LINEAR ALGEBRA AND ITS APPLICATIONS

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Department of Science and Humanities

CLASS-8

SYMMETRIC MATRICES AND DIAGONALIZATION OF A MATRIX

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Statement: If A is a square matrix of order 'n' has 'n' linearly independent vectors, then a matrix 'S' can be found such that $S^{-1}AS$ is a diagonal matrix.

Proof:

Let A be a square matrix of order 3. Let λ_1, λ_2 and λ_3 be its Eigen Value and $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ and $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be the corresponding Eigen vectors.

Let the square matrix $S = [X_1 \; X_2 \; X_3]$ i.e

$$S = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

Multiplying by A, $AS = A[X_1 X_2 X_3] = [AX_1, AX_2, AX_3]$

$AS = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$

$$AS = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = SD$$

$\therefore AS = SD$ where D is the diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$\therefore \quad AS = SD$

Multiplying both sides by S^{-1}

$\boxed{S^{-1}AS = D}$ or $S^{-1}AS = \Lambda$

or $\boxed{A = SDS^{-1}}$

S is invertible because its columns (the eigen vectors) were assumed to be independent.

NOTE:

1. Any matrix with distinct Eigen values can be diagonise.

2. Not all matrices has 'n' Linearly independent Eigen vectors. Therefore, all the matrices can't be diagonalized.

Eg: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, here $\lambda_1 = \lambda_2 = 0$. There is only one independent vector. Hence we cannot construct S.

3. If Eigen vectors $x_1, x_2, \ldots x_k$ correspond to distinct Eigen values $\lambda_1, \lambda_2 \ldots \lambda_k$ then those Eigen vectors are linearly independent.

4. Eigen vector matrix is not unique since if x is an Eigen vector corresponding to λ then kx is also an Eigen vector.

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· Problem

1. Factor the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ into $S \Lambda S^{-1}$ & also find $S A S^{-1}$.

Ans: Consider $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 1 = 0$$

$$1 + \lambda^2 - 2\lambda - 1 = 0 \Rightarrow \lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0 \Rightarrow \lambda_1 = 0 \;\&\; \lambda_2 = 2$$

$$\therefore \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Now to find Eigen vectors:-

Consider $(A-\lambda I)X=0$

$\Rightarrow \quad (1-\lambda)x+y=0$

$x+(1-\lambda)y=0$

Case 1:- When $\lambda=2 \Rightarrow \left.\begin{array}{l}-x+y=0\\ x-y=0\end{array}\right\} \Rightarrow x=y$

$\Rightarrow$ 1 equations with 2 unknowns $\therefore$ y be free variable

$\therefore x=y=k$, let $k=1$

Eigen vector $X_1=\begin{bmatrix}1\\1\end{bmatrix}$

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Case 2:- When $\lambda = 0$

$$\left.\begin{array}{l} x+y=0 \\ x+y=0 \end{array}\right\} \Rightarrow x=-y$$

$\Rightarrow$ 2 unknowns so one free variable

$x=k \Rightarrow y=-k$ or $x=-k \Rightarrow y=k$

$$\therefore X_2 = \begin{bmatrix} -k \\ k \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\therefore S = [X_1 \quad X_2] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, S^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore A = S\Lambda S^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Hence verified.

To find SAS^{-1}

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \Lambda$$

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PROBLEMS ON DIAGONALIZATION OF A MATRIX

Diagonalization of a matrix

1. Find the matrix A whose Eigen values are 2, 5 and Eigen vectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Ans: $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ & $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

$\therefore A = S \Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

2. Find the Eigen values and Eigen vectors of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and write 2 different diagonalizing matrices.

Ans: $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)\left[(1-\lambda)^2 - 1\right] - 1\left[1-\lambda-1\right] + 1\left(1-(1-\lambda)\right) = 0$$

$$\Rightarrow \lambda^2(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 0, 0, 3 \Rightarrow \text{Eigen values}$$

Consider $[A-\lambda I]X=0$

$$(1-\lambda)x+y+z=0$$
$$x+(1-\lambda)y+z=0$$
$$x+y+(1-\lambda)z=0$$

Case 1:- When $\lambda=0 \Rightarrow$

$$x+y+z=0$$
$$x+y+z=0$$
$$x+y+z=0$$

$x=-y-z$. Let $y=k_1$ & $z=k_2$ $\therefore x=-k_1-k_2$

For $\lambda=0$ Let $k_1=k_2=1 \Rightarrow x=-2, y=1, z=1 \Rightarrow X_1=\begin{bmatrix}-2\\1\\1\end{bmatrix}$

$\lambda=0$, let $k_1=0, k_2=1 \Rightarrow y=0, z=1, x=-1 \Rightarrow X_2=\begin{bmatrix}-1\\0\\1\end{bmatrix}$

Case 2: When $\lambda = 3$

$$-2x + y + z = 0$$
$$x - 2y + z = 0$$
$$x + y - 2z = 0$$

$$\Rightarrow \frac{x}{1+2} = \frac{y}{-2-1} = \frac{z}{4-1}$$

$$\Rightarrow X_3 = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

$$\therefore S_1 = \begin{bmatrix} -2 & -1 & 3 \\ 1 & 1 & -3 \\ 1 & 0 & 3 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} -2 & -1 & 3 \\ 1 & 0 & -3 \\ 1 & 1 & 3 \end{bmatrix} \quad (\text{taking } k_2 = 0,\ k_1 = 1)$$

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POWERS AND PRODUCTS OF MATRICES

Computation of Powers of a square matrix:

Diagonization of a square matrix A also helps to find the Powers of $A, A^2, \ldots$ etc.

We have $D = S^{-1}AS$

$$D^2 = (S^{-1}AS)(S^{-1}AS)$$
$$D^2 = S^{-1}A^2S$$

Pre-multiplying by S and Post-multiplying by S^{-1}

$$SD^2S^{-1} = SS^{-1}A^2SS^{-1} = I\,A^2I = A^2$$
$$A^2 = SD^2S^{-1}$$

Powers and Products of matrices

In general, $A^n = S D^n S^{-1}$

where $D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 & 0 & \cdots & \\ 0 & \lambda_2^n & 0 & 0 & \cdots & \\ 0 & 0 & \lambda_3^n & 0 & \cdots & \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & \lambda_n^n \end{bmatrix}$

Problems:-

1. Diagonalize ~~the~~ matrix $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ and find one of its square roots. How many square roots will be there?

Ans: $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)^2 - 16 = 0$

$\Rightarrow \lambda = 1, 9$

consider $[A - \lambda I]x = 0$

$\Rightarrow (5-\lambda)x + 4y = 0$

$4x + (5-\lambda)y = 0$

Case-1:- When $\lambda = 1$

$4x + 4y = 0$ $\quad \therefore X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$4x + 4y = 0$

Case 2:- When $\lambda = 9$

$-4x + 4y = 0$ $\quad \therefore X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$4x - 4y = 0$

$\therefore S = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

We know that $A = S \Lambda S^{-1}$

$$A^{1/2} = S \Lambda^{1/2} S^{-1}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}^{1/2} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^{1/2} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

We have 4 square roots $\because \Lambda^{1/2} = \sqrt{1} = \pm 1$ & also $\sqrt{9} = \pm 3$.

We get different values when we consider $\pm 1, \pm 3$.

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2) Diagonalize $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ & hence find A^{100}.

Show that $A^{100} = A$.

