Linear Algebra & its Applications

Unit 1 - Matrices & Gaussian Elimination

Topic 1 - Introduction

Equivalent Matrices

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & -3 & 4 \end{bmatrix} \xrightarrow{R_2 + \frac{1}{2}R_1} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & 1 \\ 0 & -3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & 1 \\ 0 & -3 & 4 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\boxed{A \equiv B}$$

Echelon form of a matrix :-

$$A \rightarrow \begin{bmatrix} a & b & c & d & e \\ 0 & 0 & f & g & h \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4 \times 5}$$

RREF of a matrix :-

$$\begin{bmatrix} a & b & c & d & e \\ 0 & 0 & f & g & h \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & b/a & c/a & d/a & e/a \\ 0 & 0 & 1 & g/f & h/f \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & b/a & c/a & d/a & e/a \\ 0 & 0 & 1 & g/f & h/f \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - \frac{c}{a}R_2} \begin{bmatrix} 1 & b/a & 0 & d/a - \frac{cg}{af} & e/a \\ 0 & 0 & 1 & g/f & h/f \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

②

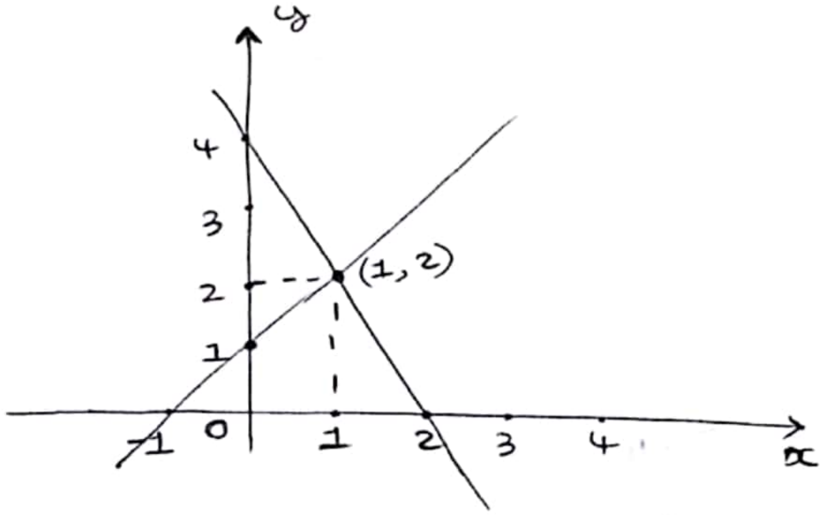
$$A = \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & -1 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & -6 & -16 & -15 \end{array}\right]$$

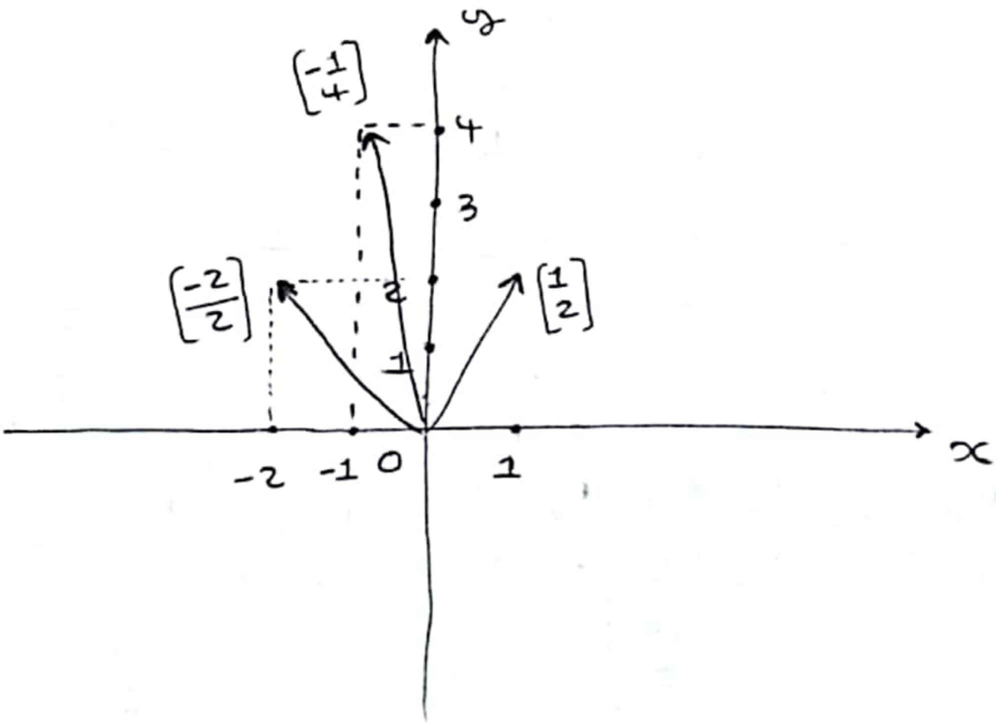
$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & 5 \\ 0 & 0 & -20 & -30 & -25 \end{array}\right]$$

Topic 2 &3 - The Geometry of Linear Equations - Row & column Pictures

Slide 4

$$x - y = -1$$
$$2x + y = 4$$

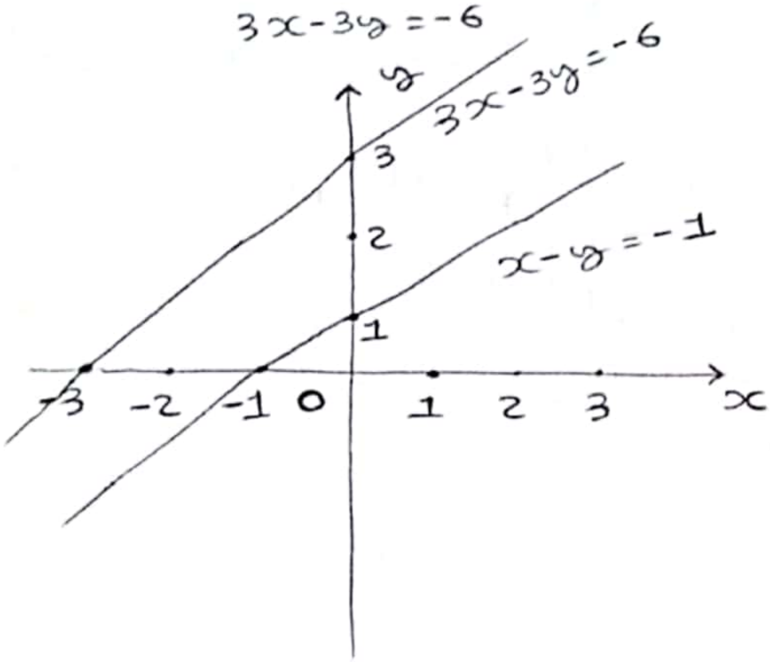




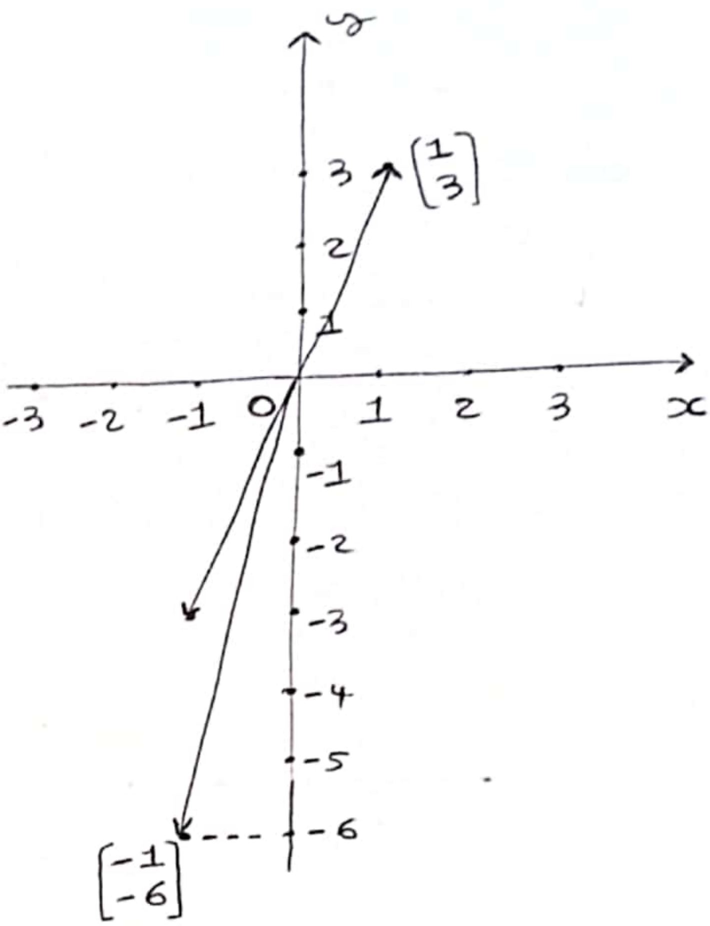
Slide 5

$x - y = -1$

$3x - 3y = -6$



$$x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-1\\-3\end{bmatrix} = \begin{bmatrix}-1\\-6\end{bmatrix}$$

