Linear Algebra - unit 3

Orthogonality

Lectures 1 & 2 - Linear Transformations, Examples, Transformations represented by Matrices :-

Linear Transformations - Definition :-

Let A be a matrix of order n. When A multiplies a n-dimensional vector x, it transforms x to a n-dimensional vector Ax. This happens at every x in $\mathbb{R}^n$. The whole space $\mathbb{R}^n$ is transformed or mapped into itself by the matrix A. The matrix A induces a transformation of $\mathbb{R}^n$.

Example 1 Let $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

If $x = (x, y)$ then $Ax = A(x, y)$.

A multiple of the identity matrix $A = cI$ streches every vector by the scale factor c. The whole space expands or contracts.

Example 2 Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

If $x = (x, y)$ then $Ax = (-y, x)$

$\therefore$ The matrix A rotates every vector about the origin through a right angle in the counter clock wise direction.

(2)

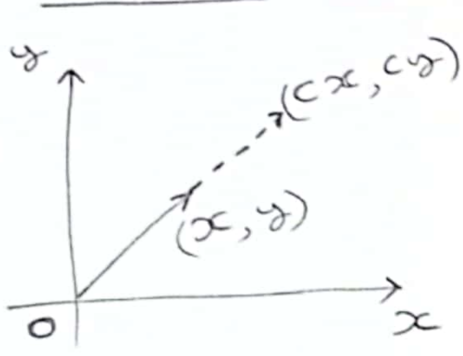
Example 3 Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

If $x = (x, y)$ then $Ax = (y, x)$

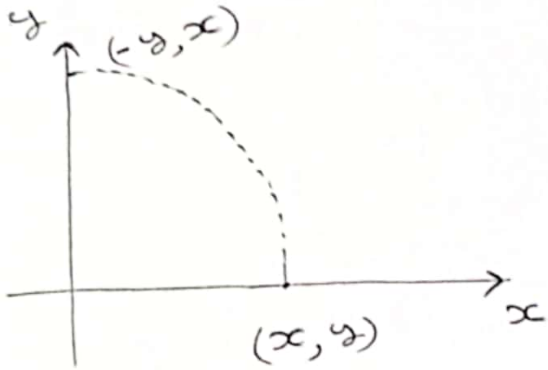
The matrix A reflects every vector on the $y = x$. It is also a permutation matrix.

Example 4 Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

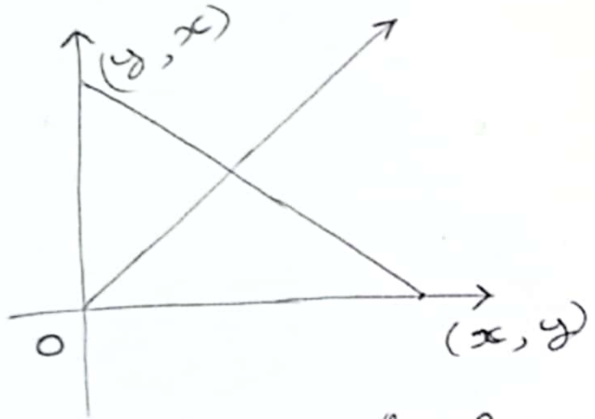
If $x = (x, y)$ then $Ax = (x, 0)$. Then matrix A projects every vector onto the x-axis.



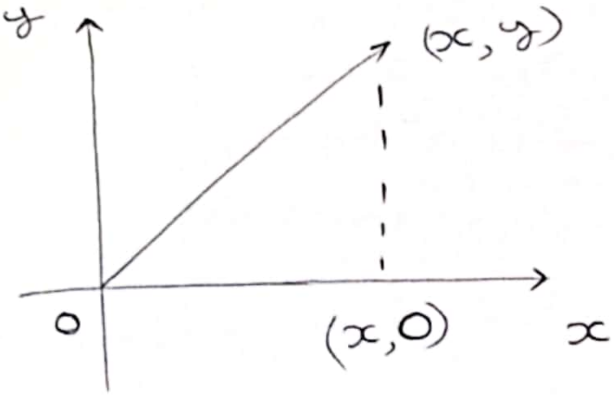
streching



90° rotation



Reflection (45° mirror)



Projection on x-axis

Note A transformation can now be understood as a function (or a mapping) $f : A \to B$ defined by $f(x) = y$. In terms of matrices, we have the rule :-

$A : R^n \to R^m$ defined by $Ax = b$.

A transformation T on $\mathbb{R}^n$ is said to be linear if it satisfies the rule of linearity.

$$T\{cx+dy\} = c.T_x + d.T_y$$

$\forall$ scalars c, d & vectors x & y respectively

Note 1) If T is linear then $T(0)=0$ i.e T preserves origin. The converse may or may not be true.

2) If A is a matrix of order $m \times n$ then A induces a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Examples

Let $v=(v_1, v_2)$, then

1. $T(v) = (v_2, v_1)$ is linear
2. $T(v) = (v_1, v_1)$ is not linear
3. $T(v) = (0, v_1)$ is not linear
4. $T(v) = (0, 1)$ is not linear
5. $T(v) = (v_1, v_2)$ is linear

Note If a transformation preserves origin, it may or may not be linear.

A transformation T is said to be linear if it satisfy the rule of linearity.

i.e $A(cx+dy) = c.A(x) + d.A(y)$

Example In a linear system of equations $Ax=b$, matrix A is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Note :- consider a transformation $T: A \to B$, where A & B are subspaces.

1) A is the domain of the transformation.

2) B is the co-domain of the transformation

3) For any x in A, there exists Tx in B. Here T_x is the image of T and x is the pre image of T_x.

4) The set of all images in the subset of B is called the range of the transformation.

5) For all x in such that $Tx = 0$ is called the kernel of the transformation.

6) Dimension of the range is called rank and dimension of kernel is called nullity.

Example The space of all polynomials in t of degree n is a vector space called the polynomial space denoted by P_n.

$$P_n = \{\text{Its basis is } 1, t, t^2 \ldots t^n \text{ and dimension is } n+1\}$$

Example The operation of differentiation is linear. It takes P_{n+1} to P_n. The column space is the whole of P_n and the null space is P_0, the one dimensional space of all constants.

Example The operation of integration is linear. It takes P_n to P_{n+1}. The column space is a subspace of P_{n+1} and the null space is just the zero vector.

Example Multiplication by a fixed polynomial say $3+4t$ is also a linear transformation.

Let $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n \cdot t^n$, then

$$A \cdot p(t) = (3+4t)\, p(t)$$
$$= 3a_0 + \ldots\ldots + 4a_n \cdot t^{n+1}$$

This A sends P_n to P_{n+1}.

Transformation represented by Matrices :-

→ The matrix of a linear transformation is a matrix for which $T(x) = Ax$, for a vector 'x' in the domain T. Such matrix is called the standard matrix for the transformation.

Note :- 1) such matrix can be found for any linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$.

2) standard basis for $\mathbb{R}^n$:- $e_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ldots e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

e_i's are columns of Identity matrix order 'n'.

⑥

The standard matrix of transformation :-

$T : \mathbb{R}^n \to \mathbb{R}^m$ has columns

$T(e_1), T(e_2), \dots, T(e_n)$ where $e_1, e_2 \dots e_n$

represents the standard basis. i.e

$$T(x) = Ax \Leftrightarrow A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}$$

<u>Example 1</u> $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_3 \end{bmatrix}$

Here $T : \mathbb{R}^3 \to \mathbb{R}^2$

Basis for $\mathbb{R}^3$:- $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$T(e_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 - 0 \\ 2(0) \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$T(e_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 - 1 \\ 2(0) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$T(e_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ 2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

The standard matrix for T is :-

$$T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Verification :- $Ax = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_3 \end{bmatrix}$

Compare this to the rule for T from the problem :- $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_3 \end{bmatrix}$

Matrix representation of Differentiation :-

→ Consider differentiation that goes from P_3 to P_2.

