VECTOR SPACES

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CLASS 2 : CONTENT

- Echelon form of a matrix
- Row reduced Echelon form of the matrix
- Pivot variables and Free variables
- Special solution

ECHELON FORM U AND ROW REDUCED FORM

A rectangular matrix is said to be in echelon form if it has following characterizations

i. All the zero rows are below the non zero rows.

ii. Each pivot (non-zero leading entry) lies to the right to the pivot in the row above(This produces the stair-case pattern).

iii. All the entries in a column below the pivot entry are zero.

The matrix is said to be in row reduced form R, if in addition to the above, the matrix has following additional characterization

iv. Pivot (it should be 1) is only non-zero entry in its column

PIVOT VARIABLES AND FREE VARIABLES

Consider $Rx = 0$

$$i.e., \quad \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The unknowns are divided into two groups

i. Pivot variables: which corresponds to columns with pivots.

ii. Free variables: which corresponds to columns without pivots.

From the above example

First and third columns contain the pivots, so u & w are the pivot variables.

Second and fourth columns do not contains pivots, so v & y are free variables.

RANK OF A MATRIX

Rank of a Matrix Definition: The rank of a matrix A is the number of nonzero rows in the echelon form U of A and is denoted by ρ (A) or simply r.

Note : If A is a matrix of order m x n then its rank r ≤ min (m, n).

Ex: 1 $$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\end{bmatrix}\sim\begin{bmatrix}1 & 2 & 3\\ 0 & 0 & -1\end{bmatrix};\rho(A)=2$$

Ex: 2 $$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{bmatrix}\sim\begin{bmatrix}1 & 2 & 3\\ 0 & 0 & 0\end{bmatrix};\rho(A)=1$$

ECHELON FORM U AND ROW REDUCED FORM

Eg:

$$A=\begin{bmatrix}1 & 2 & 3 & 4\\ 0 & 1 & 4 & 5\\ 0 & 0 & 1 & 2\end{bmatrix}=U,$$

Echelon form

$$A=\begin{bmatrix}1 & 4 & 5\\ 0 & 1 & 6\\ 0 & 0 & 0\end{bmatrix}=U,$$

Echelon form

$$A=\begin{bmatrix}1 & 0 & 0 & 10\\ 0 & 1 & 0 & 5\\ 0 & 0 & 1 & 2\end{bmatrix}=R,$$

Row reduced Echelon form

Marked elements are the Pivots.

PIVOT VARIABLES AND FREE VARIABLES

To find the most general solution to $Rx = 0$ (or equivalently to $Ax = 0$)

we may assign arbitrary values to free variables.

The pivot variables are completely determined in terms of free variables

v and y . $\Rightarrow Rx = 0 \Rightarrow u + 2v - y = 0 \Rightarrow u = -2v + y$

$$w + y = 0 \Rightarrow w = -y$$

Special solutions

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

PIVOT VARIABLES AND FREE VARIABLES

Most general solution or complete solution

The best way to find all solutions to is from the special solutions

$$x = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = v\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

The complete solution is the linear combination of 2 special solutions.

ECHELON FORM U AND ROW REDUCED FORM

E.g. : For every c, find R and special solutions to $Ax = 0$, where

$$A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

Solution: If $c = 1$ then $A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$

$$R_2 \to R_2 - 2R_1$$

$$A \sim \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = U$$

$$R_1 \to \frac{1}{2} R_1$$

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$R_x = 0 \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = 0$ & x is free variable

ECHELON FORM U AND ROW REDUCED FORM

$$x = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

If $c = 2$ then

$$A \sim \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} = U; R_1 \to -R_1$$

$$\sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

y is the free variable

ECHELON FORM U AND ROW REDUCED FORM

Now $x - 2y = 0 \Rightarrow x = 2y$

Special solution : $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

If $c \neq 1{,}2$ then no special solution

For E.g.: If $c = 0$ then

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = U; R_1 \to R_1 - R_2$$

$$A \sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R$$

$$Rx = 0 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = 0, y = 0 \quad (No\ special\ solution)$$

THANK YOU

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