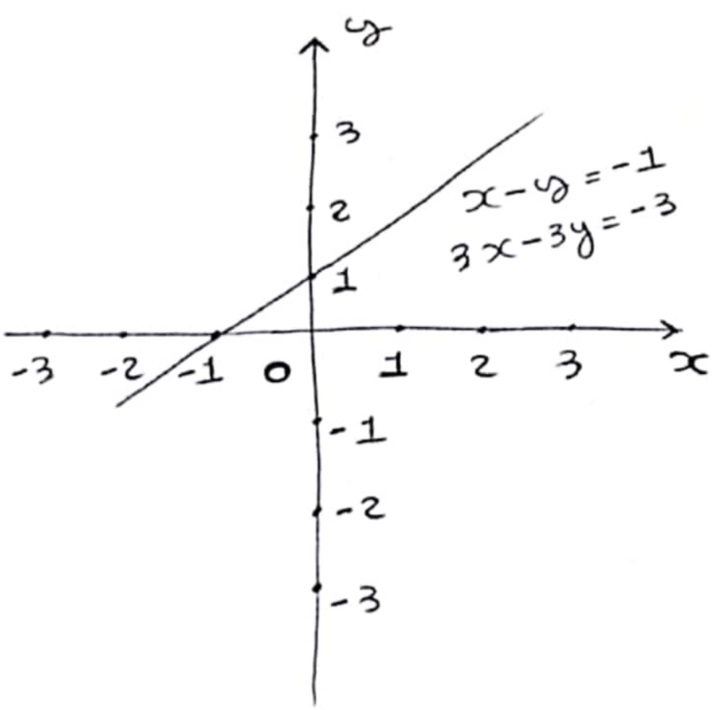


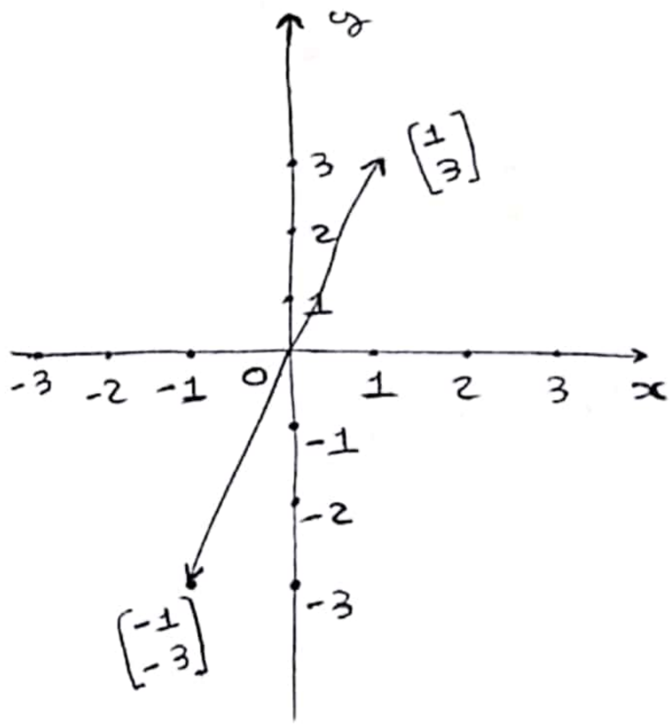
Slide 6

$x - y = -1$

$3x - 3y = -3$

$$x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-1\\-3\end{bmatrix} = \begin{bmatrix}-1\\-3\end{bmatrix}$$





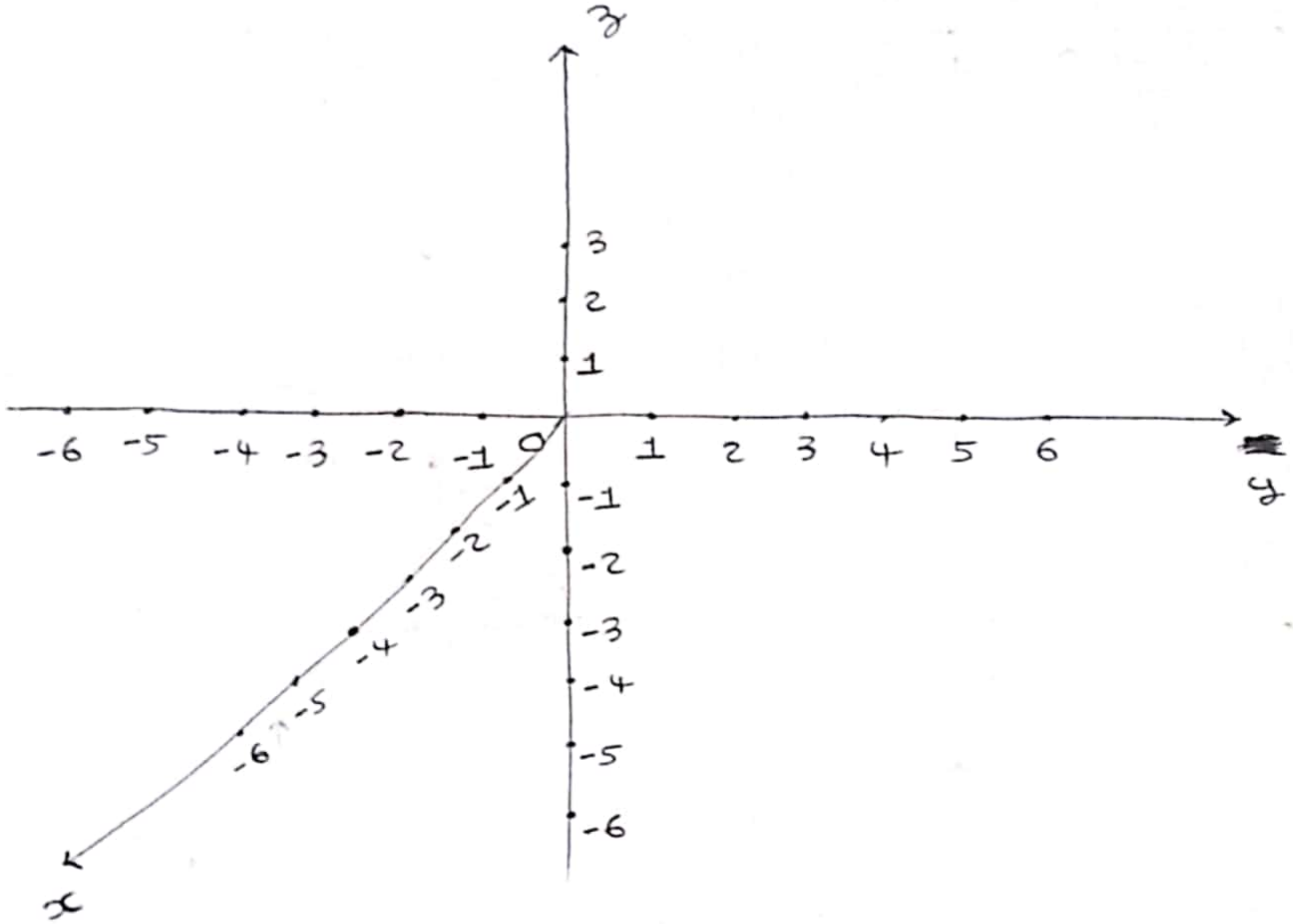
④

slide 7

$$x+y+2z = 1$$
$$x+2y-z = -2$$
$$x+3y+z = 5$$

on solving, we get $(-6,3,2)$

Row picture



column picture

$$x\begin{bmatrix}1\\1\\1\end{bmatrix}+y\begin{bmatrix}1\\2\\3\end{bmatrix}+z\begin{bmatrix}2\\-1\\1\end{bmatrix}=\begin{bmatrix}1\\-2\\5\end{bmatrix}$$

$$-6\begin{bmatrix}1\\1\\1\end{bmatrix}+3\begin{bmatrix}1\\2\\3\end{bmatrix}+2\begin{bmatrix}2\\-1\\1\end{bmatrix}=\begin{bmatrix}1\\-2\\5\end{bmatrix}$$

Scanned with CamScanner

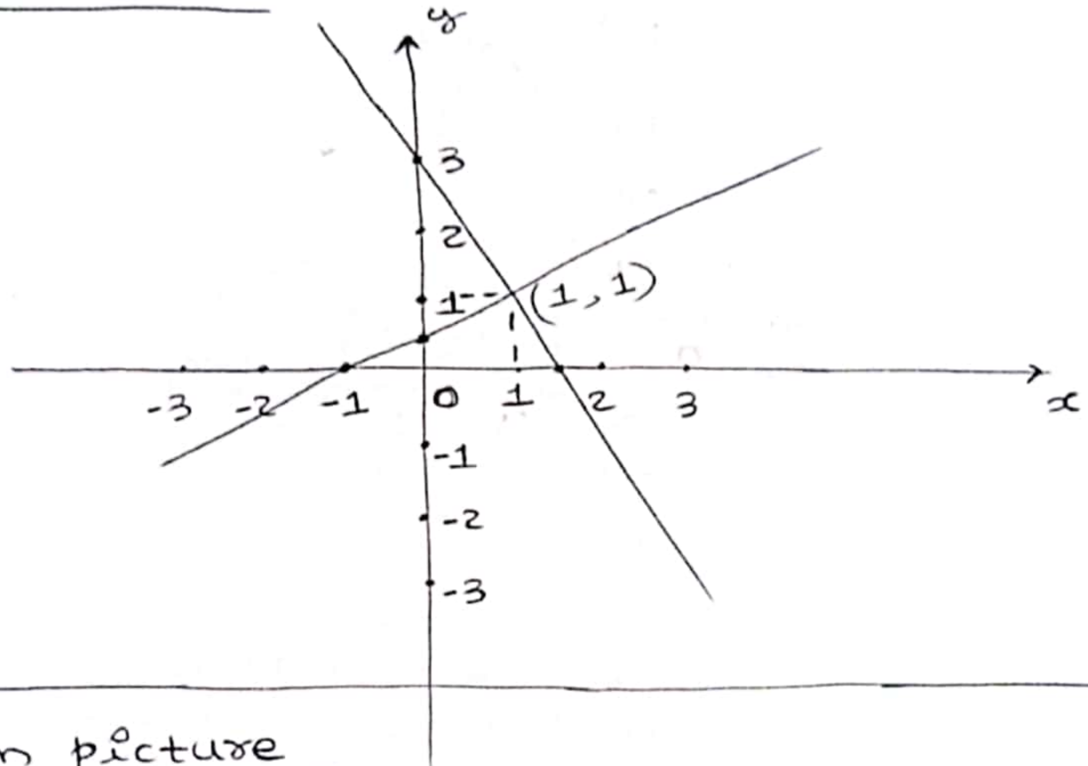
1.3

slide 3

Example 1 Solve the system of Equations and draw the row picture & column picture ?

