CLASS 4 : CONTENT

- Linear independence and dependence of vectors
- Basis
- Dimension
- Span of set of vectors

LINEAR COMBINATION

Let V be a vector space and $v_1, v_2, v_3, \ldots\ldots v_n$ be the vectors in V. Then the form $c_1v_1 + c_2v_2 + \ldots. + c_nv_n$, where $c_1, c_2, \ldots\ldots. c_n$ are scalars, is called a *linear combination of the vectors* .

Linear combination of vectors involve scalar multiplication and vector addition of vectors .

To decide on linear independence of vectors we need to look for their linear combination .

The trivial combination with all scalars 'c'=0 ,produces $0v_1 + 0v_2 + \ldots. + 0v_n = 0$

If this is the only way to produce zero ,given vectors are independent

if any other combination produces zero then vectors are dependent

LINEAR INDEPENDENCE AND DEPENDENCE

A set of vectors $\{v_1, v_2, v_3, ...v_n\}$ of a vector space, is said to be linearly independent if the linear combination $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$,

where $c_1 = c_2 = c_3 = ... = c_n = 0$

A set of vectors $\{v_1, v_2, v_3, ...v_n\}$ of a vector space, is said to be linearly dependent if there exists scalars $c_1, c_2, ...c_n \in R,$ not all zero such that

$c_1v_1 + c_2v_2 + ... + c_nv_n = 0$ either all $c_1, c_2, ...c_n$ are nonzero of few scalars are zero and few are non zero .

(A set of vectors $\{v_1, v_2, v_3, ...v_n\}$ is said to be linearly independent if one vector cannot be written as the combination of the other vectors.

LINEAR INDEPENDENCE AND DEPENDENCE

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To decide whether given set of vectors are independent or dependent apply the following procedure :

1) Place the vectors $\{v_1, v_2, v_3, \ldots.. v_n\}$ of the given set of vectors as columns of matrix A .
2) Apply Gauss Elimination on the Matrix A
3) If all the columns of the matrix is with pivot then the set of vectors are linearly independent
4) If certain columns of the matrix A do not hold pivot then the set of vectors are linearly dependent
5) If $\rho(A) = n =$ number of columns then vectors are linearly independent

LINEAR COMBINATION

E.g.: Check whether $\begin{bmatrix}1\\2\\1\\3\end{bmatrix},\begin{bmatrix}1\\1\\2\\3\end{bmatrix},\begin{bmatrix}1\\3\\7\\1\end{bmatrix}$ are independent in $\mathbb{R}^3$.

Solution

$$\begin{bmatrix}1&1&1\\2&1&3\\1&2&7\\3&3&1\end{bmatrix}\sim\begin{bmatrix}1&1&1\\0&-1&1\\0&1&6\\0&0&-2\end{bmatrix}\sim\begin{bmatrix}1&1&1\\0&-1&1\\0&0&7\\0&0&-2\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&1\\0&-1&1\\0&0&7\\0&0&0\end{bmatrix};\rho(A)=3=n$$

Hence the given vectors are linearly independent.

LINEAR COMBINATION

E.g.: Check whether $\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}3\\1\\0\end{bmatrix},\begin{bmatrix}4\\3\\1\end{bmatrix}$ are independent in $\mathbb{R}^3$.

Solution

$$A=\begin{bmatrix}1 & 3 & 4\\2 & 1 & 3\\1 & 0 & 1\end{bmatrix}\sim\begin{bmatrix}1 & 2 & 3\\0 & -5 & -5\\0 & -3 & -3\end{bmatrix}\sim\begin{bmatrix}1 & 3 & 4\\0 & -5 & -5\\0 & 0 & 0\end{bmatrix}$$

$$\rho(A)=2, n=3$$

Therefore they are dependent

LINEARLY INDEPENDENT VECTORS

Note : 1. The columns of a square invertible matrix are always independent.

2. The columns of a matrix A of order m x n with $m < n$ are always dependent.

3. The columns of A are independent exactly when N(A) = Z

(Z means 0)

4. The ‘r ‘nonzero rows of an echelon matrix U and a reduced matrix R are always independent and so are the ‘r’ columns that contain the pivots.

LINEARLY INDEPENDENT VECTORS

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A subset $S=\{v_1,v_2,v_3,.....v_n\}$ of a vector space is called a basis for

Vector space V if

i. S is a linearly independent set.

ii. S *spans the vector space* V.

The *dimension of a vector space* is the number of basis vector.

BASIS

A Basis for Vector space V is a sequence of vectors having properties :

1. The vectors are linearly independent (not too many vectors)
2. They span the space V (not too few vectors) .
3. every vector in the space is a combination of the basis vectors ,because they span (s.t. combination is unique)
4. if $\{v_1, v_2, v_3, ...v_n\}$ and $\{w_1, w_2, w_3, ...w_n\}$ are both the bases for the same vector space then m=n ,the number of vectors is same .
5. basis is maximal independent set and minimal spanning set.

BASIS

6. If The vectors $\{v_1, v_2, v_3, ...v_n\}$ and $\{w_1, w_2, w_3, ...w_n\}$ are both bases for the same vector space ,then m = n , i.e. the number of vectors is the same .
7. Any linearly independent set in V can be extended to a basis ,by adding more vectors if necessary ,any spanning set in V can be reduced to a basis ,by discarding vectors if necessary .
8. there exists one and only one way to write any vector ‘v’ in Vector space V as a combination of the bases vectors of that vector space .

Examples of Basis :

1. Basis of R^2 vector space is {(1,0) ,(0,1)} .

2. Basis of R^3 vector space is {(1,0,0) ,(0,1,0),(0,0,1)} .

3. Basis of R^nvector space is {(1,0,0,0,.......,0) ,(0,1,0,......,0),(0,0,......,1)}

4.Marices $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ is the basis for vector space of 2*2 matices .

5. The set $\left\{1, t, t^2, ..., t^n\right\}$ is a basis of space of Polynomials P_n

SPAN OF A SET

Let $W = \{v_1, v_2, v_3, ...v_n\}$ be a set of vectors belonging to vector space V,

then *span* of W is the set of all linear combinations of vectors of W .

i.e., span of $W = span(W) = c_1v_1 + c_2v_2 + ... + c_nv_n$

= subspace of V

SPAN OF A SET

E.g. : What do these vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ span

Solution

The given vectors span a 2D subspaces of $\mathbb{R}^2$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}; \quad \rho(A) = 2$$

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CLASS 2 : CONTENT

- Echelon form of a matrix
- Row reduced Echelon form of the matrix
- Pivot variables and Free variables
- Special solution

ECHELON FORM U AND ROW REDUCED FORM

A rectangular matrix is said to be in echelon form if it has following characterizations

i. All the zero rows are below the non zero rows.

ii. Each pivot (non-zero leading entry) lies to the right to the pivot in the row above(This produces the stair-case pattern).

iii. All the entries in a column below the pivot entry are zero.

The matrix is said to be in row reduced form R, if in addition to the above, the matrix has following additional characterization

iv. Pivot (it should be 1) is only non-zero entry in its column

PIVOT VARIABLES AND FREE VARIABLES

Consider $Rx = 0$

$$i.e., \quad \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The unknowns are divided into two groups

i. Pivot variables: which corresponds to columns with pivots.

ii. Free variables: which corresponds to columns without pivots.

From the above example

First and third columns contain the pivots, so u & $w$are the pivot variables.

Second and fourth columns do not contains pivots, so v & y are free variables.

RANK OF A MATRIX

Rank of a Matrix Definition: The rank of a matrix A is the number of nonzero rows in the echelon form U of A and is denoted by ρ (A) or simply r.

Note : If A is a matrix of order m x n then its rank r ≤ min (m, n).

