LINEAR ALGEBRA AND ITS APPLICATIONS

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CLASS-8

SYMMETRIC MATRICES AND DIAGONALIZATION OF A MATRIX

Statement: If A is a square matrix of order 'n' has 'n' linearly independent vectors, then a matrix 'S' can be found such that $S^{-1}AS$ is a diagonal matrix.

Proof:

Let A be a square matrix of order 3. Let λ_1, λ_2 and λ_3 be its Eigen value and $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ and $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be the corresponding Eigen vectors.

Let the square matrix $S = [X_1\ X_2\ X_3]$ i.e

$$S = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

Multiplying by A, $AS = A[X_1\ X_2\ X_3] = [AX_1, AX_2, AX_3]$

$$AS = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$$

$$AS = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = SD$$

$\therefore AS = SD$ where D is the diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$\therefore \quad AS = SD$

Multiplying both sides by S^{-1}

$\boxed{S^{-1}AS = D}$ or $S^{-1}AS = \Lambda$

or $\boxed{A = SDS^{-1}}$

S is invertible because its columns (the eigen vectors) were assumed to be independent.

NOTE:

1. Any matrix with distinct Eigen values can be diagonise.

2. Not all matrices has 'n' linearly independent Eigen vectors. Therefore, all the matrices can't be diagonalized.

Eg: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, here $\lambda_1 = \lambda_2 = 0$. There is only one independent vector. Hence we cannot construct S.

3. If Eigen vectors $x_1, x_2, \ldots x_k$ correspond to distinct Eigen values $\lambda_1, \lambda_2 \ldots \lambda_k$ then those Eigen vectors are linearly independent.

4. Eigen vector matrix is not unique since if x is an Eigen vector corresponding to λ then kx is also an Eigen vector.

· Problem

1. Factor the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ into $S \Lambda S^{-1}$ & also find $S A S^{-1}$.

Ans: Consider $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 1 = 0$$

$$1 + \lambda^2 - 2\lambda - 1 = 0 \Rightarrow \lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0 \Rightarrow \lambda_1 = 0 \ \& \ \lambda_2 = 2$$

$$\therefore \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Now to find Eigen vectors:-

Consider $(A-\lambda I)X=0$

$\Rightarrow \quad (1-\lambda)x+y=0$

$x+(1-\lambda)y=0$

Case 1:- When $\lambda=2 \Rightarrow \left.\begin{array}{l} -x+y=0 \\ x-y=0 \end{array}\right\} \Rightarrow x=y$

$\Rightarrow$ 1 equations with 2 unknowns $\therefore$ y be free variable

$\therefore x=y=k$, let $k=1$

Eigen vector $X_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Case 2:- When $\lambda = 0$

$$\left.\begin{array}{l} x+y=0 \\ x+y=0 \end{array}\right\} \Rightarrow x = -y$$

$\Rightarrow$ 2 unknowns no one free variable

$x = k \Rightarrow y = -k$ or $x = -k \Rightarrow y = k$

$$\therefore X_2 = \begin{bmatrix} -k \\ k \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\therefore S = [X_1 \;\; X_2] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \; \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \; S^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore A = S \Lambda S^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Hence verified.

To find SAS^{-1}

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \Lambda$$

THANK YOU

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