Unit 2 - Vector Spaces

Topic 1 - Vector spaces and Subspaces, column space & Null space, examples

→ A real vector space V is a non-empty set of objects called vectors, together with (Scalar multiplication & vector addition) satisfying the following axioms :-

i) If $u, v \in V$, then $(u+v) \in V \Rightarrow V$ is closed under ~~all~~ vector addition.

ii) If $c \in \mathbb{R}$ & $u \in V$, then $cu \in V \Rightarrow V$ is closed under ~~sca~~ scalar multiplication.

These operations satisfy the following properties for $u, v, w \in V$ & c_1, c_2 are scalars.

a) $u+v = v+u$ (commutative law)

b) $u+(v+w) = (u+v)+w$ (Associative law)

c) there is a unique zero vector i.e 0 such that $0+u = u+0 = u$ (identitive law)

d) for each u, there is a unique vector $(-u)$ such that $u+(-u) = (-u)+u = 0$ (Inverse law)

e) $c_1(u+v) = c_1 u + c_1 \cdot v$

f) $(c_1+c_2)u = c_1 u + c_2 u$

g) $1u = u$; 1 is a multiplicative identity

<u>Example 1</u> Prove that the set of all 2 by 2 matrices associated with the matrix addition and the scalar multiplication of a matrices is a vector space.

<u>Solution</u> consider: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $A' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ such that

$A, A' \in V$ and $r, s \in R$

Let V be the set of all 2 by 2 matrices

1) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$$

Adding any 2 by 2 matrices gives a 2 by 2 matrix and therefore the result of addition $\in V$.

2) $$r \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$$

Multiply any 2*2 matrix by a scalar and the result is a 2*2 matrix $\in V$.

3) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

$$\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} = \begin{bmatrix} a'+a & b'+b \\ c'+c & d'+d \end{bmatrix}$$

$$= \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

4)
$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}$$

$$\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}$$

$$\begin{bmatrix} (a+a')+a'' & (b+b')+b'' \\ (c+c')+c'' & (d+d')+d'' \end{bmatrix}$$

$$\begin{bmatrix} a+(a'+a'') & b+(b'+b'') \\ c+(c'+c'') & d+(d'+d'') \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \underbrace{\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}}$$

5)
$$r\left[s\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right] = r\begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}$$

$$= \begin{bmatrix} rsa & rsb \\ rsc & rsd \end{bmatrix} = \begin{bmatrix} (rs)a & (rs)b \\ (rs)c & (rs)d \end{bmatrix}$$

$$= (rs)\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

6)
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+0 & b+0 \\ c+0 & d+0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

7)
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} a+(-a) & b+(-b) \\ c+(-c) & d+(-d) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

8) $r\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right\} = \begin{bmatrix} r(a+a') & r(b+b') \\ r(c+c') & r(d+d') \end{bmatrix}$

$$= \begin{bmatrix} ra+ra' & rb+rb' \\ rc+rc' & rd+rd' \end{bmatrix} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} + \begin{bmatrix} ra' & rb' \\ rc' & rd' \end{bmatrix}$$

$$= r\begin{bmatrix} a & b \\ c & d \end{bmatrix} + r\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

9) Distributivity of sums of real numbers :-

$$(r+s)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (r+s)a & (r+s)b \\ (r+s)c & (r+s)d \end{bmatrix}$$

$$= \begin{bmatrix} ra+sa & rb+sb \\ rc+sc & rd+sd \end{bmatrix} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} + \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}$$

$$= r\begin{bmatrix} a & b \\ c & d \end{bmatrix} + s\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

10) $1\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a & 1b \\ 1c & 1d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

<u>Example 2</u> Show that the set of all real polyno-
-mials with a degree $n \leq 3$ associated with the addition of polynomials and the multiplication of polynomials by a scalar form a vector space ?

<u>Solution :-</u> The addition of two polynomials of degree less than or equal to 3 is a polynomial of degree less than or equal to 3.

The multiple of a polynomial of degree less than or equal to 3 by a real number results in a polynomial of degree less than or equal to 3.

Hence the set of polynomials of degree less than or equal to 3 is closed under addition & Scalar multiplication (the first two conditions).

The remaining 8 rules are automatically satisfied since the polynomials are real.

<u>Example 3</u> Show that the set of integers associated with addition and multiplication by a real number is not a vector space

<u>solution</u> : The multiplication of an integer by a real number may not be an integer.

Example :- Let $x = -2$. If you multiply x by the real number $\sqrt{3}$, the result is not an integer.

<u>Few examples :-</u> Vector space

1. R = the set of all real numbers
2. $R^2 = \{(x, y) / x, y \in R\}$
3. $R^3 = \{(x, y, z) / x, y, z \in R\}$
4. $R^n = \{(x_1, x_2, \dots, x_n) / x_i \in R\}$
5. $R^\infty = \{(x_1, x_2 \dots x_\infty) / x_i \in R\}$

<u>Example 4</u> Verify whether the following is a vector space ? $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} ; x \geq 0, y \geq 0, x, y \in r \right\}$

under usual vector addition & scalar multiplication ?

→ closure property holds good

→ Associative property holds good.

→ $0 \in V \Rightarrow u+0 = 0+u = u$

→ $\forall u \in V \ \exists -u \notin V \quad u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in V \quad -u = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \notin V$

→ Inverse law does not hold

→ V is not a vector space.

<u>Example 5 :-</u> $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 / \ x+y = 0 \right\} \ u, v \in V$

We can verify that all the properties holds good. Hence V is a vector space.

Precisely, we can add any two vectors and we can multiply all vectors by scalars. In other words, we can take linear combinations.

Definition of Subspace :-

A non-empty subset of a vector space is called a subspace of V, if it is itself a vector space under the same operations of vector addition & scalar multiplication as defined in vector space.

The following are the properties satisfied by a subspace of V.

i) $0 \in W$ [zero vector always belong to a subspace]

ii) If $u, v \in W$, Then $u+v \in W$

iii) If c is a scalar and $u \in W$ then $c \cdot u \in W$

→ If W is a subset of Vector space V and if W is itself a Vector Space under the inherited operations of addition & scalar multiplication from V, then W is called a subspace.

