# VECTOR SPACES

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# CLASS 9 : CONTENT

- Four Fundamental subspaces

# FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

Let $A$ be the matrix of order $m \text{ x } n$ . Associated with it are four subspaces which are defined as follows

1. $C(A)$ is the column space of $A$ is a subspace of $\mathbb{R}^m$and contains all the linear combination of the column vectors of $A$ .

If then $\rho(A) = k$ then $\dim\big(C(A)\big) = k$

A basis of $C(A)$corresponds to the columns having the pivots in echelon form of $A$ .

2. $C(A^T)$ are the row space of $A$ is a subspace of $\mathbb{R}^n$ and contains all the linear combinations of the rows of $A$.

If $\rho(A) = k$ then $\dim\big(C(A^T)\big) = k$

## FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

A basis for $C(A^T)$ is the set of row vectors in $A$ or in the echelon form corresponding to the pivots in the echelon form.

3. $N(A)$ are the null space of $A$ consists of all the solutions of the system $Ax = 0$. It is a subspace of $\mathbb{R}^n$.

If $\rho(A) = k$ then $\dim\big(N(A)\big) = n - k$

A basis for $N(A)$ is obtained by solving the system $Ux = 0$, identifying the pivot variables and free variables where $U$ is the row reduced echelon form of $A$ $\mathrm{i.e. special\ solutions\ to\ Ux = 0\ forms\ the\ basis\ of\ N(A)}$

4. $N(A^T)$ are the left null spaces of $A$ is a subspace of $\mathbb{R}^n$ and consists of all the solution to the system$A^T x = 0$.

## FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

If $\rho(A) = k$ then $\dim\left(N(A^T)\right) = m - k$.

A basis for $N(A^T)$ is obtained by looking at the zero rows of $U$ and then tracing back to the corresponding rows of $A$.

## FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

Note :

1. The row space of Amxn is the column space of $A^T$ .
   It is spanned by the rows of A.

2. The left null space contains all vectors y for which $A^T y$ = 0.

3. N(A) and C(AT ) are subspaces of $R^m$

4. N($A^T$) and C(A) are subspaces of $R^n$

5. Dim C(A) = Dim C(AT ) = r = rank of A

6. Dim N(A) = n-r and Dim N(AT ) = m- r.
7. The dimension of the null space of a matrix is called its nullity.

## FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

The rank- nullity theorem :

For any matrix Amxn ,

dim C(A) + dim N(A) = no. of columns i.e

r + ( n-r ) = n

This law applies to as well.

