LINEAR ALGEBRA AND ITS APPLICATIONS

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CLASS-7

PROBLEMS ON EIGEN VALUES AND EIGEN VECTORS AND CAYLEY-HAMILTON THEOREM

Problem:-

1. Find the Eigen values & Eigen vectors for the given matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Ans: Characteristic eqn is $|A-\lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (6-\lambda)[(3-\lambda)^2 - 1] + 2[-2(3-\lambda)+2] + 2(2-2(3-\lambda)) = 0$

$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\lambda = 8$ is one of the roots.

$$\begin{array}{r|rrrr} 8 & 1 & -12 & 36 & -32 \\ & 0 & 8 & -32 & 32 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$

$\Rightarrow \lambda = 2, 2$

$\therefore \lambda_1 = 8, \lambda_2 = 2, \lambda_3 = 2.$

Consider $|A-\lambda I|X = 0$

$$\Rightarrow \quad (6-\lambda)x - 2y + 2z = 0$$
$$-2x + (3-\lambda)y - z = 0$$
$$2x - y + (3-\lambda)z = 0$$

When $\lambda = 8$,
$$\left.\begin{aligned} -2x - 2y + 2z &= 0 \\ -2x - 5y - z &= 0 \\ 2x - y - 5z &= 0 \end{aligned}\right\} \text{ On solving } \quad X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

When $\lambda = 2$,
$$\left.\begin{aligned} 4x - 2y + 2z &= 0 \\ -2x + y - z &= 0 \\ 2x - y + z &= 0 \end{aligned}\right\} \text{ On solving } \quad X_2 = \begin{bmatrix} \frac{k_1}{2} - \frac{k_2}{2} \\ k_1 \\ k_2 \end{bmatrix} = X_3$$

Where k_1 & k_2 are arbitrary & can take any value.

$\therefore$ Eigen vector is $X = [X_1 \; X_2 \; X_3]$

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2. Find the Eigen values & Eigen vectors for the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

Ans: The characteristic equation is $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & -\lambda & 0 \\ -1 & 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 + \lambda = 0$$

$$\Rightarrow \lambda = 0, \ \lambda^2 + 1 = 0$$

$$\lambda = 0, \pm i$$

$\therefore \lambda_1 = 0, \lambda_2 = i, \lambda_3 = -i$ are the required eigen values.

Now consider $|A - \lambda I| X = 0 \Rightarrow$

$$(1-\lambda)x - y + z = 0$$

$$x - \lambda y = 0$$

$$-x + y + (-1-\lambda)z = 0$$

Case 1:- When $\lambda = 0$

$$\left.\begin{aligned} x - y + z &= 0 \\ x &= 0 \\ -x + y - z &= 0 \end{aligned}\right\}$$

On solving $X_1 = \begin{bmatrix} 0 \\ k_1 \\ k_2 \end{bmatrix}$. Let $k_1 = k_2 = 1$

$$\therefore X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case 2:- When $\lambda = i$ $\Rightarrow$

$$(1-i)x - y + z = 0$$
$$x - iy + 0.z = 0$$
$$-x + y + (-1-i)z = 0$$

$$X_2 = \begin{bmatrix} i \\ 1 \\ -i \end{bmatrix}$$

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Case 3:- When $\lambda = -i$,

$$\left.\begin{aligned} (1+i)x - y + z &= 0 \\ x + iy + 0z &= 0 \\ -x + y + (-1+i)z &= 0 \end{aligned}\right\} \quad X_3 = \begin{bmatrix} -i \\ 1 \\ i \end{bmatrix}$$

$\therefore$ Eigen vectors corresponding to $\lambda = 0, i, -i$ is

$$X = [X_1 \; X_2 \; X_3] = \begin{bmatrix} 0 & i & -i \\ 1 & 1 & 1 \\ 1 & -i & i \end{bmatrix}$$

Problem:- If $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$. Evaluate A^{-1} & A^{-2} using Cayley-Hamilton theorem or (C-H) theorem.

Ans:- $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

According to C-H theorem $(\lambda \rightarrow A)$

$$A^3 - 4A^2 - A + 4I = 0 \longrightarrow (1)$$

Now to find A^{-1} & A^{-2}

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To find A^{-1}

Multiplying eqn (1) by A^{-1},

$$\Rightarrow A^2 - 4A - I + 4A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{4}\left[-A^2 + 4A + I\right]$$

$$A^{-1} = \frac{1}{4}\left[-\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} + 4\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}$$

To find A^{-2}:

Multiplying eqn (1) by A^{-2}

$$A - 4I - A^{-1} + 4A^{-2} = 0$$

$$\Rightarrow A^{-2} = \frac{1}{4}\left[-A + A^{-1} + 4I\right]$$

$$A^{-2} = \frac{1}{16}\begin{bmatrix} 1 & -18 & -18 \\ -5 & 10 & -6 \\ 5 & 6 & 22 \end{bmatrix}$$

THANK YOU

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