→ $P_3 = p(t) = \{a_0 + a_1 t + a_2 t^2 + a_3 \cdot t^3 ; a_0, a_1, a_2 \in \mathbb{R}\}$

→ Basis is $\{v_1 = 1, v_2 = t, v_3 = t^2, v_4 = t^3\}$

→ $P_2 = \{ q(t) = b_0 + b_1 t + b_2 \cdot t^2, b_0, b_1, b_2 \in \mathbb{R}\}$

→ Basis is $\{u_1 = 1, u_2 = t, u_3 = t^2\}$

→ $\frac{d}{dt} = A_{diff} : P_3 \to P_2$

→ **To find A_{diff} :-**

→ $\frac{d}{dt}\{v_1\} = 0 = 0 \cdot u_1 + 0 \cdot u_2 + 0 \cdot u_3 = (0, 0, 0)$

→ $\frac{d}{dt}\{v_2\} = \frac{d}{dt}(t) = 1 = 1 \cdot u_1 + 0 \cdot u_2 + 0 \cdot u_3 \to (1, 0, 0)$

→ $\frac{d}{dt}\{v_3\} = \frac{d}{dt}\{t^2\} = 2t = 0 \cdot u_1 + 2 \cdot u_2 + 0 \cdot u_3 \to (0, 2, 0)$

⑧

$$\frac{d}{dt}(v_3) = \frac{d}{dt}(t^3) = 3t^2 = 0 \cdot u_1 + 0 \cdot u_2 + 3 \cdot u_3 \rightarrow (0, 0, 3)$$

We thus get the matrix of differentiation as :-

$$A_{diff} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Verification :- Let $p(t) = 3 + 6t - 7t^2 + 2t^3$

$$x = \begin{bmatrix} 3 \\ 6 \\ -7 \\ 2 \end{bmatrix}$$

$$A_{diff}(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -7 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -14 \\ 6 \end{bmatrix}$$

$$\frac{d}{dt}[p(t)] = 6 - 14t + 6t^2 \rightarrow \begin{bmatrix} 6 \\ -14 \\ 6 \end{bmatrix}$$

Integration Matrix :-

Consider the integration of a quadratic polynomial from 0 to 1. This transformation is linear which transforms P_2 to P_3.

$$P_2 = \{p(t) = a_0 + a_1 \cdot t + a_2 \cdot t^2, a_0, a_1, a_2 \in \mathbb{R}\}$$

Bases : $\{v_1 = 1; v_2 = t; v_3 = t^2\}$

$$P_3 = \{q(t) = b_0 + b_1 t + b_2 t^2 + b_3 \cdot t^3, b_i \in \mathbb{R}\}$$

Bases $= \{u_1 = 1, u_2 = t, u_3 = t^2, u_4 = t^3\}$

$A_{int} : P_2 \rightarrow P_3$

Images of v_i's are :-

$$\int_0^t v_1 \cdot dt = \int_0^t 1 \cdot dt = t$$
$$= 0 \cdot u_1 + 1 \cdot u_2 + 0 \cdot u_3 + 0 \cdot u_4$$
$$= (0, 1, 0, 0)$$

$$\int_0^t v_2(t) = \int_0^1 t \cdot dt = \frac{t^2}{2} = 0 \cdot u_1 + 0 \cdot u_2 + \frac{1}{2} \cdot u_3 + 0 \cdot u_4$$
$$= (0, 0, 1/2, 0)$$

$$\int_0^t v_3 \cdot dt = \int_0^1 t^2 \cdot dt = \frac{t^3}{3} = 0 \cdot u_1 + 0 \cdot u_2 + 0 \cdot u_3 + 1/3\, u_4$$
$$= (0, 0, 0, 1/3)$$

$$\therefore A_{int} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}_{4\times 3}$$

$$A_{diff} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4}$$

<u>Note</u> $A_{diff} \cdot A_{int} = I_3$

$$\underset{3\times 4}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}_{4\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Example For the space of all 2x2 matrices. Find the standard basis. For the linear transformation of transposing, find the matrix A with respect to this basis. Why is $A^2 = I$?

Solution :- $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a,b,c,d \in \mathbb{R} \right\}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Basis :- $A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $A_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$A_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; $A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$T : M_{2\times 2} \to M_{2\times 2}$$

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

To find A_T :-

$$T[A_{11}] = T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A_{11}$$

$$= 1\cdot A_{11} + 0\cdot A_{12} + 0\cdot A_{21} + 0\cdot A_{22}$$

$$= [1, 0, 0, 0]$$

$$T[A_{12}] = T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = A_{21}$$

$$= 0\cdot A_{11} + 0\cdot A_{12} + 1\cdot A_{21} + 0\cdot A_{22}$$

$$= [0, 0, 1, 0]$$

$$T[A_{21}] = T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A_{12}$$

$$= 0 \cdot A_{11} + 1 \cdot A_{12} + 0 \cdot A_{21} + 0 \cdot A_{22}$$

$$= [0, 1, 0, 0]$$

$$T[A_{22}] = T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = A_{22}$$

$$= 0 \cdot A_{11} + 0 \cdot A_{12} + 0 \cdot A_{21} + 1 \cdot A_{22}$$

$$= [0, 0, 0, 1]$$

$\therefore$ Matrix for transposing :-

$$A_T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(A_T)^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $(A_T)^T = A_T \Rightarrow A_T^2 = I$

Example From the cubic polynomial to the fourth degree polynomial, what matrix represents multiplication by $2+3t$?

Solution :- Basis for $P_3 = \{v_1 = 1, v_2 = t, v_3 = t^2, v_4 = t^3\}$

Basis for $P_4 = \{u_1 = 1, u_2 = t, u_3 = t^2, u_4 = t^3, u_5 = t^4\}$

To find Matrix :-

$$(2+3t)v_1 = (2+3t)1 = 2 \cdot u_1 + 3 \cdot u_2 + 0 \cdot u_3 + 0 \cdot u_4 + 0 \cdot u_5$$

$$= (2, 3, 0, 0, 0)$$

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$(2+3t)v_2 = (2+3t)t = 2t+3t^2$

$= 0\cdot u_1 + 2\cdot u_2 + 3\cdot u_3 + 0\cdot u_4 + 0\cdot u_5$

$= (0,2,3,0,0)$

$(2+3t)v_3 = (2+3t)t^2 = 2t^2+3t^3$

$= 0\cdot u_1 + 0\cdot u_2 + 2\cdot u_3 + 3\cdot u_4 + 0\cdot u_5$

$= (0,0,2,3,0)$

$(2+3t)v_4 = (2+3t)t^3 = 2t^3+3t^4$

$= 0\cdot u_1 + 0\cdot u_2 + 0\cdot u_3 + 1\cdot u_4 + 1\cdot u_5$

$= (0,0,0,2,3)$

∴ The matrix A_T for this transformation is:-

$$A_T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{5\times 4}$$

Lectures 3 & 4 - Rotations, Reflections & Projections

Rotation Matrices Q :-

→ The linear system of equations $Ax=b$ can be represented as a Linear transformation.

$$T_A(x) = Ax \quad \text{where } T_A : \mathbb{R}^n \to \mathbb{R}^m$$

→ The matrix that rotates (left) every point in $\mathbb{R}^2$ about origin through θ is given by Q_θ.

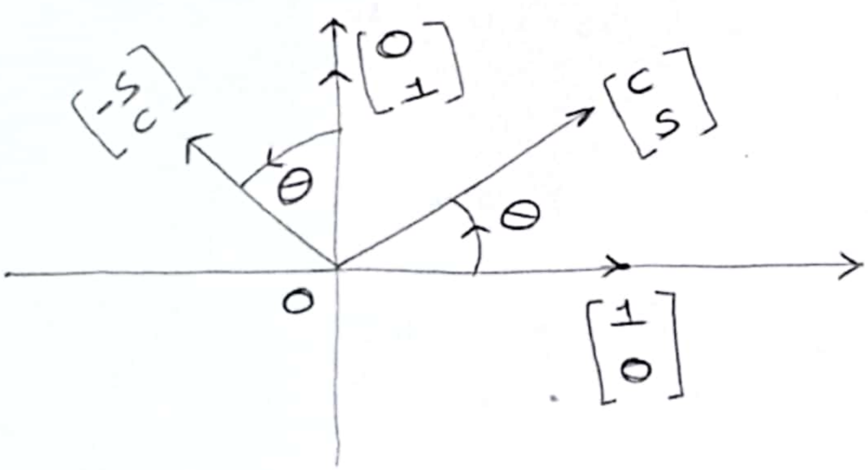
→ $Q_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