Hence, dim C(AT ) + dim N( $A^T$ ) = m i. e

r + ( m- r ) = m

# FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

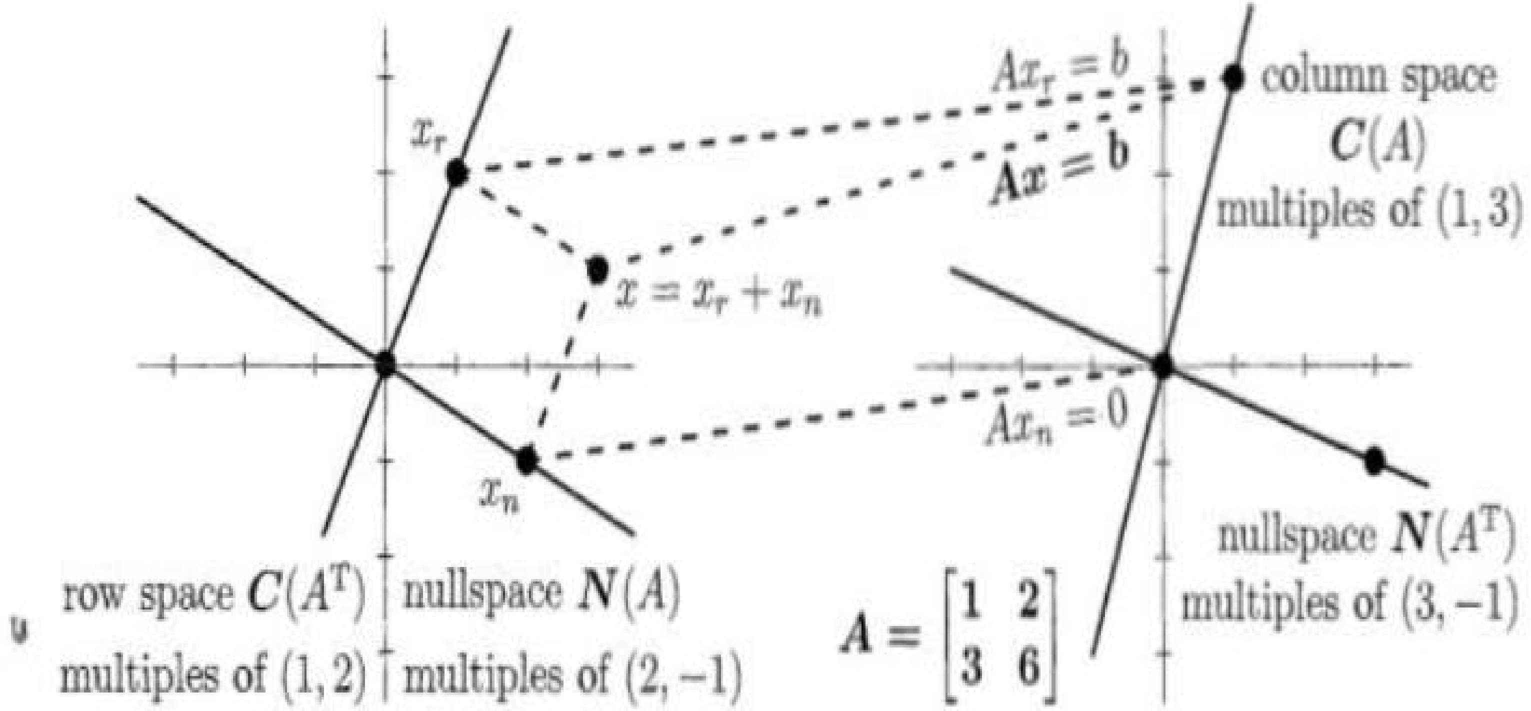
Let $$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Then, m = n= 2 and rank r = 1.

1. C(A) is the line through ( 1, 3 )
2. C($A^T$ ) is the line through ( 1, 2 )
3. N(A) is the line through ( -2 , 1 )
4. N( $A^T$) is the line through ( -3, 1 )

## VECTOR SPACES

# FOUR FUNDAMENTAL  SUBSPACES OF A MATRICES



## FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

E.g. : Find the dimensions and a basis each for the four fundamental subspaces of the matrix.

**Solution**

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} / c_1, c_2 \in R \right\}$$

## FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

Basis for $C(A)=\left\{\begin{bmatrix}1\\1\\3\end{bmatrix},\begin{bmatrix}2\\3\\7\end{bmatrix}\right\}$; $\dim C(A) = \text{Rank of } A$

$$c(A^T)=\left\{c_1\begin{bmatrix}1\\2\\1\\2\end{bmatrix}+c_2\begin{bmatrix}1\\2\\1\\3\end{bmatrix} / c_1,c_2\in R\right\}$$

(or)

$$c(A^T)=\left\{c_1\begin{bmatrix}1\\2\\1\\2\end{bmatrix}+c_2\begin{bmatrix}0\\0\\0\\1\end{bmatrix} / c_1,c_2\in R\right\}$$

## FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

Basis for $c(A^T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ ; $\dim C(A^T) = \text{Rank of } A$

$$Ax = 0 \Rightarrow Ux = 0$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y + z + 2t = 0 \Rightarrow x = -2y - z$$

$$t = 0$$

Special solution

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$N(A) = \left\{ c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} / \, c_1, c_2 \in R \right\}$$

Basis for $N(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ ; $\dim N(A) = 2$

## FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

$$A^T x = 0$$

$$A^T = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 1 & 1 & 3 \\ 2 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U^l$$

$$U^l y = 0 \Rightarrow x + y + 3z = 0 \Rightarrow x = -y - 3z$$

$$y + z = 0 \Rightarrow y = -z$$

## FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

Special solution

$$\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$$N(A^T) = \left\{ c_1 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} / \, c_1 \in R \right\}$$

Basis for $N(A^T) = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$ ; $\dim N(A^T) = 1$

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# THANK YOU

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