LINEAR ALGEBRA

UE19MA251

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Agenda

- Proofs of SVD (Alternate proof)
- Matrices and SVD
- Singular vectors and fundamental subspaces

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Alternate proof of SVD

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We show $U^TAV = \Sigma$ which leads to SVD $A = U\Sigma V^T$. For $j = 1 \ldots r$, we have $u_j = \frac{Av_j}{\sigma_j}$, so $Av_j = \sigma_j u_j$ and

$$(U^TAV)_{ij} = u_i^T A v_j = u_i^T(\sigma_j u_j) = \sigma_j \delta_{ij}, \;\; i = 1, \ldots, m$$

For $j = r+1 \ldots n$, we have $\left(A^TA\right) v_j = 0$, so $Av_j = 0$ and

$$(U^TAV)_{ij} = u_i^T A v_j = 0, \;\; i = 1, \ldots, m$$

Combining the results, we get

$$U^TAV = \Sigma$$

Hence

$$A = U\Sigma V^T$$

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Matrices and SVD

For a matrix of order $m \times n$ with SVD $\boldsymbol{A} = \boldsymbol{U\Sigma V}^T$, the column vectors of $\boldsymbol{U}$ (resp. $\boldsymbol{V}$) is an orthonormal basis of $\mathbb{R}^m$ (resp. $\mathbb{R}^n$).

$\boldsymbol{U}$ must be an eigenvector matrix of $\boldsymbol{AA}^T$.

$$\boldsymbol{AA}^T = \left(\boldsymbol{U\Sigma V}^T\right)\left(\boldsymbol{V\Sigma}^T\boldsymbol{U}^T\right) = \boldsymbol{U}\overbrace{\left(\boldsymbol{\Sigma\Sigma}^T\right)}^{\text{diagonal}}\boldsymbol{U}^T$$

Similarly, $\boldsymbol{V}$ must be an eigenvector matrix of $\boldsymbol{A}^T\boldsymbol{A}$.

$$\boldsymbol{A}^T\boldsymbol{A} = \left(\boldsymbol{V\Sigma}^T\boldsymbol{U}^T\right)\left(\boldsymbol{U\Sigma V}^T\right) = \boldsymbol{V}\overbrace{\left(\boldsymbol{\Sigma}^T\boldsymbol{\Sigma}\right)}^{\text{diagonal}}\boldsymbol{V}^T$$

Singular Vectors and Fundamental Subspaces

> The right (resp. left) singular vectors in an SVD of a matrix form an orthonormal basis of the row space (resp. column space) of the matrix.

Row space. $\{v_{r+1}, \ldots, v_n\}$ contains eigenvectors of $\left(A^T A\right)$ with eigenvalue 0, so it is a basis of $\mathcal{N}(A^T A) = \mathcal{N}(A)$. This implies $\{v_1, \ldots, v_r\}$ is a basis of the orthogonal complement of $\mathcal{N}(A)$, i.e. the row space of A.

Column space. We have $AV = U\Sigma$. The first r columns are

$$Av_i = \sigma_i u_i, \ i = 1, \ldots, r$$

So $\{u_1, \ldots, u_r\}$ is a linearly independent set in the column space of A. Hence, it is a basis of $\mathcal{C}(A)$.

THANK YOU

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