Contents

Vector Spaces and Subspaces (definitions only), Linear Independence, Basis Dimensions and Span, The Four Fundamental Subspaces.

Self Learning Component: Examples of Vector Spaces and Subspaces.

Definition of Vector space:

A **real vector space** is a nonempty set V of vectors together with rules for vector addition and multiplication by scalars.

Addition and multiplication must produce vectors in the space and they must satisfy the following conditions:

For all $x, y, z \in V$ and $c, c_1, c_2 \in R$,

1. Closure : $x + y \in V$ for all $x, y \in V$
2. Commutativity : $x + y = y + x$
3. Associativity : $x + (y + z) = (x + y) + z$
4. Identity: There exists a unique *zero vector* "0" such that $x + 0 = 0 + x$
5. Inverse : For each x there is a unique vector $-x$ such that $x + (-x) = 0$
6. Closure: $c.x \in V$
7. $1.x = x$
8. $(c_1 c_2)x = c_1(c_2 x)$
9. $c(x + y) = cx + cy$
10. $(c_1 + c_2)x = c_1 x + c_2 x$

Few examples:

1. R = the set of all real numbers
2. $R^2 = \{ (x, y) \quad / x, y \in R \}$
3. $R^3 = \{ (x, y, z) \quad / x, y, z \in R \}$
4. $R^n = \{ (x_1, x_2, \ldots, x_n) / x_i \in R \}$
5 $R^\infty = \{ (x_1, x_2, \ldots\ldots,) / x_i \in R \}$
6. The space of all m x n matrices

Subspace

Definition:

A *subspace* S of a vector space V is a nonempty subset that satisfies the following two conditions:

For all x, y ∈ S and c ∈ R

i) $x + y \in S$

ii) $cx \in S$

Note: The smallest subspace Z contains only one vector, the zero element. It is the zero dimensional space containing only the point at the origin. At the other extreme, the largest subspace is the whole of the original space.

Examples

1. For V = R , the set of reals, the possible subspaces are

(i) $Z = \{ 0 \}$

(ii) R itself

2. For $V = R^2$, the possible subspaces are

(i) Z = { (0 , 0) }

(ii) all straight lines passing through (0, 0)

(iii) R^2 itself

3. For $V= R^3$, the possible subspaces are

(i) Z={ (0 , 0, 0) }

(ii) all lines passing through (0, 0, 0)

(iii) all planes passing through (0, 0, 0)

(iv) R^3 itself

In general, if $V = R^n$, the possible subspaces are Z , lines through origin, 2-d planes through origin, 3-d planes through origin, ….. , (n-1)- d planes through origin and the space R^n itself.

Column space of A

The column space of A is the set of all linear combinations of the independent columns of A and it is denoted by C(A). Thus,

$C(A) = \{ b \in R^m / Ax = b \text{ is solvable} \}$

Note: C (A)is a subspace of R^m.

Examples

1. The smallest possible column space comes from the zero matrix A = 0. The only combination of the columns is b = 0.

2. If A is a 5 x 5 identity matrix then C(A) is the whole of R^5, the 5 columns of A can combine to produce any 5 dimensional vector b. In fact, any 5 x 5 nonsingular matrix A will have R^5 as its column space!!

3. $A=\begin{bmatrix}1 & 0\\ 2 & 3\\ 1 & 6\end{bmatrix}$

$C(A)$ is linear combination of the columns (1, 2, 1) and (0, 3, 6). Geometrically the C(A) is a 2- d plane in R^3.

4. $B=\begin{bmatrix}1 & 0 & 2\\ 2 & 3 & 4\\ 1 & 6 & 2\end{bmatrix}$

C(B) is the linear combination of independent columns of the vectors (1, 2, 1) and (0, 3, 6) .Here the third column is a linear combination of the other two columns.

The Null Space of A

The null space of A is the set of all solutions of the homogeneous system of equations Ax = 0 denoted by N (A) . Thus,

N(A) ={x ∈ R^n /Ax= 0 }

Note: 1) N(A) is a subspace of R^n.

2) The dimension of the null space of a matrix is called its nullity

Examples

1. $A=\begin{bmatrix}1 & 0\\ 2 & 3\\ 1 & 6\end{bmatrix}$

Now consider $Ax=0$

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$x = 0, y = 0$$

Thus Null space is zero vector or (0, 0). Geometrically N(A) is an origin in 2 dimensional vector space.

2. $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix}$

Now consider

$$Ax = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0;$$

$$x + 2z = 0;\ x = -2z$$

$$3y = 0$$

Ax=0 has infinitely many solutions (-2k, 0 ,k) all of which lie on a line that obviously passes through the origin.

Note: The matrices $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix}$

have the same column space but different null space!!

