LINEAR ALGEBRA AND ITS APPLICATIONS
UE19MA251

Cosines And projections Onto Lines

Definition :

If $a = (a_1, a_2, \ldots a_n)$, $b = (b_1, b_2, \ldots, b_n)$ include an angle θ between them the ***cosine formula*** states that

$$\cos\theta = \frac{a^T b}{\|a\| \, \|b\|}$$

Projections Onto A Line

To find the projection of b onto the line through a given vector ‘ a ‘, we find the point p on the line that is closest to b.

This point must be some multiple of ‘ a ‘ say $p = \hat{x}\, a$.

Now, the line from b to the closest point p is perpendicular to the vector a and hence

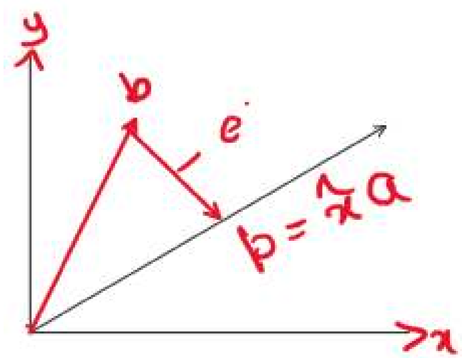
$e = b - p$ since $a \perp e$

$\Rightarrow a^T e = 0 \Rightarrow a^T (b - p) = 0$

$\Rightarrow a^T b - a^T (\hat{x} a) = 0$

$$\Rightarrow \hat{x} = \frac{a^T b}{a^T a}$$

$\therefore \; p = \hat{x} a$



Schwarz Inequality

All vectors a and b in R^n satisfy the ***Schwarz Inequality*** which is

$$\left| a^T b \right| \leq \|a\| \, \|b\|$$

Note :

1. Equality holds if and only *if a and b are dependent vectors. The angle is* $\theta = 0^o$ or *180^o*. In this case, b is identical with its projection p and the distance between b and p is zero.

2. Schwarz inequality is also stated as *$|\cos\theta| \leq 1$*

Projection Matrix of Rank 1

Projections onto a line through a given vector ' a ' is carried out by a ***Projection Matrix*** given by

$$P = \frac{a\, a^T}{a^T a}$$

This matrix multiplies b and produces p.

That is,

$$Pb = \frac{a\, a^T}{a^T a} b = a \frac{a^T b}{a^T a} = a\, \hat{x} = p$$

Projection Matrix of Rank 1

Note :

1. P is a symmetric matrix.
2. $P^n = P$ for n = 1, 2, 3, ……
3. The rank of P is one.
4. The trace of P is one.
5. If ' a' is a n- dimensional vector then P is a square matrix of order n.
6. If 'a ' is a unit vector then $P = a\,a^T$.

Problems

What multiple of $a = (1,1,1)$ is closest to $b = (2,4,4)$? Find also the point on the line throug 'b' that is closest to a?

Solution: Let 'p' be the point on the line through $a = (1,1,1)$ is closest to $b = (2,4,4)$.

$$\therefore\ p = \hat{x} a = \frac{a^T b}{a^T a} \cdot a = \frac{10}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let p_1 be the point on the line through 'b' is closest to a

$$\text{So } p_1 = \hat{x}_1 b = \frac{a^T b}{b^T b} \cdot b = \frac{10}{36} \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$$

Problems

Find the matrix that projects every point in $\mathbb{R}^3$ onto the line of intersection of the planes $x+y+z=0$ and $x-z=0$. What are the column space and row space of this matrix?

Solution:- The line of intersection of these planes is

i.e. $x+y+z=0$, $x-z=0$ $\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \underset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} = U$$

$x = -y - z$ and $y = -2z$

Problems

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$\Rightarrow$ The line passing through $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is the line of intersection.

Let $a = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and projection matrix through 'a' is $P = \dfrac{aa^T}{a^Ta} = \dfrac{1}{6} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

P is a symmetric matrix of Rank 1.

Therefore column space and row space are $c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ where $c_1 \in R$.

THANK YOU