LINEAR ALGEBRA AND ITS APPLICATIONS
UE19MA251

Reflection matrix H

The matrix H reflects every vector in $\mathbb{R}^2$ onto any 'θ' line.

From the figure

$$\overrightarrow{OA'} + \overrightarrow{A'B} = \overrightarrow{OB} \quad — (1)$$

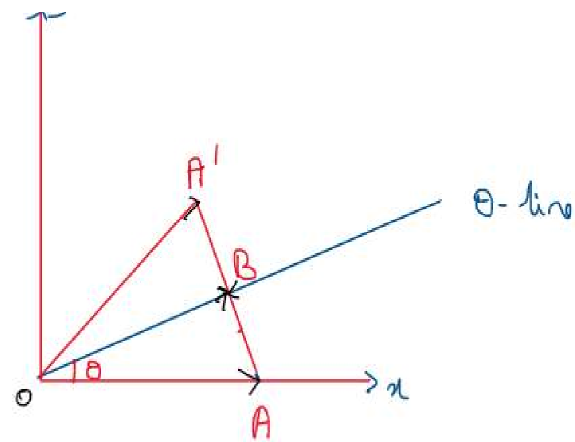
and $$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \quad — (2)$$

$$(1)+(2) \Rightarrow \overrightarrow{OA'} + \overrightarrow{OA} = 2\overrightarrow{OB}$$

$$(\text{Since } \overrightarrow{A'B} = -\overrightarrow{AB})$$

$$\Rightarrow \quad x + Hx = 2Px \Rightarrow Hx = 2Px - Ix.$$

$$\Rightarrow \quad H = 2P - I$$



Reflection matrix H

$$H = \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1 \end{bmatrix}$$

Note:

- Two reflection brings back the original.

$$H = 2P - I \Rightarrow H^2 = (2P - I)^2 = 4P^2 - 4P + I = I$$

since $P^2 = P$.

- A reflection is its own inverse.

Reflection matrix H

To conclude....

Product of two transformations is another transformation by itself. Matrix multiplication is so defined that product of matrices corresponds to the product of the transformations that they represent.

Problems

Find the matrix S that reflects every vector in R^2 on the line y = x. Also find the matrix T which projects every vector in R^2 on to the line y = x. Explain why ST = TS.

Solution: The reflection of every vector in $\mathbb{R}^2$ onto the line $y = x$ is given by

$$S = H_\theta = H_{45^\circ} = \begin{bmatrix} 2\cos^2(45) - 1 & 2\cos(45^\circ)\sin(45) \\ 2\cos(45^\circ)\sin(45^\circ) & 2\sin^2(45) - 1 \end{bmatrix}$$

Problems

$$S = \begin{bmatrix} 2\left(\frac{1}{\sqrt{2}}\right)^2 - 1 & 2\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} \\ 2\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} & 2\left(\frac{1}{\sqrt{2}}\right)^2 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Problems

The projection of every vector in $\mathbb{R}^2$ onto the $y = x$ line is given by

$$T = P_{45} = \begin{bmatrix} \cos^2(45) & \cos(45)\sin(45) \\ \cos(45)\sin(45) & \sin^2(45) \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{\sqrt{2}}\right)^2 & \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}} & \left(\frac{1}{\sqrt{2}}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Problems

$$S \cdot T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$TS = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

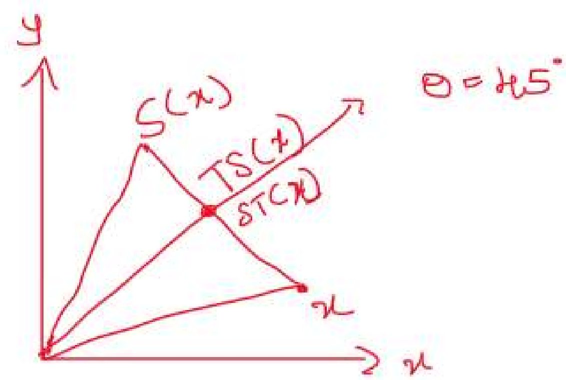
$$\Rightarrow ST = TS.$$

Problems

ST is the composition of projecting any vector of $\mathbb{R}^2$ onto $y = x$ line then reflecting it onto $y = x$ line.

TS is the composition of reflecting any vector of $\mathbb{R}^2$ onto $y = x$ line then projecting it onto $y = x$ line.

Both transformation produces the same output.



Problems

Determine the new point after applying the transformation to the given point

a. Project $x = (-2, 1)$ on the y-axis and then rotate by 45° counter-clockwise

b. Rotate $x = (-2, 1)$, 60° counter-clockwise and then project on the x-axis

Solution:

a) Projection matrix to project any vector of R^2 on the y-axis is given by

$$P(90°) = \begin{bmatrix} \cos^2(90°) & \cos(90°)\sin(90°) \\ \cos(90°)\sin 90° & \sin^2(90°) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Problems

Rotation matrix to rotate any vector of $\mathbb{R}^2$, by 45° counter clockwise about the origin is given by

$$Q(45^\circ) = \begin{bmatrix} \cos(45) & -\sin(45) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

The projection of $x = (-2, 1)$ on the y-axis and then rotating about $\theta = 45^\circ$, counter clockwise is given by

$$Q_{(45)} P_{(90)} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Problems

6. Rotation matrix to rotate any vector of $\mathbb{R}^2$

60° counter clockwise is given by

$$Q(60) = \begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

Projection matrix to project any vector of $\mathbb{R}^2$ onto

x-axis is given by

$$P(0) = \begin{bmatrix} \cos^2 0 & \cos 0 \sin 0 \\ \cos 0 \sin 0 & \sin^2 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Problems

Rotating $x = (-2, 1)$, 60° counterclockwise and then projecting on the x-axis is given by

$$P_{(0)} Q_{(60)}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-\sqrt{3}/2 \\ 0 \end{bmatrix}$$

THANK YOU