Ans: $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1/2 - \lambda & 1/2 \\ 1/2 & 1/2 - \lambda \end{vmatrix} \Rightarrow \lambda^2 - \lambda = 0$

$\lambda = 0, \lambda = 1$

Consider $[A - \lambda I]X = 0 \Rightarrow \left(\frac{1}{2} - \lambda\right)x + \frac{1}{2}y = 0$

$\frac{1}{2}x + \left(\frac{1}{2} - \lambda\right)y = 0$

Case 1:- When $\lambda = 0 \Rightarrow \frac{x}{2} + \frac{y}{2} = 0 \Rightarrow x = -y \quad X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\frac{x}{2} + \frac{y}{2} = 0$

Case 2: When $\lambda = 1$

$$-\frac{x}{2} + \frac{y}{2} = 0 \Rightarrow x = y \qquad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{x}{2} - \frac{y}{2} = 0$$

$$A = S \Lambda S^{-1}$$

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^{100} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^{100} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = A$$

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Supplementary Problems

Gram-Schmidt Orthogonalization

Q) $a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $a_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

Let us convert L.I.V → O.N.V using Gram-Schmidt Process.

$$q_1 = \frac{a_1}{\|a_1\|} \qquad \|a_1\| = \sqrt{1^2+0+1^2} = \sqrt{2}$$

$$\boxed{q_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}$$

$a_2 \rightarrow q_2$

$$q_2 = \frac{e_2}{\|e_2\|}$$

$$e_2 = (b) - (q_1^T a_2)\, q_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{1}{\sqrt{2}}\right) \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

$$\|e_2\| = \sqrt{(1/2)^2 + 0 + (-1/2)^2}$$

$$= 1/\sqrt{2}$$

$$q_2 = \frac{e_2}{\|e_2\|}$$

$$= \left(\frac{1}{\sqrt{2}},\ 0,\ 1/\sqrt{2}\right)$$

$$q_3 = \frac{e_3}{\|e_3\|} \quad ; \quad e_3 = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \sqrt{2} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} - \sqrt{2} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\|e_3\| = \sqrt{0 + 1^2 + 0} = 1$$

$$q_3 = \frac{e_3}{\|e_3\|} = (0, 1, 0)$$

$$(a, b, c) \rightarrow (q_1, q_2, q_3)$$

$$q_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$q_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

$$q_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

orthonormal columns

$$q_1^T q_2 = q_2^T q_3 = q_3^T q_1 = 0$$

$$q_1^T q_1 = q_2^T q_2 = q_3^T q_3 = 1$$

$(a, b, c) \longrightarrow (q_1, q_2, q_3)$

Gram-Schmidt orthogonalization

Q) Check the matrix are orthogonal/ diagonalizable.
If not then orthogonaly diagonalize it as $A = S\Lambda S^{-1} = Q\Lambda Q^{-1}$
$= Q\Lambda Q^T$

where Q is an orthogonal matrix.

Soln:- $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^3 - 3\lambda^2 = 0$

$\lambda^2(\lambda - 3) = 0$

$\lambda = \underline{0, 0}, 3$

Consider $[A - \lambda I]x = 0$

$(1-\lambda)x + y + z = 0$; $x + (1-\lambda)y + z = 0$; $x + y + (1-\lambda)z = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$x = y = z = 0 \quad X \neq 0$

Case 1: $\lambda = 0 \Rightarrow$

$$\left.\begin{aligned} x+y+z&=0 \\ x+y+z&=0 \\ x+y+z&=0 \end{aligned}\right\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$R_2' = R_2 - R_1$

$R_3' = R_3 - R_1$

$$\begin{array}{cccc} * & FV & FV & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$y = k_1$

$z = k_2$

$x = -y - z$

$x = -k_1 - k_2$

$\lambda = 0, X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\quad [k_1 = 1, k_2 = 0]$

$\lambda = 0, X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\quad k_1 = 0, k_2 = 1$

Case 3: $\lambda = 3$

$$\left.\begin{aligned} -2x + y + z &= 0 \\ x - 2y + z &= 0 \\ x + y - 2z &= 0 \end{aligned}\right\}$$

$$\begin{array}{cccccc} & x & & y & & z \\ -2 & 1 & 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & 1 & -2 & 1 \end{array}$$

$$\frac{x}{3} = \frac{y}{3} = \frac{z}{3}$$

$$X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} -1 & -1 & 3 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{or} \quad S = \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$a^T b = b^T c = c^T a \neq 0$$

$$a^T c = 0 = b^T c$$

$$a^T b = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 1 \neq 0$$

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$q_1 = \frac{a}{\|a\|} \Rightarrow \left(-1/\sqrt{2}, 1/\sqrt{2}, 0\right)$

$q_2 = \frac{e_2}{\|e_2\|} \quad e_2 = b - (q_1^T a) q_1 \quad q_2 = \left(-1/\sqrt{6}, -1/\sqrt{6}, 2/\sqrt{6}\right)$

$q_3 = \frac{e_3}{\|e_3\|} \quad : e_3 = \left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$

$$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$q_1^T q_2 = 0$

$q_2^T q_1 = 0$

$q_1^T q_3 = 0$

$$A = Q \Lambda Q^{-1} \rightarrow Q^{-1} = Q^T$$

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A = Q \Lambda Q^{-1}$$

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Unit 5 Singular Value Decomposition

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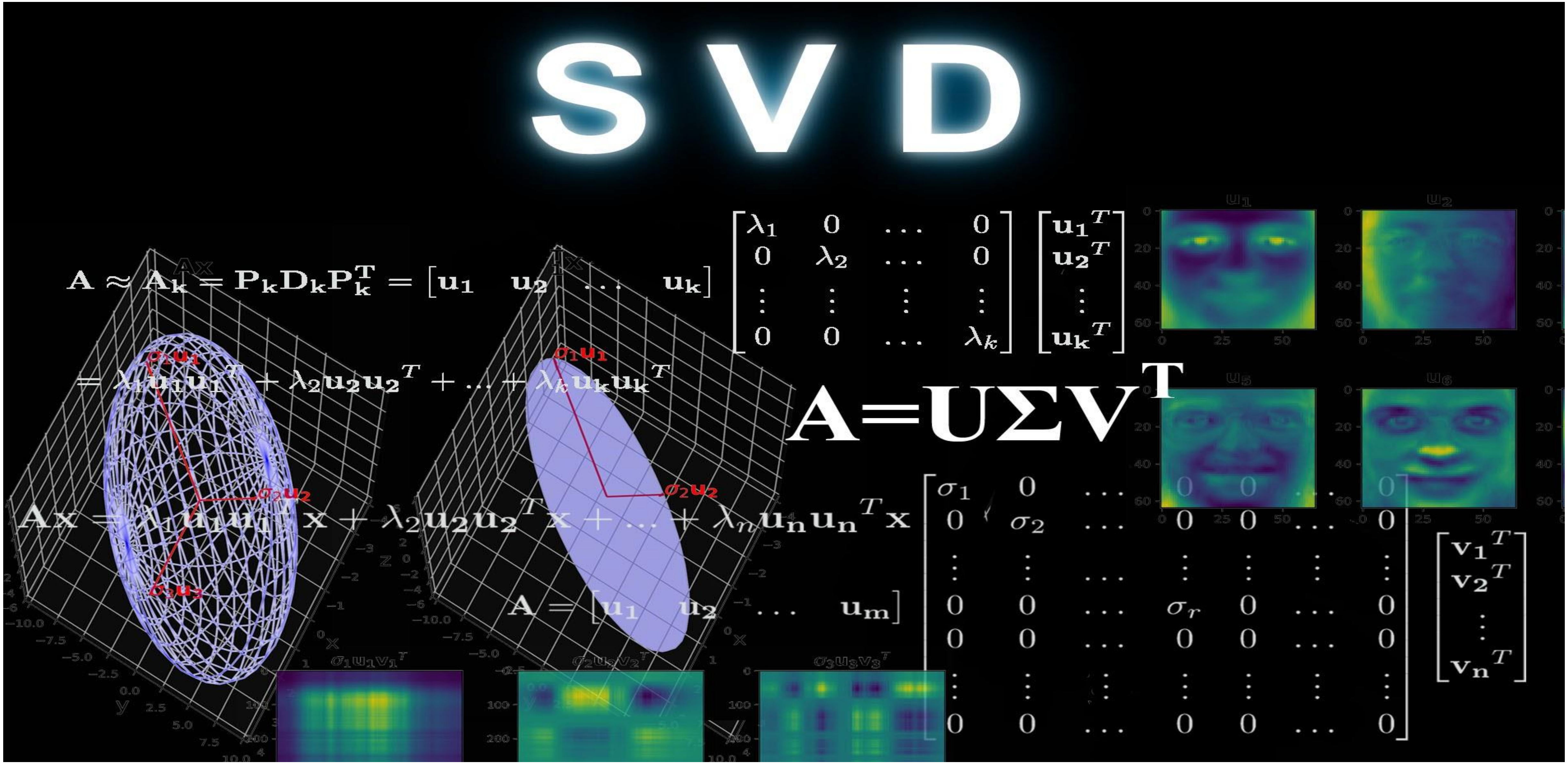

https://towardsdatascience.com/understanding-singular-value-decomposition-and-its-application-in-data-science-388a54be95d

Importance of the method:

- Intimately related to the familiar theory of diagonalizing a symmetric matrix.
- Factorizes a matrix into 3 components.
- Has interesting algebraic properties.
- Gives further geometric and theoretical insights.
- Has many applications to data science.