→ Basis for $\mathbb{R}^2 = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

→ $Q_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

→ $Q_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(90+\theta) \\ \sin(90+\theta) \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

→ $Q_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



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Note :- $Q_\theta \cdot Q_\psi = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta\cos\psi - \sin\theta\sin\psi & -\cos\theta\sin\psi - \sin\theta\cos\psi \\ \sin\theta\cdot\cos\psi + \cos\theta\cdot\sin\psi & -\sin\theta\cdot\sin\psi + \cos\theta\cdot\cos\psi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\psi) & -\sin(\theta+\psi) \\ \sin(\theta+\psi) & \cos(\theta+\psi) \end{bmatrix}$$

$$= Q_{\theta+\psi}$$

$$Q_{\theta+\theta} = Q_\theta \cdot Q_\theta = Q_{2\theta} = Q_\theta^2$$

$$\rightarrow Q_\theta \cdot Q_{-\theta} = Q_{\theta-\theta} = Q_0 = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix}$$

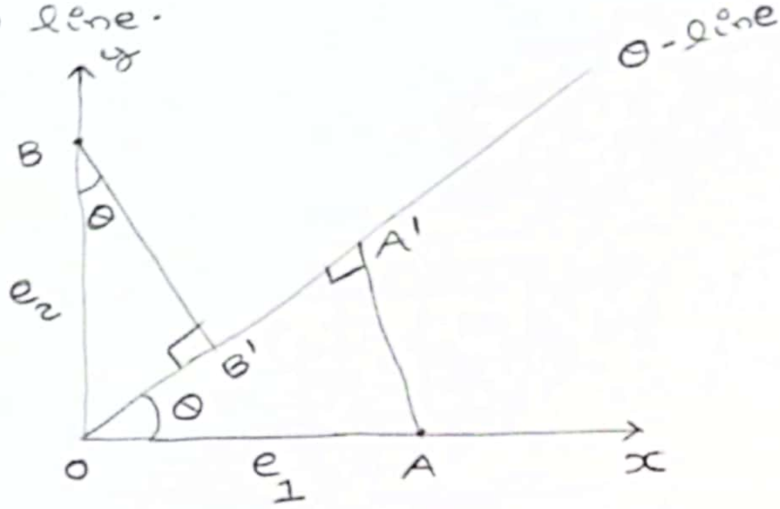
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\Rightarrow Q_\theta^{-1} = Q_{-\theta}$$

→ Rotation preserves all angles between the vectors as well as their length. so it is a reversible process.

Projection Matrices P

'P' is a matrix that projects every vector in $\mathbb{R}^2$ onto any θ line.



$$P[e_1] = P\begin{bmatrix}1\\0\end{bmatrix} = A' = \begin{bmatrix}OA' \cdot \cos\theta\\ OA' \cdot \sin\theta\end{bmatrix} = \begin{bmatrix}\cos\theta \cdot \cos\theta\\ \cos\theta \cdot \sin\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\theta\\ \cos\theta \cdot \sin\theta\end{bmatrix}$$

From right angle triangle $OA'A$, $\cos\theta = \frac{OA'}{OA}$

$\Rightarrow OA' = \cos\theta$

$$P[e_2] = P\begin{bmatrix}0\\1\end{bmatrix} = B' = \begin{bmatrix}OB' \cdot \cos\theta\\ OB' \cdot \sin\theta\end{bmatrix} = \begin{bmatrix}\sin\theta \cdot \cos\theta\\ \sin\theta \cdot \sin\theta\end{bmatrix}$$

$$P[e_2] = \begin{bmatrix}\sin\theta \cdot \cos\theta\\ \sin^2\theta\end{bmatrix}$$

From right angled triangle $OB'B$, $\sin\theta = \frac{OB'}{OB}$

$\Rightarrow OB' = OB \cdot \sin\theta \Rightarrow OB' = \sin\theta$

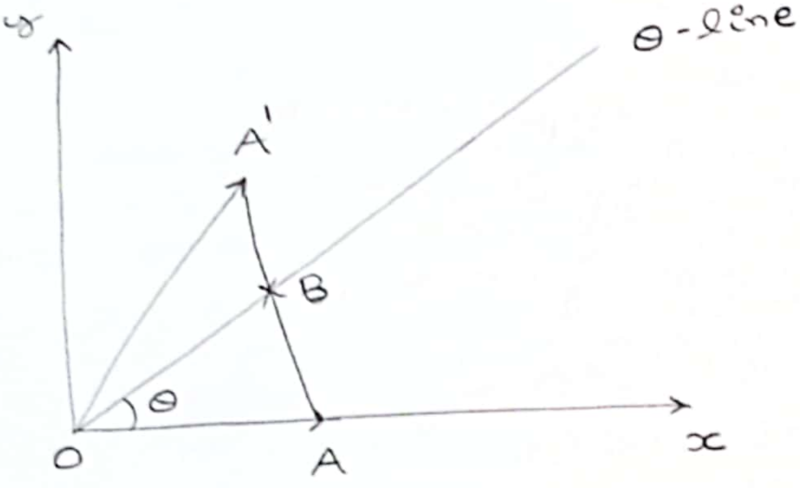
$\therefore$ The matrix that projects every vector in $\mathbb{R}^2$ onto any θ line is given by :-

$$P = \begin{bmatrix}\cos^2\theta & \cos\theta \cdot \sin\theta\\ \cos\theta \cdot \sin\theta & \sin^2\theta\end{bmatrix}$$

Note : This matrix has no inverse, beceause the transformation has no inverse. Projecting twice is the same as projecting once i.e $P^2 = P$.

Reflection Matrix H :-

→ The matrix H reflects every vector in $\mathbb{R}^2$ onto any 'θ' line.



From the figure :-

$$\overrightarrow{OA'} + \overrightarrow{A'B} = \overrightarrow{OB} \quad - (1)$$

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \quad - (2)$$

$$(1) + (2) \Rightarrow \overrightarrow{OA'} + \overrightarrow{A'B} + \overrightarrow{OA} + \overrightarrow{AB} = 2\,\overrightarrow{OB}$$

From figure, $\overrightarrow{A'B} = -\overrightarrow{AB}$

$$\therefore \quad \overrightarrow{OA'} + \overrightarrow{OA} = 2\,\overrightarrow{OB}$$

$$\Rightarrow x + Hx = 2Px$$

$$\Rightarrow Hx = 2Px - Ix$$

$$\Rightarrow \boxed{H = 2P - I}$$

$$\therefore \quad H = \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\cdot\sin\theta \\ 2\cos\theta\cdot\sin\theta & 2\sin^2\theta - 1 \end{bmatrix}$$

Note : Two reflections bring back the original

$$H = 2P - I \Rightarrow H^2 = (2P-I)^2$$
$$= 4P^2 + I^2 - 4PI$$
$$= 4P + I - 4P$$
$$= I \quad [\text{since } P^2 = P]$$

$\therefore$ A reflection is its own inverse

→ To conclude : Product of two transformations is another transformation by itself. Matrix multiplication is so defined that product of matrices corresponds to the product of the transformations they represent.

Example Find the matrix S that reflects every vector in $\mathbb{R}^2$ on the line $y = x$. Also find the matrix T which projects every vector in $\mathbb{R}^2$ on to the line $y = x$. Explain why $ST = TS$?

