LINEAR ALGEBRA AND ITS APPLICATIONS

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CLASS-4

A=QR FACTORIZATION

The factorization A=QR

- Let A be a matrix whose columns are a, b, c
- Let Q be the matrix whose columns are q_1, q_2 and q_3 which are determined using Gram − Schmidt process.
- Then to find the third matrix which connects A and Q, express a, b, c as a linear combination of q_1, q_2, q_3.

The whole factorization is

$$A = \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{bmatrix}$$

$$A = QR$$

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Note: Consider the normal equation for $Ax=b$,

$$A^TA\hat{x} = A^Tb \quad \text{But } A = QR$$

$$(QR)^T QR\hat{x} = (QR)^Tb$$

$$R^TQ^TQR\hat{x} = R^TQ^Tb$$

$$R^TR\hat{x} = R^TQ^Tb$$

$$\therefore \boxed{R\hat{x} = Q^Tb}$$

$\Rightarrow$ When $Ax=b$ is not solvable, we consider $R\hat{x} = Q^Tb$ and solve.

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⇒ If A is a square matrix of order 'n' then Q & R are also square matrix of order 'n'. But if A is a matrix of order m×n then Q is also a matrix of order m×n but R is a square matrix of order 'n'.

$$\begin{bmatrix} \uparrow & \uparrow \\ a & b \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow \\ q_1 & q_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b \\ 0 & q_2^T b \end{bmatrix}$$

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Problem 1:- Factor $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix}$ into QR,

recognizing that first column is already a unit vector.

Ans: $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix}$, $Q = [q_1 \; q_2]$

$\downarrow a \qquad \downarrow b$

$$q_1 = \frac{a}{\|a\|} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad e_2 = b - (q_1^T b)\, q_1$$

$$e_2 = \begin{pmatrix} \sin\theta(1 - \cos^2\theta) \\ -\cos\theta \sin^2\theta \end{pmatrix}$$

$$q_2 = \frac{c_2}{\|c_2\|} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

$$R = \begin{pmatrix} q_1^T a & q_1^T b \\ 0 & q_2^T b \end{pmatrix} = \begin{pmatrix} 1 & \cos\theta\sin\theta \\ 0 & \sin^2\theta \end{pmatrix}$$

$$A = QR \Rightarrow \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} 1 & \cos\theta\sin\theta \\ 0 & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \cos^2\theta\sin\theta + \sin^3\theta \\ \sin\theta & \cos\theta\sin^2\theta - \cos\theta\sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix}$$

Problem: Find an orthonormal set q_1, q_2, q_3 for which q_1, q_2 span the column space of $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$. Which fundamental subspace contains q_3. What is the least squares solution of $Ax = b$ if $b = [1 \; 2 \; 7]^T$?

Ans: $a = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \Rightarrow q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{9}} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$q_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$, $\quad q_2 = \frac{e_2}{\|e_2\|}$ where $e_2 = b - (q_1^T b) q_1$

$e_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$\therefore q_2 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$

Let $e_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. To find q_3, let the vector e_3 be assumed as orthogonal to the plane spanned by q_1 & q_2 such that $q_1^T e_3 = 0$ & $q_2^T e_3 = 0$.

$$\Rightarrow \begin{bmatrix} 1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0, \quad \begin{bmatrix} 2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow x + 2y - 2z = 0, \quad 2x + y + 2z = 0$$

On solving, $e_3 = (-2, 2, 1)$

$$\therefore q_3 = \frac{e_3}{\|e_3\|} = \left(-2/3, 2/3, 1/3\right).$$

Consider $Ax = b \Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

The above system is inconsistent. i.e $Ax = b$ is not solvable & hence we consider $R\hat{x} = Q^T b$ where

$$R = \begin{bmatrix} q_1^T a & q_1^T b \\ 0 & q_2^T b \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}$$

$$\therefore R\hat{x} = Q^T b \Rightarrow \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} \hat{x} = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} \hat{x} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} \Rightarrow \hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

THANK YOU

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