Agenda for the chapter

- Tests for positive definiteness
- Positive Definite Matrices and Least Squares
- Semi definite Matrices
- Singular Value Decomposition
- Applications of the SVD.

Agenda for the class

- Quadratic form
- Examples on quadratic form
- Quadratic form – Going the other way.
- Quadratic forms for a non-symmetric matrix
- Graphs of quadratic forms
- Examples

Recall

- If A is a real symmetric matrix of order n by n, then there exists an orthonormal matrix V whose columns are the eigenvectors of A and a diagonal matrix D , having its diagonal entries as the eigenvalues of A, such that : $A = VDV^T$

- The above process gives the eigenvalue decomposition of the matrix A.

- The singular value decomposition (SVD) is intimately related to the eigenvalue decomposition.

Any $n \times n$ real symmetric matrix A determines a quadratic form q_A

in n variables by the formula

$$q_A(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \mathbf{x}^{\mathrm{T}} A \mathbf{x}.$$

Conversely, given a quadratic form in n variables, its coefficients

can be arranged into an $n \times n$ symmetric matrix.

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Quadratic Form

✓Note : $x^t A x$ is a scalar

$$\therefore \; x^T A x = (x_1 \; x_2 \; \cdots \; x_m) \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

$$x_1(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m)$$

$$+ \; x_2(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m)$$

$$+ \; \vdots$$

$$+ \; x_m(a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mm}x_m)$$

$$= \sum_{i \le j}^{m} a_{ij}\, x_i\, x_j$$

Quadratic Form – Example 1

Let $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute $\mathbf{x}^T A\mathbf{x}$ for the following matrices.

a. $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

Solution:

a. $\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix} = 4x_1^2 + 3x_2^2.$

b. $\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{bmatrix}$

$$= x_1(3x_1 - 2x_2) + x_2(-2x_1 + 7x_2)$$

$$= 3x_1^2 - 2x_1x_2 - 2x_2x_1 + 7x_2^2$$

$$= 3x_1^2 - 4x_1x_2 + 7x_2^2$$

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Quadratic Form – Example 2

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$$

We could rewrite this in the form $Q(x) = 3x_1^2 - 4x_1x_2 + 7x_2^2$.

$$= x^T A x.$$

Quadratic Form – Going the other way

Question: Is any function of the form $Q(x) = ax_1^2 + bx_1x_2 + cx_2^2$ a quadratic form?

Answer: Yes. Set $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$.

Question: What about $Q(x) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$?

Answer: Set $A = \begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$.

Quadratic Form – What if A isn't symmetric?

If A isn't symmetric, the function $Q(x) = x^T Ax$ is still a quadratic form:

Define $\hat{A} = \frac{A+A^T}{2}$ then

$$
\begin{aligned}
x^T \hat{A} x &= x^T \left(\frac{A + A^T}{2} \right) x \\
&= \frac{1}{2} \left(x^T Ax + x^T A^T x \right) \\
&= \frac{1}{2} \left(x^T Ax + x^T Ax \right) = x^T Ax.
\end{aligned}
$$

Because of this, it is safe to assume that A is symmetric when we examine quadratic forms.

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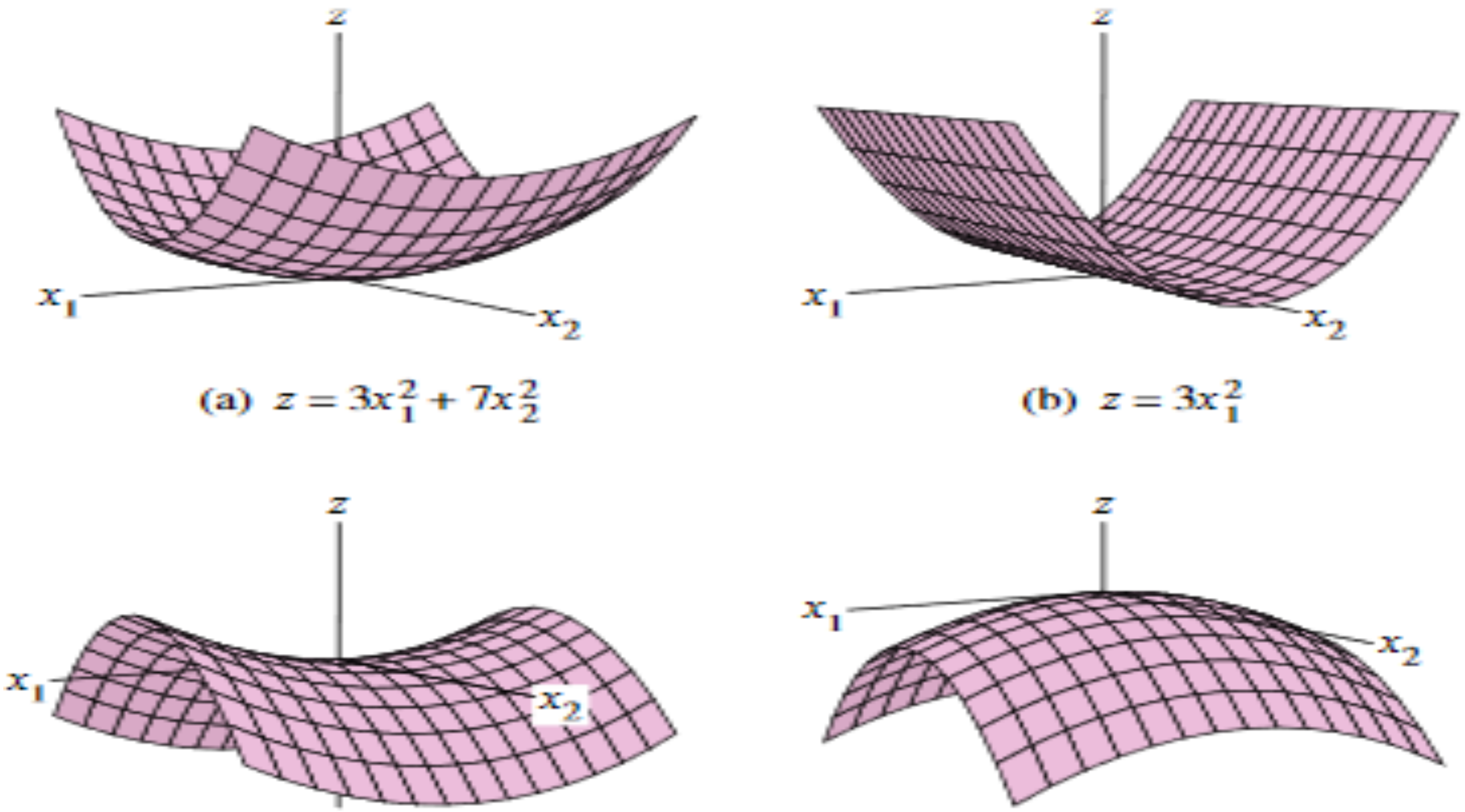
Graphs of Quadratic forms



(a) $z = 3x_1^2 + 7x_2^2$

(b) $z = 3x_1^2$

(c) $z = 3x_1^2 - 7x_2^2$

(d) $z = -3x_1^2 - 7x_2^2$

Graphically, the graph $z = Q(x)$ is

- convex up if Q is positive definite,
- concave down if Q is negative definite,
- A "saddle" if Q is indefinite.