Ex: 1 $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}; \rho(A) = 2$$

Ex: 2 $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}; \rho(A) = 1$$

ECHELON FORM U AND ROW REDUCED FORM

Eg:

$$A=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}=U,$$

Echelon form

$$A=\begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix}=U,$$

Echelon form

$$A=\begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}=R,$$

Row reduced Echelon form

Marked elements are the Pivots.

PIVOT VARIABLES AND FREE VARIABLES

To find the most general solution to $Rx = 0$ (or equivalently to $Ax = 0$)

we may assign arbitrary values to free variables.

The pivot variables are completely determined in terms of free variables

v and y . $\Rightarrow Rx = 0 \Rightarrow u + 2v - y = 0 \Rightarrow u = -2v + y$

$$w + y = 0 \Rightarrow w = -y$$

<u>**Special solutions**</u>

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

PIVOT VARIABLES AND FREE VARIABLES

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Most general solution or complete solution

The best way to find all solutions to is from the special solutions

$$x=\begin{bmatrix}u\\v\\w\\y\end{bmatrix}=v\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+y\begin{bmatrix}-1\\0\\-1\\0\end{bmatrix}$$

The complete solution is the linear combination of 2 special solutions.

ECHELON FORM U AND ROW REDUCED FORM

E.g. : For every c, find R and special solutions to $Ax = 0$, where

$$A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

Solution: If $c = 1$ then $A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$

$$R_2 \to R_2 - 2R_1$$

$$A \sim \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = U$$

$$R_1 \to \frac{1}{2} R_1$$

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$R_x = 0 \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = 0$ & x is free variable

ECHELON FORM U AND ROW REDUCED FORM

$$x=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

If $c = 2$ then

$$A\sim\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}=U;R_1\rightarrow -R_1$$

$$\sim\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

y is the free variable

ECHELON FORM U AND ROW REDUCED FORM

Now $x - 2y = 0 \Rightarrow x = 2y$

Special solution : $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

If $c \neq 1,2$ then no special solution

For E.g.: If $c = 0$ then

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = U; R_1 \to R_1 - R_2$$

$$A \sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R$$

$$Rx = 0 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = 0, y = 0 \quad (No\ special\ solution)$$

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CLASS 5 : CONTENT

- Problems on Linear independence and dependence of vectors
- Problems on Basis and Dimension

LINEAR DEPENDENCE AND INDEPENDENCE

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Problem 1: Decide the dependence or independence of the following vectors:

(1) vectors $(1,3,2)$, $(2,1,3)$, and $(3,2,1)$

Solution: Write the given set of vectors as columns of matrix A.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix} \xrightarrow[R_3-2R_1]{R_2-3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{pmatrix} \xrightarrow{R_3-\frac{1}{5}R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -\frac{18}{5} \end{pmatrix}$$

$\because$ All the columns of matrix A is with pivots

$\rho(A) = n = 3$ Given vectors are linearly independent.

LINEAR DEPENDENCE AND INDEPENDENCE

2) vectors $(1,-3,2)$, $(2,1,-3)$ and $(-3,2,1)$

$$A = \begin{pmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 + 3R_1} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ $\exists$ columns without pivots also in A.

Given vectors are Linearly Dependent.

3) vectors $(1,1)$, $(2,3)$, $(1,2)$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

vectors are linearly Dependent

LINEAR DEPENDENCE AND INDEPENDENCE

Note: A set of 'n' vectors of R^m must be Linearly dependent if $n > m$.

4) vectors $(1\ 1\ 0\ 0)^T, (1\ 0\ 1\ 0)^T, (0\ 0\ 1\ 1)^T, (0\ 1\ 0\ 1)^T$

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_4 - R_3} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\because$ All columns of A are with pivots

Given set of vectors are Linearly Independent.

LINEAR DEPENDENCE AND INDEPENDENCE

Problem 2: If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$, $v_3 = w_1 - w_2$ are dependent.

Solution: $\because$ w_1, w_2, w_3 are independent

$$\alpha w_1 + \beta w_2 + \gamma w_3 = 0 \Rightarrow \alpha = \beta = \gamma = 0 \text{ only}$$

To check if v_1, v_2 and v_3 are independent check for the combination $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

LINEAR DEPENDENCE AND INDEPENDENCE

$\because \quad c_1(w_2 - w_3) + c_2(w_1 - w_3) + c_3(w_1 - w_2) = 0$

$\Rightarrow \quad c_1 w_2 - c_1 w_3 + c_2 w_1 - c_2 w_3 + c_3 w_1 - c_3 w_2 = 0$

$\Rightarrow \quad w_1(c_2 + c_3) + w_2(c_1 - c_3) + w_3(c_3 - c_1) = 0$

$\Rightarrow \quad \alpha w_1 + \beta w_2 + \gamma w_3 = 0$

$\because$ w_1, w_2, w_3 are linearly independent

$\alpha = \beta = \gamma = 0$ only

$c_2 + c_3 = 0 \Rightarrow c_2 = -c_3$

$c_1 - c_3 = 0 \Rightarrow c_1 = c_3$

$c_3 - c_1 = 0 \Rightarrow c_1 = c_3$

$\Rightarrow c_1 = c_3 = -c_2$

c_i's can assume values other than zero also.

Hence dependent.

LINEAR DEPENDENCE AND INDEPENDENCE

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Problem 3: Find the Basis and hence find the Dimension of subspaces of R^4

1) All vectors whose components are equal.

∴ vectors are in R^4 it should contain 4 components.

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in R^4 \sim \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix} = \left\{ x \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} ; x \in R \right\} \text{ Subspace}$$

Basis $(1 \ 1 \ 1 \ 1)^T$; Dimension = 1

BASIS

2) All vectors whose components add up to zero

$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4$ s.t. $x+y+z+t=0$

$\Rightarrow \quad x = -y-z-t$

$$\begin{pmatrix} -y-z-t \\ y \\ z \\ t \end{pmatrix} \sim \left\{ y\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} ; y, z, t \in \mathbb{R} \right\}$$

Subspace of $\mathbb{R}^4$

Basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ Dimension $= 3$

BASIS

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Problem 4 :

Let 'V' be a subspace of four dimensional space R^4

Set $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in R^4 ; \; x_1 - x_2 + x_3 - x_4 = 0 \right\}$

Find the Basis and Dimension of V.

Solution :- $\because x_1 - x_2 + x_3 - x_4 = 0 \Rightarrow x_1 = x_2 - x_3 + x_4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \sim \begin{pmatrix} x_2 - x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \sim x_2 \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{V_1} + x_3 \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{V_2} + x_4 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{V_3}$$

Any vector in subspace V of R^4 can be obtained as a Linear combination of Vectors (V_1, V_2, V_3).

BASIS

Problem5: Find a Basis for each of the following subspaces of 2 by matrices.

1) All diagonal matrices
2) All symmetric matrices ($A^T = A$)
3) All skew symmetric matrices ($A^T = -A$)

Solution: Any 2 by 2 matrix is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $a,b,c,d \in \mathbb{R}$

Basis of 2 by2 matrix is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$\because \exists$ 4 vectors in Basis set

Dimension of 2 by 2 matrices subspace is 4

BASIS

1) All diagonal matrices (2 by 2)

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \;\middle|\; a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ; a, b \in R \right\}$$

Basis is $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ Dimension = 2

2) All symmetric matrices ($A^T = A$) (2 by 2)

$$\left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \;\middle|\; a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ; a, b, c \in R \right\}$$

Basis is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ Dimension = 3

BASIS

3) All Skew symmetric matrices $(A^T = -A)$ (2 by 2)

$$\left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \middle/ a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; a \in R \right\}$$

Basis $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ Dimension $= 1$

Problem 6: Find a basis for subspace of polynomials $p(a)$ of degree ≤ 3.

