# LINEAR ALGEBRA

# UE19MA251

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## Unit 5 Singular Value Decomposition

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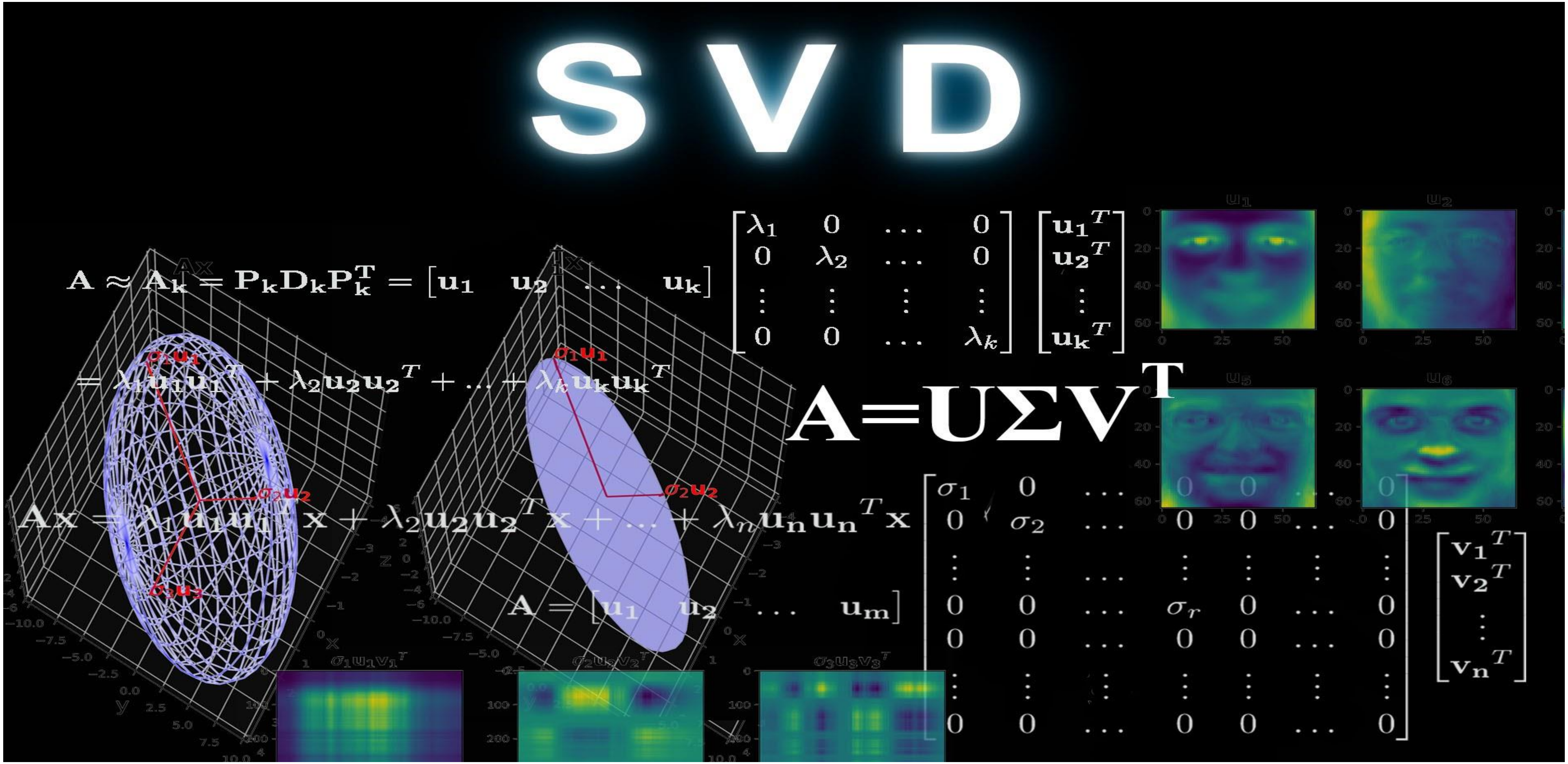


https://towardsdatascience.com/understanding-singular-value-decomposition-and-its-application-in-data-science-388a54be95d

# Importance of the method:



- Intimately related to the familiar theory of diagonalizing a symmetric matrix.
- Factorizes a matrix into 3 components.
- Has interesting algebraic properties.
- Gives further geometric and theoretical insights.
- Has many applications to data science.

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## Agenda for the chapter

- Tests for positive definiteness
- Positive Definite Matrices and Least Squares
- Semi definite Matrices
- Singular Value Decomposition
- Applications of the SVD.

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## Agenda for the class

- Quadratic form
- Examples on quadratic form
- Quadratic form – Going the other way.
- Quadratic forms for a non-symmetric matrix
- Graphs of quadratic forms
- Examples

## Recall

- ❑ If A is a real symmetric matrix of order n by n, then there exists an orthonormal matrix V whose columns are the eigenvectors of A and a diagonal matrix D , having its diagonal entries as the eigenvalues of A, such that : $A = VDV^T$

- ❑ The above process gives the eigenvalue decomposition of the matrix A.

- ❑ The singular value decomposition (SVD) is intimately related to the eigenvalue decomposition.

## Quadratic Form

Any $n \times n$ real symmetric matrix $A$ determines a quadratic form $q_A$

in $n$ variables by the formula

$$q_A(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \mathbf{x}^{\mathrm{T}} A \mathbf{x}.$$

Conversely, given a quadratic form in $n$ variables, its coefficients

can be arranged into an $n \times n$ symmetric matrix.