EXAMPLE

The quadratic form of $f(x,y) = ax^2 + 2abx + by^2$ may be represented as $X^T A X$ where

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \quad ; \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

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Tests for Positive definiteness

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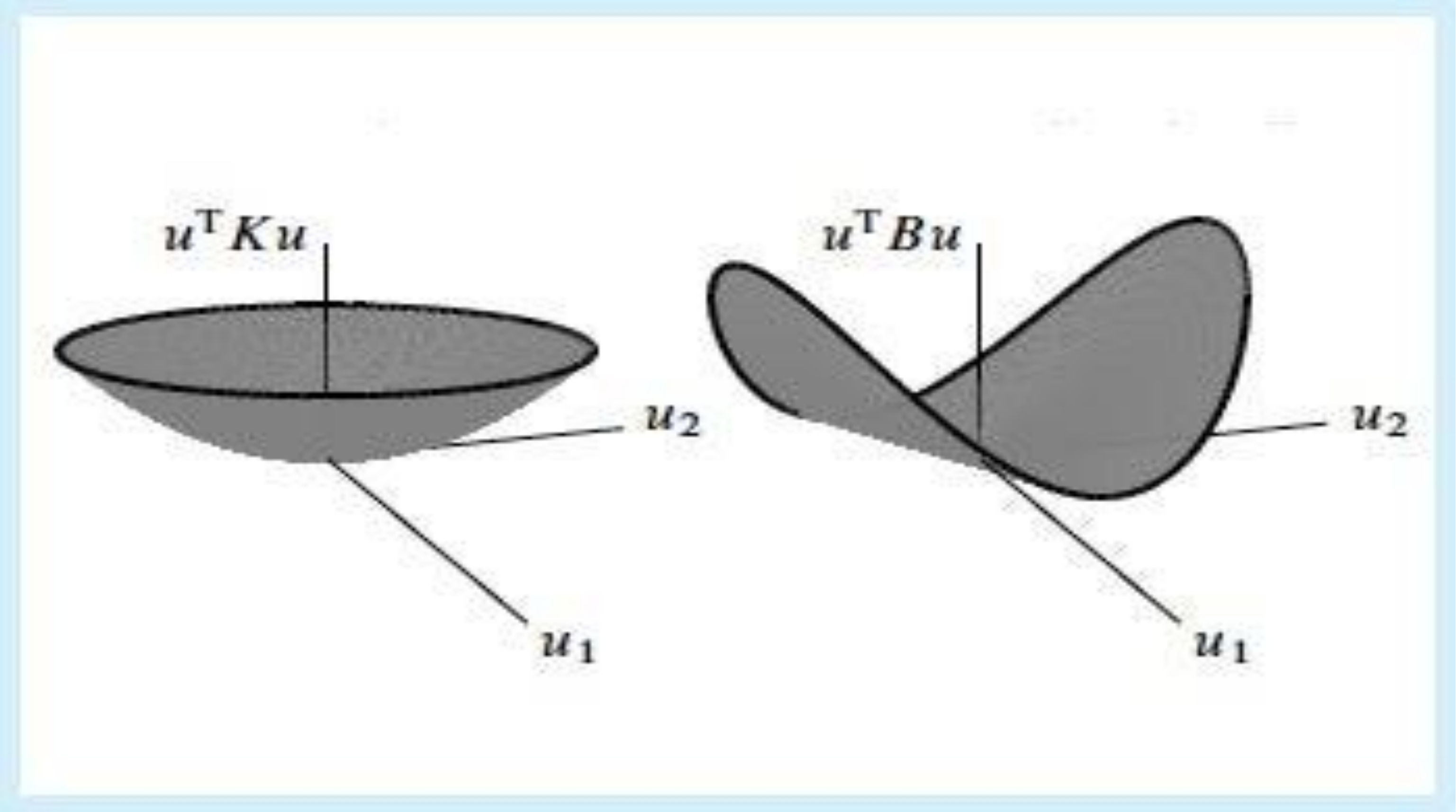
Agenda

- Quadratic form - Classification
- Positive Definite Matrix – A+B, $A^T A$
- Positivity of Eigenvalues
- Equivalent statements for positive definiteness
- Examples

Quadratic Form: Classification

Positive definite and semidefinite: graphs of x'Ax.



https://ocw.mit.edu/courses/mathematics/18-06sc-linear-algebra-fall-2011/positive-definite-matrices-and-applications/

Based on the sign of the quadratic form, SVD can be classified into 3 catagories:

Positive definite if (Quadratic form) > 0

Positive semi definite if (Quadratic form) ≥ 0

Negative definite if (Quadratic form) < 0

A^TA is Positive Definite

A^TA is symmetric and square.

Quadratic form: $x^TA^TAx = (Ax)^T(Ax) = \|Ax\|^2 > 0$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} ; \quad A^TA = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix}$$

Positive Definiteness Of A+B

If A and B are positive definite, then so is A+B.

$$x^T(A+B)x = x^TAx + x^TBx > 0$$

$\because$ $x^TAx > 0$ and $x^TBx > 0$.

POSITIVITY OF EIGENVALUES

Every eigenvalue of a positive definite matrix is positive.

Proof. Suppose $\boldsymbol{A}$ is a positive definite matrix. Let λ be an eigenvalue of $\boldsymbol{A}$, and $\boldsymbol{s}$ be an eigenvector of $\boldsymbol{A}$ corresponding to λ. We have

$$\boldsymbol{A}\boldsymbol{s} = \lambda\boldsymbol{s}$$

It follows that

$$\boldsymbol{s}^T\boldsymbol{A}\boldsymbol{s} = \lambda(\boldsymbol{s}^T\boldsymbol{s})$$

Hence

$$\lambda = \frac{\boldsymbol{s}^T\boldsymbol{A}\boldsymbol{s}}{\boldsymbol{s}^T\boldsymbol{s}} > 0$$

A matrix is positive definite if every eigenvalue of the matrix is positive.

Proof. Suppose every eigenvalue of $\boldsymbol{A}$ is positive. By spectral theorem, $\boldsymbol{A}$ has an eigenvalue decomposition $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$. It follows that

$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \overbrace{\boldsymbol{x}^T\boldsymbol{Q}}^{\boldsymbol{y}^T}\boldsymbol{\Lambda}\overbrace{\boldsymbol{Q}^T\boldsymbol{x}}^{\boldsymbol{y}} = \boldsymbol{y}^T\boldsymbol{\Lambda}\boldsymbol{y} = \sum_i \lambda_i y_i^2$$

Hence, the quadratic form $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ is positive for any $\boldsymbol{x} \neq 0$, and $\boldsymbol{A}$ is positive definite.

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Equivalent statements for Positive definiteness

There are many ways to say a matrix is positive definite.

1. A is positive definite.
2. Every eigenvalue of A is positive.
3. The determinant of every leading principal sub-matrices of A is positive.
4. A has full positive pivots.

What we have shown in the previous slides are

$$① \Leftrightarrow ②$$

and

$$① \Rightarrow ③ \Rightarrow ④ \Rightarrow ①$$

Examples

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ The quadratic form of A is

The quadratic form of A is

$$\begin{aligned} x^T A x &= 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 \\ &= 2\left(x_1 - \frac{1}{2}x_2\right)^2 + \frac{3}{2}\left(x_2 - \frac{2}{3}x_3\right)^2 + \frac{4}{3}x_3^2 \end{aligned}$$

The eigenvalues, the determinants, and the pivots are

$$\text{spectrum}(A) = \{2, 2 \pm \sqrt{2}\}, \ |A_1| = 2, \ |A_2| = 3, \ |A_3| = 4$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE

For what numbers 'b' is the following matrix positive semi-definite?

$$A = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}.$$

Solution :

$$\begin{vmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{vmatrix} = -2b^2+2b+4 \geq 0$$

$$\Rightarrow \quad b^2-b-2 \leq 0$$

$$(b-1)(b-2) \leq 0 \qquad -1 \leq b \leq 2$$

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Agenda

- Semi Definite Matrices

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Semi Definite Matrices

If A is a symmetric matrix the it is said to be positive semi-definite if

1) $x^TAx \geq 0 \quad \forall x$

2) All eigenvalues of A are greater than or equal to zero i.e $\lambda_i \geq 0$.

