

CLASS 4 : CONTENT



- Linear independence and dependence of vectors
- Basis
- Dimension
- Span of set of vectors

VECTOR SPACES

LINEAR COMBINATION



Let V be a vector space and $v_1, v_2, v_3, \dots, v_n$ be the vectors in V . Then the form $c_1v_1 + c_2v_2 + \dots + c_nv_n$, where c_1, c_2, \dots, c_n are scalars, is called a *linear combination of the vectors*.

Linear combination of vectors involve scalar multiplication and vector addition of vectors .

To decide on linear independence of vectors we need to look for their linear combination .

The trivial combination with all scalars ' c '=0 ,produces $0v_1 + 0v_2 + \dots + 0v_n = 0$

If this is the only way to produce zero ,given vectors are independent

if any other combination produces zero then vectors are dependent

VECTOR SPACES

LINEAR INDEPENDENCE AND DEPENDENCE



A set of vectors $\{v_1, v_2, v_3, \dots, v_n\}$ of a vector space, is said to be **linearly independent** if the linear combination $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$,

where $c_1 = c_2 = c_3 = \dots = c_n = 0$

A set of vectors $\{v_1, v_2, v_3, \dots, v_n\}$ of a vector space, is said to be **linearly dependent** if there exists scalars $c_1, c_2, \dots, c_n \in R$, not all zero such that

$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$ either all c_1, c_2, \dots, c_n are nonzero or few scalars are zero and few are non zero .

(A set of vectors $\{v_1, v_2, v_3, \dots, v_n\}$ is said to be **linearly independent** if one vector cannot be written as the combination of the other vectors.

VECTOR SPACES

LINEAR INDEPENDENCE AND DEPENDENCE



To decide whether given set of vectors are independent or dependent apply the following procedure :

- 1) Place the vectors $\{v_1, v_2, v_3, \dots, v_n\}$ of the given set of vectors as columns of matrix A .
- 2) Apply Gauss Elimination on the Matrix A
- 3) If all the columns of the matrix is with pivot then the set of vectors are linearly independent
- 4) If certain columns of the matrix A do not hold pivot then the set of vectors are linearly dependent
- 5) If $\rho(A) = n =$ number of columns then vectors are linearly independent

VECTOR SPACES

LINEAR COMBINATION



E.g.: Check whether $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 7 \\ 1 \end{bmatrix}$ are independent in \mathbb{R}^3 .

Solution

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 7 \\ 3 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}; \rho(A) = 3 = n$$

Hence the given vectors
are linearly independent.

VECTOR SPACES

LINEAR COMBINATION



E.g.: Check whether $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ are independent in \mathbb{R}^3 .

Solution

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 2, n = 3$$

Therefore they are dependent

VECTOR SPACES

LINEARLY INDEPENDENT VECTORS



Note : 1. The columns of a square invertible matrix are always independent.

2. The columns of a matrix A of order $m \times n$ with $m < n$ are always dependent.

3. The columns of A are independent exactly when $N(A) = Z$
(Z means 0)

4. The ' r ' nonzero rows of an echelon matrix U and a reduced matrix R are always independent and so are the ' r ' columns that contain the pivots.

VECTOR SPACES

LINEARLY INDEPENDENT VECTORS



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VECTOR SPACES

BASIS



A subset $S = \{v_1, v_2, v_3, \dots, v_n\}$ of a vector space is called a **basis** for

Vector space V if

- i. S is a **linearly independent** set.
- ii. S *spans the vector space V .*

The ***dimension of a vector space*** is the number of basis vector.

A Basis for Vector space V is a sequence of vectors having properties :

1. The vectors are linearly independent (not too many vectors)
2. They span the space V (not too few vectors) .
3. every vector in the space is a combination of the basis vectors
,because they span (s.t. combination is unique)
4. if $\{v_1, v_2, v_3, \dots, v_n\}$ and $\{w_1, w_2, w_3, \dots, w_n\}$ are both the bases for the same vector space then $m=n$,the number of vectors is same .
5. basis is maximal independent set and minimal spanning set.

VECTOR SPACES

BASIS



6. If The vectors $\{v_1, v_2, v_3, \dots, v_n\}$ and $\{w_1, w_2, w_3, \dots, w_n\}$ are both bases for the same vector space ,then $m = n$, i.e. the number of vectors is the same .
7. Any linearly independent set in V can be extended to a basis ,by adding more vectors if necessary ,any spanning set in V can be reduced to a basis ,by discarding vectors if necessary .
8. there exists one and only one way to write any vector 'v' in Vector space V as a combination of the bases vectors of that vector space .

VECTOR SPACES

BASIS



Examples of Basis :

1. Basis of R^2 vector space is $\{(1,0) ,(0,1)\}$.
2. Basis of R^3 vector space is $\{(1,0,0) ,(0,1,0),(0,0,1)\}$.
3. Basis of R^n vector space is $\{(1,0,0,0,.....,0) ,(0,1,0,.....,0),(0,0,.....,1)\}$
4. Matrices $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is the basis for vector space of 2×2 matrices .
5. The set $\{1, t, t^2,, t^n\}$ is a basis of space of Polynomials P_n

VECTOR SPACES

SPAN OF A SET



Let $W = \{v_1, v_2, v_3, \dots, v_n\}$ be a set of vectors belonging to vector space V , then *span* of W is the set of all linear combinations of vectors of W .

i.e., span of $W = \text{span}(W) = c_1v_1 + c_2v_2 + \dots + c_nv_n$
= subspace of V

VECTOR SPACES

SPAN OF A SET



E.g. : What do these vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ span

Solution

The given vectors span a 2D subspaces of \mathbb{R}^2 .

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}; \quad \rho(A) = 2$$



THANK YOU

Deepthi Rao

Department of Science & Humanities