LINEAR ALGEBRA

UE19MA251

APARNA B S
Department of Science and Humanities

LINEAR ALGEBRA
UE19MA251

Tests for Positive definiteness

Aparna B S
Department of Science and Humanities

LINEAR ALGEBRA – UE19MA251

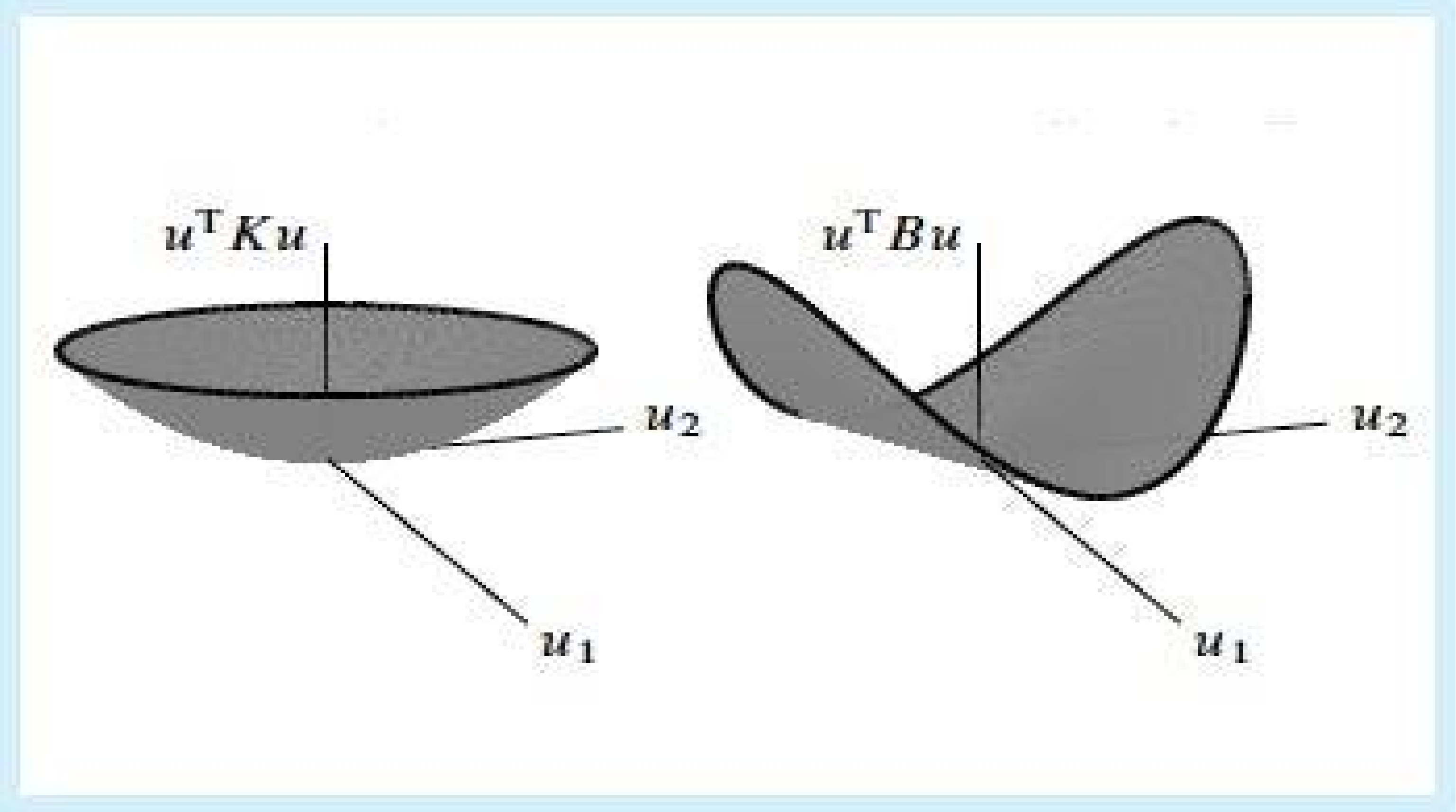
Agenda

- Quadratic form - Classification
- Positive Definite Matrix – A+B, $A^T A$
- Positivity of Eigenvalues
- Equivalent statements for positive definiteness
- Examples