3) None of the principal submatrices have negative determinants.

4) Pivots are non-negative (≥ 0)

5) There exists a matrix R, possibly with dependent columns such that $A = R^T R$

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Example on semi-definite matrix

1. Test the matrix for positive semi-definiteness.

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

Eigenvalues of A are $\lambda = 4, 4+2\sqrt{3}, 4-2\sqrt{3} > 0$

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Example on semi-definite matrix

$$A \sim \begin{bmatrix} 5 & 2 & 1 \\ 0 & 6/5 & 8/5 \\ 0 & 8/5 & 24/5 \end{bmatrix} \sim \begin{bmatrix} 5 & 2 & 1 \\ 0 & 6/5 & 8/5 \\ 0 & 0 & 8/3 \end{bmatrix}$$

1) The pivots are: $0, \frac{6}{5}, \frac{8}{3} > 0$

2) Determinants of submatrices: $|A_1| = 5 > 0$

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Example on semi-definite matrix

3) $|A_2| = \begin{vmatrix} 5 & 2 \\ 2 & 2 \end{vmatrix} = 6 > 0$

4) $|A_3| = 6 > 0$

The above submatrices have positive determinants.

$\therefore$ The given matrix is positive definite.

2) Test the positive semi-definiteness of the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$R_2 \to R_2 + \frac{1}{2} R_1$

$R_3 \to R_3 + \frac{1}{2} R_1$

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Example on semi-definite matrix

Solution: Characteristic Equation is $|A-\lambda I| = 0$

$$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 9\lambda - 18 = 0$$

$$\Rightarrow \quad \lambda = 0, 3, 3 \quad (\geq 0)$$

$$A \sim \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Example on semi-definite matrix

Pivots : $2, \frac{3}{2}, 0$ are non-negative

Determinants of the sub-matrices are:

$|A_1| = 2 \geq 0$; $|A_3| = 0 \geq 0$

$|A_2| = 2 \geq 0$

Hence the given matrix is positive semi-definite.

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Agenda

- Least square problem – Normal equations.
- Positive Definiteness and the Least squares
- A simple example

The Normal Equations

Theorem: *Any solution x of the least squares problem is a solution of the linear system*

$$A^T A x = A^T b.$$

The system is nonsingular if and only if A has linearly independent columns.

Proof.

- Since $b - Ax \in \ker(A^T)$, we have $A^T(b - Ax) = 0$ or $A^T A x = A^T b$.
- $A^T A$ is nonsingular. Suppose $A^T A x = 0$ for some $x \in \mathbb{R}^n$. Then $0 = x^T A^T A x = (Ax)^T A x = \|Ax\|_2^2$. Hence $Ax = 0$ which implies that $x = 0$ if and only if A has linearly independent columns.

The linear system $A^T A x = A^T b$ is called the **normal equations.**

We have learned that **least square** comes from **projection** :

$$b - p = e \Rightarrow, A^T(b - A\hat{x}) = 0 \Rightarrow, A^T A\hat{x} = A^T b$$

Consequently, only if $A^T A$ is invertible, then we can use linear regression to find approximate solutions $\hat{x} = (A^T A)^{-1} A^T b$ to unsolvable systems of linear equations.

According to the reasoning before, we know as long as all columns of $A_{m\times,n}$ are mutual independent, then A^TA is invertible. At the same time we ought to notice that the columns of A are guaranteed to be independent if they are orthoganal and even orthonormal.

In another prospective, if A^TA is positive definite, then $A_{m\times,n}$ has rank n (independent columns) and thus A^TA is invertible.

Overall, if $A^T A$ is positive definite or invertible, then we can find approximate solutions of least square.

Find the function of the form

$$F(x) = c_1 + c_2 x \lg x + c_3 e^x$$

that is the best least-squares fit to the data points

$$(1,1), (2,1), (3,3), (4,8).$$

First we form the A matrix

$$A = \begin{pmatrix} 1 & 0 & e \\ 1 & 2 & e^2 \\ 1 & 3\lg 3 & e^3 \\ 1 & 8 & e^4 \end{pmatrix}$$

We compute the pseudoinverse, then multiply it by y, to obtain the coefficient vector

$$c = \begin{pmatrix} 0.411741 \\ -0.20487 \\ 0.16546 \end{pmatrix}.$$

- Given $A^{m,n}$ and $b \in \mathbb{R}^m$.
- The system $Ax = b$ is over-determined if $m > n$.
- This system has a solution if $b \in \operatorname{span}(A)$, the column space of A, but normally this is not the case and we can only find an approximate solution.
- A general approach is to choose a vector norm $\|\cdot\|$ and find x which minimizes $\|Ax - b\|$.
- We will only consider the Euclidian norm here.

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Positive Definite Matrices and Least Squares

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- Given $A^{m,n}$ and $b \in \mathbb{R}^m$ with $m \geq n \geq 1$. The problem to find $x \in \mathbb{R}^n$ that minimizes $\|Ax - b\|_2$ is called the least squares problem.
- A minimizing vector x is called a least squares solution of $Ax = b$.
- Several ways to analyze:
- Quadratic minimization
- Orthogonal Projections
- SVD

- Define function $E : \mathbb{R}^n \to \mathbb{R}$ by $E(\boldsymbol{x}) = \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2$
- $E(\boldsymbol{x}) = (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) = \boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} - 2\boldsymbol{c}^T\boldsymbol{x} + \alpha$, where
- $\boldsymbol{B} := \boldsymbol{A}^T\boldsymbol{A}$, $\boldsymbol{c} := \boldsymbol{A}^T\boldsymbol{b}$ and $\alpha := \boldsymbol{b}^T\boldsymbol{b}$.
- $\boldsymbol{B}$ is positive semidefinite and positive definite if $\boldsymbol{A}$ has rank n.
- Since the Hessian $\boldsymbol{H}E(\boldsymbol{x}) := \left(\frac{\partial^2 E(\boldsymbol{x})}{\partial x_i \partial x_j}\right) = 2\boldsymbol{B}$ we can find minimum by setting partial derivatives equal zero.
- $\nabla E(\boldsymbol{x}) := \left(\frac{\partial E(\boldsymbol{x})}{\partial x_i}\right) = 2(\boldsymbol{B}\boldsymbol{x} - \boldsymbol{c}) = \boldsymbol{0}$
- Normal equations $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$.

A simple example

- $$x_1 = 1 \\ x_1 = 1, \quad \boldsymbol{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{x} = [x_1], \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \\ x_1 = 2$$
- Quadratic minimization problem:
  $$\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2 = (x_1 - 1)^2 + (x_1 - 1)^2 + (x_1 - 2)^2.$$
- Setting the first derivative with respect to x_1 equal to zero we obtain $2(x_1 - 1) + 2(x_1 - 1) + 2(x_1 - 2) = 0$ or $6x_1 - 8 = 0$ or $x_1 = 4/3$
- The second derivative is positive (it is equal to 6) and $x = 4/3$ is a global minimum.

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Agenda

- Singular value and singular vector
- Positivity of a Singular value
- Number of Singular values
- Singular value decomposition(SVD)
- Proofs of SVD
- Matrices and SVD
- Singular vectors and fundamental subspaces

A **singular value** of a real matrix $\boldsymbol{A}$ is the square root of a non-zero eigenvalue of $\left(\boldsymbol{A}^T\boldsymbol{A}\right)$.

It means to find the singular values of $\boldsymbol{A}$, one needs to find the non-zero eigenvalues of $\left(\boldsymbol{A}^T\boldsymbol{A}\right)$.

Singular vector. If σ is a singular value of $\boldsymbol{A}$, then there exists $\boldsymbol{v} \neq \mathbf{0}$ such that

$$\left(\boldsymbol{A}^T\boldsymbol{A}\right)\boldsymbol{v} = \sigma^2\boldsymbol{v}$$

Such a $\boldsymbol{v}$ is called a right singular vector of $\boldsymbol{A}$ with singular value σ. It is an eigenvector of $\left(\boldsymbol{A}^T\boldsymbol{A}\right)$ with eigenvalue σ^2.