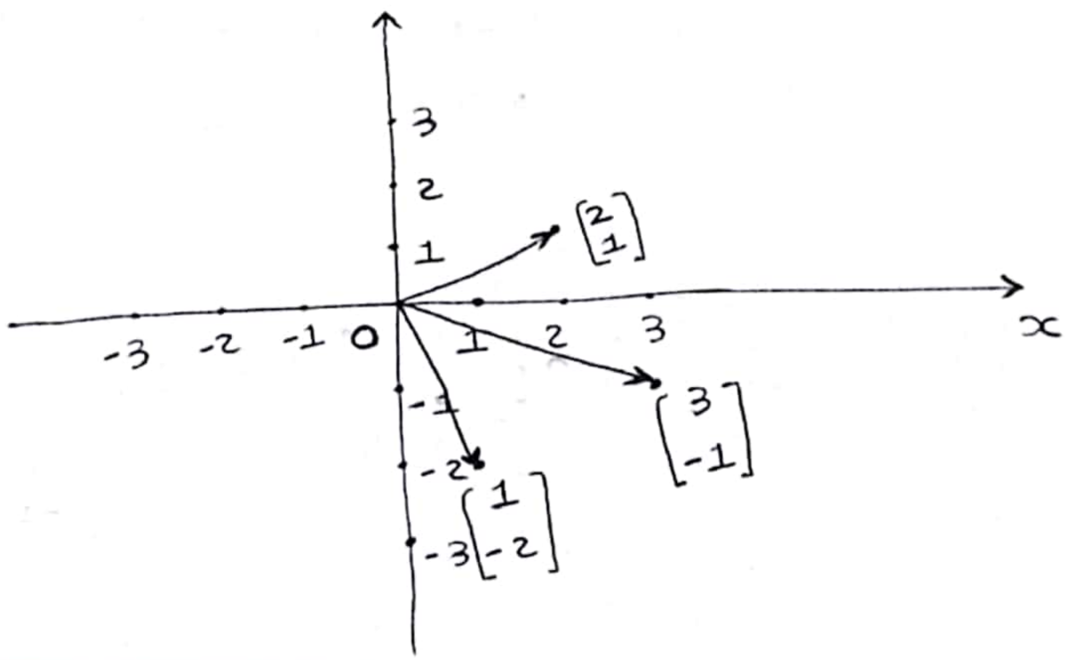
$$2x + y = 3$$
$$x - 2y = -1$$

Raw picture :-



column picture

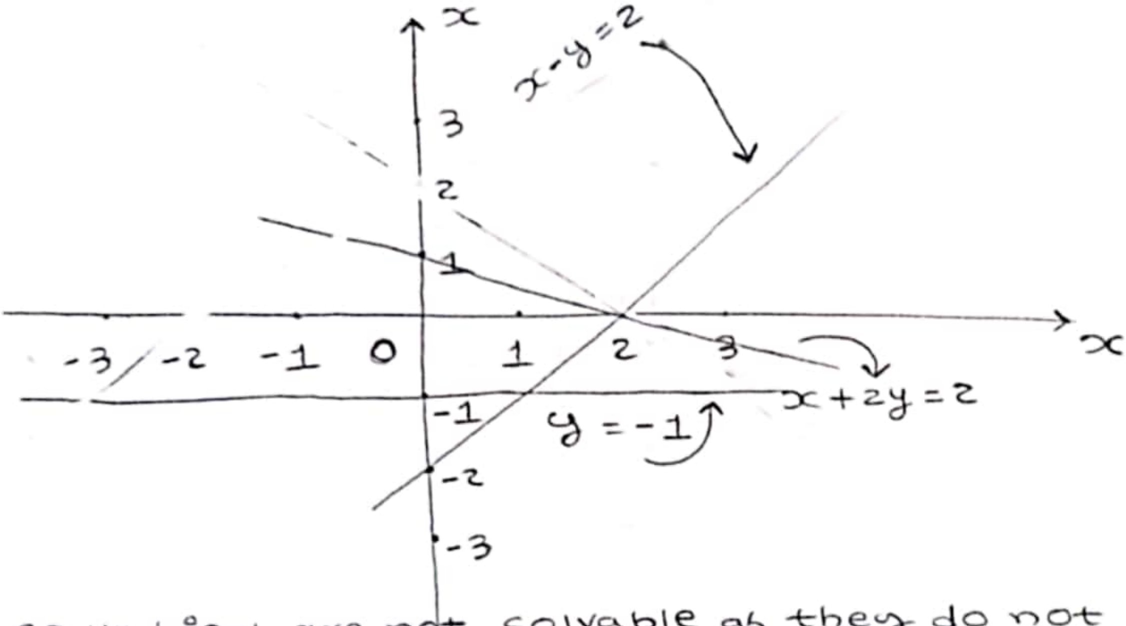
$$x\begin{bmatrix}2\\1\end{bmatrix} + y\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix}$$



⑥

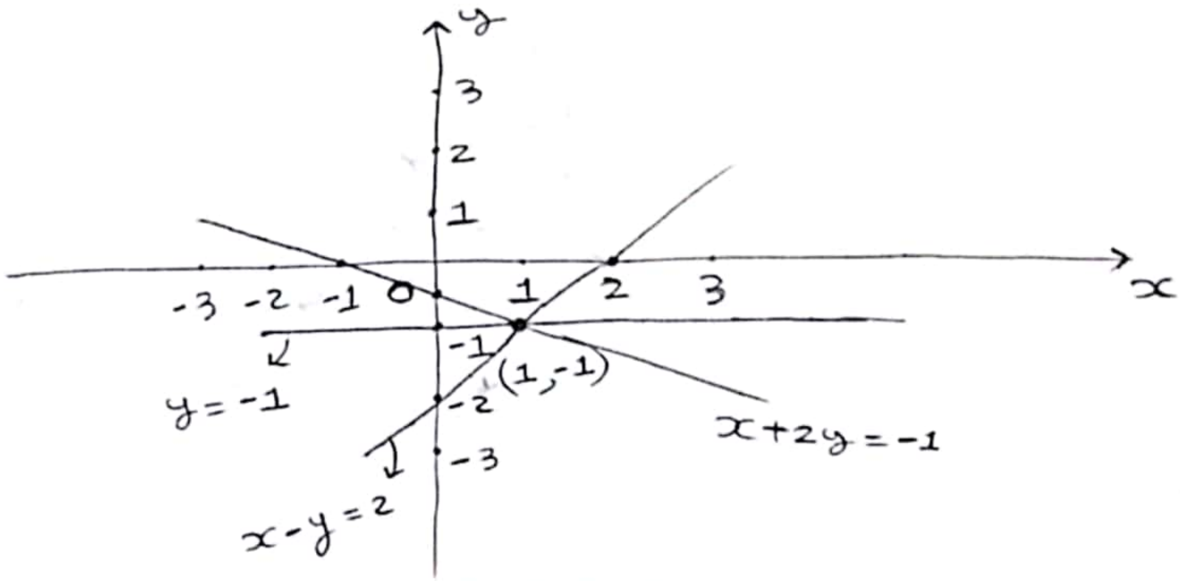
2) Sketch the three lines $x+2y=2$; $x-y=2$; $-y=1$ (row picture only) and decide if the three equations are solvable. What happens if all right hand sides are zero? Is there any non-zero choice of right hand sides that allow the three lines to intersect at a common point of intersection?