Solution: $\{(1, t, t^2, t^3)\}$ Basis, Dimension $= 4$

SPAN OF VECTORS

Problem 7: Describe the subspace of R^3 spanned by:

1) the two vectors $(1,1,-1)$ and $(-1,-1,1)$

Solution: step 1: check the independence of given vectors before drawing conclusions w.r.t. subspace.

$$\therefore \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\because$ Only 1 column with pivot, these vectors are dependent

Hence, the span of the given vectors is a Line.

2) $(0,1,1)$, $(1,1,0)$ and $(0,0,0)$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{R_3-R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{R_3+R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

SPAN OF VECTORS

$\therefore$ Only 2 columns with pivots, the given set of three vectors span a 2 dimensional plane in R^3.

3) The columns of a 3 by 5 Echelon form matrix with two pivots

2 Dimensional plane in R^3

4) all vectors with positive components

$$\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \sim a, b, c \in R \right\} \sim \left\{ a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; a, b, c \in R \right\}$$

Whole of R^3 i.e. 3 Dimensional space.

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CLASS 3 : CONTENT

- Problems on Echelon form ,free and pivot variables

ECHELON FORM OF A MATRIX

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Problem 1 :

For each of the following matrices find

1. Echelon Form 'U'
2. Row reduced Echelon Form 'R'
3. Rank of the matrix

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 6 & -2 \\ 3 & -2 & 8 \end{pmatrix} \quad C = \begin{pmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad E = \begin{pmatrix} -2 & 3 & 1 \end{pmatrix}$$

ECHELON FORM OF A MATRIX

Solution :-

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \xrightarrow{R_2 + 2R_1} \begin{pmatrix} 1 & 3 \\ 0 & 8 \end{pmatrix} = U = \text{Echelon Form}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & 8 \end{pmatrix} \xrightarrow{R_2/8} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 - 3R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = R \text{ Rowreduced Echelon form}$$

Rank of matrix $A = \rho(A) = 2$ (Number of non zero rows in echelon form of the matrix)

ECHELON FORM OF A MARIX

$$B = \begin{pmatrix} 2 & 6 & -2 \\ 3 & -2 & 8 \end{pmatrix} \xrightarrow{R_2 - \frac{3}{2}R_1} \begin{pmatrix} 2 & 6 & -2 \\ 0 & -11 & 11 \end{pmatrix} = U = \text{Echelon form}$$

$$\begin{pmatrix} 2 & 6 & -2 \\ 0 & -11 & 11 \end{pmatrix} \xrightarrow{R_1/2} \begin{pmatrix} 1 & 3 & -1 \\ 0 & -11 & 11 \end{pmatrix} \xrightarrow{R_2/(-11)} \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{R_1 - 3R_2} R = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

(Row reduced Echelon form)

Rank of $B = \rho(B) = 2$

ECHELON FORM OF A MATRIX

$$C = \begin{pmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1 \text{ and}} \begin{pmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 5 & -10 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 5 & -10 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{pmatrix} = U \quad \text{Echelon form}$$

$$\begin{pmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1/2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2/5} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Also $\rho(C) = 2$.

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xleftarrow{R_1 + R_2}$$

ECHELON FORM OF A MATRIX

$$D = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \xrightarrow{R_2 + 3/2\, R_1} \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} = U$$

$$\begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{R_1/(-2)} R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\rho(D) = 1$$

$$E = \begin{pmatrix} -2 & 3 & 1 \end{pmatrix} \xrightarrow{R_1/-2} \begin{pmatrix} 1 & -3/2 & -1/2 \end{pmatrix} = U = R$$

$$\rho(E) = 1$$

PIVOT VARIABLE AND FREE VARIABLE

Problem 2:

Solve the following system of linear eqns by identifying pivot variables and free variables:

$$x + 2y + 3z = 9$$
$$2x - 2z = -2$$
$$3x + 2y + z = 7$$

Solution: Apply Gauss Elimination on the augmented matrix

$$[A \vdots b] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & 0 & -2 & -2 \\ 3 & 2 & 1 & 7 \end{array}\right]$$

PIVOT AND FREE VARIABLE

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & 0 & -2 & -2 \\ 3 & 2 & 1 & 7 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \\ 0 & -4 & -8 & -20 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \quad U = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -2 & -8 & -20 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivots $= (1, -2)$;

$\rho(A) = 2$

$\rho(A:b) = 2$

$n = 3$

$\rho(A:b) = \rho(A) < n \rightarrow$ Infinitely Many Solution

System of Linear eqns given has Many solution

PIVOT AND FREE VARIABLES

Solving system of Eqns further we obtain

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -20 \\ 0 \end{pmatrix}$$

$$x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix} + z \begin{pmatrix} 3 \\ -8 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ -20 \\ 0 \end{pmatrix}$$

$x, y \rightarrow$ Pivot variables $\rightarrow$ associated to columns with pivots

$z \longrightarrow$ <u>free variable</u> $\rightarrow$ associated to column without pivot

Solution $\{k-1,\ 5-2k,\ k\}$ where $z = k \in \mathbb{R}$

SPECIAL SOLUTIONS

Problem 3 :

Reduce the matrix $A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{pmatrix}$ to its Echelonform and hence find special solution.

Solution:

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \sim R_3} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

if $\underline{c = 1}$ then $A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R$

Solving for $Rx = 0 \Rightarrow \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

SPECIAL SOLUTIONS

$$Rx = 0 \quad \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Pivot var (x)

Free variables (y, z, t)

$$x + y + 2z + 2t = 0 \Rightarrow x = -y - 2z - 2t$$

Sol.

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \sim \begin{pmatrix} -y - 2z - 2t \\ y \\ z \\ t \end{pmatrix} \sim y \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Where $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is the special solution

SPECIAL SOLUTIONS

if $c \neq 1$ then

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - \frac{1}{c-1} R_2} \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 / c-1} \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Pivot var (x, y)

Free var (z, t)

Solving for $Rx = 0$

$$x + 2z + 2t = 0 \Rightarrow x = -2z - 2t, \quad y = 0$$

SPECIAL SOLUTIONS

Special Solutions $\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Note : Echelon form is sufficient to look for back substition to obtain solution, but need for Row reduced Echelon form R is the matrix R readily gives Special solutions.

Given $A \longrightarrow U \longrightarrow R$ (to obtain special solutions)

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CLASS 6 : CONTENT

- Column Space
- Row Space

The column Space

Definition: Let A be a m x n matrix. The column space of A is the set of all linear combinations of the columns of A denoted by C(A). Thus,

C(A) = { b Є R m / Ax = b is solvable }

Note : C(A) is a subspace of R^m

CLASS 6 : CONTENT

- In other words, The space spanned by linear combination of linearly independent columns of matrix A spans the Column Space of Matrix A .
- Column Space is denoted by C(A)
- C(A) can lie anywhere in between the zero space and the whole space R^m
- The system of Linear equations Ax=b is solvable iff the vector ‘b’ can be expressed a combination of columns of A ,then ‘b’ is in C(A).

The column Space

Few examples…. 1. The smallest possible column space comes from the zero matrix A = 0. The only combination of the columns is b = 0.

2. If A is a 5 x 5 identity matrix then C(A) is the whole of R^5 the 5 columns of A can combine to produce any 5 dimensional vector b. In fact, any 5 x 5 nonsingular matrix A will have R^5 as its column space !!

COLUMN SPACE

Let $$A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix}$$

then C(A) is the subspace of R^3 consisting of vectors b that are linear combinations of the vectors (1, 5, 2) and (0, 4, 3). Geometrically the subspace is a 2- d plane.

Let $$A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix}$$

Then C(B) is the subspace of R^3 consisting of vectors b that are linear combinations of the vectors (1, 5, 2) , (0, 4, 3) and (1, 9, 5).

