LINEAR ALGEBRA AND ITS APPLICATIONS

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CLASS-2

RECTANGULAR MATRICES WITH ORTHONORMAL COLUMNS

Rectangular matrices with orthonormal columns

- If Q has orthonormal columns, the least-squares problem becomes easy.
- $Q^T Qx = Q^T b$ are the normal equations for the best solution -in which $Q^T Q = I$.
- $x = Q^T b$
- $p=Qx$ the projection of b is $(q_1^T b)q_1+....+(q_n^T b)q_n$
- $p = QQ^T b$, the projection matrix is $P = Q Q^T$.

Special cases:-

1. If Q is a square matrix, $\hat{x} = Q^T b$

$\hat{x} = Q^{-1} b \quad [Q^T = Q^{-1}]$

2. If Q is a square matrix, $P = Q\hat{x}$

$= QQ^{-1} b$

$P = b$

Problems:

1. Project $b = (0, 3, 0)$ onto each of the orthonormal vectors $a_1 = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$, $a_2 = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ & then find its projection p onto the plane of a_1 & a_2.

Soln The Projection of b onto a_1 is given by

$$P_1 = \left(\frac{a_1^T b}{a_1^T a_1}\right) a_1 \quad \text{but } a_1^T a_1 = \|a_1\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1$$

$$P_1 = \left(\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\right) \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix} \Rightarrow P_1 = \begin{bmatrix} 4/3 \\ 4/3 \\ -2/3 \end{bmatrix}$$

Problems:

Similarly, $P_2 = \frac{(a_2^T b)\, a_2}{a_2^T a_2} = \begin{bmatrix} -2/3 \\ 4/3 \\ 4/3 \end{bmatrix}$

$\therefore$ P is the projection onto the plane containing a_1 & a_2 i.e. $P = P_1 + P_2$

$$= \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix}$$

Problem 2:- Find a third column so that the matrix

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & - \\ 1/\sqrt{3} & 2/\sqrt{14} & - \\ 1/\sqrt{3} & -3/\sqrt{14} & - \end{bmatrix}$$

is orthogonal. verify that the rows automatically become orthonormal at the same time.

Ans: By the definition of orthogonal matrix, first and second columns are orthogonal to the required third column. Let the third column be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$B = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & x \\ 1/\sqrt{3} & 2/\sqrt{14} & y \\ 1/\sqrt{3} & -3/\sqrt{14} & z \end{bmatrix} \Rightarrow a^T c = 0 \Rightarrow \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = 0 \Rightarrow x + y + z = 0 \longrightarrow (1)$$

Similarly, $b^T c = 0 \Rightarrow \begin{bmatrix} 1/\sqrt{14} & 2/\sqrt{14} & -3/\sqrt{14} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$$\Rightarrow x + 2y - 3z = 0 \longrightarrow (2)$$

Solving ① & ②, $(x, y, z) = (-5, 4, 1)$

$\therefore \quad Q = (a \quad b \quad c)$

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & -5/\sqrt{42} \\ 1/\sqrt{3} & 2/\sqrt{14} & 4/\sqrt{42} \\ 1/\sqrt{3} & -3/\sqrt{14} & 1/\sqrt{42} \end{bmatrix}$$

For the rows to be orthonormal, the norm must be equal to 1.

i.e. $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{5}{\sqrt{42}}\right)^2 = 1$

$\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{4}{\sqrt{42}}\right)^2 = 1$

& $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{-3}{\sqrt{14}}\right)^2 + \left(\frac{1}{\sqrt{42}}\right)^2 = 1$

THANK YOU

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