→ To show that W is a subspace of V, it is enough to show that :-

i) W is a subset of V.

ii) The zero vector of V is in W.

iii) For any vectors u & v in W, u+v is in W. W is closed under addition.

iv) For any vector u and scalar in r; r·u is in W. ∴ W is closed under scalar multiplication

Example 6 The set W of vectors of the form $(x,0)$ where $x \in \mathbb{R}$ is a subspace of $\mathbb{R}^2$ beceause W is a subset of $\mathbb{R}^2$ whose vectors are of the form (x,y) where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

→ The zero vector is in W.

→ $(x_1,0) + (x_2,0) = (x_1+x_2,0)$ → closure under addition

→ $r \cdot (x,0) = (rx,0)$ → closure under scalar multiplication.

Example 7 The set W of vectors of the form (x, y) such that $x \geq 0$ and $y \geq 0$ is not a sub space of $\mathbb{R}^2$ beceause it is not closed under scalar multiplication.

u $(2,2)$ is in W but $-1(2,2) = (-2,-2)$ is not in W.

→ If U & W are two subspaces of a vector space V, intersection $U \cap W$ is also a subspace of V.

→ $0 \in U$ and $0 \in W$ since U and W are subspaces they must contain '0'. $0 \in U \cap W$.

→ ∴ The intersection of any number of subspaces of a vector space V is a subspace of V.

Subspace of $\mathbb{R}^3$:-

i) $\mathbb{R}^3$ itself ii) zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

iii) Line passing through origin

iv) Plane passing through origin

v) In general, if $V = \mathbb{R}^n$, the possible subspaces are lines through origin, 2-d planes through origin, 3-d planes through origin, ... $(n-1)$d planes through origin & the space itself.

Lectures 3 & 4 - Echelon form, Row reduced Echelon form, Pivot variables, Free variables

→ A rectangular matrix is said to be in echelon form if it has the following characterization:-

i) All the zero rows are below the non-zero rows

ii) Each pivot lies to the right to the pivot in the row above.

iii) All the entries in a column below the pivot entry are zero.

The matrix is said to be in row reduced echelon form R, if in addition to the above, the matrix has the following additional characterization.

iv) Pivot (it should be 1) is the only non-zero entry in its column.

Pivot variables & Free variables

→ consider $Rx = 0$

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i) Pivot variables → which corresponds to columns with pivots.

ii) Free variables → which corresponds to columns without pivots.

→ From the above example :-

→ First & third columns contain the pivots, so u & w are pivot variables.

→ Second & fourth columns do not contain the pivots, so v & y are called free variables.

→ Rank of a Matrix :- The rank of a matrix A is the number of non-zero rows in the echelon form U of A and is denoted by $\rho(A)$.

→ If a matrix A is of order $m \times n$, then its rank $r \leq \min(m, n)$.

→ Example 1 :- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$; $\rho(A) = 2$

→ Example 2 :- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$; $\rho(A) = 1$

Echelon form U and Row Reduced Echelon form R

Eg

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} = U \rightarrow \text{Echelon form}$$

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix} = U, \text{ Echelon form}$$

Row Reduced Echelon form R :-

$$A = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} = R$$

Pivot Variables & Free Variables :-

→ To find the most general solution to $Rx=0$ or $Ax=0$, we may assign arbitrary values to free variables.

→ The pivot variables are completely determined in terms of free variables;

→ consider matrix
$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u+2v-y=0 \Rightarrow u=-2v+y; \quad w+y=0 \Rightarrow w=-y$$

Pivot variables $= u$ & w

Free Variables $= v$ & y

If $\begin{cases} y=0; \text{ then } w=0; u=-2 \\ v=1 \end{cases}$ $\therefore \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

If $\begin{cases} y=1; \text{ then } w=-1; u=-1; \\ v=0 \end{cases}$ $\therefore \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

These two are called Special solutions.

→ The best way to find all solutions is to form the special solutions.

$$x = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = v \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

The complete solution is the linear combination of 2 special solutions.

Example 8 For every c, find R and special solutions to $Ax = 0$, where $A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$

Solution If $c = 1$, $A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{2R_2 - R_1} \underbrace{\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}}_{U}$$

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2} R_1} \underset{R}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

$$Rx = 0 \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$y = 0$ & x is free variable

If $c=2$, then $A = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to -R_1} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here y is the free variable

$x - 2y = 0$

$\boxed{x = 2y}$ special solution $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

If $c \neq [1, 2]$ or $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ no special solution.

If $c=0$, then

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \xrightarrow{R_2 \to \frac{R_2}{2}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{R}$$

$$Rx = 0 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x = 0$ & $y = 0 \Rightarrow$ NO special solutions

Problems on Echelon form, free and Pivot variables

Example 9 For each of the following matrices, find i) U ii) R iii) ρ

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 6 & -2 \\ 3 & -2 & 8 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix}$$

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$$D = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \qquad E = \begin{bmatrix} -2 & 3 & 1 \end{bmatrix}$$

Solution i) $A = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} \xrightarrow{R_2+2R_1} \underbrace{\begin{bmatrix} 1 & 3 \\ 0 & 8 \end{bmatrix}}_{U}$

$$\begin{bmatrix} 1 & 3 \\ 0 & 8 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{8}R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1-3R_2} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{R}$$

Number of non-zero rows $= \rho(A) = 2$

ii) $B = \begin{bmatrix} 2 & 6 & -2 \\ 3 & -2 & 8 \end{bmatrix} \xrightarrow{R_2 - \frac{3}{2}R_1} \begin{bmatrix} 2 & 6 & -2 \\ 0 & -11 & 11 \end{bmatrix}$

$3 - x \cdot 2 = 0 \qquad -2-9 ; \; 8+3$

$3 = 2x$

$x = 3/2$

$$\underbrace{\begin{bmatrix} 2 & 6 & -2 \\ 0 & -11 & 11 \end{bmatrix}}_{U} \xrightarrow{R_1 \to \frac{R_1}{2}} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -11 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & -11 & 11 \end{bmatrix} \xrightarrow{R_3 \to \frac{R_3}{-11}} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1-3R_2} \underbrace{\begin{bmatrix} 1 & 0 & +2 \\ 0 & 1 & -1 \end{bmatrix}}_{R}$$

$$\boxed{\rho(B) = 2}$$

iii) $C = \begin{bmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 5 & -10 \end{bmatrix}$

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 5 & -10 \end{bmatrix} \xrightarrow{R_3-R_2} \underbrace{\begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1/2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{R_2}{5}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+R_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}}_{R}$$

$$\boxed{\therefore \; \rho(C) = 2}$$

iv) $D = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \xrightarrow{R_2+3/2R_1} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{2R_3+R_1} \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ U

$$\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{R_1}{-2}} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{R} \qquad \boxed{\rho(D) = 1}$$

v) $E = \begin{bmatrix} -2 & 3 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{R_1}{-2}} \underbrace{\begin{bmatrix} 1 & -3/2 & -1/2 \end{bmatrix}}_{U = R}$