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Positivity of a Singular Value

A singular value is always positive.

The matrix $\left(\boldsymbol{A}^T\boldsymbol{A}\right)$ is positive semi-definite:

$$\boldsymbol{x}^T\left(\boldsymbol{A}^T\boldsymbol{A}\right)\boldsymbol{x} = (\boldsymbol{A}\boldsymbol{x})^T(\boldsymbol{A}\boldsymbol{x}) = \|\boldsymbol{A}\boldsymbol{x}\|^2 \geq 0$$

so the eigenvalues of $\left(\boldsymbol{A}^T\boldsymbol{A}\right)$ must be non-negative, and the non-zero eigenvalues must be positive. Hence a singular value is positive.

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Number of Singular Values

A matrix of **rank** r has exactly r singular values.

Proof. Note $\left(A^T A\right) x = 0 \Leftrightarrow x^T A^T A x = 0 \Leftrightarrow A x = 0$, so $\mathcal{N}\left(A^T A\right) = \mathcal{N}(A)$. Let A be of order $m \times n$ with rank r. Then $\dim \mathcal{N}(A) = n - r = \dim \mathcal{N}\left(A^T A\right)$. Matrix $\left(A^T A\right)$ is non-defective, so the algebraic multiplicity of eigenvalue 0 is $(n - r)$. It follows that the total algebraic multiplicities of the non-zero eigenvalues of $\left(A^T A\right)$ is

$$n - (n - r) = r$$

Notation. Singular values are denoted by $\sigma_1, \ldots, \sigma_r$.

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Singular Value Decomposition (SVD)

A real matrix can be decomposed by its singular values and singular vectors. This is called singular value decomposition.

A matrix of order $m \times n$ has SVD

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$$

where $\boldsymbol{\Sigma}$ is an $m \times n$ "diagonal" matrix with the singular values of $\boldsymbol{A}$ as the leading diagonal elements, $\boldsymbol{U}$ is an $m \times m$ orthogonal matrix with the eigenvectors of $\left(\boldsymbol{A}\boldsymbol{A}^T\right)$ as columns, and $\boldsymbol{V}$ is an $n \times n$ orthogonal matrix with the eigenvectors of $\left(\boldsymbol{A}^T\boldsymbol{A}\right)$ as columns.

Proof of SVD

Let r be the rank of A. Let $\sigma_1 \ldots \sigma_r$ be the singular values of A. Let $v_1 \ldots v_r$ be orthonormal eigenvectors of $\left(A^T A\right)$ with positive eigenvalues σ_i^2 and $u_1 \ldots u_r$ be defined by $u_i = \frac{Av_i}{\sigma_i}$.
Note

$$\left(AA^T\right) u_i = \frac{AA^T Av_i}{\sigma_i} = \frac{A\sigma_i^2 v_i}{\sigma_i} = \sigma_i^2 u_i, \;\; i = 1, \ldots, r$$

So u_i is an eigenvector of $\left(AA^T\right)$ with the same eigenvalue σ_i^2. Let $v_{r+1} \ldots v_n$ be orthonormal eigenvectors of $\left(A^T A\right)$ with eigenvalue 0, and $u_{r+1} \ldots u_m$ be eigenvectors of $\left(AA^T\right)$ with eigenvalue 0. Construct matrices U and V by

$$U = \begin{bmatrix} u_1 & \ldots & u_m \end{bmatrix}, \; V = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$$

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Agenda

- Proofs of SVD (Alternate proof)
- Matrices and SVD
- Singular vectors and fundamental subspaces

Alternate proof of SVD

We show $U^TAV = \Sigma$ which leads to SVD $A = U\Sigma V^T$. For $j = 1 \ldots r$, we have $u_j = \frac{Av_j}{\sigma_j}$, so $Av_j = \sigma_j u_j$ and

$$(U^TAV)_{ij} = u_i^T A v_j = u_i^T(\sigma_j u_j) = \sigma_j \delta_{ij}, \;\; i = 1, \ldots, m$$

For $j = r+1 \ldots n$, we have $\left(A^TA\right) v_j = 0$, so $Av_j = 0$ and

$$(U^TAV)_{ij} = u_i^T A v_j = 0, \;\; i = 1, \ldots, m$$

Combining the results, we get

$$U^TAV = \Sigma$$

Hence

$$A = U\Sigma V^T$$

> For a matrix of order $m \times n$ with SVD $\boldsymbol{A} = \boldsymbol{U\Sigma V}^T$, the column vectors of $\boldsymbol{U}$ (resp. $\boldsymbol{V}$) is an orthonormal basis of $\mathbb{R}^m$ (resp. $\mathbb{R}^n$).

$\boldsymbol{U}$ must be an eigenvector matrix of $\boldsymbol{AA}^T$.

$$\boldsymbol{AA}^T = \left(\boldsymbol{U\Sigma V}^T\right)\left(\boldsymbol{V\Sigma}^T\boldsymbol{U}^T\right) = \boldsymbol{U}\overbrace{\left(\boldsymbol{\Sigma\Sigma}^T\right)}^{\text{diagonal}}\boldsymbol{U}^T$$

Similarly, $\boldsymbol{V}$ must be an eigenvector matrix of $\boldsymbol{A}^T\boldsymbol{A}$.

$$\boldsymbol{A}^T\boldsymbol{A} = \left(\boldsymbol{V\Sigma}^T\boldsymbol{U}^T\right)\left(\boldsymbol{U\Sigma V}^T\right) = \boldsymbol{V}\overbrace{\left(\boldsymbol{\Sigma}^T\boldsymbol{\Sigma}\right)}^{\text{diagonal}}\boldsymbol{V}^T$$

Singular Vectors and Fundamental Subspaces

> The right (resp. left) singular vectors in an SVD of a matrix form an orthonormal basis of the row space (resp. column space) of the matrix.

Row space. $\{\boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_n\}$ contains eigenvectors of $\left(\boldsymbol{A}^T\boldsymbol{A}\right)$ with eigenvalue 0, so it is a basis of $\mathcal{N}(\boldsymbol{A}^T\boldsymbol{A}) = \mathcal{N}(\boldsymbol{A})$. This implies $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ is a basis of the orthogonal complement of $\mathcal{N}(\boldsymbol{A})$, i.e. the row space of $\boldsymbol{A}$.

Column space. We have $\boldsymbol{AV} = \boldsymbol{U\Sigma}$. The first r columns are

$$\boldsymbol{A}\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i, \quad i = 1, \ldots, r$$

So $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r\}$ is a linearly independent set in the column space of $\boldsymbol{A}$. Hence, it is a basis of $\mathcal{C}(\boldsymbol{A})$.

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Agenda

- Problems on SVD
- SVD and Rank one matrices
- SVD and Pseudoinverse

Find SVD of

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

The eigenvalues of

$$A^T A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

are $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0$. Hence the singular values of A are

$$\sigma_1 = \sqrt{3}, \ \sigma_2 = 1$$

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Problems on SVD

Orthonormal eigenvectors of $\left(A^T A\right)$ are

$$v_1 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix},\ v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix},\ v_3 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

The corresponding left singular vectors of A are

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix},\ u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

So

$$A = U\Sigma V^T = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\sqrt{3} & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}$$

Rank One matrices and SVD

Every real matrix of rank r is the sum of r real matrices of rank 1 based on singular values and singular vectors.

By SVD

$$\begin{aligned} \boldsymbol{A} = \boldsymbol{U}\Sigma\boldsymbol{V}^T &= \sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^T + \cdots + \sigma_r\boldsymbol{u}_r\boldsymbol{v}_r^T \\ &= \boldsymbol{A}_1 + \cdots + \boldsymbol{A}_r \end{aligned}$$

Image approximation. For an image of size 1000×1000, a compression rate of 90% is achieved if 50 terms are used.

Let $A = U\Sigma V^T$ be an SVD of A. For a rectangular system of linear equations $Ax = b$, the least-squares solution with the minimum length is $x^+ = V\Sigma^+U^Tb$.

