Lecture 11 - Inverse by Gauss Jordan Method,

Transposes :-

Inverses & Transposes :-

→ Let A be a square matrix of order n, the inverse of A is the matrix B such that $AB = I = BA$.

→ Here $B = A^{-1}$.

→ Inverse of a matrix is unique. i.e $AB = BA = I$ & $AC = CA = I$, then $B = C$

→ Inverse of the product is the product of inverses.

$$(ABCD)^{-1} = D^{-1} \cdot C^{-1} \cdot B^{-1} \cdot A^{-1}.$$

→ If $A = LU$, then $A^{-1} = U^{-1} \cdot L^{-1}$

→ Since $E_{32} \cdot E_{31} \cdot E_{21} \cdot A = U$, we have

$$E_{21}^{-1} \cdot E_{31}^{-1} \cdot E_{32}^{-1} \cdot U = A$$

$$A^{-1} = U^{-1} \cdot E_{32} \cdot E_{31} \cdot E_{21}$$

$$\Rightarrow L = E_{21}^{-1} \cdot E_{31}^{-1} \cdot E_{32}^{-1}$$

→ A matrix A is invertible if and only if elimination produces n pivots with or without row exchanges. Elimination solves $Ax = b$ without explicitly finding A^{-1}.

→ If A is invertible, the only & one solution to $Ax = b$ is $x = A^{-1} \cdot b$.

Gauss Jordan method of computing A^{-1} :-

→ The inverse of an invertible matrix is obtained by a set of row operations that transforms A to I & I to A^{-1}. This process is known as Gauss Jordan method.

→ Consider the augumented matrix $[A : I]$. Then perform row operations on it so that A reduces to echelon form U and at the same time I reduces to C.

→ Further reduce U to I using elementary row transformations which reduces C to A^{-1}.

→ $[A : I] \longrightarrow [U : C] \rightarrow [I : A^{-1}]$

Example 22 : Compute A^{-1} using Gauss Jordan method given $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -2 & 2 & 2 \end{bmatrix}$

Solution

$$[A : I] = \left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & 4 & 0 & 1 & 0 \\ -2 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-2R_1} \left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 3 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_3-3R_2} \left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right] \xrightarrow[R_2+\frac{2}{3}R_3]{R_1+\frac{1}{3}R_3} \left[\begin{array}{ccc:ccc} 2 & 1 & 0 & 10/3 & -1 & 1/3 \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right]$$

$-2+\frac{14}{3}$;

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 0 & \frac{10}{3} & -1 & \frac{1}{3} \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{ccc:ccc} 2 & 0 & 0 & 2/3 & 0 & -1/3 \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc:ccc} 2 & 0 & 0 & 2/3 & 0 & -1/3 \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right] \xrightarrow[R_3=\frac{-1}{3}R_3]{R_1=1/2 R_1} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/3 & 0 & -1/6 \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & 1 & -7/3 & 1 & -1/3 \end{array}\right]$$

$[I : B]$

$$\therefore B = A^{-1} = \begin{bmatrix} 1/3 & 0 & -1/6 \\ 8/3 & -1 & 2/3 \\ -7/3 & 1 & -1/3 \end{bmatrix}$$

Transpose of a matrix A^T :-

→ If $A = [a_{ij}]_{m\times n}$ is an $m\times n$ matrix, then its transpose is obtained by interchanging its rows & columns and is denoted by $A^T = [a_{ji}]_{n\times m}$

→ If $A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 2 & 0 \end{bmatrix}$ then $A^T = \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ -3 & 0 \end{bmatrix}$

Properties

→ The transpose of a lower triangular matrix is an upper triangular matrix.

→ $(A^T)^T = A$; $(AB)^T = B^T \cdot A^T$; $(A^{-1})^T = (A^T)^{-1}$

→ $(A \pm B)^T = A^T \pm B^T$; $(A^{-1})^T \cdot A^T = (A \cdot A^{-1})^T = I$

Symmetric matrices :-

→ If A is a matrix of order n, then it is said to be symmetric matrix of order n if $A^T = A$.

→ If $A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \Rightarrow \boxed{A^T = A}$

Properties :-

→ If A is symmetric then A^{-1} may or may not exist.

→ If for a symmetric matrix A^{-1} exists, then A^{-1} is also symmetric.

→ For a symmetric matrix A, we have $(A^{-1})^T = A^{-1}$ [$\because A^T = A$].

Symmetric products AA^T, A^TA, LDL^T :-

→ If A is a matrix of order $m \times n$, then AA^T and A^TA are both symmetric.

Example[23] :- If $A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 2 & 0 \end{bmatrix}$ $AA^T = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ -3 & 0 \end{bmatrix}$

$= \begin{bmatrix} 14 & 10 \\ 10 & 20 \end{bmatrix}$

$$A^T = \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ -3 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 2 & 0 \end{bmatrix} \Rightarrow A^T \cdot A = \begin{bmatrix} 20 & 10 & -6 \\ 10 & 5 & -3 \\ -6 & -3 & 9 \end{bmatrix}$$

If A is symmetric and if $A = LDU$ then,

$$A = A^T = LDL^T \quad [\because U = L^T \ \& \ L = U^T]$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix} \xrightarrow[R_3 + R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 7 \end{bmatrix} = U$$

$$LDU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 24

→ From which three numbers "c" is this matrix not invertible and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix} \xrightarrow[R_3 - 4R_1]{R_2 - c/2 R_1} \begin{bmatrix} 2 & c & c \\ 0 & c - \frac{c^2}{2} & c - \frac{c^2}{2} \\ 0 & 7 - 4c & -3c \end{bmatrix}$$

$$7 - 4c - x\left(c - \frac{c^2}{2}\right) = 0$$

$$\Rightarrow x\left(c - \frac{c^2}{2}\right) = 7 - 4c \Rightarrow x = \left[\frac{7 - 4c}{2c - c^2}\right] 2R_2$$

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$$\begin{bmatrix} 2 & c & c \\ 0 & c-\frac{c^2}{2} & c-\frac{c^2}{2} \\ 0 & 7-4c & -3c \end{bmatrix} \xrightarrow{R_3 - \left[\frac{7-4c}{2c-c^2}\right] 2R_2} \begin{bmatrix} 2 & c & c \\ 0 & c-\frac{c^2}{2} & c-\frac{c^2}{2} \\ 0 & 0 & c-7 \end{bmatrix}$$

$$-3c - \left[\frac{7-4c}{2\cancel{c}-c^2}\right]\cancel{2} \qquad \left[\frac{2\cancel{c}-c^2}{\cancel{2}}\right]$$

$$-3c-7+4c$$
$$= c-7$$

→ Matrix A is not invertible for $c = 0, 2, 7$

→ For $c = 0, 2, 7$ elimination gives one zero row. Hence A will be singular and so A will not be invertible.

<u>Example 25</u> Find $PA = LU$ and $PA = LDU$ for A given by $A = \begin{bmatrix} 3 & -1 & 0 \\ 6 & -2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$