$\boxed{\rho(E) = 1}$

Example 10 Solve the following system of equations by identifying pivot variables & free variables

$x + 2y + 3z = 9$

$2x - 2z = -2$

$3x + 2y + z = 7$

Solution $[A:b] = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 2 & 0 & -2 & -2 \\ 3 & 2 & 1 & 7 \end{array}\right]$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 2 & 0 & -2 & -2 \\ 3 & 2 & 1 & 7 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \\ 0 & -4 & -8 & -20 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \\ 0 & -4 & -8 & -20 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivots : 1, -4

$\rho(A) = 2$; $\rho(A:b) = 2 < n = 3$

$\therefore$ system of equations are consistent & have infinitely many solutions.

Pivot variables = x, y

Free variables = z

$$x\begin{bmatrix}1\\0\\0\end{bmatrix}+y\begin{bmatrix}2\\-4\\0\end{bmatrix}+z\begin{bmatrix}3\\-8\\0\end{bmatrix}=\begin{bmatrix}9\\-20\\0\end{bmatrix}$$

x, y are pivot variables $\rightarrow$ Associated to columns with pivots.

z is a free variable $\rightarrow$ Associated to columns without pivots.

Let $\boxed{z=K}$

$$-4y-8z=-20$$
$$-4y-8K=-20$$
$$-4y=-20+8K$$
$$\boxed{y=5-2K}$$

$$x+2y+3z=9$$
$$x+2\{5-2K\}+3K=9$$
$$x+10-4K+3K=9$$
$$x-K=-1$$
$$\boxed{x=K-1}$$

$\therefore (x,y,z)=(K-1, 5-2K, K)$

Example 11 Reduce the matrix $A=\begin{bmatrix}1&1&2&2\\2&2&4&4\\1&c&2&2\end{bmatrix}$ to its echelon form and hence find special solution :-

Solution

$$\begin{bmatrix}1&1&2&2\\2&2&4&4\\1&c&2&2\end{bmatrix}\xrightarrow[R_3-R_1]{R_2-2R_1}\begin{bmatrix}1&1&2&2\\0&0&0&0\\0&c-1&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&1&2&2\\0&0&0&0\\0&c-1&0&0\end{bmatrix}\xrightarrow{R_2\leftrightarrow R_3}\begin{bmatrix}1&1&2&2\\0&c-1&0&0\\0&0&0&0\end{bmatrix}$$

If $c=1$

$$\begin{bmatrix}1&1&2&2\\0&0&0&0\\0&0&0&0\end{bmatrix}=R$$

Solving for $Rx=0$

$$\begin{bmatrix}1&1&2&2\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\\t\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

(18) Pivot variable $= x$

Free variables $= y, z, t$

$$x+y+2z+2t=0$$

$$\Rightarrow \quad x = -y-2z-2t$$

$$\therefore \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -y-2z-2t \\ y \\ z \\ t \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$y=1$	$z=1$	$t=1$
$z=0$	$y=0$	$z=0$
$t=0$	$t=0$	$y=0$

If $c \neq 1$, then $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - \frac{R_2}{c-1}} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2/c-1} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Pivot variables $= x, y$

Free variables $= z, t$

$$x+2z+2t=0 \Rightarrow x = -2z-2t$$

$$y=0$$

If $t=0$ & $z=1$; $x=-2, y=0$

$$\therefore \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

If $t=1$ & $z=0$; $x=-2$ & $y=0$

$$\therefore \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

These two are special solutions.

Lectures 5 & 6 - Linear Independence, Basis & Dimensions

Linear combination :-

Let V be a vector space and $v_1, v_2, v_3 \ldots v_n$ be the vectors in V. Then, the form

$$c_1 \cdot v_1 + c_2 \cdot v_2 + \ldots + c_n \cdot v_n \text{ , where } c_1, c_2 \ldots c_n$$

are scalars is called a linear combination of the vectors.

→ Linear combination of vectors involve scalar multiplication and vector addition of vectors.

→ To decide on linear independence of vectors, we need to look for their linear combination.

→ The trivial combination with all scalars 'c'=0, produces $0 \cdot v_1 + 0 \cdot v_2 + \ldots + 0 \cdot v_n = 0$. If this is the only way to produce zero, given vectors are independent. If any other combination

→ Produces zero, then vectors are dependent.

Linear Independence & Linear Dependence :-

→ A set of vectors $\{V_1, V_2 .. V_n\}$ of a vector space is said to be linearly independent if their linear combination $c_1 \cdot V_1 + c_2 \cdot V_2 + \dots + c_n \cdot V_n = 0$, where $c_1 = c_2 = c_3 = \dots = c_n = 0$.

→ A set of vectors $\{V_1, V_2 \dots V_n\}$ of a vector space V, is said to be linearly dependent if there exists scalars $c_1, c_2 \dots c_n \in R$, not all zero such that $c_1 \cdot V_1 + c_2 \cdot V_2 + \dots + c_n \cdot V_n = 0$;

Either all $c_1, c_2 \dots c_n$ are non-zero **OR** Few scalars are zero and few are non-zero.

→ A set of vectors $\{V_1 \dots V_n\}$ is said to be linearly independent if one vector cannot be written as the combination of the other vectors.

→ To decide whether given set of vectors are independent or dependent, apply the following procedure :-

1) Place the vectors $\{V_1, V_2 \dots V_n\}$ of the given set of vectors as columns of matrix A.

2) Apply Gaussian Elimination on the matrix A.

3) If all the columns of the matrix is with pivot, then the set of vectors are linearly independent.

4) If certain columns of the matrix A do not hold pivot, then the set of vectors are linearly dependent.

5) If $\rho(A) = n$ = number of columns, then vectors are linearly independent.

Example 12 Check whether $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 3 \\ 7 \\ 1 \end{bmatrix}$ are independent in $\mathbb{R}^3$?

Solution :-

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 7 \\ 3 & 3 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 - R_1 \\ R_4 - 3R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{2R_4 + R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3 = n$; Vectors are linearly independent.

<u>Example 13</u> check whether $\begin{bmatrix}1\\2\\1\end{bmatrix} \begin{bmatrix}3\\1\\0\end{bmatrix} \begin{bmatrix}4\\3\\1\end{bmatrix}$ are independent in $\mathbb{R}^3$?