<u>Solution</u> $x+2y=2$; $x-y=2$; $y=-1$



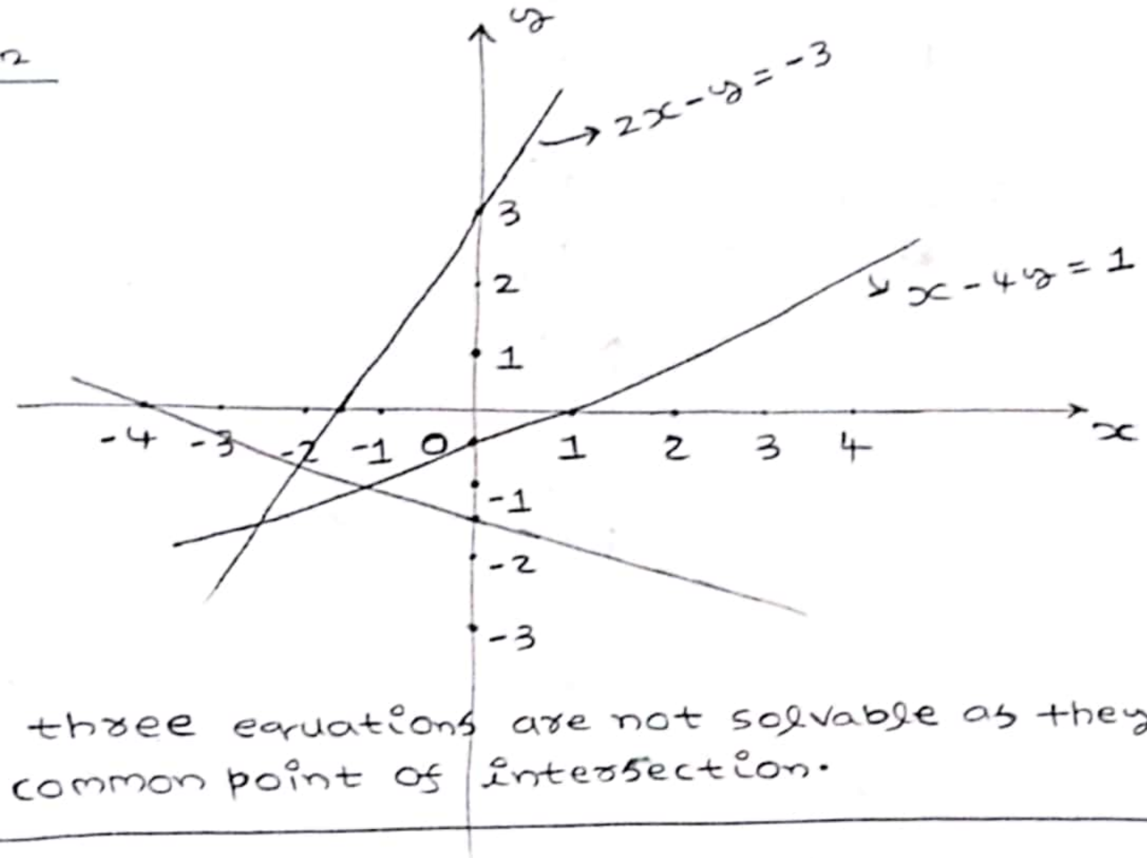
These equations are not solvable as they do not have a common point of intersection. If all RHS are zero, we get a trivial solution i.e $x=0$ & $y=0$.

If RHS of equation (1) is -1, then the three lines $x+2y=-1$; $x-y=2$ and $y=-1$ intersect at a common point of intersection $(1,-1)$ as shown below:-



3) sketch the three lines $x-4y=1$; $2x-y=-3$; $-x-3y=4$ and decide if the three equations are solvable. Do they have a common point of intersection? Explain.

solution

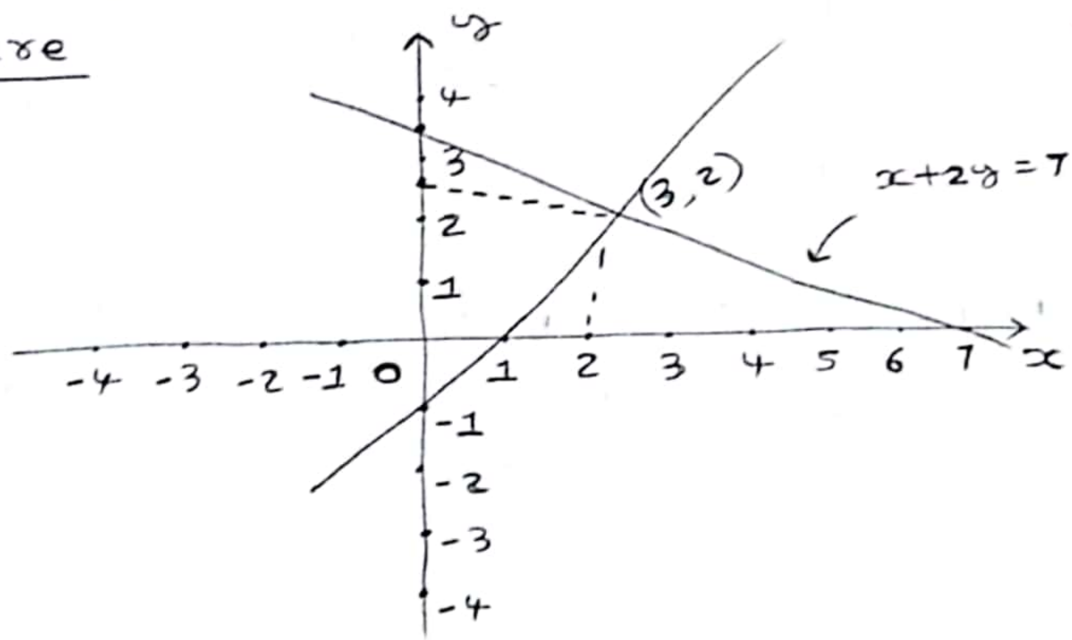


Given three equations are not solvable as they have no common point of intersection.

4) Draw the rowpicture and column picture for the following system of equations and discuss its consistency, singularity and existence of the solution :-

i) $x+2y=7$; $x-y=1$

Row picture



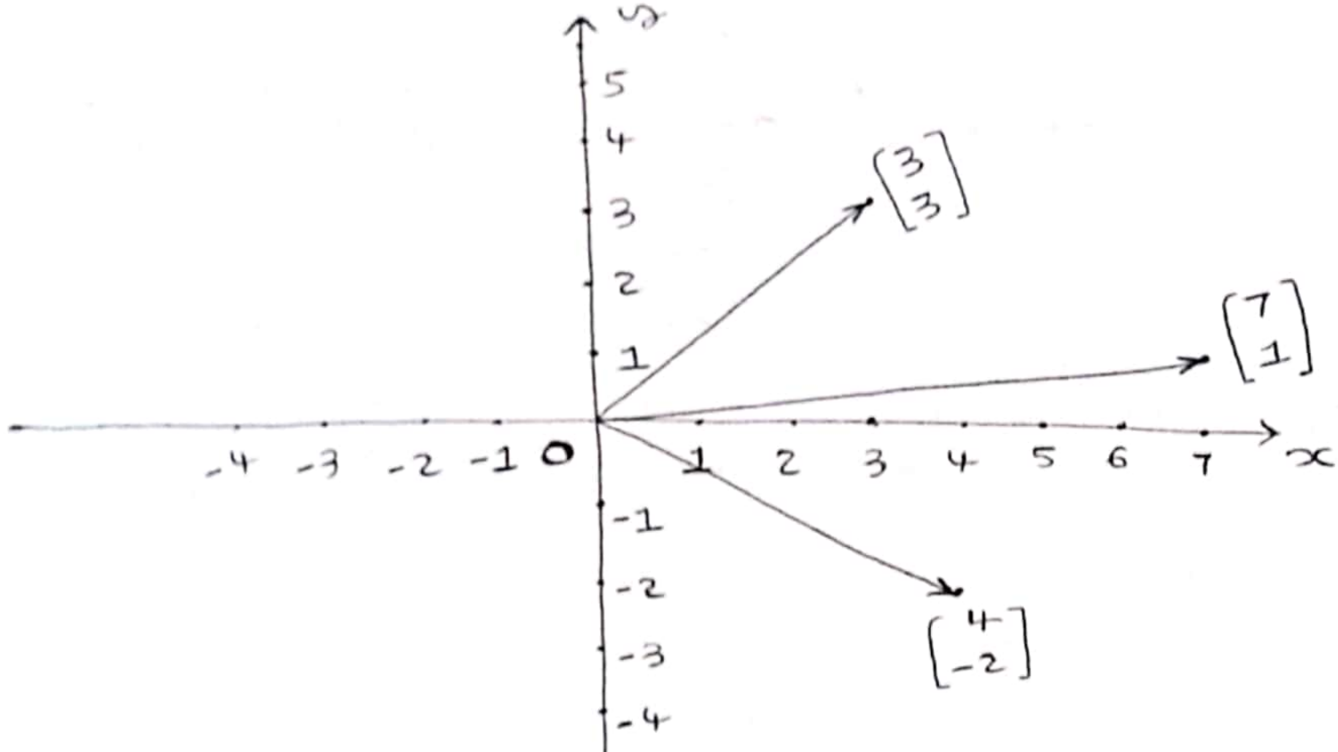
⑧

Column picture :-

$x + 2y = 7$; $x - y = 1$

$$x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}7\\1\end{bmatrix}$$

$$3\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}7\\1\end{bmatrix}$$



$$A = \begin{bmatrix}1 & 1 & -2 & 3 & : & 4\\2 & 3 & 3 & -1 & : & 3\\5 & 7 & 4 & -1 & : & 5\end{bmatrix} \xrightarrow[R_3-5R_1]{R_2-2R_1} \begin{bmatrix}1 & 1 & -2 & 3 & : & 4\\0 & 1 & 7 & -7 & : & -5\\0 & 2 & 14 & -16 & : & -15\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & -2 & 3 & : & 4\\0 & 1 & 7 & -7 & : & -5\\0 & 2 & 14 & -16 & : & -15\end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix}1 & 1 & -2 & 3 & : & 4\\0 & 1 & 7 & -7 & : & -5\\0 & 0 & 0 & -2 & : & -5\end{bmatrix}$$

$$\left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & -2 & -5 \end{array}\right] \xrightarrow{R_3/-2} \left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 1 & 5/2 \end{array}\right]$$

$$\left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 1 & 5/2 \end{array}\right] \xrightarrow[R_2+7R_3]{R_1-3R_3} \left[\begin{array}{cccc:c} 1 & 1 & -2 & 0 & -7/2 \\ 0 & 1 & 7 & 0 & 25/2 \\ 0 & 0 & 0 & 1 & 5/2 \end{array}\right]$$

$$\left[\begin{array}{cccc:c} 1 & 1 & -2 & 0 & -7/2 \\ 0 & 1 & 7 & 0 & 25/2 \\ 0 & 0 & 0 & 1 & 5/2 \end{array}\right] \xrightarrow{R_1-R_2} \underbrace{\left[\begin{array}{cccc:c} 1 & 0 & -9 & 0 & -16 \\ 0 & 1 & 7 & 0 & 25/2 \\ 0 & 0 & 0 & 1 & 5/2 \end{array}\right]}_{R}$$

