VECTOR SPACES

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CLASS 5 : CONTENT

- Problems on Linear independence and dependence of vectors
- Problems on Basis and Dimension

LINEAR DEPENDENCE AND INDEPENDENCE

Problem 1: Decide the dependence or independence of the following vectors:

(1) vectors (1, 3, 2), (2, 1, 3), and (3, 2, 1)

Solution: Write the given set of vectors as columns of matrix A.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{pmatrix} \xrightarrow{R_3 - \frac{1}{5}R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -\frac{18}{5} \end{pmatrix}$$

$\because$ All the columns of matrix A is with pivots

$\rho(A) = n = 3$ Given vectors are linearly independent.

LINEAR DEPENDENCE AND INDEPENDENCE

2) vectors $(1,-3,2)$, $(2,1,-3)$ and $(-3,2,1)$

$$A = \begin{pmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 + 3R_1} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ $\exists$ columns without pivots also in A.

Given vectors are Linearly Dependent.

3) vectors $(1,1)$, $(2,3)$, $(1,2)$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

vectors are linearly Dependent

LINEAR DEPENDENCE AND INDEPENDENCE

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Note: A set of 'n' vectors of R^m must be Linearly dependent if $n > m$.

4) vectors $(1\ 1\ 0\ 0)^T, (1\ 0\ 1\ 0)^T, (0\ 0\ 1\ 1)^T, (0\ 1\ 0\ 1)^T$

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_4 - R_3} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\because$ All columns of A are with pivots

Given set of vectors are Linearly Independent.

LINEAR DEPENDENCE AND INDEPENDENCE

Problem 2: If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$, $v_3 = w_1 - w_2$ are dependent.

Solution: $\because$ w_1, w_2, w_3 are independent

$$\alpha w_1 + \beta w_2 + \gamma w_3 = 0 \Rightarrow \alpha = \beta = \gamma = 0 \text{ only}$$

To check if v_1, v_2 and v_3 are independent check for the combination $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

LINEAR DEPENDENCE AND INDEPENDENCE

$\therefore \quad c_1(w_2 - w_3) + c_2(w_1 - w_3) + c_3(w_1 - w_2) = 0$

$\Rightarrow \quad c_1 w_2 - c_1 w_3 + c_2 w_1 - c_2 w_3 + c_3 w_1 - c_3 w_2 = 0$

$\Rightarrow \quad w_1(c_2 + c_3) + w_2(c_1 - c_3) + w_3(c_3 - c_1) = 0$

$\Rightarrow \quad \alpha w_1 + \beta w_2 + \gamma w_3 = 0$

$\therefore$ w_1, w_2, w_3 are linearly independent

$\alpha = \beta = \gamma = 0$ only

Hence dependent.

$c_2 + c_3 = 0 \Rightarrow c_2 = -c_3$

$c_1 - c_3 = 0 \Rightarrow c_1 = c_3$

$c_3 - c_1 = 0 \Rightarrow c_1 = c_3$

$\Rightarrow c_1 = c_3 = -c_2$

c_i's can assume values other than zero also.

LINEAR DEPENDENCE AND INDEPENDENCE

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Problem 3: Find the Basis and hence find the Dimension of Subspaces of R^4

1) All vectors whose components are equal.

∴ vectors are in R^4 it should contain 4 components.

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in R^4 \sim \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix} = \left\{ x\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix};\ x \in R \right\} \text{ Subspace}$$

Basis $(1\ 1\ 1\ 1)^T$; Dimension $= 1$

BASIS

2) All vectors whose components add up to zero

$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4$ s.t. $x+y+z+t=0$

$\Rightarrow x = -y-z-t$

$$\begin{pmatrix} -y-z-t \\ y \\ z \\ t \end{pmatrix} \sim \left\{ y\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} ; y, z, t \in \mathbb{R} \right\}$$

Subspace of $\mathbb{R}^4$

Basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ Dimension $= 3$

BASIS

Problem 4 :

Let 'V' be a subspace of four dimensional space R^4

Sot $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in R^4 ; \; x_1 - x_2 + x_3 - x_4 = 0 \right\}$

Find the Basis and Dimension of V.

Solution :- $\because x_1 - x_2 + x_3 - x_4 = 0 \Rightarrow x_1 = x_2 - x_3 + x_4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \sim \begin{pmatrix} x_2 - x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \sim x_2 \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{V_1} + x_3 \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{V_2} + x_4 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{V_3}$$

Any vector in subspace V of R^4 can be obtained as a Linear combination of Vectors (V_1, V_2, V_3).

BASIS

Problem5: Find a Basis for each of the following subspaces of 2 by matrices.

1) All diagonal matrices

2) All symmetric matrices ($A^T = A$)

3) All skew symmetric matrices ($A^T = -A$)

Solution: Any 2 by 2 matrix is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $a, b, c, d \in \mathbb{R}$

Basis of 2 by2 matrix is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$\therefore \exists$ 4 vectors in Basis set

Dimension of 2 by 2 matrices subspace is 4

BASIS

1) All diagonal matrices (2 by 2)

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \;\middle|\; a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ;\ a, b \in R \right\}$$

Basis is $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ Dimension = 2

2) All symmetric matrices $(A^T = A)$ (2 by 2)

$$\left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \;\middle|\; a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ;\ a, b, c \in R \right\}$$

Basis is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ Dimension = 3

BASIS

3) All Skew symmetric matrices $(A^T = -A)$ (2 by 2)

$$\left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \middle/ a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; a \in \mathbb{R} \right\}$$

Basis $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ Dimension $= 1$

Problem 6 : Find a basis for subspace of polynomials $p(x)$ of degree ≤ 3.

Solution : $\{ (1, t, t^2, t^3) \}$ Basis, Dimension $= 4$

SPAN OF VECTORS

Problem 7: Describe the subspace of R^3 spanned by:

1) the two vectors $(1,1,-1)$ and $(-1,-1,1)$

Solution: step 1: check the independence of given vectors before drawing conclusions w.r.t. subspace.

$$\therefore \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\because$ Only 1 column with pivot, these vectors are dependent

Hence, the span of the given vectors is a Line.

2) $(0,1,1)$, $(1,1,0)$ and $(0,0,0)$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \sim R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{R_3-R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{R_3+R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

SPAN OF VECTORS

$\therefore$ Only 2 columns with pivots,
the given set of three vectors span a
2 dimensional plane in R^3.

3) The columns of a 3 by 5 Echelon form matrix
with two pivots
2 Dimensional plane in R^3

4) all vectors with positive components.

$$\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \sim a, b, c \in R \right\} \sim \left\{ a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ; a, b, c \in R \right\}$$

Whole of R^3 i.e. 3 Dimensional space.

THANK YOU

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