COLUMN SPACE

Note : The column spaces of A and B are same though the matrices are different. This is because the new column is a linear combination of the other two columns. Hence, appending a dependent column does not alter the column space of a matrix .

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CLASS 7 : CONTENT

- Null Space

NULL SPACE

Definition :

Let A be a matrix of order m x n.

The ***null space of A*** is the set of all solutions of the homogeneous system of equations

Ax = 0 denoted by N (A) .

Thus,

N(A) = { x ∈ R^n / Ax = 0 }

Note : N(A) is a subspace of R^n .

NULL SPACE

- Null Space of a matrix A is denoted by N(A)
- Null Space of A is spanned by Special solutions to Ax=0which is same as solving for Rx=0 where R is the row reduced echelon form of A
- Null space of A is a subspace of vector space R^n
- Dimension of Null space is ‘n-r’
- For a system of linear equations to be nonsingular and matrix A to be invertible N (A) = 0
- Special solutions are the basis of N(A) .

NULL SPACE

Example :

Let $A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(x, y both are pivot variables)

No Free variables.

Then gives x = y = 0 as the only solution.

The null space of this matrix thus contains only the zero vector (0, 0).

Null space of this matrix is 'origin' in R^2.

NULL SPACE

$$\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

$(x = -y = z)$ hence

Gives infinitely many solutions (c, -c c) all of which lie on a line that obviously passes through the origin.

The matrices $A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 3 & 5 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 3 \end{bmatrix}$

have the same column space but different null space !!

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CLASS 8 : CONTENT

- Left Null Space

LEFT NULL SPACE

Left Null Space

- Null Space of A^T is left null space

- Solutions to $A^T y = 0 \Rightarrow y^T A = 0$ spans the left null space

- $N(A^T) \subseteq R^m$, LEFT NULL IS A SUBSPACE OF R^m
- $Dimension\ of\ N(A^T) = m - r$

- LINEAR COMBINATION OF ROWS WHICH GIVES ZERO ROWS FORMS THE BASIS FOR LEFT NULL SPACE

LEFT NULL SPACE

Obtain the left null space for the following :

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}$$

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 2 & 6 & 3 & b_2 \\ 0 & 2 & 5 & b_3 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 2 & 1 & b_2 - 2b_1 \\ 0 & 2 & 5 & b_3 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 2 & 1 & b_2 - 2b_1 \\ 0 & 0 & 4 & b_3 - b_2 + 2b_1 \end{array}\right]$$

$\because$ No zero rows ; Left Null Space {zero vector}

Basis $N(A^T) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

Dim $N(A^T) = 0$

$N(A^T)$ is origin in R^3 .

LEFT NULL SPACE

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 4 & b_2 \\ 2 & 4 & 8 & b_3 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & 2 & 6 & b_3 - 2b_1 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - 2(b_2 - b_1) \end{array}\right]$$

Combination of Rows which gives zero rows is

$(b_3 - 2b_2 + 0 \cdot b_1)$

LEFT NULL SPACE

· Solutions to $A^T y = 0$ or $y^T A = 0$ gives $N(A^T)$

Basis of $N(A^T) = \left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}$ Dimension $N(A^T) = 1$

$N(A^T)$ line spanned by $(0, -2, 1)$ in R^3 .

$\because \exists$ one zero row, $N(A^T)$ Basis has one vector .

LEFT NULL SPACE

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 2 & 6 & 9 & 7 & b_2 \\ -1 & -3 & 3 & 4 & b_3 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 3 & b_2 - 2b_1 \\ 0 & 0 & 6 & 6 & b_3 + b_1 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 3 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_1 - 2(b_2 - 2b_1) \end{array}\right]$$

$b_3 + b_1 - 2b_2 + 4b_1$

$\Rightarrow b_3 - 2b_2 + 5b_1$

Combination of rows which produces zero rows is $b_3 - 2b_2 + 5b_1$

LEFT NULL SPACE

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

Basis $N(A^T) = \left\{ \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \right\}$

Dimension of $N(A^T) = 1$

$N(A^T)$ is a line spanned by $\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$

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CLASS 9 : CONTENT

- Four Fundamental subspaces

FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

Let A be the matrix of order $m \text{ x } n$. Associated with it are four subspaces which are defined as follows

1. $C(A)$ is the column space of A is a subspace of $\mathbb{R}^m$and contains all the linear combination of the column vectors of A .

If then $\rho(A) = k$ then $\dim\big(C(A)\big) = k$

A basis of $C(A)$corresponds to the columns having the pivots in echelon form of A .

2. $C(A^T)$ are the row space of A is a subspace of $\mathbb{R}^n$ and contains all the linear combinations of the rows of A.

If $\rho(A) = k$ then $\dim\big(C(A^T)\big) = k$

FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

A basis for $C(A^T)$ is the set of row vectors in A or in the echelon form corresponding to the pivots in the echelon form.

3. $N(A)$ are the null space of A consists of all the solutions of the system $Ax = 0$. It is a subspace of $\mathbb{R}^n$.

If $\rho(A) = k$ then $\dim\big(N(A)\big) = n - k$

A basis for $N(A)$ is obtained by solving the system $Ux = 0$, identifying the pivot variables and free variables where U is the row reduced echelon form of A $\mathrm{i.e. special\ solutions\ to\ Ux = 0\ forms\ the\ basis\ of\ N(A)}$

4. $N(A^T)$ are the left null spaces of A is a subspace of $\mathbb{R}^n$ and consists of all the solution to the system$A^T x = 0$.

FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

If $\rho(A) = k$ then $\dim\left(N(A^T)\right) = m - k$.

A basis for $N(A^T)$ is obtained by looking at the zero rows of U and then tracing back to the corresponding rows of A.

FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

Note :

1. The row space of Amxn is the column space of A^T .
   It is spanned by the rows of A.

2. The left null space contains all vectors y for which $A^T y$ = 0.

3. N(A) and C(AT) are subspaces of R^m

4. N(A^T) and C(A) are subspaces of R^n

5. Dim C(A) = Dim C(AT) = r = rank of A

6. Dim N(A) = n-r and Dim N(AT) = m- r.
7. The dimension of the null space of a matrix is called its nullity.

FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

The rank- nullity theorem :

For any matrix Amxn ,

dim C(A) + dim N(A) = no. of columns i.e

r + (n-r) = n

This law applies to as well.

Hence, dim C(AT) + dim N(A^T) = m i. e

r + (m- r) = m

FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Then, m = n= 2 and rank r = 1.