Lecture 4 - Singular cases in two & three dimensions

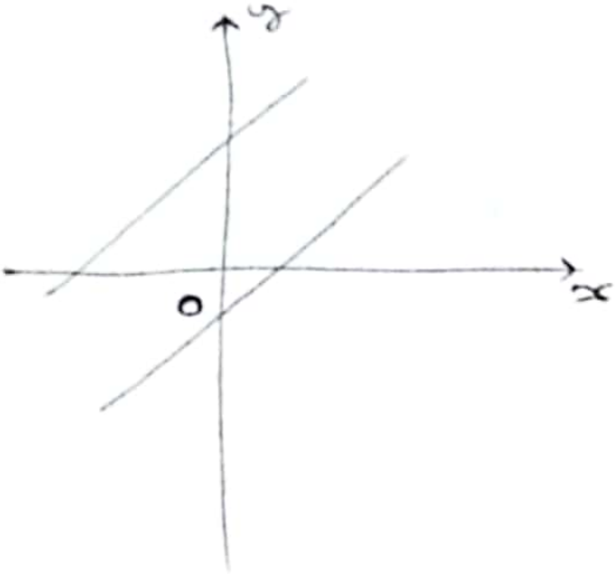
Singular cases in two dimensions

A system of linear equations is said to be singular ($|A|=0$) if it has no solution or has infinite number of solutions.

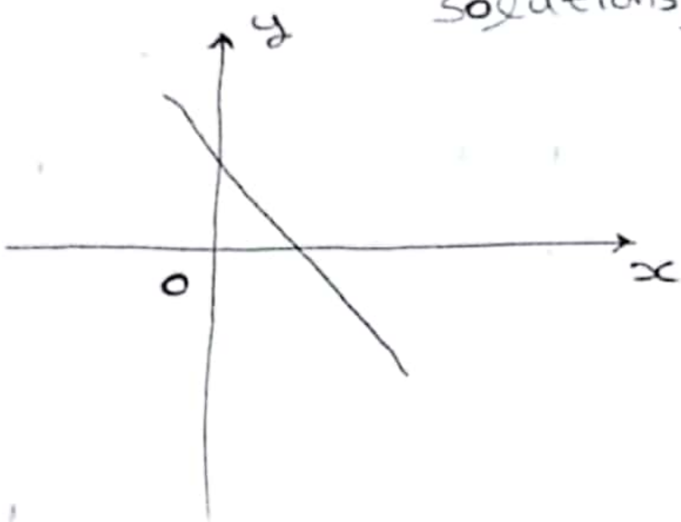
i) Row picture (Two dimensions) :-

In two dimensions, the lines are parallel if they have no solution and coincident if they have infinite number of solutions. In such a case, the matrix A will have dependent row/column and $\det\{A\}=0$. Such a matrix is called singular matrix.

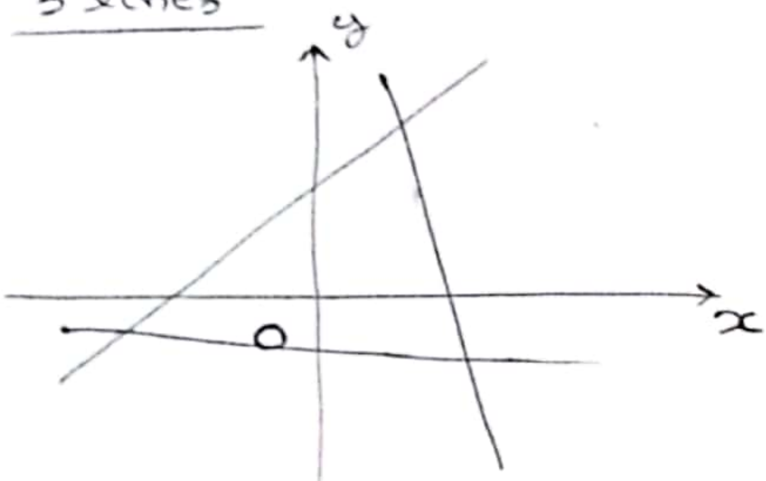
Lines Parallel [No solution]



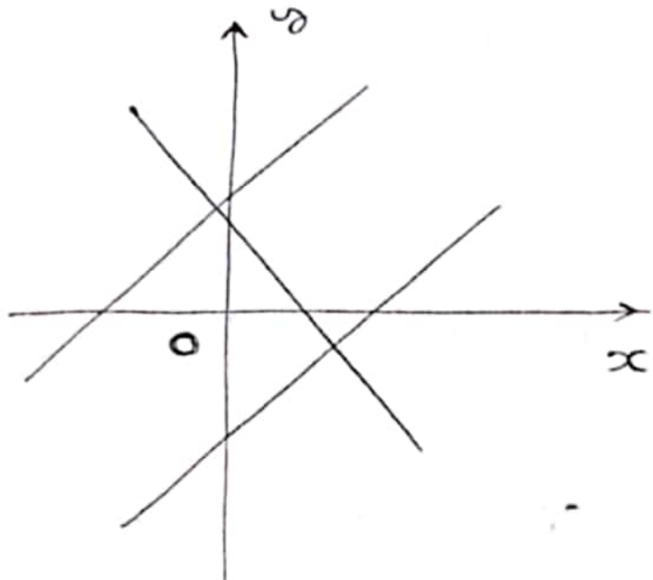
Lines coincident [infinite no. of solutions]



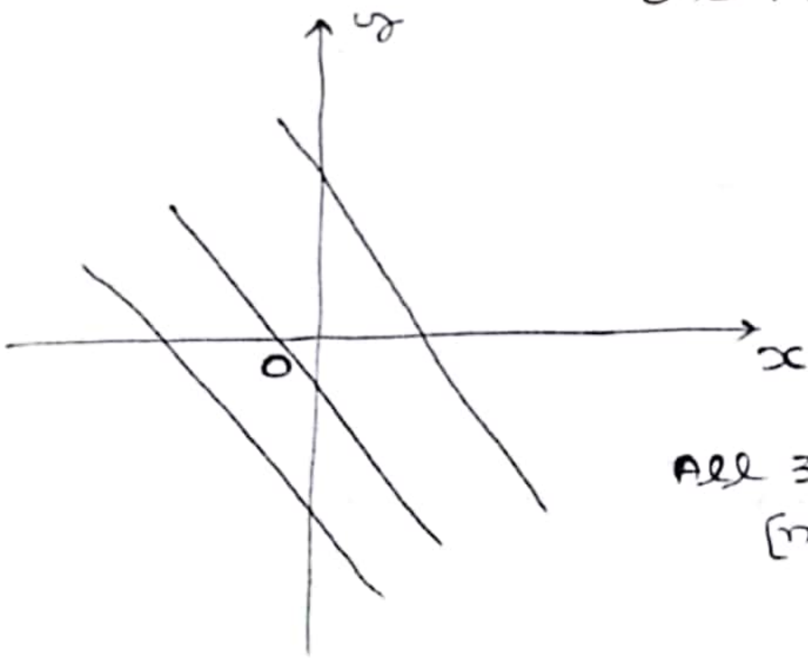
3 lines



Lines Intersecting in pairs [no solution]



Two lines parallel & one intersecting [no solution]



All 3 Parallel lines [no solution]

Row picture [Three Dimensions] :-

In three dimensions, if the 3 planes do not intersect then we have the following cases :-

Case 1 Every pair of planes intersecting in a line [no solution] :-

Case 2 2 planes intersecting in a line and 3^{rd} plane is parallel to this line [no solution].

Case 3 All three parallel planes [no solution].

Case 4 All 3 overlapping planes [infinite no. of solutions]

Case 5 All 3 planes intersecting in a line

Case 6 Two parallel planes and a third plane intersecting them [no solution].

iii) Column picture [Three dimensions] :-

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + a_{13} \cdot x_3 = b_1$$

$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3 = b_2$$

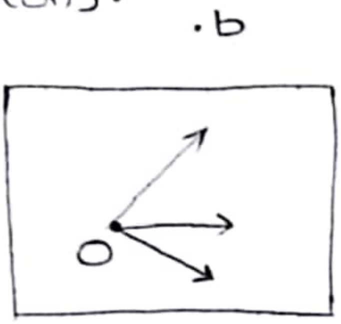
$$a_{31} \cdot x_1 + a_{32} \cdot x_2 + a_{33} \cdot x_3 = b_3$$

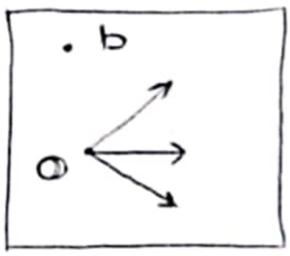
$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Each of these column vectors is a position vector with origin as a point, These column vectors lie in a plane (as they pass through the origin). Then every combination of these vectors on LHS lie in the same plane.

If the vector b is not in that plane, then solution is impossible. This system is singular &, has no solution.

If the vector b lies in the plane, (b is also coplanar) there are too many solutions. The 3 columns combine in many ways to produce b. This system is singular and has infinite no. of solutions.