Echelon Form of a Matrix

A matrix A of order mxn is said to be in **echelon form U** if

i) Pivots are the first nonzero entries in their rows

ii) Below each pivot is a column of zeros

iii) Each pivot lies to the right of the pivots in the rows above

iv) Zero rows, if any, come last.

Row Reduced Form of a Matrix

Let A be a matrix of order m x n and U be its echelon form. Then the matrix A is said to be in row **reduced echelon form R** if in U

i) the pivots are all 1 and

ii) there are zeros above the pivots

Rank of a Matrix

The rank of a matrix A is the number of nonzero rows in the echelon form U of A and it is denoted by ρ (A) or simply r.

Note:

If A is a matrix of order m x n then its rank r ≤ m.

Pivot variables & Free Variables

Consider a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & -1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is the Row echelon form of A.

$$U = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

is the Row Reduced form of A

The solutions of Rx = 0 (or Ux = 0 or Ax = 0) are

$$U = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$UX = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0$$

$$X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -2y-3z-t \\ 5z+3t \\ z \\ t \end{bmatrix} = z\begin{bmatrix} 1 \\ 5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -11 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

The variables x and y whose columns contain the pivots are called **pivot variables** and the remaining variables z and t are called **free variables**. The vectors (1, 5, 1, 0) and (-11, 5, 0, 1) are called the **special solutions** of Ax = 0. All the other solutions are linear combinations of these two.

Note:

If Ax = 0 has more unknowns than the equations (n > m) it has at least one special solution. There are more solutions than the trivial x = 0.

Linear Independence, Basis and Dimension

A set of vectors v_1, v_2, …., v_k of a vector space V is said to be **linearly independent** if the equation

$c_1v_1 + c_2v_2 + ... + c_kv_k = 0$, $c_i \in R$

holds if and only if $c_i = 0$ for all i.

If any of the $c_i \neq 0$ then the set is **linearly dependent.**

Examples:

1. The set containing only the zero vector is dependent. For, we choose some c ≠ 0.

2. The columns of the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 6 \end{bmatrix}$ are linearly independent

3.The columns of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 6 \end{bmatrix}$ are linearly dependent

4. The columns of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$ are linearly dependent

Note:

1. The columns of a square invertible matrix are always independent.
2. The columns of a matrix A of order m x n with m < n are always dependent.
3. The columns of A are independent exactly when N(A) = Z.
4. The r nonzero rows of an echelon matrix U and a reduced matrix R are always independent and so are the r columns that contain the pivots.
5. Columns with pivots are linearly independent columns
6. Columns without pivots are linearly pendent columns

Span of set

A set of vectors $v_1, v_2, \ldots, v_k$ of a vector space V is said to **span** V if every $v \in V$ is a linear combination of these v_i's.

Note:

(i) The columns of A span C(A).

(ii) The columns (rows) of a square invertible matrix A of order n x n span the whole of R^n.

(iii) The columns of the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 6 \end{bmatrix}$ span the two dimensional plane

Bases

A **basis** for a vector space V is a set of vectors having the following two properties at once:

i. the vectors are linearly independent
ii. the vectors span the space V

Note:

a) Every vector v in V is a unique combination of the base vectors.
b) A basis for V is not unique.
c) The columns of A that contain the pivots form a basis for C(A).
d) A basis for V is a maximal independent set and also a minimal spanning set.

Dimension

Any two bases for V have the same number of vectors. This number which is common to all the bases is called the **dimension** of the vector space V.

Note: The dimension of a vector space is unique!!

Examples:

1. $\{(1,2,3),(2,1,4)\}$ is the basis for the 2d plane in R^3

   Dimension=2

2. $\{(2,3,1),(1,1,4),(1,4,2)\}$ is the basis for R^3

   Dimension=3

3. The columns of the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ form a basis for R^2

   Dimension=2

4. $\{(1,0,1)\}$ is basis for a line in R^3

   Dimension=1

5. $\{(1,0),(0,1)\}$ is the standard basis for R^2

6. $\{(1,0,0),(0,1,0),(0,0,1)\}$ is the standard basis for R^3

The Four Fundamental Subspaces:

Let A be a matrix of order m x n. The following are called the four fundamental subspaces of A

1. The column space of A denoted by C(A)
2. The null space of A denoted by N(A)
3. The row space of A denoted by $C(A^T)$
4. The left null space of A denoted by $N(A^T)$

Note: 1) N(A) and $C(A^T)$ are subspaces of R^n.

2) $N(A^T)$ and C(A) are subspaces of R^m.

3) Dim C(A) = Dim $C(A^T)$ = r = rank of A.

4) Dim N(A) = n-r and Dim $N(A^T)$ = m- r.