<u>Solution</u> :- $A = \begin{bmatrix}1 & 3 & 4\\2 & 1 & 3\\1 & 0 & 1\end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix}1 & 3 & 4\\0 & -5 & -5\\0 & -3 & -3\end{bmatrix}$

$$-3 - x(-5) = 0$$
$$-3 + 5x = 0$$
$$5x = 3$$
$$x = 3/5$$

$$\begin{bmatrix}1 & 3 & 4\\0 & -5 & -5\\0 & -3 & -3\end{bmatrix} \xrightarrow{R_3 - 3/5 R_2} \begin{bmatrix}1 & 3 & 4\\0 & -5 & -5\\0 & 0 & 0\end{bmatrix}$$

Here $\rho = 2, n = 3$. $\therefore$ vectors are linearly dependent.

<u>Note</u> :- 1) The columns of a square invertible matrix are always independent.

2) The columns of a matrix A of order $m \times n$ with $m < n$ are always dependent.

3) The columns of A are independent exactly when $N(A) = Z$ (Z means 0).

4) The 'r' nonzero rows of an echelon matrix U and a reduced matrix R are always independent and so are the 'r' columns that contain the pivots.

Basis

→ A subset $S = \{v_1, v_2 \ldots v_n\}$ of a vector space is called a basis for vector space V if

i) S is a linearly independent set

ii) spans the vector space V

→ The dimension of a vector space is the number of basis vectors.

Properties of Basis :-

1) The vectors are linearly independent (not too many vectors).

2) They span the space V (not too few vectors).

3) Every vector in the space is a combination of these basis vectors beceause they span V.

4) If $\{v_1, v_2 \ldots v_n\}$ and $\{\omega_1, \omega_2 \ldots \omega_n\}$ are both the bases for the same vector space, then the number of vectors is same.

5) Basis maximal independent set & minimal spanning set.

6) Any linearly independent set in V can be extended to a basis in V by adding more vectors if necessary. Any spanning set in V can be reduced to a basis, by discarding vectors if necessary.

7) There exists one and only one way to write any vector 'V' in vector space as a combination of the bases vectors of that vector space.

Examples Of Basis :-

1) Basis of $\mathbb{R}^2$ vector space $\{(1,0)\ (0,1)\}$
2) Bases of $\mathbb{R}^3$ vector space is $\{(1,0,0)\ (0,1,0)\ (0,0,1)\}$
3) Bases of $\mathbb{R}^n$ vector space is $\{(1,0,0,0\ldots 0)\ (0,1,0\ldots 0), (0,0,\ldots,1)\}$
4) Matrices $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$ is the bases for vector space of $2*2$ matrices.
5). The set $\{1, t, t^2, \ldots t^n\}$ is a basis of space of polynomials P_n.

Span of a set :-

→ Let $W = \{V_1, V_2 \cdots V_n\}$ be a set of vectors belonging to a vector space V, then span of W is the set of all linear combinations of vectors in W

i.e span of $W = c_1 \cdot V_1 + c_2 \cdot V_2 + \ldots + c_n \cdot V_n$

$= \text{subspace of } V$

Example 14 What do these vectors $\begin{bmatrix}1\\2\end{bmatrix} \begin{bmatrix}2\\4\end{bmatrix} \begin{bmatrix}3\\5\end{bmatrix}$ span?

Solution The given vectors span a 2D subspaces of $\mathbb{R}^2$. $A = \begin{bmatrix}1 & 2 & 3\\2 & 4 & 5\end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix}1 & 2 & 3\\0 & 0 & -1\end{bmatrix}$; $\rho(A) = 2$

Problems on Linear Dependence, Independence, Basis & Dimension :-

Example 15 Decide the dependence or independence of the following vectors :- i) (1,3,2) (2,1,3) (3,2,1)

Solution Write the given set of vectors as columns of matrix A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix}$$

$$-1-x(-5)=0$$
$$-1+5x=0$$
$$5x=1$$
$$\boxed{x=1/5}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \xrightarrow{R_3-1/5R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix}$$

$$\frac{-5}{1}+\frac{7}{5} = \frac{-18}{5}$$

∴ All the columns of matrix A is with pivots, $\rho(A)=n=3$. ∴ Given vectors are linearly independent.

ii) (1,-3,2) (2,1,-3) (-3,2,1)

$$\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \xrightarrow[R_2-2R_1]{R_2+3R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

Beceause there exists columns without pivots in A, Given vectors are linearly dependent.

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iii) Vectors (1, 1) (2, 3) (1, 2)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

A set of 'n' vectors of $\mathbb{R}^m$ must be linearly dependent if $n > m$. ∴ above set of vectors are linearly dependent.

iv) Vectors $(1100)^T, (1010)^T, (0011)^T, (0101)^T$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) < n$. ∴ vectors are linearly dependent

<u>Example 16</u> If W_1, W_2, W_3 are independent vectors, show that the differences $V_1 = W_2 - W_3$, $V_2 = W_1 - W_3$, $V_3 = W_1 - W_2$ are dependent ?

<u>Solution :-</u> $\because$ W_1, W_2, W_3 are independent

$$\alpha W_1 + \beta \cdot W_2 + \gamma \cdot W_3 = 0 \Rightarrow \alpha = \beta = \gamma = 0$$

To check if V_1, V_2 and V_3 are independent, check for the combination $c_1 \cdot V_1 + c_2 V_2 + c_3 \cdot V_3 = 0$

$$c_1(W_2 - W_3) + c_2(W_1 - W_3) + c_3(W_1 - W_2) = 0$$

$$c_1 \cdot W_2 - c_1 \cdot W_3 + c_2 \cdot W_1 - c_2 \cdot W_3 + c_3 \cdot W_1 - c_3 W_2 = 0$$

$$W_1(c_2 + c_3) + W_2(c_1 - c_3) - W_3(c_1 + c_2) = 0$$

$$\alpha W_1 + \beta \cdot W_2 + \gamma \cdot W_3 = 0$$

Since W_1, W_2 & W_3 are linearly independent

$$\alpha = \beta = \gamma = 0$$

$$\Rightarrow \quad c_2 + c_3 = 0 \Rightarrow c_2 = -c_3$$

$$c_1 - c_3 = 0 \Rightarrow c_1 = c_3$$

$$c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$$

Since scalars are non zero, V_1, V_2 & V_3 are dependent.