Solution :- The reflection of every vector in $\mathbb{R}^2$ onto the line $y = x$ is given by :-

$$S = H_\theta = H_{45} = \begin{bmatrix} 2\cos^2(45) - 1 & 2\cos(45)\cdot\sin(45) \\ 2\cos(45)\sin(45) & 2\sin^2(45) - 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 2\left(\frac{1}{2}\right) - 1 & 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) \\ 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) & 2\left(\frac{1}{2}\right) - 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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→ The projection of every vector in $\mathbb{R}^2$ on to the line $y = x$ line is given by :-

$$T = P_{45} = \begin{bmatrix} \cos^2(45) & \cos(45)\sin(45) \\ \cos(45)\sin(45) & \sin^2(45) \end{bmatrix}$$

$$T = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

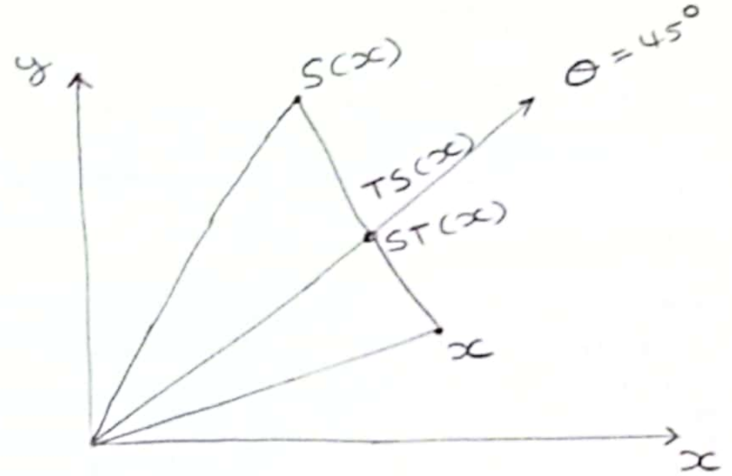
$$ST = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$TS = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$\therefore$ $\boxed{ST = TS}$

ST is the composition of projecting any vector of $\mathbb{R}^2$ on to $y = x$ line then reflecting it onto $y = x$ line.

TS is the composition of reflecting any vector of $\mathbb{R}^2$ onto $y = x$ line, then projecting it on to $y = x$ line. Both transformations produce the same output.



Example Determine the new point after applying the transformation to the given point :-

a) Project $x = (-2, 1)$ on the y-axis and then rotate by $45°$ counter clockwise.

b) Rotate $x = (-2, 1)$ $60°$ counter clockwise and then project on the x-axis.

Solution :-

a) Projection matrix to project any vector of $\mathbb{R}^2$ on the y-axis is given by $P_{90°} = \begin{bmatrix} \cos^2 90° & \cos 90° \cdot \sin 90° \\ \cos 90° \cdot \sin 90° & \sin^2 90° \end{bmatrix}$

$$\therefore \quad P_{90°} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Rotation matrix to rotate any values of $\mathbb{R}^2$ by $45°$ counter clockwise about the origin is given by :-

$$Q_{45°} = \begin{bmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(20)

The projection of $x = (-2, 1)$ on the y-axis and then rotating about $\theta = 45°$ counter clockwise is given by :-

$$Q_{45°} P_{90°} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$Q_{45°} \cdot P_{90°} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

b) Rotation matrix to rotate any vector of $\mathbb{R}^2$ by $60°$ counter clockwise is given by :-

$$Q_{60} = \begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix}$$

$$Q_{60} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

Projection matrix to project any vector of $\mathbb{R}^2$ onto x-axis is given by

$$P_0 = \begin{bmatrix} \cos^2 0 & \cos 0 \cdot \sin 0 \\ \cos 0 \cdot \sin 0 & \sin^2 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rotating $x = (-2, 1)$ $60°$ counter clock wise and then projecting on the x-axis is given by :-

$$P_0 \, Q_{60}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 - \sqrt{3}/2 \\ 0 \end{bmatrix}$$

Lectures 5 & 6 - Orthogonal vectors & subspaces, orthogonal bases

Example Find a vector x orthogonal to the row space of A, a vector y orthogonal to the column space of A and a vector z orthogonal to the null space of A. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}$

Solution Null space of A is orthogonal to row space of A. $\therefore$ $x \in N(A)$ and $z \in C(A^T)$

Left null space is orthogonal to the column space of A. $\therefore$ $y \in N(A^T)$

To find the vectors x, y and z, we need to find $N(A)$, $N(A^T)$ & $C(A^T)$.

(22)

$$[A:b] = \begin{bmatrix} 1 & 2 & 3 & : b_1 \\ 2 & 3 & 4 & : b_2 \\ 2 & 4 & 6 & : b_3 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & 3 & : b_1 \\ 0 & -1 & -2 & : b_2-2b_1 \\ 0 & 0 & 0 & : b_3-2b_1 \end{bmatrix}$$

$$\underbrace{\qquad\qquad}_{U\,:\,c}$$

To find $N(A)$:- $Ux = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_2 - 2x_3 = 0 \qquad\qquad x_1 + 2x_2 + 3x_3 = 0$$

$$x_2 = -2x_3 \qquad\qquad x_1 = -2x_2 - 3x_3$$

$$= -2\{-2x_3\} - 3x_3$$

$$= x_3$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix} \Rightarrow x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$N(A) = \left\{ x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} ;\ x_3 \in \mathbb{R} \right\}$$

$$x = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$C(A^T) = \left\{ c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} ;\ c_1, c_2 \in \mathbb{R} \right\}$$

$$\therefore z = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{since} \quad x^T z = 0$$

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$Ax = b$ is solvable only if $b_3 - 2b_1 = 0$

[Only then $\rho(A) = \rho(A:b) = 2$]

$$\therefore N(A^T) = c_3\begin{pmatrix}-2\\0\\1\end{pmatrix};\ c_3 \in \mathbb{R}$$

$$\therefore y = \begin{pmatrix}-2\\0\\1\end{pmatrix}$$

Example Let P be the plane whose equation is $x - 2y + 9z = 0$. Find a vector perpendicular to P. What matrix has the plane P as its null space and what matrix has P as its row space?

Solution :- Let $\{P = x, y, z$ such that $x - 2y + 9z = 0\}$

$$\begin{bmatrix}1 & -2 & 9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = 0$$

The vector perpendicular to P is $\begin{bmatrix}1\\-2\\9\end{bmatrix}$

So the matrix $A = \begin{bmatrix}1 & -2 & 9\end{bmatrix}$ has the plane P as its null space.

To find B, find the null space of P :-

$$\begin{bmatrix}1 & -2 & 9\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2y - 9z\\y\\z\end{pmatrix} = \left\{ y\begin{pmatrix}2\\1\\0\end{pmatrix} + z\begin{pmatrix}-9\\0\\1\end{pmatrix};\ x, y \in \mathbb{R} \right\}$$

$$B = \begin{bmatrix} 2 & 1 & 0 \\ -9 & 0 & 1 \end{bmatrix}$$ so that $C(B^T) = P$

For verification if we solve this, we get P.

$$\begin{bmatrix} 2 & 1 & 0 \\ -9 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 + 9/2 R_1} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 9/2 & 1 \end{bmatrix}$$

$-9 - 2x = 0$

$-9 = 2x$

$x = -9/2$

$x, y \rightarrow$ Pivot variables

$z \rightarrow$ Free variable

$\boxed{z = k}$; $9/2\, y + z = 0$; $9/2\, y = -z$

$\boxed{y = -2k/9}$

$2x + y = 0$

$2x = -y$

$2x = +2k/9$

$\boxed{x = +k/9}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k/9 \\ -2k/9 \\ k \end{pmatrix}$$

If $k = 9 \Rightarrow$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 9 \end{pmatrix}$

Orthogonal vectors & subspaces :-

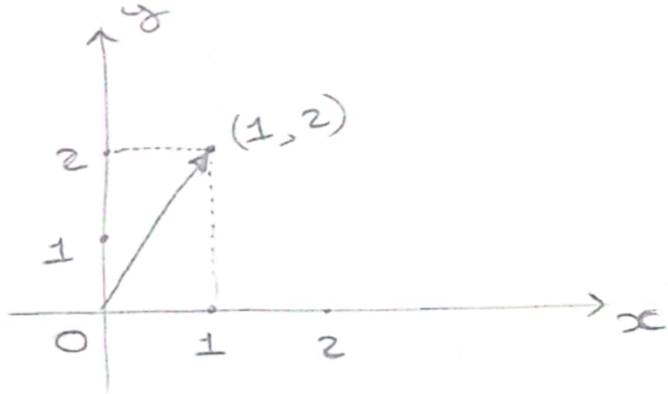
Let $x = (x_1, x_2 \dots x_n)$

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots x_n^2}$$

Norm

$$\|x\|^2 = x^T \cdot x$$

zero is the only vector whose Norm is 0.