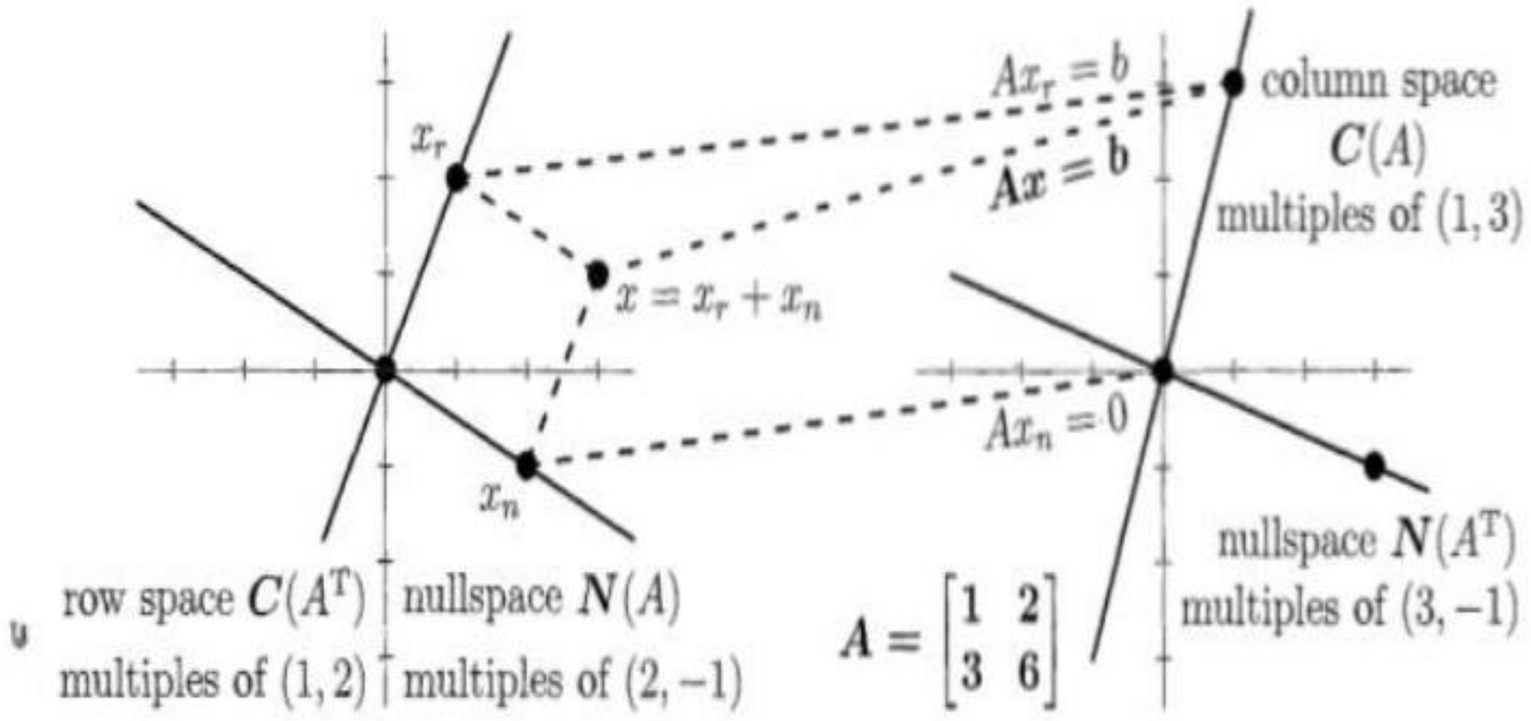
1. C(A) is the line through (1, 3)

2. C(A^T) is the line through (1, 2)

3. N(A) is the line through (-2 , 1)

4. N(A^T) is the line through (-3, 1)

FOUR FUNDAMENTAL SUBSPACES OF A MATRICES



FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

E.g. : Find the dimensions and a basis each for the four fundamental subspaces of the matrix.

Solution

$$A=\begin{bmatrix}1 & 2 & 1 & 2\\ 1 & 2 & 1 & 3\\ 3 & 6 & 3 & 7\end{bmatrix}\sim\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix};C(A)=\left\{c_1\begin{bmatrix}1\\1\\3\end{bmatrix}+c_2\begin{bmatrix}2\\3\\7\end{bmatrix}/\,c_1,c_2\in R\right\}$$

FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

Basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}$; $\dim C(A) = \text{Rank of } A$

$$c(A^T) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} / c_1, c_2 \in R \right\}$$

(or)

$$c(A^T) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} / c_1, c_2 \in R \right\}$$

FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

Basis for $c(A^T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$; $\dim C(A^T) = \text{Rank of } A$

$$Ax = 0 \Rightarrow Ux = 0$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y + z + 2t = 0 \Rightarrow x = -2y - z$$

$$t = 0$$

Special solution

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$N(A) = \left\{ c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} / \, c_1, c_2 \in R \right\}$$

Basis for $N(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$; $\dim N(A) = 2$

FOUR FUNDAMENTAL SUBSPACES OF A MATRICES

$$A^T x = 0$$

$$A^T = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 1 & 1 & 3 \\ 2 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U^l$$

$$U^l y = 0 \Rightarrow x + y + 3z = 0 \Rightarrow x = -y - 3z$$

$$y + z = 0 \Rightarrow y = -z$$

Special solution

$$\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$$N(A^T) = \left\{ c_1 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} / \, c_1 \in R \right\}$$

Basis for $N(A^T) = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$; $\dim N(A^T) = 1$

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CLASS 10 : CONTENT

- Problems on Four Fundamental Subspaces

FUNDAMENTAL SUBSPACES

Problem 1: Find the Basis and Dimension for the four fundamental subspaces.

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix}$$

Solution :- StepI :- Apply Gauss Elimination and reduce the matrix to Echelon form.

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - R_1} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

FUNDAMENTAL SUBSPACES

Column space : C(A)

$$C(A) = \left\{ c_1 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \Big/ \ c_1, c_2 \in R \right\}$$

C(A) is a 2 Dimensional space spanned by $(1, 1, 3)$ and $(2, 3, 7)$ in R^3.

$$\text{Basis of } C(A) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \right\}$$

Dim of C(A) = 2.

Note: Select columns of A and not from 'U' to span C(A)
$\because$ Columns are not Preserved in elementary Row operations.

CLASS 10 : CONTENT

Row space : $C(A^T)$

$$C(A^T) = \left\{ c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} \Big/ \ c_1, c_2 \in R \right\}$$

or

$$C(A^T) = \left\{ c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Big/ \ c_1, c_2 \in R \right\}$$

$C(A^T)$ is a 2 Dim Plane spanned by the linear combination of vectors $(1,2,1,2)$ and $(0,0,0,1)$ in R^4

Basis $C(A^T) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} \right\}$ Dimension of $C(A^T) = 2$

Note: Either non zero rows of A or U can be selected to span $C(A^T)$

FUNDAMENTAL SUBSPACES

Null Space : $N(A)$

Special solutions to $Ax=0 \Rightarrow Ux=0$ forms the basis of $N(A)$.

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x + 2y + y + 2t = 0 \Rightarrow x = -2y - z, \quad t = 0$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \sim \begin{pmatrix} -2y - z \\ y \\ z \\ 0 \end{pmatrix} \sim y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

FUNDAMENTAL SUBSPACES

N(A) is a 2 Dim Plane spanned by linear combination of vectors $(-2, 1, 0, 0)$ and $(-1, 0, 1, 0)$ in R^4.

Basis of N(A) : $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ Dimension = 2

$$N(A) = \left\{ y\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \; y, z \in R \text{ (as } y, z \text{ are free variables)} \right\}$$

Left Null space : $N(A^T)$:-

To find $N(A^T)$ Solve for $A^T y = 0 \Rightarrow y^T A = 0$.

FUNDAMENTAL SUBSPACES

$$A = \begin{array}{c} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \qquad \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 1 & 2 & 1 & 3 & b_2 \\ 3 & 6 & 3 & 7 & b_3 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$R_2 - R_1, \quad R_3 - 3R_1 \qquad \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 1 & b_3 - 3b_1 \end{array}\right)$$

$$R_3 - R_2 \qquad \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & (b_3 - 3b_1) - (b_2 - b_1) \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right)$$

FUNDAMENTAL SUBSPACES

$$N(A^T) = \left\{ c_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} ; c_1 \in R \right\}$$

$N(A^T)$ is a line spanned by $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ in R^3.

Basis for $N(A^T) = \left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$

Dimension for $N(A^T) = 1$

Note:- To find the left Null space, find the combination of the rows of A which produces zero rows. (instead of finding A^T)

FUNDAMENTAL SUBSPACES

Problem 2 : Describe the Column space and the Null space for the following matrices.

