LINEAR ALGEBRA AND ITS APPLICATIONS

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Course Content: Cholesky Decomposition Or Factorization:

Cholesky decomposition or Cholesky factorization is decomposition of a Hermitian positive definite matrix which is factored into lower triangular matrix L and its conjugate transpose L^T. In this L has real positive diagonal entries.

- A Hermitian positive definite matrix A can be factored as $A=LL^T$ where $L=\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$

$$A=LL^T \rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}$$

Algorithm:

Let $A_{3x3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a symmetric positive definite matrix.

Then A can be factored as LL^T

$A=LL^T \rightarrow$
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}$$

$$= \begin{pmatrix} l_{21}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{31} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{pmatrix}$$

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This gives $l_{11}^2 = a_{11} \Longrightarrow l_{11} = \sqrt{a_{11}}$;

$$l_{21} = \frac{a_{21}}{l_{11}}; \qquad l_{22} = \sqrt{a_{22} - l_{21}^2}$$

$$l_{31} = \frac{a_{31}}{l_{11}}; \qquad l_{32} = \frac{a_{32} - l_{31}l_{21}}{l_{22}}; \qquad l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2}$$

Example: Factorize A using Cholesky Decomposition given A= $\begin{pmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 22 \end{pmatrix}_3$

A= $\begin{pmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 22 \end{pmatrix}$ $l_{11} = \sqrt{a_{11}}$ =2 ; $l_{21} = \frac{a_{21}}{l_{11}}$ =1 ; $l_{22} = \sqrt{a_{22} - l_{21}^2}$ =1

$l_{31} = \frac{a_{31}}{l_{11}}$ =3 ; $l_{32} = \frac{a_{32} - l_{31} l_{21}}{l_{22}}$ =2 ; $l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2}$ =3

Hence L=$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 3 \end{pmatrix}$ L^T= $\begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

$\therefore$ A= LL^T =$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

Factorize $A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix}$ using Cholesky factorization.

Answer: $L=\begin{pmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{pmatrix}$

THANK YOU