$$\|x\| = \sqrt{1^2+2^2} = \sqrt{5}$$

$$\|y\| = \sqrt{1^2+2^2+3^2} = \sqrt{14} \quad \text{if } y = (1,2,3)$$

If $x = (x_1, x_2 \dots x_n)$ & $y = (y_1, y_2 \cdots y_n)$
then the inner product $= \langle x, y \rangle = x^T y$

$$x^T y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

$$x^T y = y^T x$$

If $x^T y = 0$, vectors are orthogonal

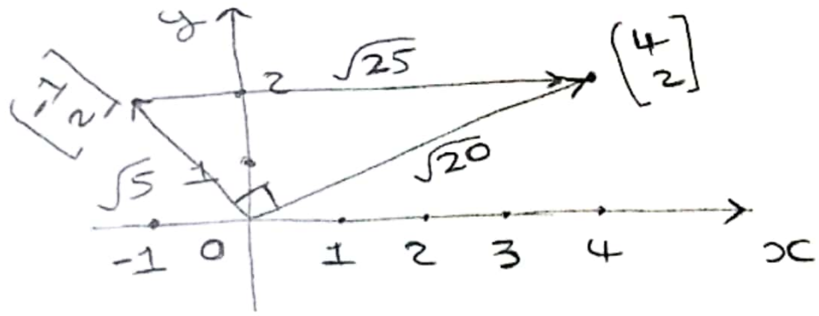
$$\Rightarrow x^T y = y^T x = 0$$

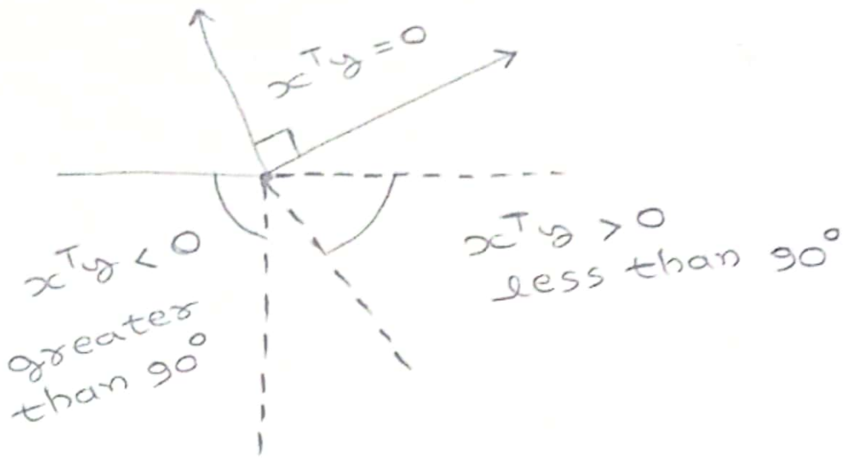
Note Zero is the only vector that is orthogonal to itself & zero is the only vector that is orthogonal to every other vector.

Example :- consider $x = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $y = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$x^T y = \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0$$

$\therefore$ vectors x & y are orthogonal.





Examples

1. The coordinate vectors $(1,0,\dots 0), (0,1,0 \dots 0)$ $\dots (0,0,\dots,1)$ are mutually orthogonal in $\mathbb{R}^n$.

2. The vectors $(c,s), (s,-c)$ are orthogonal in $\mathbb{R}^2$.

3. The vectors $(2,1,0), (-1,2,0)$ are orthogonal in $\mathbb{R}^3$.

Theorem If the non-zero vectors $v_1, v_2 \dots v_k$ are mutually orthogonal then these vectors are linearly independent but converse need not be true.

Example Vectors $(2,1)$ and $(1,2)$ are linearly independent but they are not mutually orthogonal.

Two subspaces S & T of a vector space V are orthogonal if every vector x in S is orthogonal to every vector y in T. Thus $x^T y = 0$ for all $x \in S$ and $y \in T$.

Examples

1. $Z = \{0\}$ is orthogonal to all subspaces.

2. In $\mathbb{R}^2$, a line can be orthogonal to another line.

3) In $\mathbb{R}^3$, a line can be orthogonal to another line or a plane. But, a plane cannot be orthogonal to another plane.

Note If S & T are orthogonal in V then

$$\dim\{S\} + \dim\{T\} \leq \dim\{V\}$$

Fundamental theorem of orthogonality :-

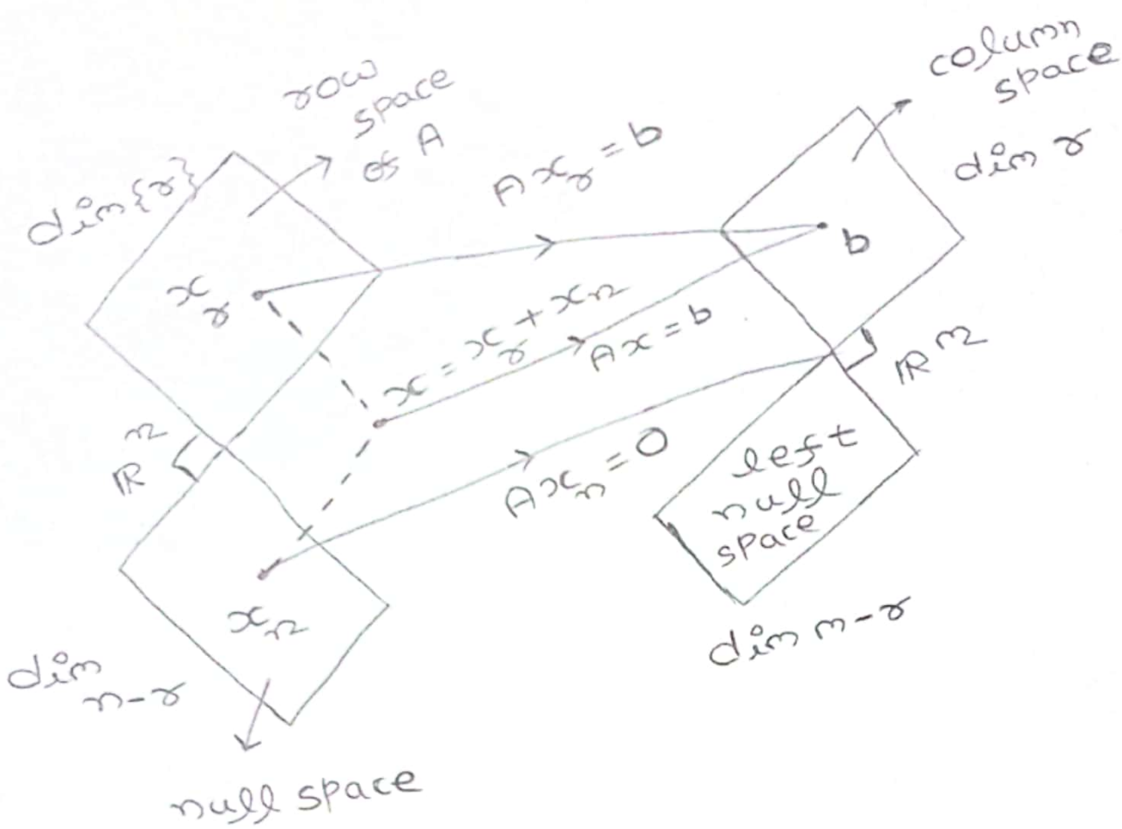
Let A be an $m \times n$ matrix then row space of A is orthogonal to its null space in $\mathbb{R}^n$ and the column space is orthogonal to left null space in $\mathbb{R}^m$.

Orthogonal complement :- Given a subspace V of $\mathbb{R}^n$, the space of all vectors orthogonal to V is called the orthogonal complement of V written as $V^\perp$ and read as "V perp". The orthogonal complement of a subspace V is unique.

Fundamental theorem of Linear Algebra - Part II

The null space is the orthogonal complement of the row space in $\mathbb{R}^n$ and the column space is the orthogonal complement of the left null space in $\mathbb{R}^m$.

If S & T are orthogonal complements in $\mathbb{R}^n$, then it is always true that $\dim\{S\} + \dim\{T\} = n$.



→ Splitting $\mathbb{R}^n$ into orthogonal parts V & W will split every vector into $x = v + w$.

→ The vector v is the projection of x onto the subspace V and the orthogonal complement w is the projection of x onto W.