1. $\begin{bmatrix} 0 \end{bmatrix}$

$C(A) = Z$; $N(A) = R$

2. $\begin{bmatrix} 0 & -3 \end{bmatrix}$

$C(A) = R$; $N(A) = \{x \text{ axis in } R^2\}$

3. $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$C(A) = \{y \text{ axis in } R^2\}$; $N(A) = Z$

FUNDAMENTAL SUBSPACES

4. $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$C(A) = \{x \text{ axis in } R^2\}$; $N(A) = \{\text{line } x = y \text{ in } R^2\}$

5. $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$C(A) = \{x \text{ axis in } R^2\}$; $N(A) = \left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} ; x \in R \right\}$

$N(A)$ is line spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ in R^3

6. $\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix}$

$C(A) = \{\text{whole of } R^2\}$; $N(A) = \{\text{origin}(0,0) \text{ in } R^2\}$

FUNDAMENTAL SUBSPACES

Problem 3: Find the Column Space and the Null space of $A = \begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 5 & 3 \end{pmatrix}$. Give an example of a matrix whoose $C(A)$ is same as that of A but the Null space is different.

Solution:- $C(A)$ is 2 Dim Plane in R^3

$N(A)$ is origin in R^2.

$A' = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 7 & 9 \\ 5 & 3 & 8 \end{pmatrix}$ has same $C(A')$ as $C(A)$

$N(A')$ is line spanned by $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ in R^3.

FUNDAMENTAL SUBSPACES

Problem 4: Let $V = \{(a,b,c,d) / b+c+d=0\}$ and $W = \{(a,b,c,d) / a+b=0 \text{ and } c=2d\}$ be subspaces of R^4. Find the dimension of $V \cap W$.

Solution: $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}; b+c+d=0, \sim \begin{pmatrix} a \\ -c-d \\ c \\ d \end{pmatrix} \sim a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

V is a 3 Dim Plane in R^4

$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}; a+b=0 \text{ and } c=2d \quad \begin{pmatrix} -b \\ b \\ 2d \\ d \end{pmatrix} \sim b\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$

W is a 2 Dim Plane in R^4.

FUNDAMENTAL SUBSPACES

$V \cap W$

$b+c+d=0$ and $a+b=0$, $c=2d$

$b+3d=0 \Rightarrow b=-3d$, $\Rightarrow a=3d$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \sim \begin{pmatrix} 3d \\ -3d \\ 2d \\ d \end{pmatrix} \sim d\begin{pmatrix} 3 \\ -3 \\ 2 \\ 1 \end{pmatrix}$$

$V \cap W$ is $\therefore$ a line spanned by $(3,-3,2,1)$ in R^4.

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VECTOR SPACES

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CLASS 11 : CONTENT

- Uniqueness, Existence of right and left inverse
- Matrix of rank 1

EXISTENCE OF INVERSE FOR A RECTANGULAR MATRIX

Existence of Inverses

Definition:

Let Amxn (m ≤ n) be a matrix such that rank of A = m. Then Ax = b has at least one solution x for every b if and only if the columns span R^m.

In this case, A has a right inverse C such that AC= I (m*m).

Let Amxn (m ≥ n) be a matrix such that rank of A = n. Then Ax = b has at most one solution x for every b if and only if the columns are linearly independent. In this case, A has a left inverse B such that BA= I (n*n).

EXISTENCE AND UNIQUENESS

A has a left inverse if BA = I

A has a right inverse if AC= I

Rank always satisfies $r \leq m$ and $r \leq n$ .An m by n matrix cannot have More than 'm' independent rows or 'n' independent columns .there is not a space for more than m pivots or more than n .

When r= m there is right inverse and AX=b always has a solution

When r=n there is a left inverse and the solution (if it exists) is unique .

Only a square matrix has both r=m=n hence a square matrix has both existence and uniqueness achieved ,so only square matrix has two sided inverse.

EXISTENCES OF INVERSES

Case (i) If $\rho(A) = m$, (m is the number of rows) then A will has right inverse of order m x n such that $A_{m\times n}\, C_{n\times m} = I_{m\times m}$

Case (ii) If $\rho(A) = n$, (n is the number of columns) then A will has left inverse of order n x m such that $B_{n\times m}\, A_{m\times n} = I_{n\times n}$

[Best right inverse, $C = A^T\,(A\,A^T)^{-1}$]

[Best left inverse, $B = (A^T A)^{-1} A^T$]

LEFT AND RIGHT INVERSE

E.g. : Obtain left inverse or a right inverse if it exists for the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3\times 2}$$

Solution: Here $\rho(A) = 2 = n$

Therefore A has left inverse, say B

$$B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3}$$ then we have

EXISTENCES OF INVERSES

$$BA = I$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a = 1, b = 0, d = 0, e = 1$$

$c = 1, f = 1\}$ free variables

$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is the required left inverse.

EXISTENCES OF INVERSES

Example:

Let $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

Then, a right inverse of A is $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \\ a & b \end{bmatrix}$

Since the third row is arbitrary, there are infinitely many right inverses for A

MATRICES OF RANK 1

Every matrix of rank 1 has the simple form $A = uv^T =$ column times row

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \rho(A) = 1$$

Every row is a multiple of first row, so row space is one dimensional. In fact , we can write the whole matrix as the product of a column vector and row vector

MATRIX OF RANK 1

i.e, $A = (column)(row)$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$$

Where the rows are all multiples of the vectors v^T

columns are all multiples of the vector u

MATRIX OF RANK 1

Matrices Of Rank One:

When the rank of a matrix is as small as possible,

a complicated system of equations can be broken into simple pieces.

Those simple pieces are matrices of rank one.

The matrix has rank r = 1.

We can write such matrices as a column times row.

That is $$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \\ 8 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$$

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VECTOR SPACES

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CLASS 12 : CONTENT

- Problems on Uniqueness, Existence of right and left inverse

  Matrix of rank 1

RIGHT INVERSE

Problem 1: Find the left or Right Inverse for the following matrices, whichever exists:-

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution :- step I : Apply Gauss elimination and obtain rank

step II : check if rank 'ρ' of the matrix is equal to 'm' or 'n' i.e. number of rows or columns of the matrix A.

step III : if $\rho(A) = m$ then (Right Inverse exists) $A_{m\times n} B_{n\times m} = I_{m\times m}$

if $\rho(A) = n$ then (left inverse exists) $C_{n\times m} A_{m\times n} = I_{n\times n}$

RIGHT INVERSE

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$\rho(A) = 2 = m \quad \therefore$ Right inverse exists

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Best Right Inverse is $A^T (A A^T)^{-1}$

$$AA^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} ; \quad (AA^T)^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

LEFT INVERSE

$$A^T(AA^T)^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix} \text{ Ans}$$

2) Find inverse for the matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$

Solution : $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\rho(A) = 2 = n =$ number of columns

A has left inverse ; Best left inverse is $(A^TA)^{-1}A^T$

$$(A^TA)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} ; \quad (A^TA)^{-1}A^T = \begin{pmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix} \text{ Ans}$$

MATRIX OF RANK 1

Matrices Of Rank One:

When the rank of a matrix is as small as possible,

a complicated system of equations can be broken into simple pieces.

Those simple pieces are matrices of rank one.

The matrix has rank r = 1.

We can write such matrices as a column times row.

That is $\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \\ 8 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$ $\Rightarrow A = uv^T$

$u = (1, 2, 3, 4)$ $v = (2, 1, 1)$

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LINEAR ALGEBRA AND ITS APPLICATIONS

Algebra A(V) of Linear operators

Let V be a vector space over a field K. The linear mappings of the form $T: V \to V$ are called linear operators and linear transformations on V .

Note:

1. If dim V = n then dim A(V) = n^2.
2. For any mapping F ,G from A(V), the composition G.F exists and also belongs to A(V).