<u>Example 17</u> Find the basis and hence find the dimension of subspaces of $\mathbb{R}^4$?

i) All vectors whose components are equal :-

$\because$ vectors are in $\mathbb{R}^4$, it should contain 4 components.

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \in \mathbb{R}^4 = \begin{bmatrix} x \\ x \\ x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} ; x \in \mathbb{R}^4$$

(28)

Basic = $[1\ 1\ 1\ 1]^T$; Dimension = 1

(ii) All vectors whose components add up to zero.

Solution :-

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \in \mathbb{R}^4 \text{ s.t } \quad x+y+z+t=0$$

$$\Rightarrow x = -y-z-t$$

$$\begin{bmatrix} -y-z-t \\ y \\ z \\ t \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis :- $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Dimension = 3

Example 18 Let V be a subspace of four dimensional space $\mathbb{R}^4$ such that :-

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 ;\ x_1 - x_2 + x_3 - x_4 = 0 \right\}$$

Find the bases & dimensions of V ?

Solution :- $\because$ $x_1 - x_2 + x_3 - x_4 = 0 \Rightarrow x_1 = x_2 - x_3 + x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{V_1} + x_3 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{V_2} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{V_3}$$

$\therefore$ Any vector in subspace V of $\mathbb{R}^4$ can be obtained as a linear combination of vectors (V_1, V_2, V_3).

Example 19 Find a basis for each of the following subspaces of 2 by 2 matrices

i) All diagonal matrices

ii) All symmetric matrices $[A^T = A]$

iii) All skew symmetric matrices $[A^T = -A]$

Solution :- Any 2×2 matrices is given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Basis of 2×2 matrices is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Dimensions of 2 by 2 matrices subspace is 4.

i) All diagonal matrices [2 by 2]

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \; a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} ; \; a, b \in \mathbb{R} \right\}$$

Basis is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Dimension $= 2$

ii) All symmetric matrices $(A^T = A)$

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ Basis $= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Dimension $= 3$

iii) All skew symmetric matrices $[A^T = -A]$

$$\left\{ \begin{matrix} 0 & a \\ -a & 0 \end{matrix} \right\} \middle| \; a \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Big\} \quad \text{Basis} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Example 20 Find a bases for subspace of polynomials $P(x)$ of degree $= 3$?

Solution :- $\{(1, t, t^2, t^3)\}$ Basis, Dimension $= 4$

Example 21 Describe the subspace of $\mathbb{R}^3$ spanned by i) the two vectors $(1, 1, -1)$ and $(-1, -1, 1)$

Solution :- check the independence of given vectors before drawing conclusions w.r.t subspace.

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow[R_3 + R_1]{R_2 - R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ only 1 column with pivot, these vectors are dependent. Hence, the span of the given vectors is a line.

ii) $(0, 1, 1), (1, 1, 0)$ and $(0, 0, 0)$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ only two columns with two pivots, the given set of vectors span a two dimensional plane in $\mathbb{R}^3$.

iii) The columns of a 3 by 5 echelon matrix form with two pivots.

→ This is a two dimensional plane in $\mathbb{R}^3$.

iv) All vectors with positive components :-

$$\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} ; (a, b, c) \in \mathbb{R} \right\} \quad a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

whole of $\mathbb{R}^3$. i.e 3 dimensional space.

Column space

Definition Let A be a $m \times n$ matrix. The column space of A is the set of all linear combinations of the columns of A denoted by $C(A)$. Thus,

$$C(A) = \{ b \in \mathbb{R}^m \,/\, Ax = b \text{ is solvable} \}$$

Note $C(A)$ is a subspace of $\mathbb{R}^m$.

→ The space spanned by linear combination of linearly independent columns of matrix A spans the column space of matrix A.

→ $C(A)$ can lie anywhere in between the zero space & the whole space $\mathbb{R}^m$.

→ The system of Linear equations $Ax=b$ is solvable iff the vector b can be expressed as a combination of columns of A, then 'b' is in $C(A)$.

→ The smallest possible column space comes from the zero matrix $A=0$. The only combination of columns is $b=0$.

→ If A is a 5×5 identity matrix then $C(A)$ is the whole of $\mathbb{R}^5$. The 5 columns of A can combine to produce any 5 dimensional vector b. In fact, any 5×5 non singular matrix A will have $\mathbb{R}^5$ as its column space.

→ Let $A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix}$, then $C(A)$ is the subspace of $\mathbb{R}^3$ consisting of vectors b that are linear combination of the vectors $(1,5,2)$ and $(0,4,3)$. Geometrically, the subspace is a 2D plane.

→ Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix}$, then $C(B)$ is the subspace of $\mathbb{R}^3$ consisting of vectors b that are linear combination of the vectors $(1,5,2)(0,4,3)$ and $(1,9,5)$

Note: The column spaces of A & B are same though the matrices are different. This is beceause the new column is a linear combination of the other two columns. Hence, appending a dependent column does not alter the column space of a matrix.

Definition of Null space :-

→ Let A be a matrix of order $m \times n$

→ The null space of A is the set of all solutions of the homogeneous system of equations.

→ $Ax = 0$ denoted by $N(A)$.

→ $N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$

→ $N(A)$ is a subspace of $\mathbb{R}^n$.

→ Null space of A is spanned by special solutions to $Ax = 0$ which is same as solving for $Rx = 0$, where R is the row reduced echelon form of A.

→ $N(A)$ is the subspace of vector space $\mathbb{R}^n$.

→ Dimension of Null space is '$n - r$'.

→ For a system of linear equations to be non-singular and matrix A to be invertible, N(A) = 0.

→ special solutions are the basis of N(A).

<u>Example 22</u> Let $A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix}$

$$Ax = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 5R_1} \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 3 \end{bmatrix} \xrightarrow{R_3 - \frac{3}{4}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$$

$$3 - x(4) = 0$$

$$3 = 4x$$

$$x = 3/4$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

x & y are both pivot variables. There are no free variables.

This gives $x = 0$ & $y = 0$ as the only solution.

Thus the null space of this matrix thus contains only the zero vector; [0,0].

Null space of this matrix is origin in $\mathbb{R}^2$.