→ The true effect of matrix multiplication is that every Ax is in $C(A)$. The null space goes to zero. The row space component goes to $C(A)$. Nothing is carried to the left null space.

→ $Ax = b$ is solvable if and only if $y^Tb = 0$ whenever $y^TA = 0$.

→ From the row space to the column space A is actually invertible, every vector 'b' in the column space comes from exactly one vector in the row space.

→ Every vector transforms its row space onto its column space.

Lecture 7 - Algebra of Linear Transformations, Invertible Maps, Isomorphism

Algebra of Linear operators

Let V be a vector space over a field K. The linear mappings of the form $T: V \to V$ are called linear operators and linear transformations on V.

Note: 1) If $\dim\{V\} = n$ then $\dim\{A(V)\} = n^2$

2) For any mapping F, G from $A(V)$, the composition $G \cdot F$ exists and also belongs to $A(V)$.

→ An algebra a over a field K is a vector space over K in which an operation of multiplication is defined satisfying, for $F, G, H \in A$ and every $k \in K$

1. $F(G+H) = FG + FH$
2. $(G+H)F = GF + HF$
3. $k(GF) = (kG)F = G(kF)$

Invertible Maps & Isomorphism

→ A mapping $f: A \to B$ is said to be one-to-one or 1-1 or injective if different elements of A have distinct images; that is if $a \neq a'$, then $f(a) \neq f(a')$

→ Equivalently if $a = a'$, then $f(a) = f(a')$.

→ A mapping $f: A \to B$ is said to be onto or surjective if every $b \in B$ is the image of atleast one $a \in A$.

→ A mapping $f: A \to B$ is said to be one-to-one correspondence between A & B or bijective if f is both one-to-one & onto.

Invertible Maps & Isomorphism

→ A mapping $f: A \to B$ is said to be invertible if f is one-to-one & onto.

Example: Let $f: R \to R$ be defined by $f(x) = 2x - 3$. Now f is one-to-one & onto. Let y be the image of x under the mapping f i.e $y = 2x - 3$.

Interchange x & y to obtain $x = 2y - 3 \Rightarrow y = \frac{x+3}{2}$

$$\Rightarrow \boxed{f^{-1} = \frac{x+3}{2}}$$

→ A mapping $F: V \to U$ is called Isomorphism if F is linear & bijective, i.e one-to-one & onto.

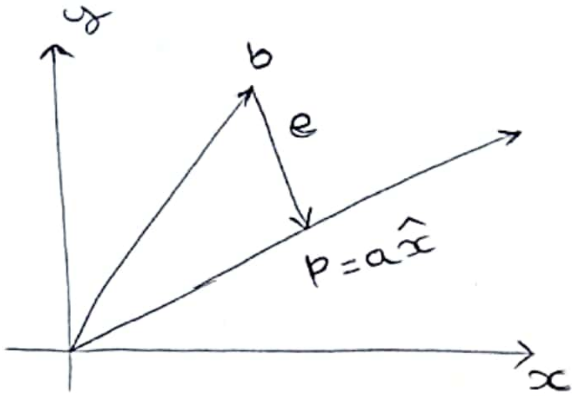
→ Example: The mapping $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $T(x, y) = (x + 4y, y - 3x)$ is isomorphism.

Lecture 8 - Cosines & Projections onto lines

Definition: If $a = (a_1, a_2 \dots a_n)$, $b = (b_1, b_2 \dots b_n)$ include an angle θ between them, then the cosine formula states that

$$\boxed{\cos\theta = \frac{a^T \cdot b}{\|a\| \cdot \|b\|}}$$

To find the projection of b onto the line through a given vector 'a', we find the point p on the line that is closest to b. This point must be some multiple of 'a' say $p = a\hat{x}$. Now the line from b to the closest point p is perpendicular to the vector a and hence



$e = b - p$ since $a \perp e$

$\Rightarrow a^T e = 0 \Rightarrow a^T(b-p) = 0$

$\Rightarrow a^T b - a^T \cdot a \cdot \hat{x} = 0$

$\Rightarrow a^T b = a^T \cdot a \cdot \hat{x}$

$$\therefore \hat{x} = \frac{a^T b}{a^T a}$$

$$\boxed{\therefore p = a\hat{x}}$$

→ All vectors a and b in $\mathbb{R}^n$ satisfy the schwarz inequality which is

$$|a^T b| \leq \|a\| \cdot \|b\|$$

→ Note 1) Equality holds if and only if a and b are dependent vectors. The angle is $\theta = 0^\circ$ or 180°. In this case, b is identical with its projection p and the distance between b & p is zero.

2) Schwarz inequality is also stated as $|\cos\theta| \leq 1$.

→ Projection onto a line through a given vector 'a' is carried out by a projection matrix given by :-

$$P = \frac{a \cdot a^T}{a^T a}$$

→ This matrix multiplies b and produces p.

→ that is, $Pb = \frac{a \cdot a^T}{a^T \cdot a} \cdot b = a \cdot \frac{a^T b}{a^T a}$

$= a\hat{x} = p$

Note :-

1) P is a symmetric matrix

2) $P^n = P$ for $n = 1, 2, 3 \ldots$

3) The rank of P is one.

4) The trace of P is one.

5) If a is a n-dimensional vector, then P is a square matrix of order n.

6) If 'a' is a unit vector, then $P = a \cdot a^T$

Example What multiple of $a = (1, 1, 1)$ is closest to $b = (2, 4, 4)$? Find also the point on the line through 'b' that is closest to a?

Solution Let 'p' be the point on the line through $a = (1, 1, 1)$ closest to $b = (2, 4, 4)$

$\therefore p = a\hat{x} = a \cdot \frac{a^T b}{a^T a} \Rightarrow p = \frac{10}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$a^T b = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = 10$

$a^T a = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3$

Let 'p_1' be the point on the line through 'b' that is closest to 'a'.

$$\Rightarrow \quad p_1 = b \cdot \hat{x} = b \cdot \frac{b^T a}{b^T b}$$

$$= \frac{b^T a}{b^T b} \cdot b$$

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad b^T = \begin{bmatrix} 2 & 4 & 4 \end{bmatrix} \quad a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow b^T a = 10$$

$$b^T b = \begin{bmatrix} 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = 36$$

$$\therefore \quad p_1 = \frac{10}{36} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 5/9 \\ 10/9 \\ 10/9 \end{bmatrix}$$

Example Find the matrix that projects every point in $\mathbb{R}^3$ onto the line of intersection of the planes $x+y+z=0$ and $x-z=0$. What are the column space and row space of this matrix?

Solution:- The line of intersection of these planes is

$$\begin{aligned} x + y + z &= 0 \\ x \quad - z &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}$$

$$x = -y - z \; ; \quad -y - 2z = 0$$
$$y = -2z$$

(34) $x = -y - z$; $-y - 2z = 0 \Rightarrow y = -2z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$\therefore$ The line passing through $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is the point of intersection.

Let 'a' $= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and the projection matrix through 'a' is $P = \dfrac{a \cdot a^T}{a^T a}$

$$a \cdot a^T = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$a^T \cdot a = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 6$$

$$\therefore \quad P = \frac{1}{6} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

P is a symmetric matrix of rank 1.

$\therefore$ columnspace & rowspace are $\left\{ c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \text{ where } c_1 \in \mathbb{R} \right\}$

Lecture 9 - Projections & Least Squares

→ The failure of Gaussian elimination is almost certain when we have several equations in one unknown.

$$a_1 x = b_1 \,;\, a_2 x = b_2 \,;\, \dots \,;\, a_m x = b_m$$

→ This system is solvable if $b = (b_1, b_2 \dots b_m)$ is a multiple of $a = (a_1, a_2 \dots a_m)$

→ If the system is inconsistent, then we choose that value of '<u>a</u>' that minimizes an average error E in the '<u>m</u>' equations. The most convenient average comes from the sum of squares.

→ Squared Error $E^2 = \sum_{i=1}^{m} (a_i x - b_i)^2$

→ If there is an exact solution, the minimum error is $E = 0$. If not, the minimum error occurs when $\frac{dE^2}{dx^2} = 0$.

