LINEAR ALGEBRA AND ITS APPLICATIONS

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CLASS-6

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS AND CAYLEY-HAMILTON THEOREM

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS:

- If λ is an Eigen value of A with x as the corresponding Eigen vector then λ^2 is an Eigen value of A^2 with the same Eigen vector x.
- For a given Eigen vector x, there corresponds only one Eigen value λ.
- For a given Eigen value there corresponds infinitely many Eigen vectors.

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS (Contd……..)

- $\lambda = 0$ is an Eigen value of A , if and only if A is singular i.e det(A)=0.
- If λ is an Eigen value of A with x as the Eigen vector then $1/\lambda$ is an Eigen value of A^{-1} provided A^{-1} exists.
- A and its transpose A^T have the same Eigen values.

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- The Eigen values of a diagonal matrix are just the diagonal elements of the matrix.
- The Eigen values of an idempotent matrix are either zero or unity.
- The sum of the Eigen values of a matrix is the sum of the elements of the principal diagonal.
- The product of the Eigen values of a matrix A is equal to its determinant.

CAYLEY-HAMILTON THEOREM:

"Every square matrix satisfies its own characteristic equation"

Proof: Let A be an n-square matrix. Let $D(\lambda)$ be the characteristic polynomial of A given by,

$$D(\lambda) = |\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0 \longrightarrow 1$$

Let $B(\lambda)$ be the adjoint of $(\lambda I - A)$. The elements of $B(\lambda)$ are cofactors of the matrix $(\lambda I - A)$ and are polynomials in λ of degree not exceeding $n-1$. Thus

$$B(\lambda) = B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + \cdots + B_1\lambda + B_0 \longrightarrow 2$$

where B_i are n-square matrices whose elements are functions of the elements of A and independent of λ.

We know that $(\lambda I - A) \cdot \text{adj}(\lambda I - A) = (\lambda I - A) I$

$$(\lambda I - A) \cdot B(\lambda) = |\lambda I - A| I$$

From (1) & (2), we have

$$(\lambda I - A)(B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + \cdots + B_1\lambda + B_0)$$

$$= I(\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0) \longrightarrow (3)$$

Equating the like powers of λ on both sides of (3), we get

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$B_{n-1} = I$

$B_{n-2} - AB_{n-1} = C_{n-1}I$

$B_{n-3} - AB_{n-2} = C_{n-2}I$

$\cdots \cdots \cdots$

$B_0 - AB_1 = C_1 I$

$-AB_0 = C_0 I$

Multiplying both sides of the above matrix equations by $A^n, A^{n-1}, A^{n-2}, \ldots, A, I$, respectively, we have

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$$A^n B_{n-1} = A^n$$

$$A^{n-1} B_{n-2} - A^n B_{n-1} = c_{n-1} A^{n-1}$$

$$A^{n-2} B_{n-1} - A^{n-1} B_{n-2} = c_{n-2} A^{n-2}$$

$$\ldots\ldots\ldots$$

$$AB_0 - A^2 B_1 = c_1 A$$

$$-AB_0 = c_0 I$$

By adding all the above equations, we get

$$0 = A^n + c_{n-1} A^{n-1} + c_{n-2} A^{n-2} + \cdots + c_1 A + c_0 I \longrightarrow (4)$$

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Since all the terms on the L.H.S cancel each other. Thus A satisfies its own characteristic equation.

Inverse by Cayley-Hamilton theorem:-

Multiplying (4) by A^{-1},

$$0 = A^{n-1} + c_{n-1}A^{n-2} + c_{n-2}A^{n-3} + \cdots\cdots + c_1 I + c_0 A^{-1}$$

Solving for A^{-1}, we get

$$A^{-1} = -\frac{1}{c_0}\left[A^{n-1} + c_{n-1}A^{n-2} + \cdots\cdots + c_1 I\right]$$

Note:- A^{-1} exists only iff c_0 = determinant of $A \neq 0$.

THANK YOU

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