Example 23 Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix}$

$$Ax = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix} \xrightarrow[R_4 - 2R_1]{R_2 - 5R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow{R_3 - \frac{3}{4} R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$3 - 4x = 0$$

$$3 = 4x$$

$$x = 3/4$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4y + 4z = 0$ $\qquad$ $x + z = 0$

$y = -z$ $\qquad$ $x = -z$

let $z = c$; $y = -c$; $x = -c$

Gives infinitely many solutions $(-c, -c, c)$ all of which lie on a line that obviously passes through the origin.

The matrices $\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix}$

have the same column space but different null space.

Topic 7 - The four fundamental subspaces, Left null space

Left null space :-

→ Null space of A^T is called left null space.

→ solutions to $A^T y = 0 \Rightarrow y^T \cdot A = 0$ spans the left null space.

→ $N(A^T) \subseteq \mathbb{R}^m$, Left null space is a subspace of $\mathbb{R}^m$.

→ Dimensions of $N(A^T) = m - r$

→ Linear combination of rows which gives zero rows forms the basis for left null space.

Example 24 Obtain the left null space for the following:-

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

∴ No zero rows; Left null space is a zero vector. Basis $\{N(A^T)\} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad$ Dim $\{N(A^T)\} = 0$

∴ $N(A^T)$ is origin in $\mathbb{R}^3$.

Example 25 :- Obtain the left null space for the following :-

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

Solution :- $\left[\begin{array}{ccc:c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 4 & b_2 \\ 2 & 4 & 8 & b_3 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2-b_1 \\ 0 & 2 & 6 & b_3-2b_1 \end{array}\right]$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2-b_1 \\ 0 & 2 & 6 & b_3-2b_1 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc:c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2-b_1 \\ 0 & 0 & 0 & b_3-2b_1-\{2b_2-2b_1\} \end{array}\right]$$

combination of rows which gives zero rows is $b_3-2b_2+0\cdot b_1$.

solutions to $A^T\cdot y=0$ or $y^T\cdot A=0$ gives $N(A^T)$.

Basis of $N(A^T) = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$; Dimension $\{N(A^T)\} = 1$

$\therefore$ $N(A^T)$ is a line spanned by $\{0,-2,1\}$ in $\mathbb{R}^3$.

Bec*e*ause there exists one zero row, $N(A^T)$ Basis has one vector.

Example 26 Obtain the left null space for the following :-

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

Solution:- $\left[\begin{array}{cccc:c} 1 & 3 & 3 & 2 & b_1 \\ 2 & 6 & 9 & 7 & b_2 \\ -1 & -3 & 3 & 4 & b_3 \end{array}\right] \xrightarrow[R_3+R_2]{R_2-2R_1} \left[\begin{array}{cccc:c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 3 & b_2-2b_1 \\ 0 & 0 & 6 & 6 & b_3+b_1 \end{array}\right]$

$$\left[\begin{array}{cccc:c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 3 & b_2-2b_1 \\ 0 & 0 & 6 & 6 & b_3+b_1 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{cccc:c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 3 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & \end{array}\right]$$

$b_3+b_1-2b_2+4b_1$

Combination of rows which produces zero rows is $b_3-2b_2+5b_1$.

Basis $N(A^T) = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$; Dimension of $N(A^T) = 1$

$N(A^T)$ is a line spanned by $\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$.

Four Fundamental Subspaces :-

→ Let A be a matrix of order $m \times n$. Associated with it are four subspaces which are defined as follows :-

→ 1) $C(A)$ is the column space of A which is a subspace of $\mathbb{R}^m$ and contains all linear combinations of vectors of A. If $\rho(A) = K$, then dim of Basis of $C(A) = K$. A basis of $C(A)$ corresponds to the columns having the pivots in echelom form of A.

→ 2) $C(A^T)$ also called the row space of A is a subspace of $\mathbb{R}^n$ and contains all the linear combinations of the rows of A. If $\rho(A) = K$, then dim of $C(A^T) = K$. A basis for $C(A^T)$ is the set of row vectors in A or in the echelon form corresponding to the pivots in the echelon form.

→ 3) $N(A)$ are called the null space of A which consists of all the solutions of the system $Ax = 0$. It is a subspace of $\mathbb{R}^n$. If $\rho(A) = K$, then $\dim\{N(A)\} = n - K$. A basis for $N(A)$ is obtained by solving $Ux = 0$, identifying the pivot variables and free variables. Special solutions to $Ux = 0$ forms the basis of $N(A)$.

→ 4) $N(A^T)$ are the left null spaces of A and is a subspace of $\mathbb{R}^m$. It consists of all the solutions to the system $A^T \cdot x = 0$. If $\rho(A) = K$, then $\dim\{N(A^T)\} = m - K$.

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→ A basis for $N(A^T)$ is obtained by looking at the zero rows of U and then tracing back to the corresponding rows of A.

Note :

→ The row space of $A_{m\times n}$ is the columnspace of A^T. It is spanned by the rows of A.

2) The left null space contains all vectors y for which $A^T \cdot y = 0$.

3) $N(A)$ & $C(A^T)$ are subspaces of $\mathbb{R}^n$

4) $N(A^T)$ & $C(A)$ are subspaces of $\mathbb{R}^m$

5) $\mathrm{Dim}\{C(A)\} = \mathrm{Dim}\ C(A^T) = r = $ rank of A

6) $\mathrm{Dim}\{N(A)\} = n-r$ & $\mathrm{Dim}\{N(A^T)\} = m-r$

7) The Dimension of null space of a matrix is called its nullity.

The rank-nullity theorem :-

→ For any matrix $A_{m\times n}$

$$\dim\{C(A)\} + \dim\{N(A)\} = \text{no. of. columns}$$
$$r + (n-r) = n.$$

→ $$\dim\{C(A^T)\} + \dim\{N(A^T)\} = m$$
$$r + (m-r) = m$$

→ Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ $\qquad C(A) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

→ column space of A : $\alpha \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

<u>N(A)</u>

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \xrightarrow{R_3-3R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x + 2y = 0$

let $y = 1 \Rightarrow x = -2$

$\therefore$ N(A) is the line through $(-2, 1)$.

<u>N(AT)</u>

$$\left[\begin{array}{cc:c} 1 & 2 & b_1 \\ 3 & 6 & b_2 \end{array}\right] \xrightarrow{R_3-3R_1} \left[\begin{array}{cc:c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 3b_1 \end{array}\right]$$

$\therefore$ N(AT) is the line through $(-3, 1)$

<u>C(AT)</u> :

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

C(AT) is a line through $(1, 2)$

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Example 27 Find bases & dimensions for each of the four fundamental subspaces of a matrix ?