Definitions:

Column space of A

The column space of A is the set of all linear combinations of the independent columns of A and it is denoted by C(A). Thus,

$C(A) = \{ b \in R^m / Ax = b \text{ is solvable} \}$

Note: C (A)is a subspace of R^m.

Examples

1. The smallest possible column space comes from the zero matrix A = 0. The only combination of the columns is b = 0.

2. If A is a 5 x 5 identity matrix then C(A) is the whole of R^5, the 5 columns of A can combine to produce any 5 dimensional vector b. In fact, any 5 x 5 nonsingular matrix A will have R^5 as its column space!!

3. $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 6 \end{bmatrix}$

$C(A)$ is linear combination of the columns (1, 2, 1) and (0, 3, 6). Geometrically the C(A) is a 2- d plane in R^3.

4. $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 1 & 6 & 2 \end{bmatrix}$

C(B) is the linear combination of independent columns of the vectors (1, 2, 1) and (0, 3, 6) .Here the third column is a linear combination of the other two columns.

The Null Space of A

The null space of A is the set of all solutions of the homogeneous system of equations Ax = 0 denoted by N (A) . Thus,

$N(A) = \{x \in R^n / Ax = 0\}$

Note: 1) N(A) is a subspace of R^n.

2) The dimension of the null space of a matrix is called its nullity

Examples

1. $$A=\begin{bmatrix}1&0\\2&3\\1&6\end{bmatrix}\sim\begin{bmatrix}1&0\\0&3\\0&6\end{bmatrix}\sim\begin{bmatrix}1&0\\0&3\\0&0\end{bmatrix}$$

Now consider $Ax=0$

$$\begin{bmatrix}1&0\\0&3\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0$$

$$x=0, y=0$$

Thus Null space is zero vector or (0, 0). Geometrically N(A) is an origin in 2 dimensional vector space.

2. $$B=\begin{bmatrix}1&0&2\\2&3&4\\1&6&2\end{bmatrix}\sim\begin{bmatrix}1&0&2\\0&3&0\\0&6&0\end{bmatrix}\sim\begin{bmatrix}1&0&2\\0&3&0\\0&0&0\end{bmatrix}$$

Now consider

$$Ax=0\Rightarrow\begin{bmatrix}1&0&2\\0&3&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=0;$$

$$x+2z=0;\ x=-2z$$

$$3y=0$$

Since z is free variable, Ax=0 has infinitely many solutions

i.e x=-2k, y=0 and z=k

Note: The matrices $A=\begin{bmatrix}1 & 0\\ 2 & 3\\ 1 & 6\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 2\\ 2 & 3 & 4\\ 1 & 6 & 2\end{bmatrix}$

have the same column space but different null space!!

The row space of A

The row space of A_{mxn} is the column space of A^T. It is spanned by the rows of A.

For example

$$A=\begin{bmatrix}1 & 0 & 2\\ 2 & 3 & 4\\ 1 & 6 & 2\end{bmatrix}$$

The $C(A^T)$ is the linear combination of independent rows of the matrix A. ie (1, 0, 2) and (2, 3, 4)

The Left Null space of A

The left null space contains all vectors y for which ATy = 0.

For example:

$$A=\begin{bmatrix}1 & 0 & 1\\ 2 & 3 & 4\\ 2 & 0 & 2\end{bmatrix}$$

Consider

$$A^T y = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0; \quad \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0;$$

$$x + 2y + 2z = 0; \; x = -2y - 2z; \; x = -2k$$

$$3y = 0; \quad y = 0$$

$$z = k;$$

Therefore Null space contains the vectors (-2k, 0, k)

1. $N(A)$ and $C(A^T)$ are subspaces of R^n.
2. $N(A^T)$ and $C(A)$ are subspaces of R^m.
3. Dim $C(A)$ = Dim $C(A^T)$ = r = rank of A.
4. Dim $N(A)$ = n-r and Dim $N(A^T)$ = m- r.
5. The dimension of the null space of a matrix is called its ***nullity***.
6. The rank- nullity theorem: For any matrix A_{mxn},

   dim C(A) + dim N(A) = no. of columns

   i.e r + (n-r)= n

   This law applies to A^T as well.

