = S^{-1} \cdot S \cdot A^2}$$

$$D^2 = S^{-1} \cdot A \cdot \underbrace{S \cdot S^{-1}}_{I} \cdot A \cdot S$$

$$D^2 = S^{-1} \cdot A^2 \cdot S$$

Premultiplying by S and Post multiplying by S^{-1}

$$S D^2 S^{-1} = S \cdot S^{-1} \cdot A^2 \cdot S \cdot S^{-1}$$

$$= I \cdot A^2 \cdot I$$

$$= A^2$$

$$\therefore \boxed{A^2 = S \cdot D^2 \cdot S^{-1}}$$

In general, $A^n = SD^nS^{-1}$

where
$$D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 & 0 & \cdots \\ 0 & \lambda_2^n & 0 & 0 & \cdots \\ 0 & 0 & \lambda_3^n & 0 & \cdots \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \lambda_n^n \end{bmatrix}$$

<u>Example</u> Diagonalize the matrix $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ and find one of its square roots. How many square roots will be there ?

<u>Solution</u> :- $|A - \lambda I| = 0$ $\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = 0$

$$(5-\lambda)^2 - 16 = 0$$
$$5^2 + \lambda^2 - 10\lambda - 16 = 0$$
$$\lambda^2 - 10\lambda + 9 = 0$$
$$\lambda^2 - 9\lambda - \lambda + 9 = 0$$
$$\lambda(\lambda - 9) - 1(\lambda - 9) = 0$$
$$(\lambda - 9)(\lambda - 1) = 0$$
$$\lambda = 9 \text{ or } \lambda = 1$$

consider $[A - \lambda I]x = 0$

$$(5-\lambda)x + 4y = 0$$
$$4x + (5-\lambda)y = 0$$

case 1) for $\lambda = 1$

$$4x + 4y = 0$$
$$4x + 4y = 0$$
$$x = -y$$

let $y = K ; x = -K$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = K \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

let $K = 1$

$$x_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

<u>case 2</u> for $\lambda = 9$

$$-4x + 4y = 0$$
$$4x - 4y = 0$$
$$x = y$$

let $y = K = x$

let $K = 1$

$$\therefore x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$S = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \wedge = 0$$

$$\wedge = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$$

$$S^{-1} = \frac{1}{-2} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

we know that $A = SDS^{-1}$

or

$$A = S\Lambda S^{-1}$$

$$A^{1/2} = S\Lambda^{1/2}S^{-1}$$

$$A^{1/2} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^{1/2} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

we have 4 square roots $\because \Lambda^{1/2} = \sqrt{1} = \pm 1$ and also $\sqrt{9} = \pm 3$. we got different values when we consider $\pm 1, \pm 3$.

<u>Example</u> Diagonalize $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and hence find A^{100}. Show that $A^{100} = A$.

<u>solution:</u> $|A - \lambda I| = 0$ $\quad \begin{vmatrix} 1/2 - \lambda & 1/2 \\ 1/2 & 1/2 - \lambda \end{vmatrix} = 0$

$$\left(1/2 - \lambda\right)^2 - 1/4 = 0$$

$$\frac{1}{4} + \lambda^2 - \lambda - 1/4 = 0$$

$$\lambda^2 - \lambda = 0$$

$$\lambda(\lambda - 1) = 0$$

$$\lambda = 0 \text{ or } \lambda = 1$$

consider $[A - \lambda I]x = 0$ $\quad \left(1/2 - \lambda\right)x + 1/2\, y = 0$

$$1/2\, x + \left(1/2 - \lambda\right) y = 0$$

Case 1) $\lambda = 0$

$$\tfrac{1}{2}x + \tfrac{1}{2}y = 0$$

$$\tfrac{1}{2}x + \tfrac{1}{2}y = 0$$

$$x = -y$$

let $y = K \Rightarrow x = -K$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = K\begin{pmatrix} -1 \\ 1 \end{pmatrix}$; let $K = 1$

$\therefore X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Case 2) $\lambda = 1$

$$-\tfrac{1}{2}x + \tfrac{1}{2}y = 0$$

$$\tfrac{1}{2}x - \tfrac{1}{2}y = 0$$

$$\boxed{x = y}$$

$\therefore$ let $y = K$; $\Rightarrow x = K$; let $K = 1$

$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow S = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$$A = S \wedge S^{-1}$$

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$A^{100} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^{100}\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\boxed{\therefore A^{100} = A}$$

Spectral theorem - Statement :-

An $n \times n$ Symmetric matrix A has the following properties :-

a) A has 'n' real eigen values, counting multiplicities.

b) The dimension of the eigenspace for each eigen value λ equals the multiplicity of λ as a root of the characteristic equation.

c) The eigen spaces are mutually orthogonal, in the sense that the eigen vectors corresponding to different eigen values are orthogonal.

d) A is orthogonally diagonalizable.

Spectral Decomposition theorem - Statement :-

Let A be a $n \times n$ matrix. Let $\{u_1, u_2 \ldots u_n\}$ be an orthonormal basis for $\mathbb{R}^n$ consisting of eigen vectors of A with corresponding eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$. Then there exists symmetric matrices $P_1, P_2 \ldots P_n$ such that the following results hold :-

i) $A = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n \cdot P_n$

ii) Rank $P_i = 1$ for all i

iii) $P_i \cdot P_i = P_i$ for all i and $P_i \cdot P_j = 0$ for $i \neq j$

iv) $P_i \cdot u_i = u_i$ for all i and $P_i \cdot u_j = 0$ for $i \neq j$.

Spectral theorem example :-

Let $A = \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$

To find eigen values :- $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -4 \\ -4 & -3-\lambda \end{vmatrix} = 0 \; ; \; (3-\lambda)(-3-\lambda) - 16 = 0$$

$$-9 + 3\lambda - 3\lambda + \lambda^2 - 16 = 0$$

$$\lambda^2 - 25 = 0$$

$$\lambda^2 = 25 \; ; \quad \lambda = \pm 5$$

consider $[A-\lambda I]x=0$

$$(3-\lambda)x-4y=0$$

$$-4x+(-3-\lambda)y=0$$

case 1) $\lambda=-5$

$$8x-4y=0$$

$$-4x+2y=0$$

$$-4x=-2y$$

$$4x=2y$$

$$2x=y$$

$$x=y/2$$

$\therefore\ X_1=\begin{pmatrix}1/2\\1\end{pmatrix}$

here let $y=2$

$$X_1=\begin{pmatrix}1\\2\end{pmatrix}$$

case 2) $\lambda=5$

$$-2x-4y=0$$

$$-4x-8y=0$$

$$-4x=8y$$

$$x=-2y$$

let $y=1$

$$x=-2$$

$\therefore\ X_2=\begin{pmatrix}-2\\1\end{pmatrix}$

The orthonormal eigen vectors are $u_1=\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}$ and $u_2=\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$

$$P_1=u_1\cdot u_1^T=\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}-2 & 1\end{pmatrix}=\frac{1}{5}\begin{pmatrix}4 & -2\\-2 & 1\end{pmatrix}$$

$$P_2=u_2\cdot u_2^T=\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\end{pmatrix}=\frac{1}{5}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}$$

We can verify that :- $\boxed{A=\lambda_1\cdot P_1+\lambda_2 P_2}$