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \,/\, c_1, c_2 \in \mathbb{R} \right\}$$

Bases for $C(A) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$; $\text{Dim}\{C(A)\} = 2$

$C(A)$ is a two dimensional space spanned by $(1,1,3)$ & $(2,3,7)$ in $\mathbb{R}^3$

Note select columns of A & not from 'U' to span $C(A)$. $\because$ columns are not preserved in elementary row operations.

$$C(A^T) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \,/\, c_1, c_2 \in \mathbb{R} \right\}$$

or

$$C(A^T) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \,/\, c_1, c_2 \in \mathbb{R} \right\}$$

Bases for $C(A^T) = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$; $\text{Dim}\{C(A^T)\} = 2 = r$

$C(A^T)$ is a 2-D plane spanned by a linear combination of vectors $(1,2,1,2)$ & $(0,0,0,1)$ in $\mathbb{R}^4$

$Ax = 0 \Rightarrow Ux = 0$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + y(2) + z + 2t = 0$; $t = 0$

$x = -2y - z$; Pivot variables $= x, t$

Free variables $= y, z$

If $y = 1, z = 0 \Rightarrow x = -2$

$\therefore \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

If $y = 0, z = 1 \Rightarrow \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$$\therefore \quad N(A) = \left\{ y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} / \; y, z \in \mathbb{R} \right\}$$

$\therefore$ Basis for $N(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$; $\dim\{N(A)\} = 2$

$A^T \cdot x = 0$

Given $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{bmatrix}$; $A^T = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 1 & 1 & 3 \\ 2 & 3 & 7 \end{bmatrix} \xrightarrow[\substack{R_3 - R_1 \\ R_4 - 2R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

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$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_2} \underbrace{\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{U^T}$$

$$U^T \cdot y = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y + z = 0$; $y = -z$

$x + y + 3z = 0$; $x = -y - 3z$

Let $z = 1$; $y = -1$; $x = 1 - 3 = -2$

$\therefore$ special solution $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ $N(A^T) = \left\{ x \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} / x \in \mathbb{R} \right\}$

$\Rightarrow$ Dimension of $N(A^T) = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = 1$.

Basis $= \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

Alternate method to find N(AT):-

To find the left null space, find the combinations of rows of A which produce zero rows.

$$A = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 1 & 2 & 1 & 3 & b_2 \\ 3 & 6 & 3 & 7 & b_3 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2-b_1 \\ 0 & 0 & 0 & 1 & b_3-3b_1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2-b_1 \\ 0 & 0 & 0 & 1 & b_3-3b_1 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2-b_1 \\ 0 & 0 & 0 & 0 & \end{array}\right]$$

$$b_3-3b_1-b_2+b_1$$

$$N(AT) = \left\{ c_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} ; c_1 \in \mathbb{R} \right\}$$

N(AT) is a line spanned by $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$.

Basis for N(AT) $= \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$; Dimension of N(AT) = 1

Example 28 Describe the column space and the null space for the following matrices?

1) $[0] \rightarrow$ $C(A) = Z$; $N(A) = \mathbb{R}$

2) $[0, -3] \longrightarrow C(A) = \mathbb{R}$; $N(A) = \{x\text{-axis in } \mathbb{R}^2\}$

3) $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ $C(A) = y$ axis in R^2; $N(A) = Z$

4) $\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$; $\left[\begin{array}{l} C(A) = x\text{-axis in } \mathbb{R}^2 \\ N(A) = \text{line } x = y \text{ in } \mathbb{R}^2 \end{array}\right]$

5) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $C(A) = x\text{-axis in } \mathbb{R}^2$

$N(A) = \left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} ; x \in \mathbb{R} \right\}$

$N(A)$ is a line spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$.

6) $\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix}$

$C(A) =$ whole of $\mathbb{R}^2$

$N(A) \Rightarrow \quad 4y = 0 \quad ; \quad x - y = 0$

$y = 0 \qquad x = 0$

$\boxed{\therefore \text{ origin } (0,0) \text{ in } \mathbb{R}^2}$

Example 29 Find the column space & the null space of $A = \begin{bmatrix} 1 & 0 \\ 2 & 7 \\ 5 & 3 \end{bmatrix}$. Give an example of a matrix whose column-space is same as that of A but the null space is different.

Solution :- $C(A)$ is a 2D plane in $\mathbb{R}^3$

$N(A)$ is origin in $\mathbb{R}^2$.

Let $A' = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 7 & 9 \\ 5 & 3 & 8 \end{bmatrix}$. This has same column space as $C(A) \Rightarrow C(A) = C(A')$

$N(A')$ is a line spanned by $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ in $\mathbb{R}^3$.

To verify :-

$C(A)$ $\begin{bmatrix} 1 & 0 \\ 2 & 7 \\ 5 & 3 \end{bmatrix} \xrightarrow[R_3 - 5R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 0 \\ 0 & 7 \\ 0 & 3 \end{bmatrix} \xrightarrow{R_3 - \frac{3}{7}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 7 \\ 0 & 0 \end{bmatrix}$

$$3 - 7x = 0$$
$$3 = 7x$$
$$\boxed{x = 3/7}$$

$$\therefore \quad C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R} \right\}$$

$C(A)$ is a 2D-plane in $\mathbb{R}^3$

$$A' = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 7 & 9 \\ 5 & 3 & 8 \end{bmatrix} \xrightarrow[R_3 - 5R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 7 & 7 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow{R_3 - \frac{3}{7}R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 7 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C(A') = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} / c_1, c_2 \in \mathbb{R} \right\}$$

$$\boxed{\therefore \; C(A) = C(A')}$$

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TO find N(A) & N(A') :-

N(A) is special solutions of $Ux=0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x=0$ & $y=0$

$\therefore$ N(A) is the origin in $\mathbb{R}^2$.

N(A')

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 7 & 7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

x, y = pivot variables

z = free variable

$7y + 7z = 0$

$7y = (7z)-1$

$x + z = 0$

$x = -z$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix}$$

let $z = -1$ $\therefore \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\therefore$ N(A') is a line spanned by $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ in $\mathbb{R}^3$.

Example 30 Let $V = \{(a,b,c,d) / b+c+d=0\}$ and $W = \{(a,b,c,d) \mid a+b=0 \text{ \& } c=2d\}$ be subspaces of $\mathbb{R}^4$. Find the dimension of $V \cap W$?