Lectures 5 to 7 - Gaussian Elimination, The breakdown of Elimination

Gaussian Elimination

Rank of a Matrix

A square matrix A of order n is said to have rank r if :-

1) Atleast one minor of order r does not vanish
2) Every minor of order (r+1) vanishes.

Rank of matrix A is denoted by r i.e

$\text{rank}\{A\} = r$.

If $A = [a_{ij}]_{m\times n}$ is a rectangular matrix, then rank of the matrix is defined as the number of non-zero rows in the echelon form of A. It is also defined as the maximum number of linearly independent rows or columns of the matrix A.

Example Find the conditions on a & b so that the matrix has rank 1, 2, 3.

$$\begin{bmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 1 & 3 & 6 \end{bmatrix}$$

Solution
$$\begin{bmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 1 & 3 & 6 \end{bmatrix} \xrightarrow{R_3 - \frac{1}{a}R_1} \begin{bmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 0 & 3-\frac{1}{a} & 6-2/a \end{bmatrix}$$

i) For no values of a & b, this matrix will have rank 1.

ii) If $a = 1/3$ and $b = 4$, rank of the matrix is 2.

iii) If $a \neq 1/3$ and $b \neq 4$, rank of the matrix is 3.

Relation between Rank, consistency and solution

i) If $\text{rank}(A) = \text{rank}[A:b] = r$, system $Ax = b$ is consistent and has a solution.

ii) If $\text{rank}\{A\} = \text{rank}\{A:b\} = r = n$, system $Ax = b$ is consistent and has a unique solution.

iii) If rank{A} = rank{A:b} = r < n, system $Ax=b$ is inconsistent and infinite number of solutions.

iv) If rank{A} ≠ rank{A:b}, system $Ax=b$ is inconsistent and has no solution.

Gaussian Elimination

→ used to check for consistency & solve a system of linear equations.

→ For a given system of Equations $Ax=b$, apply elementary row transformations to the augumented matrix $[A:b]$ and reduce it to $[U:c]$ where U is an upper triangular matrix so that we get an equivalent system $Ux=c$ which can be solved by Backward substitution.

→ Here A & U are equivalent matrices and hence solution of $Ax=b$ is same as $Ux=c$.

→ The following steps are to be followed while performing elementary row operations in Gaussian Elimination.

→ No exchange of rows

→ First row should be retained as it is

→ The first non-zero element in every non zero row is called pivot.

→ The original system $\boxed{Ax=b}$ and new system obtained $\boxed{Ux = c}$ have the same solution.

Example 1 check for consistency and solve for the following system of equations if consistent

$$x_1 + x_2 - 2x_3 + 4x_4 = 5$$

$$2x_1 + 2x_2 - 3x_3 + x_4 = 3$$

$$3x_1 + 3x_2 - 4x_3 - 2x_4 = 1$$

Solution

$$\left[\begin{array}{cccc:c} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cccc:c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{array}\right]$$

$$\left[\begin{array}{cccc:c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{array}\right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{cccc:c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$r(A) = r(A:b) < n$$

$\therefore$ System is consistent & has infinitely many solutions.

$$x_1 + x_2 - 2x_3 + 4x_4 = 5$$

$$x_3 - 7x_4 = -7$$

Solution is $(x_1, x_2, x_3, x_4) = (10k_1 - k_2 - 9,\ k_2,\ 7k_1 - 7,\ k_1)$

Depending upon values of k_1 & k_2, we get infinity of solutions.

Example 2 check for consistency and solve the following system of equations if consistent ?

i) $x_1+x_2-2x_3+3x_4=4$

$2x_1+3x_2+3x_3-x_4=3$

$5x_1+7x_2+4x_3+x_4=5$

Solution :- $[A:b] = \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right]$

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right] \xrightarrow[R_3-5R_1]{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right]$$

$r(A)=2$; $r(A:b)=3$

$r(A) \neq r(A:b)$

The system is inconsistent & no solution.

Example 3 $x_1+2x_2+x_3=3$

$2x_1+5x_2-x_3=-4$

$3x_1-2x_2-x_3=5$

Solution $[A:b]$ $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{array}\right]$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{array}\right] \xrightarrow{R_3+8R_2} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{array}\right]$$

$r(A) = r(A:b)$. $\therefore$ system is consistent & has a unique solution.

$$-28\,x_3 = -84$$

$$\boxed{x_3 = 3}$$

$$x_2 - 3x_3 = -10$$

$$x_2 - 3(3) = -10$$

$$\boxed{x_2 = -1}$$

$$x_1 + 2x_2 + x_3 = 3$$

$$x_1 - 2 + 3 = 3$$

$$x_1 + 1 = 3$$

$$\boxed{x_1 = 2}$$

$$(x_1, x_2, x_3) = (2, -1, 3)$$

Example 4 $\quad 2x - 3y + 2z = 1$

$$5x - 8y + 7z = 1$$

$$y - 4z = 3$$

Solution :-

$$\left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \\ 0 & 1 & -4 & 3 \end{array}\right] \xrightarrow{R_2 - 5/2 R_2} \left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & -1/2 & 2 & -3/2 \\ 0 & 1 & -4 & 3 \end{array}\right]$$

$$-8 + \frac{15}{2} \qquad 1 - 5/2$$

$$\left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & -1/2 & 2 & -3/2 \\ 0 & 1 & -4 & 3 \end{array}\right] \xrightarrow{R_3+2R_2} \left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & -1/2 & 2 & -3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$3 - 3 \qquad r(A) = r(A:b) = 2 < n$

$\therefore$ System is consistent & has infinite no. of solutions

Example 5 Find all values of a for which the resulting linear system has a) no solution b) unique solution c) Infinitely many solutions :-

$$x+y-z=2$$
$$x+2y+z=3$$
$$x+y+(a^2-5)z=a$$

Solution $[A:b]=\begin{bmatrix}1 & 1 & -1 & : & 2\\ 1 & 2 & 1 & : & -3\\ 1 & 1 & a^2-5 & : & a\end{bmatrix}\xrightarrow[R_3-R_1]{R_2-R_1}\begin{bmatrix}1 & 1 & -1 & : & 2\\ 0 & 1 & 2 & : & -5\\ 0 & 0 & a^2-4 & : & a-2\end{bmatrix}$

a) System has no solution if $a=-2$ when $r(A) \neq r(A:b)$

b) System has a unique solution if $a \neq \pm 2$ when $r(A)=r(A:b)=3=n$

c) System has infinite no. of. solutions when $a=2$ when $r(A)=r(A:b)=2<n$.

Example 6 Find an equation relating a, b & c so that the linear system shown below is consistent for any values a, b & c that satisfy that equation. When a, b, c = (2, 3, 9), then what is the solution of the system ?

Solution $\begin{bmatrix}1 & 2 & -3 & : & a\\ 2 & 3 & 3 & : & b\\ 5 & 9 & -6 & : & c\end{bmatrix}\xrightarrow[R_3-5R_1]{R_2-2R_1}\begin{bmatrix}1 & 2 & -3 & : & a\\ 0 & -1 & 9 & : & b-2a\\ 0 & -1 & 9 & : & c-5a\end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & -3 & : & a \\ 0 & -1 & 9 & : & b-2a \\ 0 & -1 & 9 & : & c-5a \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & -3 & : & a \\ 0 & -1 & 9 & : & b-2a \\ 0 & 0 & 0 & : & c-b-3a \end{bmatrix}$$

The given linear system is consistent when

$c-b-3a=0$.

When $(a,b,c)=(2,3,9) \Rightarrow c-b-3a = 9-3-6$

$= 0$

$b-2a = 3-2(2) = -1$

$$\therefore \begin{bmatrix} 1 & 2 & -3 & : & 2 \\ 0 & -1 & 9 & : & -1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$x_1 + 2x_2 - 3x_3 = 2$$

$$-x_2 + 9x_3 = -1$$

let $\boxed{x_3 = K}$

$$-x_2 + 9K = -1$$

$$\boxed{x_2 = 9K+1}$$

$$\therefore \quad x_1 + 2(9K+1) - 3K = 2$$

$$x_1 + 18K + 2 - 3K = 2$$

$$x_1 = 2-2+3K-18K$$

$$\boxed{x_1 = -15K}$$

$$\therefore (x_1, x_2, x_3) = (-15K, 9K+1, K)$$

Breakdown of Gaussian Elimination :-

→ If a zero appears in a pivot position, elimination has to stop - either temporarily or permanently.

→ In this case, the system may or may not be singular.

→ In many cases, this problem can be cured & elimination can proceed. Such a system is non singular and has a full set of pivots.

→ In other cases, when the breakdown is unavoidable (permanent). These systems are singular and have no solution or infinitely many solutions. For such a system, a full set of pivots cannot be found.

→ CASE 1 - Non singular & curable ($|A| \neq 0$) :-

Consider the system

$$x+y+z=6$$
$$x+y+3z=10$$
$$x+2y+4z=12$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 1 & 3 & 10 \\ 1 & 2 & 4 & 12 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 1 & 3 & 6 \end{array}\right]$$

Here there is a zero in the second pivot position which can be avoided by row exchange. Thus breakdown is temporary & curable.

→ Hence, this system reduces to an upper triangular system which can be solved by back substitution and so system will become consistent and will have unique solution.

