LINEAR ALGEBRA AND ITS APPLICATIONS
UE19MA251

Rotation Matrices Q:

The linear system of equations $Ax = b$ can be represented as a linear Transformation

$T_A(x) = Ax$, where $T_A : \mathbb{R}^n \to \mathbb{R}^m$

Rotation Matrices Q:

The matrix that rotates (left) every point in R^2 about origin through θ is given by Q_θ

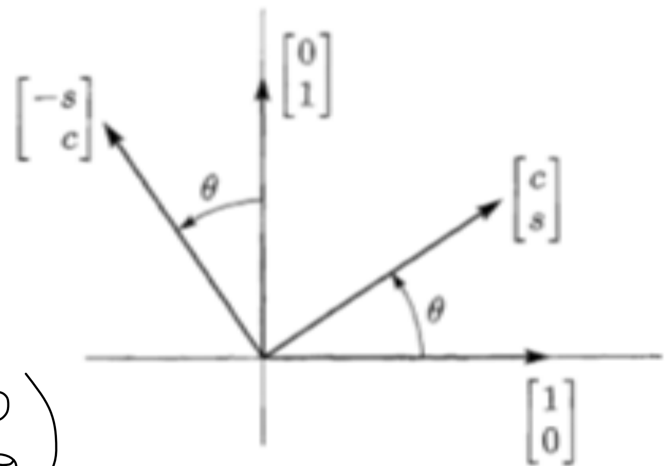
$Q_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

Basis for $\mathbb{R}^2 = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$$Q_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$Q_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(90+\theta) \\ \sin(90+\theta) \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$Q_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



Rotation Matrices Q:

Note :

- $$Q_\theta \cdot Q_\psi = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\psi - \sin\theta\sin\psi & -\cos\theta\sin\psi - \sin\theta\cos\psi \\ \sin\theta\cos\psi - \cos\theta\sin\psi & -\sin\theta\sin\psi + \cos\theta\cos\psi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\psi) & -\sin(\theta+\psi) \\ \sin(\theta+\psi) & \cos(\theta+\psi) \end{bmatrix} = Q(\theta+\psi)$$

- $$Q_\theta \cdot Q_\theta = Q(\theta+\theta) = Q_{2\theta} \Rightarrow Q_\theta^2 = Q_{2\theta}$$

Rotation Matrices Q:

- $Q_\theta \cdot Q_{-\theta} = Q(\theta-\theta) = Q_0 = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} = I$

$\Rightarrow \left[Q_\theta\right]^{-1} = Q(-\theta)$

- Rotation preserves all angles between the vectors as well as their length. So it is reversible process.

Projection Matrices P

'P' is a matrix, that projects every vector in $\mathbb{R}^2$ onto any 'θ' line.

$$P[e_1] = P\begin{bmatrix}1\\0\end{bmatrix} = A' = \begin{bmatrix}OA' \cdot \cos\theta \\ OA' \cdot \sin\theta\end{bmatrix} = \begin{bmatrix}\cos\theta \cdot \cos\theta \\ \cos\theta \cdot \sin\theta\end{bmatrix}$$

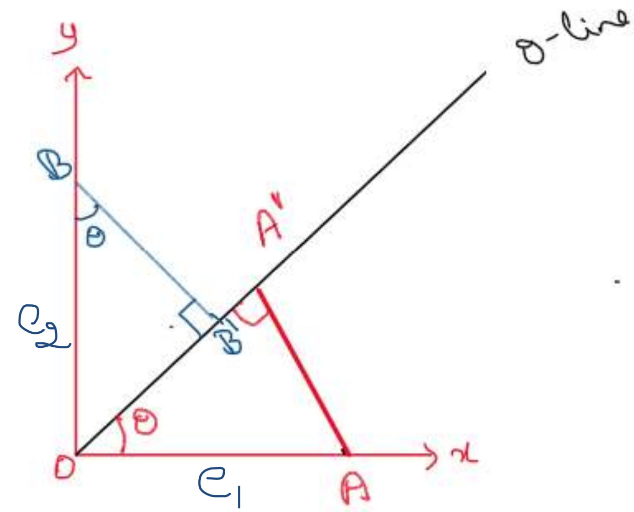
$$= \begin{bmatrix}\cos^2\theta \\ \cos\theta\sin\theta\end{bmatrix}$$

From right angle triangle $OA'A$, $OA' = \cos\theta$

$$P[e_2] = P\begin{bmatrix}0\\1\end{bmatrix} = B' = \begin{bmatrix}OB' \cos\theta \\ OB' \sin\theta\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\theta \\ \sin\theta\sin\theta\end{bmatrix}$$

$$= \begin{bmatrix}\sin\theta\cos\theta \\ \sin^2\theta\end{bmatrix}$$

From right angle triangle $OB'B$, $OB' = \sin\theta$



Projection Matrices P

The matrix that projects every vector in $\mathbb{R}^2$ onto any θ line is given by

$$P = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}.$$

Note:

- This matrix has no inverse, because the transformation has no inverse.
- Projecting twice is the same as projection once i.e $P^2 = P$.

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Reflection matrix H

The matrix H reflects every vector in $\mathbb{R}^2$ onto any 'θ' line.