→ Solving for x, the least square solution is

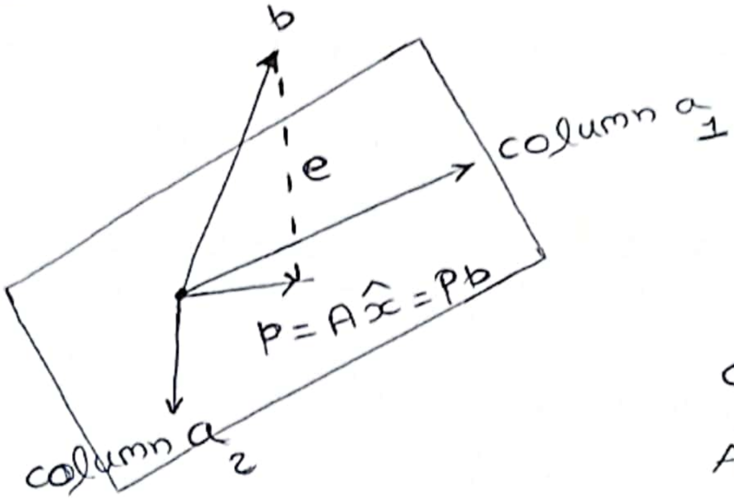
$$\boxed{\hat{x} = \frac{a^T b}{a^T a}}$$

Least Squares Problem with Several Variables :-

→ Consider a system of equations $Ax = b$ that is inconsistent.

→ The vector b lies outside C(A) and we need to project it onto C(A) to get the point 'p' in C(A) that is closest to b. The problem here is the same as to minimize the error $E = \|Ax - b\|$ and this is exactly the distance from 'b' to the point Ax in C(A).

→ Searching for the least squares solution $\hat{x}$ is the same as locating the point p that is closest to b.



$a_1^T \cdot e = 0$

$a_2^T \cdot e = 0$

Combine into

$A^T e = A^T(b - A\hat{x}) = 0$

→ The error vector $e = b - A\hat{x}$ must be perpendicular to C(A) and hence can be found in the left null space of A.

→ Thus $A^T(b - A\hat{x}) = 0$ or $A^T b - A^T \cdot A \cdot \hat{x} = 0$

→ These are called the normal equations.

→ Solving them, we get the optimal solution $\hat{x}$.

→ If b is orthogonal to C(A) then its projection is the zero vector.

Example :- Find the projection of $b = (1, 2, 7)$ onto the column space of A spanned by $(1, 1, -2)$ and $(1, -1, 4)$. Split b into $p + q$ with p in $C(A)$ and q in $N(A^T)$.

solution :- Let p be the projection of b onto $C(A)$ which is spanned by $(1, 1, -2)$ and $(1, -1, 4)$

so $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}$ and $p = A \cdot \hat{x}$

Formula to find $\hat{x}$:-

$$A^T b = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix}$$

$$A^T \cdot A \cdot \hat{x} = A^T b$$

$$\Rightarrow \quad \hat{x} = (A^T A)^{-1} \cdot A^T b$$

$$(A^T A)^{-1} = \frac{1}{44} \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix}$$

$$\Rightarrow \quad \hat{x} = \frac{1}{44} \begin{bmatrix} 18 & 8 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} -11 \\ 27 \end{bmatrix} = \begin{bmatrix} -198 + 216 \\ -88 + 162 \end{bmatrix} \frac{1}{44}$$

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$$\hat{x} = \frac{1}{44}\begin{bmatrix} 18 \\ 74 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 9/22 \\ 37/22 \end{bmatrix}$$

$$p = A \cdot \hat{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 9/22 \\ 37/22 \end{bmatrix} = \begin{bmatrix} 23/11 \\ -14/11 \\ 65/11 \end{bmatrix}$$

and $p + r = b$

$$\Rightarrow r = b - p$$

$$r = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} - \begin{bmatrix} \frac{23}{11} \\ \frac{-14}{11} \\ \frac{65}{11} \end{bmatrix}$$

$$\therefore r = \begin{bmatrix} 12/11 \\ 36/11 \\ 12/11 \end{bmatrix}$$

Example Find a basis for the orthogonal complement of the row space of $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$. Split the vector $(3,3,3)$ into a row space component x_r and a null space component x_n?

Solution :- x_r & x_n are projections of $x = (3,3,3)$ on to $C(A^T)$ and $N(A)$ respectively.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

x = pivot variable ; y, z = free variables

$$y + 2z = 0 \Rightarrow y = -2z$$

$$x + 2z = 0 \Rightarrow x = -2z$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = N(A)$$

Let $a = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

Projection of b onto a line through 'a' is x_n

$$x_n = a \cdot \hat{x} = a \cdot \frac{a^T b}{a^T a}$$

$$a^T b = \begin{bmatrix} -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = -6 - 6 + 3 = -9$$

$$a^T a = \begin{bmatrix} -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = 4 + 4 + 1 = 9$$

$$\therefore x_n = \frac{-9}{9} \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

We know that $x = x_r + x_n$

$$\Rightarrow x_r = x - x_n$$

$$= \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

Projection Matrices

→ The matrix P that projects onto $C(A)$ is given by $P = A(A^TA)^{-1} \cdot A^T$

→ Also, if P & Q are the matrices that project onto orthogonal subspaces then it is always true that $PQ = 0$ & $P+Q = I$.

Least squares fitting of data :-

→ Suppose we do a series of experiments and expect the output b to be a linear function of the input t. We look for a straight line $\boxed{b = C + Dt}$.

→ If there is no experimental error, then two measurements of b will determine the line.

→ But, if there is error, we minimize it by the method of least squares and find the optimal straight line.

→ Consider the following system of equations

$$C + Dt_1 = b_1$$

$$C + Dt_2 = b_2$$

$$\vdots$$

$$C + Dt_m = b_m$$

In matrix form :- $\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ & \vdots \\ 1 & t_m \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

→ The best solution $\hat{x}$ can be obtained by solving the normal equations.

<u>Example</u> use the method of least squares to fit the best line to the data $b = (4, 3, 1, 0)$ at $t = (-2, -1, 0, 2)$ respectively. Find the projection of $b = (4, 3, 1, 0)$ onto the column space of A. calculate the error vector 'e' and check that 'e' is orthogonal to the columns of A.

<u>Solution</u> Let $C + Dt = b$ be the best fit straight line for the given data.

Given that $b = (4, 3, 1, 0)$ at $t = (-2, -1, 0, 2)$

$$\Rightarrow \left.\begin{aligned} C + D(-2) &= 4 \\ C + D(-1) &= 3 \\ C + D(0) &= 1 \\ C + D(2) &= 0 \end{aligned}\right\} \quad \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

The system is inconsistent. To find least squares solution $\hat{x}$, we have to solve normal equation:-

i.e $A^T.A.\hat{x} = A^T b$

$$A^T.A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}$$

$$A^T.b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix} \hat{x} = \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$$\hat{x} = \frac{1}{35} \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$$\hat{x} = \frac{1}{35} \begin{bmatrix} 61 \\ -36 \end{bmatrix} \Rightarrow \hat{x} = \begin{bmatrix} 61/35 \\ -36/35 \end{bmatrix}$$

$\therefore$ The best straight line fit for the given data is $\boxed{b = \frac{61}{35} - \frac{36}{65} \cdot t}$

Let p be the projection of $b = (4, 3, 1, 0)$ onto $C(A)$. $\Rightarrow p = A\hat{x} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 61/35 \\ -36/35 \end{bmatrix} = \begin{bmatrix} 133/35 \\ 97/35 \\ 61/35 \\ -11/35 \end{bmatrix}$

$$\therefore p = \frac{1}{35} \begin{bmatrix} 133 \\ 97 \\ 61 \\ -11 \end{bmatrix}$$

The error vector $e = b - p = \begin{pmatrix} 4 \\ 3 \\ 1 \\ 0 \end{pmatrix} - \begin{bmatrix} 133/35 \\ 97/35 \\ 61/35 \\ -11/35 \end{bmatrix}$

$$= \begin{bmatrix} \frac{4}{1} - \frac{133}{35} \\ 3 - 97/35 \\ 1 - 61/35 \\ 11/35 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 8/35 \\ -26/35 \\ 11/35 \end{bmatrix}$$

→ The error vector is orthogonal to both the columns of A.

i.e $e^T \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ should be 0

$$\Rightarrow \begin{bmatrix} 1/5 & 8/35 & \frac{-26}{35} & \frac{11}{35} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 7/35 + \frac{8}{35} - \frac{26}{35} + \frac{11}{35} = 0$$

$$e^T \begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{8}{35} & \frac{-26}{35} & \frac{11}{35} \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \frac{-2}{5} - \frac{8}{35} + \frac{22}{35}$$

$$= \frac{-14 - 8 + 22}{35} = 0$$

$\therefore$ The vector $\underline{e}$ is orthogonal to $C(A)$.