Algebra A(V) of Linear operators

Remark : An algebra a over a field K is a vector space over K in which an operation of multiplication is defined satisfying , for F, G , H ∈ A and every k ∈ K

1 . F (G+H) = FG + FH

2 . (G+H) F = GF + HF

3 . K(GF) = (kG)F = G(kF)

Invertible Maps and Isomorphism

A mapping $f : A \to B$ is said to be *One-to-one* or 1-1 or *injective* if different elements of A have distinct images; that is

$$\text{IF } a \neq a', then\ f(a) \neq f(a').$$

Equivalently,

$$\text{IF } a = a', then\ f(a) = f(a').$$

A mapping f: $A \to B$ is said to be *onto* or *surjective* if every $b \in B$ is the image of at least one $a \in A$.

A mapping $f : A \to B$ is said to be *One-to-one correspondence* between A and B or bijective if f is both *one-to-one and onto.*

Invertible Maps and Isomorphism

A mapping $f : A \rightarrow B$ is said to be invertible if f is one-to-one and onto.

Example: Let $f: R \rightarrow R$ be defined by $f(x) = 2x - 3$. Now f is one-to-one and onto.

Let y be the image of x under the mapping f, i.e., y =2x-3.

Interchange x and y to obtain $x = 2y - 3 \Rightarrow y = \frac{x+3}{2} \Rightarrow f^{-1} = \frac{x+3}{2}$.

Invertible Maps and Isomorphism

A mapping $F: V \rightarrow U$ is called isomorphism if F is linear and bijective, i.e., one-to-one and onto.

Example : The mapping $T: R^2 \rightarrow R^2$ is defined as $T(x, y) = (x + 4y, y - 3x)$ is isomorphism.

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Unit 3. Linear Transformations and Orthogonality

Topics

1. **Linear Transformations**

2. **Orthogonal vectors and Subspaces**

3. **Cosines and Projections onto lines**

4. **Projections and Least Squares**

Linear Transformations

Definition:

Let A be a matrix of order n. When A multiplies a n- dimensional vector x, it transforms x to a n-dimensional vector Ax. This happens at every x in R^n. The whole space R^n is ***transformed or mapped*** into itself by the matrix A. The matrix A induces a transformation of R^n.

Linear Transformations

Few Examples…..

1. $$A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

If x = (x, y) then Ax = (cx, cy).

A multiple of the identity matrix A = *cI* ***stretches*** every vector by the scale factor c. The whole space expands or contracts.

2. $$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

If x = (x, y) then Ax = (-y, x).

The matrix A ***rotates*** every vector about the origin through a right angle in the counter clockwise direction.

Linear Transformations

3. $$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

If x = (x, y) then Ax = (y, x).

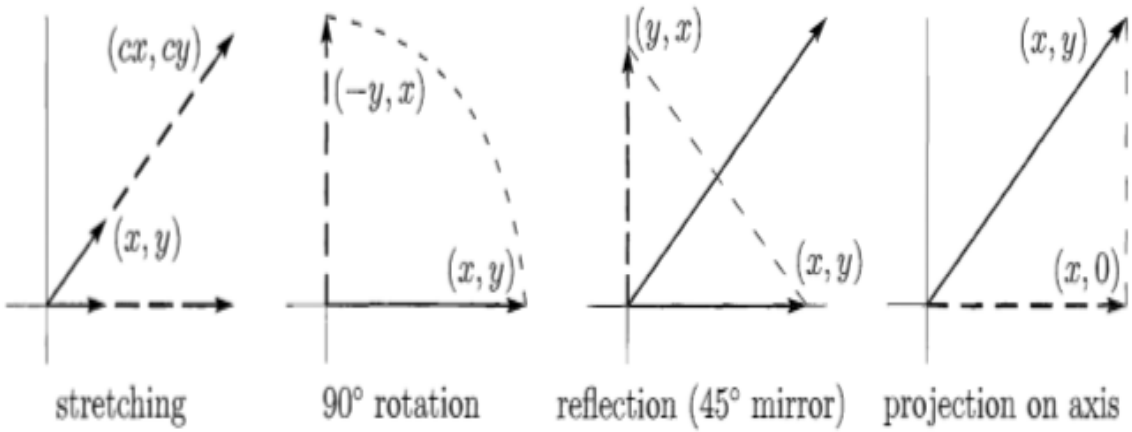
The matrix A ***reflects*** every vector on the line y = x. It is also a permutation matrix.

4. $$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

If x = (x , y) then Ax = (x , 0).

The matrix A ***projects*** every vector onto the x axis.

Linear Transformations



Linear Transformations

Note

- A transformation can now be understood as a function (or a mapping) $f : A \to B$ defined by $f(x) = y$. In terms of matrices we have the rule

$A : R^n \to R^m$ defined by $Ax = b$.

Unit 3. Linear Transformations and Orthogonality

Linear Transformations

Definition :

A transformation T on R^n is said to be ***linear*** if it satisfies the ***rule of linearity***

$$T (cx + dy) = c (Tx) + d (Ty)$$

for all scalars c, d and vectors x, y.

Note :

1. If T is linear then T (0) = 0 i.e T preserves origin. The converse may or may not be true.
2. If A is a matrix of order m x n then A induces a transformation from R^n to R^m.

Linear Transformations

Few examples…..

Let $v = (v_1, v_2)$. Then,

1. $T(v) = (v_2, v_1)$ is linear
2. $T(v) = (v_1, v_1)$ is not linear
3. $T(v) = (0, v_1)$ is not linear
4. $T(v) = (0, 1)$ is not linear
5. $T(v) = (v_1, v_2)$ is linear

Note :

If a transformation preserves origin it may or may not be linear!!

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Linear Transformations

Definition :

A transformation T is said to be linear if it satisfy the rule of linearity.

i.e., $A(cx+dy) = c\,A(x) + d\,A(y)$ for any scalar c,d are real constants.

Unit 3. Linear Transformations and Orthogonality

Linear Transformations

Example: I n linear system of equations Ax =b , Matrix A is a transformation from R^n to R^m .

Note: Consider a transformation $T : A \rightarrow B$
where A and B are subspaces .

1. A is the domain of the transformation.

2. B is the co domain of the transformation.

3. For any x in A , there exist Tx in B, here Tx is the image of T and x is the pre image of Tx

Unit 3. Linear Transformations and Orthogonality

Linear Transformations

4. The set of all images is the subset of B is called Range of the transformation.

5. For all x in A such that Tx =o is called the Kernel of the transformation.

6. Dimension of the range is called rank and dimension of Kernel is called nullity.

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Definition :

The space of all polynomials in t of degree n is a vector space called the ***polynomial space*** denoted by P_n.

P_n= { Its basis is 1 , t, t^2, …. , t^n and dimension is n+1 .

Linear Transformations

Example 1 : The operation of differentiation is linear. It takes P_{n+1} to P_n. The column space is the whole of P_n and the null space is P_0, the 1-dimensional space of all constants.

Example 2: The operation of integration is linear. It takes P_n to P_{n+1}. The column space is a subspace of P_{n+1} and the null space is just the zero vector.

Linear Transformations

Example 3 :

Multiplication by a fixed polynomial , say $3 + 4t$ is also a linear transformation.

Let $p(t) = a_0 + a_1t + a_2t^2 + \dots + a_nt^n$ then

$A\,p(t) = (3+4t)\,p(t) = 3a_0 + \dots + 4a_nt^{n+1}$.

This A sends P_n to P_{n+1}.

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Transformations Represented by Matrices

Integration Matrix:

Consider the integration of a quadratic polynomial from 0 to 1. This transformation is linear which transforms P_2 to P_3 .

