



VECTOR SPACES

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CLASS 11 : CONTENT



- Uniqueness, Existence of right and left inverse
- Matrix of rank 1

EXISTENCE OF INVERSE FOR A RECTANGULAR MATRIX



Existence of Inverses

Definition:

Let $A_{m \times n}$ ($m \leq n$) be a matrix such that rank of $A = m$. Then $Ax = b$ has at least one solution x for every b if and only if the columns span \mathbb{R}^m .

In this case, A has a right inverse C such that $AC = I$ ($m \times m$).

Let $A_{m \times n}$ ($m \geq n$) be a matrix such that rank of $A = n$. Then $Ax = b$ has at most one solution x for every b if and only if the columns are linearly independent. In this case, A has a left inverse B such that $BA = I$ ($n \times n$).

EXISTENCE AND UNIQUENESS

A has a left inverse if $BA = I$

A has a right inverse if $AC = I$

Rank always satisfies $r \leq m$ and $r \leq n$. An m by n matrix cannot have

More than 'm' independent rows or 'n' independent columns. There is not a space for more than m pivots or more than n .

When $r = m$ there is right inverse and $AX = b$ always has a solution

When $r = n$ there is a left inverse and the solution (if it exists) is unique.

Only a square matrix has both $r = m = n$ hence a square matrix has both existence and uniqueness achieved, so only square matrix has two sided inverse.

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EXISTENCES OF INVERSES



Case (i) If $\rho(A) = m$, (m is the number of rows) then A will have right inverse of order $m \times n$ such that $A_{m \times n} C_{n \times m} = I_{m \times m}$

Case (ii) If $\rho(A) = n$, (n is the number of columns) then A will have left inverse of order $n \times m$ such that $B_{n \times m} A_{m \times n} = I_{n \times n}$

[Best right inverse, $C = A^T (A A^T)^{-1}$]

[Best left inverse, $B = (A^T A)^{-1} A^T$]

LEFT AND RIGHT INVERSE

E.g. : Obtain left inverse or a right inverse if it exists for the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3 \times 2}$$

Solution: Here $\rho(A) = 2 = n$

Therefore A has left inverse, say B

$$B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2 \times 3} \quad \text{then we have}$$

$$BA = I$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2 \times 3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$$

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a = 1, b = 0, d = 0, e = 1$$

$$c = 1, f = 1\} \text{ free variables}$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ is the required left inverse.}$$

Example:

Let $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

Then, a right inverse of A is $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \\ a & b \end{bmatrix}$

Since the third row is arbitrary, there are infinitely many right inverses for A

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MATRICES OF RANK 1



Every matrix of rank 1 has the simple form $A = uv^T =$ column times row

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \rho(A) = 1$$

Every row is a multiple of first row, so row space is one dimensional. In fact , we can write the whole matrix as the product of a column vector and row vector

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MATRIX OF RANK 1



i.e, $A = (\text{column})(\text{row})$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} [1 \quad 2 \quad 3 \quad 4]$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$$

Where the rows are all multiples of the vectors v^T
columns are all multiples of the vector u

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MATRIX OF RANK 1



Matrices Of Rank One:

When the rank of a matrix is as small as possible,
a complicated system of equations can be broken into simple pieces.
Those simple pieces are matrices of rank one.

The matrix has rank $r = 1$.

We can write such matrices as a column times row.

That is

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \\ 8 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$$



THANK YOU

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