$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 0 & 2 & : & 4 \\ 0 & 1 & 3 & : & 6 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 3 & : & 6 \\ 0 & 0 & 2 & : & 4 \end{bmatrix}$$

$x + y + z = 6$

$y + 3z = 6$

$2z = 4$

$\boxed{z = 2}$

$y + 3(2) = 6$

$\boxed{y = 0}$

$x + 0 + 2 = 6$

$x = 4$

$\therefore (x, y, z) = (4, 0, 2)$

CASE 2 Singular & Incurable $[|A| = 0]$

Consider $x + y + z = 6$

$x + y + 3z = 10$

$x + y + 4z = 13$

$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 1 & 3 & : & 10 \\ 1 & 1 & 4 & : & 13 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 0 & 2 & : & 4 \\ 0 & 0 & 3 & : & 7 \end{bmatrix}$$

Here, there is zero in the second pivot position which cannot be avoided by any row exchange. Hence breakdown cannot be avoided which is incurable. The system is singular and has no solution. Here, we get $z = 2$ & $z = 7/3$

which is not possible.

CASE 3 - Singular $|A|=0$

Consider $x+y+z=6$

$x+y+3z=10$

$x+y+4z=12$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 1 & 3 & 10 \\ 1 & 1 & 4 & 12 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

Here, there is a zero in the second pivot position which cannot be avoided. But since last two equations are consistent and we get infinite number of solutions.

$x+y+z=6$ $\Rightarrow$ $\boxed{z=2}$

$2z=4$ $x+y=4$

$3z=6$ let $\boxed{y=k}$

$\boxed{x=4-k}$

$\therefore$ $x,y,z=(4-k,k,2)$

Example 7 Apply Gaussian elimination to the system of equations $u+v+w=2$; $3u+3v-w=6$; $u-v+w=-1$. When does elimination fail & at which pivot position. Is the breakdown temporary or permanent discuss. What coefficient of v in the third equation, in place of the present -1, would make it impossible to proceed and force elimination to break down?

Solution
$$\begin{bmatrix} 1 & 1 & 1 & : -2 \\ 3 & 3 & -1 & : 6 \\ 1 & -1 & 1 & : -1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 1 & 1 & : -2 \\ 0 & 0 & -4 & : 12 \\ 0 & -2 & 0 & : 1 \end{bmatrix}$$

Elimination fails at second pivot position. This breakdown is temporary which can be cured by exchanging 2^{nd} & 3^{rd} row. Then system becomes consistent and we get a unique solution.

$$\begin{bmatrix} 1 & 1 & 1 & : -2 \\ 0 & 0 & -4 & : 12 \\ 0 & -2 & 0 & : 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 & : -2 \\ 0 & -2 & 0 & : 1 \\ 0 & 0 & -4 & : 12 \end{bmatrix}$$

$-4\omega = 12$ $\quad$ $\omega = -3$

$-2v = 1$ $\quad$ $v = -1/2$

$u + v + \omega = -2$

$u - \frac{1}{2} - 3 = -2$

$u = \frac{-2}{1} + \frac{3}{1} + \frac{1}{2}$

$u = \frac{-4+6+1}{2}$

$u = 3/2$

$\therefore \quad u, v, \omega = \left(3/2, -1/2, -3\right)$

If the coefficient of v in the third equation is 1 instead of -1, then the elimination breaks down permanently and it is impossible to proceed.

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Example 8 For which three numbers a will elimination fail to give three pivots?

$$ax + 2y + 3z = b_1$$

$$ax + ay + 4z = b_2$$

$$ax + ay + az = b_3$$

$$\left[\begin{array}{ccc:c} a & 2 & 3 & b_1 \\ a & a & 4 & b_2 \\ a & a & a & b_3 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc:c} a & 2 & 3 & b_1 \\ 0 & a-2 & 1 & b_2 - b_1 \\ 0 & a-2 & a-3 & b_3 - b_1 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} a & 2 & 3 & b_1 \\ 0 & a-2 & 1 & b_2 - b_1 \\ 0 & a-2 & a-3 & b_3 - b_1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc:c} a & 2 & 3 & b_1 \\ 0 & a-2 & 1 & b_2 - b_1 \\ 0 & 0 & a-4 & b_3 - b_2 \end{array}\right]$$

$a = 0, a = 2, a = 4$ will fail to give 3 pivots

Example 9 For what values of a & b does the following system have i) unique solution ii) Infinitely many solutions iii) No solution

$$x + 2y + 3z = 2$$

$$-x - 2y + az = -2$$

$$2x + by + 6z = 5$$

Solution $$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 2 \\ -1 & -2 & a & -2 \\ 2 & b & 6 & 5 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 + R_1} \left[\begin{array}{ccc:c} 1 & 2 & 3 & 2 \\ 0 & 0 & a+3 & 0 \\ 0 & b-4 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 2 \\ 0 & 0 & a+3 & 0 \\ 0 & b-4 & 0 & 1 \end{array}\right] \xleftrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc:c} 1 & 2 & 3 & 2 \\ 0 & b-4 & 0 & 1 \\ 0 & 0 & a+3 & 0 \end{array}\right]$$

i) If $a \neq -3$ & $b \neq 4$, $r(A) = r(A:b) = 3 = n$. $\therefore$ system of equations are consistent and have unique solution.

ii) If $a = -3$ & $b \neq 4$; $r(A) = 2 = r(A:b) < n$. The system will have infinite no. of solutions

iii) If $a = -3$ & $b = 4$; $r(A) = 1$; $r(A:b) = 2$. The system of equations are inconsistent & have no solution.

Lecture 8 - Elimination Matrices

Elementary matrix E_{ij} is obtained from the Identity matrix I by using transformation $R_i - l_{ij} R_j$.

l_{ij} is multiplier. $I \rightarrow E_{ij}$

Example : E_{32} is obtained as follows :-

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = E_{32}$$

Example 10 :

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 - 1/3 R_1]{R_2 - 2/3 R_1} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 5/3 & 1/3 \end{bmatrix}$$

$$-3 - 2/3 ; -1 - \frac{4}{3} ; 2 - 1/3 ; 1 - \frac{2}{3}$$

$$E = E_{21} = R_2 - (2/3) R_1$$

(26)

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & -\frac{11}{3} & -\frac{7}{3} \\ 0 & 5/3 & 1/3 \end{bmatrix} \xrightarrow{R_3+\frac{5}{11}R_2} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -\frac{11}{3} & -\frac{7}{3} \\ 0 & 0 & -8/11 \end{bmatrix} = U$$

$$\frac{5}{3} + \frac{5}{11} \times \frac{-11}{3} = 0 \quad ; \quad \frac{1}{3} + \frac{5}{11}\left(\frac{-7}{3}\right)$$

$$\frac{1}{3} - \frac{35}{33}$$

$$\frac{11-35}{33} = \frac{-24}{33} = \frac{-8}{11}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F = E_{31} = R_3 - (1/3) R_1$$

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$$

$$G = E_{32} = R_3 + 5/11 \, R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5/11 & 1 \end{bmatrix}$$

$$\therefore \boxed{E_{32} \cdot E_{31} \cdot E_{21} \cdot A = U}$$

Example 11 Which elimination matrices put A into upper triangular matrix U ?

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_2+1/2R_1} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 + \frac{2}{3} R_2} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$2 - \frac{2}{3}$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_4 + \frac{3}{4} R_3} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} = U$$

$$2 - \frac{3}{4}$$

$\therefore$ The elimination matrices E_{21}, E_{32} and E_{43} put A into upper triangular matrix i.e

$$\boxed{E_{43} \cdot E_{32} \cdot E_{21} \cdot A = U}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A = U$$

Example 12 Which elementary matrices convert

$$A = \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix}$$ in to upper triangular matrix ?

Solution :-
$$\begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix} \xrightarrow[R_2 + 1/2 R_1]{R_3 + 1/2 R_1} \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix} \xrightarrow{R_3 - 3R_2} \begin{bmatrix} 0 & +2 & -6 & -2 & 4 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$\therefore$ Matrices E_{21}, E_{31}, E_{32} convert A into triangular form U.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

Lectures 9 & 10 - Triangular factors (LU & Cholesky method) and Row exchanges

Triangular factors

→ consider the system of Equations $Ax=b$.

→ using elementary row transformations we reduce A to U an upper triangular matrix.

→ While doing so, $Ax=b$ reduces to $Ux=c$.

→ The steps involved as :-

$$Ax=b \Rightarrow E_{32}\cdot E_{31}\cdot E_{21}\cdot A\cdot x = Ux = c \quad \therefore E_{32}\cdot E_{31}\cdot E_{21}\cdot A = U$$

→ To undo the steps of elimination, we must trace back the steps instead of subtracting we must add.

→ For this, we need inverse of elementary matrices i.e $E_{21}^{-1}, E_{31}^{-1}, E_{32}^{-1}$.

→ E_{21}^{-1} can be obtained by changing $-l$ to l in the transformation.

→ Similarly E_{31}^{-1} & E_{32}^{-1} can also be obtained.

→ But while going from U to A, the inverses should be in reverse direction i.e $E_{21}^{-1}\cdot E_{31}^{-1}\cdot E_{32}^{-1}\cdot U = A$

Triangular factorization $A=LU$:-

→ Any square matrix A can be factored as $\boxed{A=LU}$ where L is a lower triangular matrix with 1's on the diagonal. They have l_{ij}'s below the diagonal in their respective positions.