Pseudo-inverse. The minimum-length least-squares solution can be written as $x^+ = A^+b$, where $A^+ = V\Sigma^+U^T$. A^+ is called the **pseudo-inverse** of A.

The Pseudoinverse of a matrix generalizes the notion of the inverse in a similar manner that SVD generalized the diagonalization of a matrix.

Not every matrix has an inverse, but every matrix has a psuedoinverse.

Computing the pseudoinverse from SVD is simple.

If $A = U\Sigma V^*$ then $A^+ = V\Sigma^+ U^*$

where Σ^+ is formed from Σ by taking the reciprocal of all the non-zero elements, leaving all the zeros alone, and making the matrix the right shape: if Σ is an m by n matrix, then Σ^+ must be an n by m matrix.

Computing the pseudoinverse from SVD is simple.

If $A = U\Sigma V^*$ then $A^+ = V\Sigma^+ U^*$

where Σ^+ is formed from Σ by taking the reciprocal of all the non-zero elements, leaving all the zeros alone, and making the matrix the right shape: if Σ is an m by n matrix, then Σ^+ must be an n by m matrix.

Consider the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & -2 \end{bmatrix}$

The Singular value decomposition of A is

$$\begin{bmatrix} \frac{1}{\sqrt{26}} & -\frac{5}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{bmatrix} \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{11}{\sqrt{195}} & \frac{7}{\sqrt{195}} & -\sqrt{\frac{5}{39}} \\ -\frac{3}{\sqrt{26}} & 2\sqrt{\frac{2}{13}} & -\frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{30}} & \sqrt{\frac{2}{15}} & \sqrt{\frac{5}{6}} \end{bmatrix}$$

SVD and Pseudoinverse

Consider the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & -2 \end{bmatrix}$

The Singular value decomposition of A is

$$\begin{bmatrix} \frac{1}{\sqrt{26}} & -\frac{5}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{bmatrix} \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{11}{\sqrt{195}} & \frac{7}{\sqrt{195}} & -\sqrt{\frac{5}{39}} \\ -\frac{3}{\sqrt{26}} & 2\sqrt{\frac{2}{13}} & -\frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{30}} & \sqrt{\frac{2}{15}} & \sqrt{\frac{5}{6}} \end{bmatrix}$$

The Pseudoinverse may be computed from SVD using the definition explained earlier.

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More on Pseudoinverse $A^{\dagger}$.

- If A is square and nonsingular then $A^{\dagger} = A^{-1}$.
- $A^{\dagger}$ is always defined.
- Thus $A^{\dagger}$ is a generalization of usual inverse.
- If $B \in \mathbb{R}^{n,m}$ satisfies
  1. $ABA = A$
  2. $BAB = B$
  3. $(BA)^T = BA$
  4. $(AB)^T = AB$

  then $B = A^{\dagger}$.
- Thus $A^{\dagger}$ is uniquely defined by these axioms.

Example on Pseudoinverse $A^\dagger$.

Show that the pseudoinverse of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$ is $B = \frac{1}{4}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

We have $BA = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $AB = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus

1. $ABA = A$
2. $BAB = B$
3. $(BA)^T = BA$
4. $(AB)^T = AB$

and hence $A^\dagger = B$.

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Agenda – Problems on SVD

- Example 1

Example 1. Find the singular value decomposition for the matrix:

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 3 & 2 & -2 \end{bmatrix}$$

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Solution: The singular value decomposition of A is given by $\underset{2\times 3}{A} = \underset{2\times 3}{U}\;\underset{1\times 3}{\Sigma}\;\underset{2\times 3}{U^T}$

where $U = AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$, a square symmetric matrix.

Let $B = AA^T$. Eigenvalues of B can be found from: $|B - \lambda I| = 0$

$\therefore \lambda_1 = 25, \quad \lambda_2 = 9$

Corresponding Eigenvectors are:

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Note that x_1 and x_2 are orthogonal.

The orthonormal vectors are

$$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\therefore \quad U = [u_1 \;\; u_2] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Singular values are the square roots of the eigenvalues.

$$\therefore \quad \sigma_1 = \sqrt{\lambda_1} = \sqrt{25} = 5$$

and $$\sigma_2 = \sqrt{\lambda_2} = \sqrt{9} = 3$$

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Agenda – Problems on SVD

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Define $v_i = \dfrac{u_i^T A}{\sigma_i}$; $i = 1, 2$

$$\therefore \quad v_1 = \frac{u_1^T A}{\sigma_1} = \left[\tfrac{1}{\sqrt{2}} \quad \tfrac{1}{\sqrt{2}} \quad 0\right]^T$$

$$v_2 = \frac{u_2^T A}{\sigma_2} = \left[-\tfrac{1}{\sqrt{32}} \quad \tfrac{1}{\sqrt{32}} \quad -\tfrac{4}{\sqrt{32}}\right]^T$$

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Agenda – Problems on SVD

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$$V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} \phantom{1/\sqrt{2}} & \phantom{-1/\sqrt{32}} & v_3 \end{array} \\ \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{32} & * \\ 1/\sqrt{2} & 1/\sqrt{32} & * \\ 0 & -4/\sqrt{32} & * \end{bmatrix} \end{array}$$

The 3^{rd} vector v_3 must be orthogonal to both v_1 and v_2.

$$\therefore \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{32} & 1/\sqrt{32} & -4/\sqrt{32} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving, we get $\dfrac{v_3}{\|v_3\|} = \dfrac{1}{\|v_3\|}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

Thus $V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{32} & 1/\sqrt{32} & -4/\sqrt{32} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$

The SVD is

$$A = U \Sigma V^T$$

where $\Sigma = \begin{bmatrix} \sqrt{25} & 0 & 0 \\ 0 & \sqrt{9} & 0 \end{bmatrix}_{2\times 3}$

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Agenda – Problems on SVD

- Example 2

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Example 2. Find the SVD for $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & 2 \end{bmatrix}$

Soln: SVD: $A = U \Sigma V^T$

$$AA^T = \begin{bmatrix} 9 & -1 \\ -1 & 9 \end{bmatrix}$$

Agenda – Problems on SVD

Eigenvalues of AA^T are

$\lambda_1 = 10, 8$ (arranged in increasing order)

Corresponding Eigenvectors are

$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Clearly X_1 and X_2 are orthogonal.

$$V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = V^T$$

$$\Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{8} \\ 0 & 0 \end{bmatrix}$$

Agenda – Problems on SVD

$$u_i = \frac{Av_i}{\sigma_i} \quad ; \quad i\ 1, 2, 3 \ ; \quad \sigma_1 = \sqrt{10}, \ \sigma_2 = \sqrt{8}$$

$$u_1 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$u_1^T u_2 = 0 \quad ; \quad u_1 \perp u_2 .$$

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Agenda – Problems on SVD

using orthogonality condition, u_3 can be obtained. Let $u_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Agenda – Problems on SVD

$\therefore \quad u' = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, normalizing, we get

$$u_3 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} \quad ; \quad U = \begin{bmatrix} -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}$$

$\therefore \quad A = U \Sigma V^T$

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Agenda – Applications of SVD

- Covariance of a matrix (2 x 2)

- Example on constructing covariance matrix

Applications : Covariance of a 2x2 matrix

we can represent the covariance matrix by its eigenvectors and eigenvalues:

$$\Sigma\vec{v} = \lambda\vec{v}$$

where $\vec{v}$ is an eigenvector of Σ, and λ is the corresponding eigenvalue.