$P_2 = \{ p(t) = a_0 + a_1 t + a_2 t^2, \; a_0, a_1, a_2 \in R \}$

Basis $= \{ v_1 = 1, \; v_2 = t, \; v_3 = t^2 \}$

$P_3 = \{ q(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3, \; b_i \in R \}$

Basis $= \{ u_1 = 1, \; u_2 = t, \; u_3 = t^2, \; u_4 = t^3 \}$

$A_{int} : P_2 \to P_3$

Transformations Represented by Matrices

Images of v_i's are

$$\int_0^t v_1\,dt = \int_0^t 1\,dt = t = 0\cdot u_1 + 1\cdot u_2 + 0\cdot u_3 + 0\cdot u_4$$

$$\Rightarrow (0, 1, 0, 0)$$

$$\int_0^t v_2\,dt = \int_0^1 t\,dt = \frac{t^2}{2} = 0\cdot u_1 + 0\cdot u_2 + \frac{1}{2}u_3 + 0\cdot u_4$$

$$\Rightarrow (0, 0, \tfrac{1}{2}, 0)$$

$$\int_0^t v_3\,dt = \int_0^1 t^2\,dt = \frac{t^3}{3} = 0\cdot u_1 + 0\cdot u_2 + 0\cdot u_3 + \frac{1}{3}u_4$$

$$\Rightarrow (0, 0, 0, \tfrac{1}{3})$$

Transformations Represented by Matrices

Integration matrix is

$$A_{int} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}_{4\times 3}$$

$$A_{diff} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4}$$

Transformations Represented by Matrices

Note:

- → $A_{diff} \cdot A_{int} = I_3$
- → Differentiation is a left inverse of integration.
- → Integration is a right inverse of differentiation.
- → Column space i.e. range of A_{int} is a subspace of P_3
- → Kernel i.e Nullspace $= \{\vec{0} \in P_2\}$

Unit 3. Linear Transformations and Orthogonality

Transformations Represented by Matrices

Problems:

1. For the space of all 2 x 2 matrices find the standard basis. For the linear transformation of transposing, find the matrix A with respect to this basis . Why is $A^2 = I$?

Solution: $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \in R \right\}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Basis $= \left\{ A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Transformations Represented by Matrices

$T : M_{2\times 2} \to M_{2\times 2}$.

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

To find matrix A_T,

$$T[A_{11}] = T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A_{11}$$

$$= 1 \cdot A_{11} + 0 A_{12} + 0 \cdot A_{21} + 0 A_{22}$$

$$\to (1, 0, 0, 0)$$

Unit 3. Linear Transformations and Orthogonality

Transformations Represented by Matrices

$$T(A_{12}) = T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = A_{21}$$

$$= 0 \cdot A_{11} + 0 \cdot A_{12} + 1 \cdot A_{21} + 0 \cdot A_{22} \rightarrow [0, 0, 1, 0]$$

$$T(A_{21}) = T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A_{12}$$

$$= 0 \cdot A_{11} + 1 \cdot A_{12} + 0 \cdot A_{21} + 0 \cdot A_{22} \rightarrow [0, 1, 0, 0]$$

$$T[A_{22}] = T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = A_{22}$$

$$= 0 \cdot A_{11} + 0\, A_{12} + 0 \cdot A_{21} + 1\, A_{22} \rightarrow [0, 0, 0, 1]$$

Transformations Represented by Matrices

Matrix for transposing is

$$A_T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $A^2 = I$, because $(A^T)^T = A$.

Transformations Represented by Matrices

2. From the cubics P_3 to the fourth-degree polynomial P_4, what matrix represents multiplication by $2+3t$?

Solution: Basis for $P_3 = \{v_1 = 1, v_2 = t, v_3 = t^2, v_4 = t^3\}$

Basis for $P_4 = \{u_1 = 1, u_2 = t, u_3 = t^2, u_4 = t^3, u_5 = t^4\}$

To find matrix

$(2+3t)v_1 = (2+3t)\cdot 1 = 2\cdot u_1 + 3\cdot u_2 + 0u_3 + 0\cdot u_4 + 0u_5$

$\rightarrow (2, 3, 0, 0, 0)$

Transformations Represented by Matrices

Similarly:

$(2+3t)v_2 = (2+3t)t = 2t+3t^2 \rightarrow (0,2,3,0,0)$

$(2+3t)v_3 = (2+3t)t^2 = 2t^2+3t^3 \rightarrow (0,0,2,3,0)$

$(2+3t)v_4 = (2+3t)t^3 = 2t^3+3t^4 \rightarrow (0,0,0,2,3)$

The matrix A_T for this transformation is

$$A_T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{5\times 4}$$

THANK YOU

LINEAR ALGEBRA AND ITS APPLICATIONS
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Unit 3. Linear Transformations and Orthogonality

Transformations Represented by Matrices

The matrix of a linear transformation is a matrix for which $T(x)=Ax$, for a vector 'x' in the domain T. Such matrix is called standard matrix for the transformation.

Note:

1.Such matrix can be found for any Linear transformation T from R^n to R^m.

2. Standard basis for R^n = $\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$.

e_i's are columns of Identity matrix of order 'n'.

Transformations Represented by Matrices

The standard matrix of transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ has columns $T(e_1), T(e_2), \cdots T(e_n)$ where $e_1, e_2, \cdots e_n$ represents the standard basis, ie

$$T(x) = Ax \iff A = [T(e_1), T(e_2) \cdots T(e_n)]$$

Transformations Represented by Matrices

Examples: $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} x_1 - x_2 \\ 2x_3 \end{pmatrix}$

Here $T : \mathbb{R}^3 \to \mathbb{R}^2$.

Basis for $\mathbb{R}^3 = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$T(e_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$T(e_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0-1 \\ 2(0) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$T(e_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0-0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Transformations Represented by Matrices

The standard matrix for T is

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Verification!.

$$Ax = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_3 \end{bmatrix}$$

Compare this to the rule for T from the problem

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ 2x_3 \end{pmatrix}.$$

Transformations Represented by Matrices

Matrix Representation of Differentiation:

- Consider differentiation that goes from P_3 to P_2.

$P_3 = \{ p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, \ a_0, a_1, a_2, a_3 \in R \}$

Basis is $\{ v_1 = 1, v_2 = t, v_3 = t^2, v_4 = t^3 \}$

$P_2 = \{ q(t) = b_0 + b_1 t + b_2 t^2, \ b_0, b_1, b_2 \in R \}$

Basis is $\{ u_1 = 1, u_2 = t, u_3 = t^2 \}$

$\frac{d}{dt} = A_{diff} : P_3 \to P_2$

Transformations Represented by Matrices

Matrix Representation of Differentiation:

To find A_{diff}.

$$\frac{d}{dt}(v_1) = \frac{d}{dt}(1) = 0 = 0 \cdot u_1 + 0u_2 + 0u_3 \rightarrow (0,0,0)$$

$$\frac{d}{dt}(v_2) = \frac{d}{dt}(t) = 1 = 1 \cdot u_1 + 0 \cdot u_2 + 0 \cdot u_3 \rightarrow (1,0,0)$$

$$\frac{d}{dt}(v_3) = \frac{d}{dt}(t^2) = 2t = 0 \cdot u_1 + 2 \cdot u_2 + 0 \cdot u_3 \rightarrow (0,2,0)$$

$$\frac{d}{dt}(v_4) = \frac{d}{dt}(t^3) = 3t^2 = 0 \cdot u_1 + 0 \cdot u_2 + 3u_3 \rightarrow (0,0,3)$$

Transformations Represented by Matrices

We thus get the matrix of differentiation as

$$A_{diff} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3x4}$$

Transformations Represented by Matrices

Verification: Let $p(t) = 3 + 6t - 7t^2 + 2t^3$

$$x = \begin{pmatrix} 3 \\ 6 \\ -7 \\ 2 \end{pmatrix}$$

$$A_{diff}(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{pmatrix} 3 \\ 6 \\ -7 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -14 \\ 6 \end{pmatrix}$$

$$\frac{d}{dt}[p(t)] = 6 - 7 \cdot 2 \cdot t - 2 \cdot 3\, t^2 \rightarrow \begin{pmatrix} 6 \\ -14 \\ 6 \end{pmatrix}$$

THANK YOU