Solution

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} ; \; b+c+d=0$$

$$\begin{pmatrix} a \\ -c-d \\ c \\ d \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

V is a 3D plane in $\mathbb{R}^4$.

$$W = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} ; \; \begin{matrix} a+b=0 \\ c=2d \end{matrix} \qquad \begin{pmatrix} -b \\ b \\ 2d \\ d \end{pmatrix} = b\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

W is a 2D plane in $\mathbb{R}^4$.

$V \cap W$ $\quad b+c+d=0$ & $a+b=0$, $c=2d$

$$b+2d+d=0 \Rightarrow b=-3d$$

$$a-3d=0 \Rightarrow a=3d$$

$$\therefore \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3d \\ -3d \\ 2d \\ d \end{pmatrix} = d\begin{pmatrix} 3 \\ -3 \\ 2 \\ 1 \end{pmatrix}$$

$\therefore$ $V \cap W$ is a line spanned by $(3,-3,2,1)$ in $\mathbb{R}^4$.

Topic 8 - Uniqueness & Existence of Inverses, rank-nullity theorem

Existence of Inverses :-

Definition

→ Let $A_{m \times n}$ be a matrix such that rank of $A = m$. Then $Ax = b$ has atleast one solution x for every b if and only if the columns span $\mathbb{R}^m$. In this case, A has a right inverse C such that $AC = I_{m \times m}$.

→ Let $A_{m \times n}$ be a matrix such that rank of $A = n$. Then $Ax = b$ has atmost one solution x for every b if and only if the columns are linearly independent. In this case, A has a left inverse B such that $BA = I_{n \times n}$.

→ A has a left inverse if $BA = I$

→ A has a right inverse if $AC = I$

→ Rank always satisfies $r \leq m$ and $r \leq n$. An m by n matrix cannot have more than 'm' independent rows or 'n' independent columns. There is not a space for more than 'm' pivots or more than 'n'.

→ when $r=m$, there is right inverse and $Ax=b$ always has a solution.

→ when $r=n$, there is a left inverse and the solution (if it exists) is unique.

→ only a square matrix has both $m=n$.

Hence a square matrix has both existence and uniqueness achieved. So only square matrix has two sided inverses.

case i) If $\rho(A)=m$, then A will have right inverse of order $m\times n$ such that $A_{m\times n} * C_{n\times m} = I_{m\times m}$

case ii) If $\rho(A)=n$, then A will have left inverse of order $n\times m$ such that $B_{n\times m} * A_{m\times n} = I_{n\times n}$

Best right inverse, $C = A^T(AA^T)^{-1}$

Best left inverse, $B = (A^TA)^{-1}\cdot A^T$

Example 31 Obtain left inverse or a right inverse if it exists for the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3\times 2}$

Solution:- Here $\rho(A) = 2 = n$. $\therefore$ A has left inverse B, then

$$B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3}$$

$BA = I$

$$\underbrace{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{3\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$a=1,\ b=0,\ d=0,\ e=1$

$\left. c=1, f=1 \right\}$ free variables

$\therefore B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is the required left inverse.

Example 32 Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}_{2\times 3}$

Solution Here $\rho(A) = m = 2$. A has right inverse of $m\times n$ such $A_{m\times n} * C_{n*m} = I_{m*m}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}_{2\times 3} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}_{3*2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2*2}$$

$$\begin{bmatrix} 2a & 0b \\ 3c & 3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad a = 1/2 \quad d = 1/3$$

right inverse of A is $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \\ a & b \end{bmatrix}$.

→ Since the third row is arbitrary, there are infinitely many right inverses for A.

Matrices of Rank 1

→ Every matrix of rank 1 has the simple form

$A = u \cdot v^T$ = column times row.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ Every row is a multiple of first row, so row space is one dimensional. In fact, we can write the whole matrix as the product of a column vector & row vector.

$$A = (\text{column})(\text{row})$$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$$

where the rows are all multiples of the vectors v^T; the columns are all multiples of the vector u.

Matrices of rank One :-

When the rank of a matrix is as small as possible, a complicated system of equations can be broken into simple pieces. Those simple pieces are matrices of rank 1.

→ We can write such matrices as a column times row.

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \\ 8 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$$

Example 33 Find the left or right inverse for the following matrices, whichever exists :-

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution Step I Apply Gaussian elimination and obtain rank.

Step II Check if rank 'ρ' of the matrix is equal to 'm' or 'n' i.e the number of rows or columns of the matrix A.

Step III If $\rho(A) = m$ then find right inverse $A_{m \times n} * B_{n \times m} = I_{m \times m}$

If $\rho(A) = n$ then left inverse exists $C_{n \times m} \cdot A_{m \times n} = I_{n \times n}$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$$

$\rho(A) = 2 = m$ $\therefore$ right inverse exists

Best right inverse is $A^T(A\cdot A^T)^{-1}$

$$A\cdot A^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A\cdot A^T)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A^T(A\cdot A^T)^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3\times 2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}_{2\times 2}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

<u>Example 34</u> Find inverse for the matrix A as

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3\times 2}$$

<u>Solution</u>

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 = n$

$\therefore$ A has left inverse ; Best left inverse is $(A^T \cdot A)^{-1} \cdot A^T$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^T \cdot A)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

$$(A^T \cdot A)^{-1} \cdot A^T = \frac{1}{3} \underset{2\times 2}{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

Lecture 9 - Sum of Subspaces, Direct Sum :-

→ The sum of two subspaces U & W of a vector space V is defined as :-

$$U + W = \{ u \in U, \omega \in W \}$$

Definition : Let U, W be subspaces of V. Then V is said to be the direct sum of U & W and we write $V = U \oplus W$, if $V = U + W$ and $U \cap W = \{0\}$.

Let U, W be subspaces of V. Then $V = U \oplus W$ if and only if for every $v \in V$ there exists unique vectors $u \in U$ and $w \in W$ such that $v = u + w$.

Properties :-

1) The zero vector '0' of V is in $U+W$.
2) For any $u, w \in U+W$, we have $u+v \in U+W$.
3) For any $v \in U+W$ and $\alpha \in \mathbb{R}$, we have $\alpha v \in V \in U+W$.
4) $v = u+w$ must be unique.

Example consider $U = \{(a,0,0) / a \in \mathbb{R}\}$

$W = \{(0,b,c) / b,c \in \mathbb{R}\}$

Thus $V = U+W = \{(a,b,c) / a,b,c \in \mathbb{R}\}$

Hence the direct sum of subspaces U & W results in vector space $\mathbb{R}^3$.