Equation above

holds for each eigenvector-eigenvalue pair of matrix Σ. In the 2D case, we obtain

two eigenvectors and two eigenvalues. The system of two equations defined by equation be represented efficiently using matrix notation:

$$\Sigma V = V L$$

Applications : Covariance of a 2x2 matrix

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This means that we can represent the covariance matrix as a function of its eigenvectors and eigenvalues:

$$\Sigma = V L V^{-1}$$

is called the eigendecomposition of the covariance matrix and can be obtained

using a Singular Value Decomposition algorithm. Whereas the eigenvectors represent the directions of the largest variance of the data, the eigenvalues represent the magnitude of this variance in those directions. In other words, V represents a rotation matrix, while $\sqrt{L}$ represents a scaling matrix. The covariance matrix can thus be decomposed further as:

$$\Sigma = R S S R^{-1}$$

Applications : Covariance of a 2x2 matrix

where $R = V$ is a rotation matrix and $S = \sqrt{L}$ is a scaling matrix.

linear transformation $T = R\,S$. Since S is a diagonal scaling matrix, $S = S^\intercal$. Furthermore, since R is an orthogonal matrix, $R^{-1} = R^\intercal$. Therefore, $T^\intercal = (R\,S)^\intercal = S^\intercal\,R^\intercal = S\,R^{-1}$. The covariance matrix can thus be written as:

$$\Sigma = R\,S\,S\,R^{-1} = T\,T^\intercal,$$

In other words, if we apply the linear transformation defined by $T = R\,S$ to the original data D shown by figure we obtain the rotated and scaled data D' with covariance matri: $T\,T^\intercal = \Sigma' = R\,S\,S\,R^{-1}$. This is illustrated by figure below

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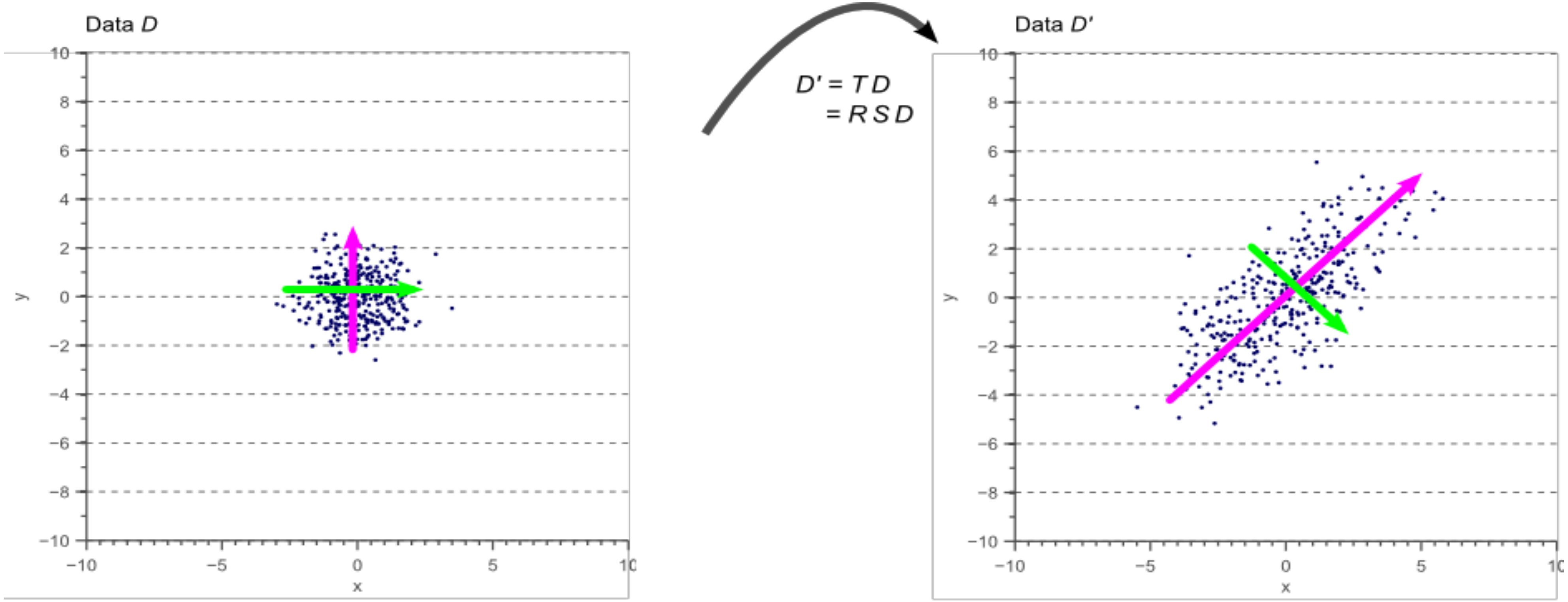
Applications : Covariance of a 2x2 matrix



$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Sigma' = \begin{bmatrix} 4.25 & 3.10 \\ 3.10 & 4.29 \end{bmatrix} = R\,S\,S\,R^{-1}$$

https://www.visiondummy.com/wp-content/uploads/2014/04/lineartrans.png

The covariance matrix represents a linear transformation of the original data.

The largest eigenvector, i.e. the

eigenvector with the largest corresponding eigenvalue, always points in the direction of the largest variance of the data and thereby defines its orientation. Subsequent eigenvectors are always orthogonal to the largest eigenvector due to the orthogonality of rotation matrices.

Example: Covariance of a matrix

1) Find the covariance matrix given that :

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 4 \\ 2 \\ 13 \end{bmatrix} \quad X_3 = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix}$$

and

$$X_4 = \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix}$$

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Example: Covariance of a matrix

step 1. Find the mean vector.

$$\text{Mean} = \frac{1}{n}\left[X_1 + X_2 + X_3 + X_4\right] = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix} = M \text{ (say)}$$

step 2: The mean deviation of the vectors X_i, $i = 1, 2, 3, 4$ are $\hat{X}_i = X_i - M$, $i = 1, 2, 3, 4$.

Example: Covariance of a matrix

Therefore $\hat{X}_1 = \begin{bmatrix} -4 \\ -2 \\ -4 \end{bmatrix}$, $\hat{X}_2 = \begin{bmatrix} -1 \\ -2 \\ 8 \end{bmatrix}$ $\hat{X}_3 = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}$ and

$$\hat{X}_4 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

Let the matrix $B = [\hat{X}_1 \ \hat{X}_2 \ \hat{X}_3 \ \hat{X}_4]$

Example: Covariance of a matrix

Then, $B = \begin{bmatrix} -4 & -1 & 2 & 3 \\ -2 & -2 & 4 & 0 \\ -4 & 8 & -4 & 0 \end{bmatrix}$

The covariance matrix is $S = \frac{1}{n-1}\left[B B^T \right]$

$$S = \frac{1}{3}\begin{bmatrix} 30 & 18 & 0 \\ 18 & 24 & -24 \\ 0 & -24 & 96 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 0 \\ 6 & 8 & -8 \\ 0 & -8 & 32 \end{bmatrix}$$

2) The table below lists the weights and heights of 5 boys. Find the covariance matrix.

Boy:	1	2	3	4	5
Weight:	120	125	125	135	150
Height:	61	60	64	68	72

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Example: Covariance of a matrix

Solution: Let $X = \begin{pmatrix} \text{Weight} \\ \text{Height} \end{pmatrix}$

Mean of X is $M = \frac{1}{5}\left[X_1 + X_2 + X_3 + X_4 + X_5\right]$

$\therefore \quad M = \frac{1}{5}\begin{bmatrix} 650 \\ 325 \end{bmatrix} = \begin{bmatrix} 130 \\ 65 \end{bmatrix}$

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Example: Covariance of a matrix

Let $\hat{X}_i = X_i - M$, $i = 1, 2, 3, 4, 5.$

$\therefore \hat{X}_1 = \begin{bmatrix} -10 \\ -4 \end{bmatrix}$, $\hat{X}_2 = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$, $\hat{X}_3 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

$\hat{X}_4 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\hat{X}_5 = \begin{bmatrix} 15 \\ 7 \end{bmatrix}$

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Example: Covariance of a matrix

Let $B = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 & \hat{X}_3 & \hat{X}_4 & \hat{X}_5 \end{bmatrix}$

$$= \begin{bmatrix} -10 & -5 & -5 & 5 & 15 \\ -4 & -5 & -1 & 3 & 7 \end{bmatrix}_{2\times 5}$$

$$S = \frac{1}{n-1}\left[BB^T\right] = \begin{bmatrix} 100 & \frac{190}{4} \\ \frac{190}{4} & 25 \end{bmatrix}$$

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