LINEAR ALGEBRA – UE19MA251

Quadratic Form: Classification

Positive definite and semidefinite: graphs of x'Ax.



https://ocw.mit.edu/courses/mathematics/18-06sc-linear-algebra-fall-2011/positive-definite-matrices-and-applications/

LINEAR ALGEBRA – UE19MA251

Quadratic Form: Classification

Based on the sign of the quadratic form, SVD can be classified into 3 catagories:

Positive definite if (Quadratic form) > 0

Positive semi definite if (Quadratic form) ≥ 0

Negative definite if (Quadratic form) < 0

A^TA is Positive Definite

A^TA is symmetric and square.

Quadratic form: $x^TA^TAx = (Ax)^T(Ax) = \|Ax\|^2 > 0$

$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$; $A^TA = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix}$

Positive Definiteness Of A+B

If A and B are positive definite, then so is A+B.

$$x^T(A+B)x = x^TAx + x^TBx > 0$$

$\because \; x^TAx > 0$ and $x^TBx > 0$.

POSITIVITY OF EIGENVALUES

Every eigenvalue of a positive definite matrix is positive.

Proof. Suppose $\boldsymbol{A}$ is a positive definite matrix. Let λ be an eigenvalue of $\boldsymbol{A}$, and $\boldsymbol{s}$ be an eigenvector of $\boldsymbol{A}$ corresponding to λ. We have

$$\boldsymbol{A}\boldsymbol{s} = \lambda \boldsymbol{s}$$

It follows that

$$\boldsymbol{s}^T\boldsymbol{A}\boldsymbol{s} = \lambda(\boldsymbol{s}^T\boldsymbol{s})$$

Hence

$$\lambda = \frac{\boldsymbol{s}^T\boldsymbol{A}\boldsymbol{s}}{\boldsymbol{s}^T\boldsymbol{s}} > 0$$

A matrix is positive definite if every eigenvalue of the matrix is positive.

Proof. Suppose every eigenvalue of $\boldsymbol{A}$ is positive. By spectral theorem, $\boldsymbol{A}$ has an eigenvalue decomposition $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$. It follows that

$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \overbrace{\boldsymbol{x}^T\boldsymbol{Q}}^{\boldsymbol{y}^T}\boldsymbol{\Lambda}\overbrace{\boldsymbol{Q}^T\boldsymbol{x}}^{\boldsymbol{y}} = \boldsymbol{y}^T\boldsymbol{\Lambda}\boldsymbol{y} = \sum_i \lambda_i y_i^2$$

Hence, the quadratic form $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ is positive for any $\boldsymbol{x} \neq 0$, and $\boldsymbol{A}$ is positive definite.

LINEAR ALGEBRA – UE19MA251

Equivalent statements for Positive definiteness

There are many ways to say a matrix is positive definite.

1. A is positive definite.
2. Every eigenvalue of A is positive.
3. The determinant of every leading principal sub-matrices of A is positive.
4. A has full positive pivots.

What we have shown in the previous slides are

$$① \Leftrightarrow ②$$

and

$$① \Rightarrow ③ \Rightarrow ④ \Rightarrow ①$$

Examples

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ The quadratic form of A is

The quadratic form of A is

$$\begin{aligned} x^T A x &= 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 \\ &= 2\left(x_1 - \frac{1}{2}x_2\right)^2 + \frac{3}{2}\left(x_2 - \frac{2}{3}x_3\right)^2 + \frac{4}{3}x_3^2 \end{aligned}$$

The eigenvalues, the determinants, and the pivots are

$$\text{spectrum}(A) = \{2, 2 \pm \sqrt{2}\}, \ |A_1| = 2, \ |A_2| = 3, \ |A_3| = 4$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE

For what numbers 'b' is the following matrix positive semi-definite?

$$A = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}.$$

Solution: $\begin{vmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{vmatrix} = -2b^2 + 2b + 4 \geq 0$

$$\Rightarrow \quad b^2 - b - 2 \leq 0$$

$$(b-1)(b-2) \leq 0 \qquad -1 \leq b \leq 2$$

THANK YOU

Aparna B. S
Department of Science & Humanities
aparnabs@pes.edu