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U is a upper triangular matrix. U is having pivots on the diagonal u_{11}, u_{22}, u_{33}

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} d_1 & u_{12} & u_{13} \\ 0 & d_2 & u_{23} \\ 0 & 0 & d_3 \end{bmatrix}$$

<u>Example 13</u> Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 - 1/3 R_1]{R_2 - \frac{2}{3}R_1} \begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{-11}{3} & -7/3 \\ 0 & 5/3 & 1/3 \end{bmatrix}$

$-3 - \frac{2}{3}$; $-1 - \frac{4}{3}$; $2 - 1/3$; $1 - \frac{2}{3}$

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{-11}{3} & \frac{-7}{3} \\ 0 & 5/3 & 1/3 \end{bmatrix} \xrightarrow{R_3 + \frac{5}{11}R_2} \begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{-11}{3} & \frac{-7}{3} \\ 0 & 0 & \frac{-8}{11} \end{bmatrix} = U$$

$\frac{5}{3} + \frac{5}{11} \times -\frac{11}{3}$; $\frac{1}{3} - \frac{5}{11}\left(\frac{7}{3}\right)$

$= 1/3 - \frac{35}{33}$

$\frac{11 - 35}{33} = \frac{-24}{33} = \frac{-8}{11}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -5/11 & 1 \end{bmatrix} \quad \therefore \boxed{A = LU}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & \frac{-5}{11} & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 2 \\ 0 & -\frac{11}{3} & -7/3 \\ 0 & 0 & \frac{-8}{11} \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & \frac{-5}{11} & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{-11}{3} & \frac{-7}{3} \\ 0 & 0 & \frac{-8}{11} \end{bmatrix}}_{U} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\frac{2}{3} - \frac{11}{3} = \frac{-9}{3} = -3 \; ; \; \frac{4}{3} - \frac{7}{3} = -1$$

$$\frac{1}{3} + \frac{5}{3} = 2 \; ; \; \frac{2}{3} + \frac{35}{33} - \frac{8}{11}$$

$$\frac{22+35-24}{33} = 1$$

$$\begin{array}{r|l} 11 & 11, 33, 3 \\ \hline 3 & 1, 3, 3 \\ \hline & 1, 1, 1 \end{array}$$

Triangular factorization :-

→ $A = LU$ factorization is unsymmetric as L has 1's on the diagonal whereas U has pivots on the diagonal. In order to make this factorization symmetric we do $A = LDU$ factorization.

→ D is a diagonal matrix with pivots d_1, d_2, d_3 on the diagonal.

→ L is a lower triangular matrix with 1's on the diagonal. & having multiplier l_{ij} below the diagonal in their respective positions.

→ U is a upper triangular matrix with 1's on the diagonal obtained by dividing each row by its pivot.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \quad U = \begin{bmatrix} 1 & \frac{u_{12}}{u_{11}} & \frac{u_{13}}{u_{11}} \\ 0 & 1 & \frac{u_{23}}{u_{22}} \\ 0 & 0 & 1 \end{bmatrix}$$

Example 14 Factorise $A = LU$ and hence $A = LDU$

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 + \frac{1}{2} R_1} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 + \frac{2}{3} R_2} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_4 + \frac{3}{4} R_3} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

$$A = LDU = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

To solve a system of equations $Ax=b$, factorize $A=LU$, then $Ax=LUx$. Let $Ux=c$.

Then $Lc=b=Ax$.

Solve $Lc=b$, using forward elimination and then find x using $Ux=c$ by backward substitution which gives x.

Example 15 Solve as triangular systems without multiplying LU to find A.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

(34) Let $Lc = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$c_1 = 2$; $c_1 + c_2 = 0 \Rightarrow c_2 = -2$

$c_1 + c_3 = 2$

$2 + c_3 = 2$

$\Rightarrow \boxed{c_3 = 0}$

$\therefore c_1, c_2, c_3 = (2, -2, 0)$

then $Ux = c \Rightarrow \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$

$2x + 4y + 4z = 2$

$y + 2z = -2$

$\boxed{z = 0}$ $\boxed{y = -2}$

$\cancel{2x + 4y = 2 \times 2}$

$\cancel{x + 2y = -2 \times 4}$

$\cancel{4x + 8y =}$

$2x + 4(-2) = 2$

$2x = 10$

$\boxed{x = 5}$

$\therefore x, y, z = (5, -2, 0)$

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Example 16 Find $A = LU$ & LDU factorization:

$$A = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix} \xrightarrow[\substack{R_3+2R_1 \\ R_4+R_1}]{R_2-\frac{1}{2}R_1} \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 4 & 13 & 0 \\ 0 & -2 & -14 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 4 & 13 & 0 \\ 0 & -2 & -14 & 11 \end{bmatrix} \xrightarrow[R_4-R_2]{R_3+2R_2} \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -10 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -10 & 12 \end{bmatrix} \xrightarrow{R_4+2R_3} \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$A = LU = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix}}_{U}$$

$$A = LDU = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

Row exchanges and Permutation Matrices :-

→ Consider the system of equations $Ax=b$. While solving the system if a zero appears in the pivot position, it calls for a row exchange.

→ This row exchange is taken care by Permutation matrix P.

→ Here if $A \neq LU$, then $PA = LU$. Where P is a permutation matrix which is an identity matrix with rows in different order.

Example 17 Consider the system $y = b_1$ & $2x + 3y = b_2$. $Ax = b \Rightarrow \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Here Gaussian Elimination fails & so calls for a row exchange.

$$\Rightarrow \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

This is same as $PAX = Pb$

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$$Pb = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$$

$$\therefore \ PAx = Pb \Rightarrow \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$$

Permutation Matrices :-

→ P is a permutation matrix with rows in different order.

→ Product of two permutation matrices is also a permutation matrix.

→ Inverse of a permutation matrix is also a permutation matrix.

→ P^{-1} is always same as P^T.

→ Permutation matrices of order 2 are $2! = 2$ in number.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad P_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

→ Permutation matrices of order 3 are $3! = 6$ in number.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P_{21}^{-1}$$

$$P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = P_{31}^{-1} \; ; \; P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = P_{32}^{-1}$$

$$P_{21} \cdot P_{31} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad P_{21} \cdot P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (P_{21} \cdot P_{32})^{-1}$$

Example 18

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 0 & -1 \\ 0 & 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 0 & -1 \\ 0 & 5 & 7 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 6 & 14 & 16 \\ 0 & 0 & -1 \end{bmatrix} \neq A$$

$$P_{23} \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} = P_{23} \cdot A$$

Example 19 Explain why A is not factorizable into LU? How A can be modified so that the new matrix can be factored into LU? Also obtain the factors L, D, U for the new matrix. What is the relation between L & U thus obtained? Explain.

Solution $A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ -2 & 5 & -4 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2-2R_1} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A is not factorizable into LU as Gaussian elimination fails and row exchange is required.

So A should be multiplied with permutation matrix P_{23} so that $PA = LU$

$$P_{23} \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ -2 & 5 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2+2R_1} \underbrace{\begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \quad L = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{L}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \boxed{L = U^T}$$

Cholesky Decomposition or Factorization :-

→ It is decomposition of a Hermitian positive definite matrix which is factored into lower triangular matrix and its conjugate transpose L^T. In this L has real positive diagonal entries.

→ A Hermitian positive definite matrix A can be factored as $A = L \cdot L^T$; where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

$$A = L \cdot L^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \cdot \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Algorithm

Let $A_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a symmetric positive definite matrix

→ Then A can be factored as $L \cdot L^T$.

$$A = L \cdot L^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \cdot \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11} \cdot l_{21} & l_{11} \cdot l_{31} \\ l_{21} \cdot l_{11} & l_{21}^2 + l_{22}^2 & l_{21} \cdot l_{31} + l_{22} \cdot l_{32} \\ l_{31} \cdot l_{11} & l_{31} \cdot l_{21} + l_{32} \cdot l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

This gives :- $l_{11}^2 = a_{11} \Rightarrow l_{11} = \sqrt{a_{11}}$

$$a_{21} = l_{21} \cdot l_{11} \Rightarrow \boxed{l_{21} = \frac{a_{21}}{l_{11}}}$$

$$a_{22} = l_{21}^2 + l_{22}^2$$

$$l_{22}^2 = a_{22} - l_{21}^2$$

$$\boxed{l_{22} = \sqrt{a_{22} - l_{21}^2}}$$

$$a_{31} = l_{31} \cdot l_{11}$$

$$\boxed{l_{31} = \frac{a_{31}}{l_{11}}}$$

$$a_{33} = l_{31}^2 + l_{32}^2 + l_{33}^2$$

$$l_{33} = a_{33} - l_{31}^2 - l_{32}^2$$

$$\boxed{l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2}}$$

$$a_{32} = l_{31} \cdot l_{21} + l_{32} \cdot l_{22}$$

$$l_{32} \cdot l_{22} = a_{32} - l_{31} \cdot l_{21}$$

$$\boxed{l_{32} = \frac{a_{32} - l_{31} \cdot l_{21}}{l_{22}}}$$

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Example 20 Factorize A using cholesky Decomposition

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 22 \end{bmatrix}$$

Solution :- $l_{11} = \sqrt{a_{11}} = 2$; $l_{21} = \dfrac{a_{21}}{l_{11}} = \dfrac{2}{2} = 1$

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{2 - 1^2} = 1$$

$$l_{31} = \frac{a_{31}}{l_{11}} = \frac{6}{2} = 3 \; ; \; l_{32} = \frac{a_{32} - l_{31} \cdot l_{21}}{l_{22}}$$

$$\Rightarrow l_{32} = \frac{5 - \{3 \times 1\}}{1} = 2$$

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{22 - 3^2 - 2^2} = 3$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 3 \end{bmatrix} ; \; L^T = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\therefore A = L \cdot L^T = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Example 21 Factorize $A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$ using cholesky method

Answer : $L = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix}$