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## Quadratic Form

✓Note : $x^t A x$ is a scalar

$$\therefore \quad x^T A x = (x_1 \ x_2 \ \cdots \ x_m)\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

$$x_1(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m)$$

$$+ \; x_2(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m)$$

$$+ \; \vdots$$

$$+ \; x_m(a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mm}x_m)$$

$$= \sum_{i \le j}^{m} a_{ij}\, x_i\, x_j$$

## Quadratic Form – Example 1

Let $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute $\mathbf{x}^T A\mathbf{x}$ for the following matrices.

a. $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

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## Quadratic Form - Example

**Solution:**

a. $$\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix} = 4x_1^2 + 3x_2^2.$$

b. $$\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{bmatrix}$$

$$= x_1(3x_1 - 2x_2) + x_2(-2x_1 + 7x_2)$$

$$= 3x_1^2 - 2x_1x_2 - 2x_2x_1 + 7x_2^2$$

$$= 3x_1^2 - 4x_1x_2 + 7x_2^2$$

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## Quadratic Form – Example 2

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$$

We could rewrite this in the form $Q(x) = 3x_1^2 - 4x_1x_2 + 7x_2^2$.

$$= x^T Ax.$$

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## Quadratic Form – Going the other way

Question: Is any function of the form $Q(x) = ax_1^2 + bx_1x_2 + cx_2^2$ a quadratic form?

Answer: Yes. Set $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$.

Question: What about $Q(x) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$?

Answer: Set $A = \begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$.

## Quadratic Form – What if A isn't symmetric?

If $A$ isn't symmetric, the function $Q(x) = x^T Ax$ is still a quadratic form:

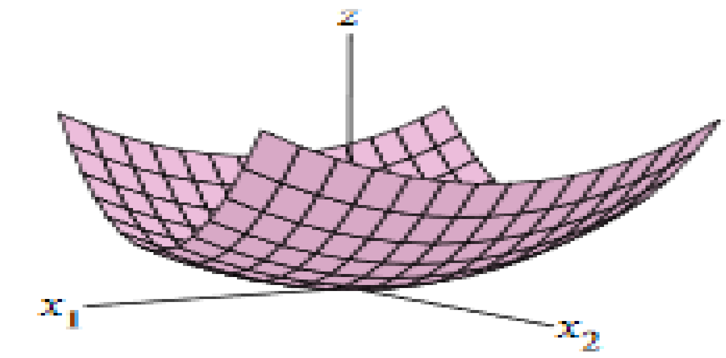
Define $\hat{A} = \frac{A+A^T}{2}$ then

$$\begin{aligned} x^T \hat{A} x &= x^T \left( \frac{A + A^T}{2} \right) x \\ &= \frac{1}{2} \left( x^T Ax + x^T A^T x \right) \\ &= \frac{1}{2} \left( x^T Ax + x^T Ax \right) = x^T Ax. \end{aligned}$$

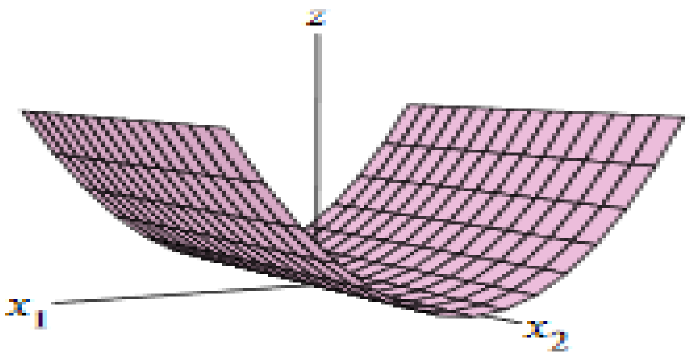
Because of this, it is safe to assume that $A$ is symmetric when we examine quadratic forms.

# Graphs of Quadratic forms

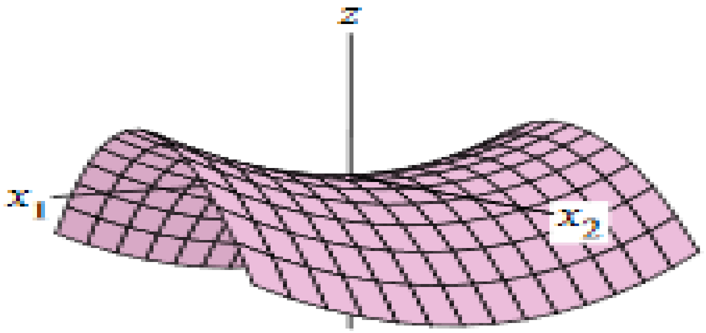




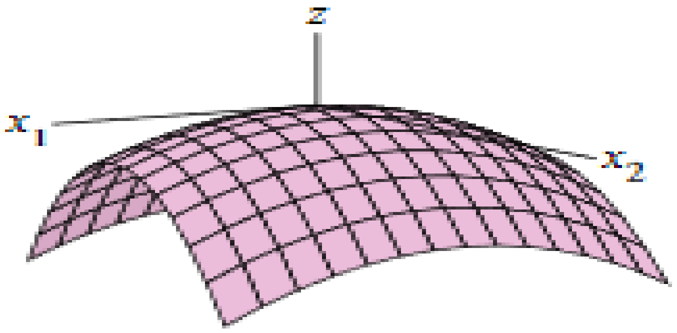

(a) $z = 3x_1^2 + 7x_2^2$




(b) $z = 3x_1^2$




(c) $z = 3x_1^2 - 7x_2^2$




(d) $z = -3x_1^2 - 7x_2^2$

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Graphically, the graph $z = Q(x)$ is

- convex up if $Q$ is positive definite,
- concave down if $Q$ is negative definite,
- A “saddle” if $Q$ is indefinite.

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## EXAMPLE

The quadratic form of $f(x,y) = ax^2 + 2abx + by^2$

may be represented as $X^T A X$ where

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \quad ; \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

# THANK YOU

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