   Hence, dim $C(A^T)$ + dim $N(A^T)$ = m

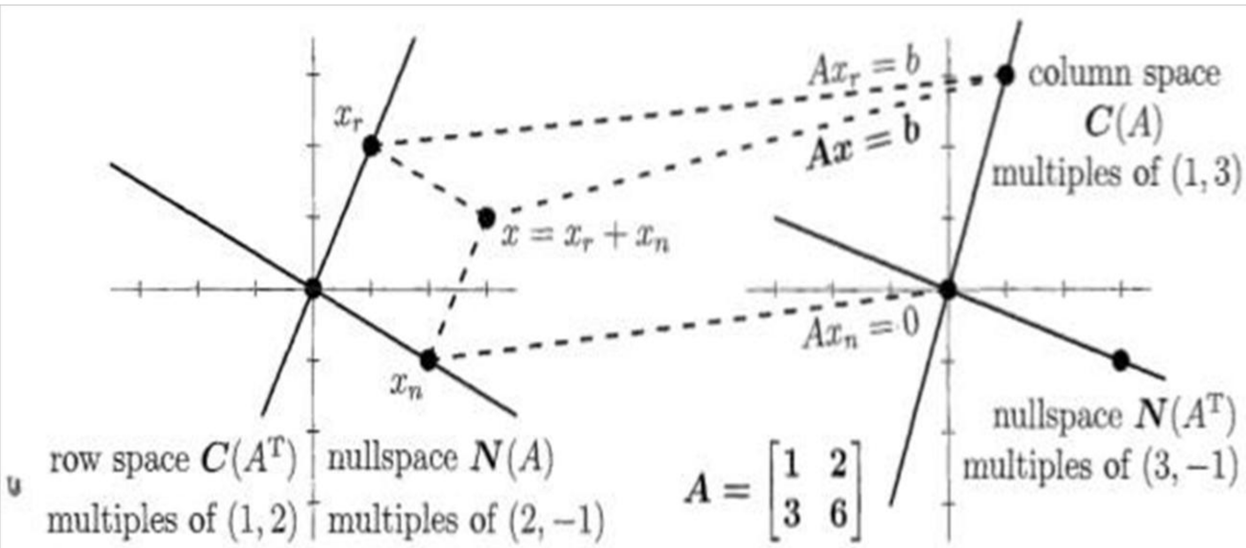
   i. e r + (m- r) = m

Examples:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Then, m = n= 2 and rank r = 1.

1. C(A) is the line through (1, 3)
2. $C(A^T)$ is the line through (1, 2)
3. N(A) is the line through (-2 , 1)
4. $N(A^T)$ is the line through (-3, 1)



Problem on fundamental subspaces:

Find the bases and dimension of the four fundamental subspaces of the matrix $A=\begin{bmatrix}1 & 2 & 1 & 2\\ 1 & 2 & 1 & 3\\ 3 & 6 & 3 & 7\end{bmatrix}$

Ans: $A=\begin{bmatrix}1 & 2 & 1 & 2\\ 1 & 2 & 1 & 3\\ 3 & 6 & 3 & 7\end{bmatrix};\begin{bmatrix}a\\ b\\ c\end{bmatrix}$

$$=\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\end{bmatrix};\begin{bmatrix}a\\ b-a\\ c-3a\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix};\begin{bmatrix}a\\ b-a\\ c-2a-b\end{bmatrix}$$

Basis of C(A)={ (1,1,3), (2,3,7)}; Dim=2

Basis of C(A^T)={ (1,2,1,2), (1,2,1,3)}; Dim=2

Basis of N(A^T)={ (-2,-1,1)}; Dim=2

To find the basis of N(A):

Consider UX=0 or AX=0

i.e

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0;$$

$$x + 2y + z + 2t = 0; x = -(2y + z + 2t)$$
$$y = 0;$$

Here y and z are free variables

Basis for N(A)={(-2,1,0,0),(-1,0,1,0)} ; Dim=2

Existence of Inverses

Let A_{mxn} (m ≤ n) be a matrix such that r = m. Then Ax = b has at least one solution x for every b if and only if the columns span R^m. In this case, A has a **right inverse** C such that AC= I_m.

Let A_{mxn} (m ≥ n) be a matrix such that r = n. Then Ax = b has at most one solution x for every b if and only if the columns are linearly independent. In this case, A has a **left inverse *B*** such that BA= I_n.

Note:

1. The right (left) inverse of a matrix is not unique.
2. When m= n, the matrix A has a unique inverse (B = C).
3. The best one sided inverses can be found using

$B = (A^T A)^{-1} A^T$, $C = A^T (AA^T)^{-1}$

Examples:

1. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}_{2\times 3}$

Rank r=2=m(m<n)

$$C = A^T\left(AA^T\right)^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/4 & 0 \\ 0 & 1/16 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$$

2. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3\times 2}$

Rank r=2=n(n<m)

$$B = \left(A^T A\right)^{-1} A^T = \begin{bmatrix} 3/2 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3/2 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

Matrices of Rank One

When the rank of a matrix is as small as possible, a complicated system of equations can be broken into simple pieces. Those simple pieces are matrices of rank one

The matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \end{bmatrix}$ has the rank 1

i.e. $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$

We can write such matrices as a column times row.