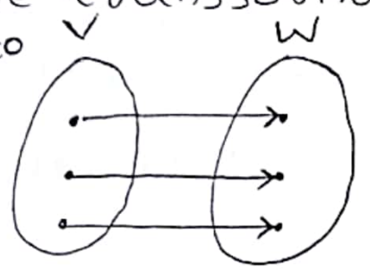
Additional reading - Algebra on Linear Transformation

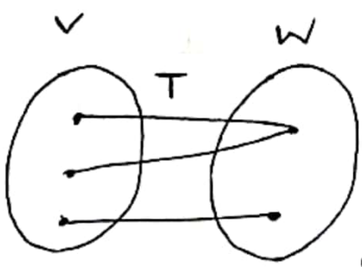
One-to-one transformation

→ Let V & W be two vector spaces. A direct transformation $T: V \to W$ is one-to-one if T maps distinct vectors in V to distinct vectors in W.

→ A one-to-one transformation is also called injective transformation.

T is one-to-one





T is not one-to-one

Properties :-

1) A linear transformation $T: V \to W$ is one-to-one iff $\ker\{T\} = \{0\}$

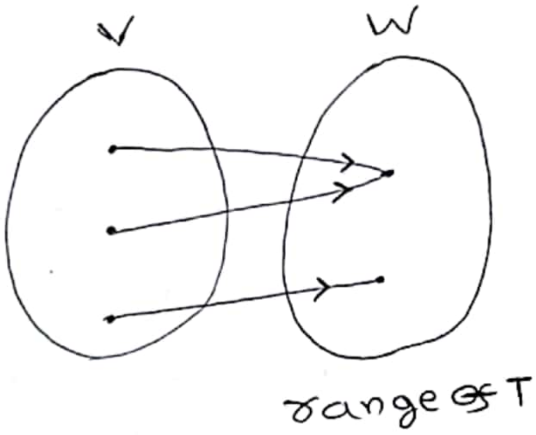
2) A linear transformation $T: V \to W$ is one-to-one iff $\dim\{\ker\{T\}\} = 0$.

3) A linear transformation $T: V \to W$ is one-to-one iff $\text{rank}\{T\} = \dim\{V\}$

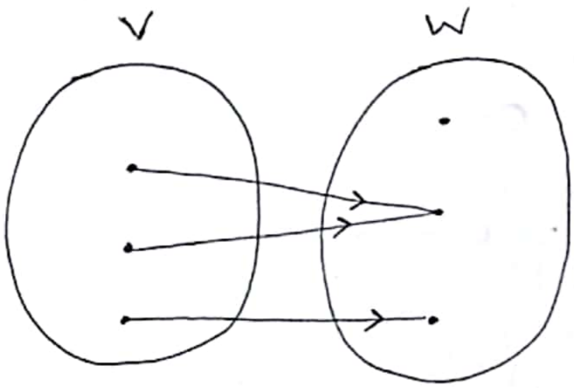
4) If A is an $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$, then T_A is one-to-one iff $\text{rank}(A) = n$.

Onto Transformation

→ Let V & W be vector spaces. A direct transformation $T: V \to W$ is onto if the range of T is W i.e, T is onto iff for every $\underline{w}$ in $\underline{W}$, there is a $\underline{v}$ in $\underline{V}$ such that $T(v) = w$. An onto transformation is also called as surjective transformation.



T is onto



T is not onto

Properties :-

1) A linear transformation $T: V \to W$ is onto iff $\text{rank}\{T\} = \dim\{W\}$

2) If A is an $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$; T_A is onto iff $\text{rank}(A) = m$

Bijective Transformation :-

→ If a transformation $T: V \to W$ is both one-to-one and on-to, then it is called bijective transformation.

Isomorphism

→ A bijective transformation from V to W is known as an isomorphism between V & W.

Properties Let V be a finite dimensional vector space. If $\dim\{V\} = n$, then this is an isomorphism from V to $\mathbb{R}^n$.

Example : $T: \mathbb{R}^2 \to \mathbb{R}^3$ where $T(x,y) = (x-y, y-x, 2x-2y)$

1) A linear transformation is one-to-one iff $\mathrm{Ker}\{T\} = 0$.

Let $T(x,y) = 0$

$$(x-y, y-x, 2x-2y) = (0,0,0)$$

$$x - y = 0 \Rightarrow x = y$$

$$y - x = 0 \Rightarrow y = x$$

$$2x - 2y = 0 \Rightarrow 2x = 2y \Rightarrow x = y$$

solving for $(x, y, z) \neq (0,0,0)$

$\therefore$ T is not one-to-one

2) Let $v(x,y)$ in $\mathbb{R}^2$ and $w = (a,b,c)$ in $\mathbb{R}^3$

where $a,b,c \in \mathbb{R}$ s.t $T(v) = w$

$$T(x,y) = a,b,c$$

$$(x-y, y-x, 2x-2y) = (a,b,c)$$

$$x - y = a ; \ y - x = b ; \ 2x - 2y = c$$

$$\left.\begin{aligned} x-y &= c/2 \\ x-y &= a \end{aligned}\right\}$$
$(-)\ (+)$

$a = c/2$

$x-y=a$

$x-y=c/2$

$-x+y=b$

we get $\boxed{a=-b=c/2}$

Thus $T(v)=\omega$ only if $a=-b=c/2$ not for all a,b,c $\therefore$ T is not onto.

2) $T:\mathbb{R}^2\to\mathbb{R}^2$; $T(x,y)=(x+y,x-y)$

consider $T(x,y)=0$

$x+y, x-y=(0,0)$

$x+y=0$ & $x-y=0$

$x-y=0$

$x+y=0$

$\Rightarrow x=0, y=0$

$\ker\{T\}=0$

$\therefore$ T is One-to-One

Let $V=(x,y)$ and $\omega=(a,b)\in\mathbb{R}^2$; $a,b\in\mathbb{R}$ s.t $T(v)=\omega$

$T(x,y)=a,b$

$x+y, x-y=a,b$

$x+y=a$

$x-y=b$

$2x=a+b$

$x=\dfrac{a+b}{2}$

$x+y=a$

$x-y=b$

$(-)\ (+)\quad(-)$

$y=\dfrac{a-b}{2}$

Thus for every $\omega=(a,b)$ in $\mathbb{R}^2$ there exists $V\left(\dfrac{a+b}{2},\dfrac{a-b}{2}\right)$ in $\mathbb{R}^2$. Hence T is onto

3) check whether A is <u>One-One</u> or <u>onto</u>?

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ 3 & -6 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$m \times n$

3×2

A is One-one iff $\text{rank}(A) = n = 2$

Here $\text{rank}(A) = 1 \neq 2$. $\therefore$ A is not One-one

A is onto iff $\text{rank}(A) = m = 3$

Here $\text{rank}(A) = 1 \neq 3$. $\therefore$ A is not onto.

Algebra of Linear transformations

→ Let V & W be Vector spaces, Let L & T be linear transformations from V to W. We can define the scalar multiple aL of L and the sum $L+T$ of L & T as linear transformation from V to W by the rules :-

$$(aL)(v) = a \cdot L(v) \quad \text{for } a \in \mathbb{R}, v \in V$$

$$(L+T)(v) = L(v) + T(v)$$

→ If M_L and M_T are the representing matrices with respect to S & T (S & T are basis of L & T) respectively then aM_L represents aL and $M_L + M_T$ represents $L+T$.

<u>Definition</u> A linear transformation is invertible iff $\text{Ker}\{T\} = \{0\}$ and $\text{rank}\{T\} = \dim\{W\}$

OR

A linear transformation is invertible iff it is injective and surjective. [Inverse Transformation]