From the figure

$$\overrightarrow{OA'} + \overrightarrow{A'B} = \overrightarrow{OB} \quad - (1)$$

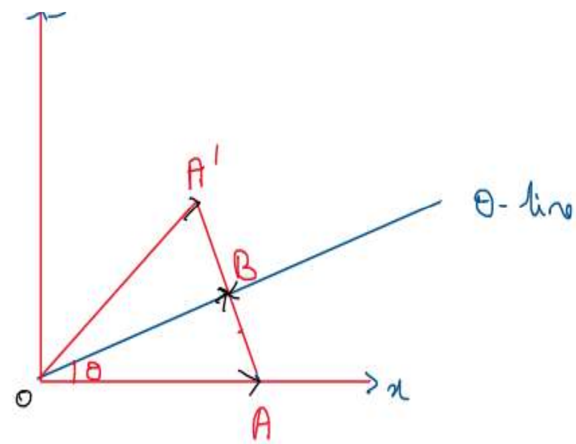
and $$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \quad - (2)$$

$$(1)+(2) \Rightarrow \overrightarrow{OA'} + \overrightarrow{OA} = 2\overrightarrow{OB}$$

$$(\text{Since } \overrightarrow{A'B} = -\overrightarrow{AB})$$

$$\Rightarrow \quad x + Hx = 2Px \Rightarrow Hx = 2Px - Ix.$$

$$\Rightarrow \quad H = 2P - I$$



Reflection matrix H

$$H = \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1 \end{bmatrix}$$

Note:

- Two reflection brings back the original.

$$H = 2P - I \Rightarrow H^2 = (2P - I)^2 = 4P^2 - 4P + I = I$$

since $P^2 = P$.

- A reflection is its own inverse.

Reflection matrix H

To conclude….

Product of two transformations is another transformation by itself. Matrix multiplication is so defined that product of matrices corresponds to the product of the transformations that they represent.

Problems

Find the matrix S that reflects every vector in R^2 on the line y = x. Also find the matrix T which projects every vector in R^2 on to the line y = x. Explain why ST = TS.

Solution: The reflection of every vector in $\mathbb{R}^2$ onto the line $y = x$ is given by

$$S = H_\theta = H_{45^\circ} = \begin{bmatrix} 2\cos^2(45) - 1 & 2\cos(45^\circ)\sin(45) \\ 2\cos(45^\circ)\sin(45^\circ) & 2\sin^2(45) - 1 \end{bmatrix}$$

Problems

$$S = \begin{bmatrix} 2\left(\frac{1}{\sqrt{2}}\right)^2 - 1 & 2\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \\ 2\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} & 2\left(\frac{1}{\sqrt{2}}\right)^2 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Problems

The projection of every vector in $\mathbb{R}^2$ onto the $y = x$ line is given by

$$T = P_{45} = \begin{bmatrix} \cos^2(45) & \cos(45)\sin(45) \\ \cos(45)\sin(45) & \sin^2(45) \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{\sqrt{2}}\right)^2 & \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} & \left(\frac{1}{\sqrt{2}}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Problems

$$S \cdot T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$TS = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

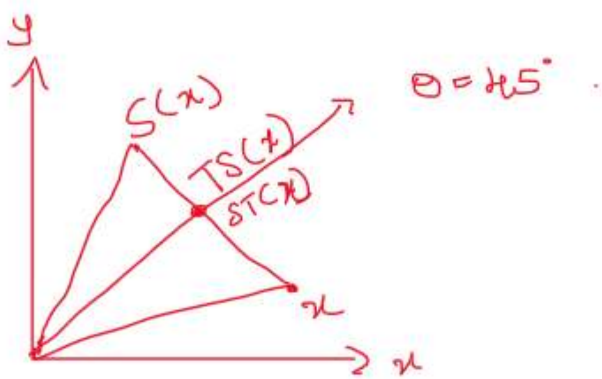
$$\Rightarrow ST = TS.$$

Problems

ST is the composition of projecting any vector of $\mathbb{R}^2$ onto $y = x$ line then reflecting it onto $y = x$ line.

TS is the composition of reflecting any vector of $\mathbb{R}^2$ onto $y = x$ line then projecting it onto $y = x$ line.

Both transformation produces the same output.



Unit 3. Linear Transformations and Orthogonality

Problems

Determine the new point after applying the transformation to the given point

a. Project $x = (-2, 1)$ on the y-axis and then rotate by 45° counter-clockwise

b. Rotate $x = (-2, 1)$, 60° counter-clockwise and then project on the x-axis

Solution:

a) Projection matrix to project any vector of R^2 on the y-axis is given by

$$P(90^\circ) = \begin{bmatrix} \cos^2(90^\circ) & \cos(90^\circ)\sin(90^\circ) \\ \cos(90^\circ)\sin 90^\circ & \sin^2(90^\circ) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Problems

Rotation matrix to rotate any vector of $\mathbb{R}^2$, by 45° counter clockwise about the origin is given by

$$Q(45^\circ) = \begin{bmatrix} \cos(45) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

The projection of $x = (-2, 1)$ on the y-axis and then rotating about $\theta = 45^\circ$ counter clockwise is given by

$$Q_{(45)} P_{(90)} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Problems

6. Rotation matrix to rotate any vector of $\mathbb{R}^2$, 60° counter clockwise is given by

$$Q(60) = \begin{bmatrix} \cos(60) & -\sin(60) \\ \sin(60^\circ) & \cos(60^\circ) \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

Projection matrix to project any vector of $\mathbb{R}^2$ onto x-axis is given by

$$P(0) = \begin{bmatrix} \cos^2 0 & \cos 0 \sin 0 \\ \cos 0 \sin 0 & \sin^2 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Problems

Rotating $x = (-2, 1)$, 60° counter clockwise and then projecting on the x-axis is given by

$$P_{(0)} Q_{(60)}(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 - \sqrt{3}/2 \\ 0 \end{bmatrix}$$

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Orthogonal Vectors & Subspaces

Definition:

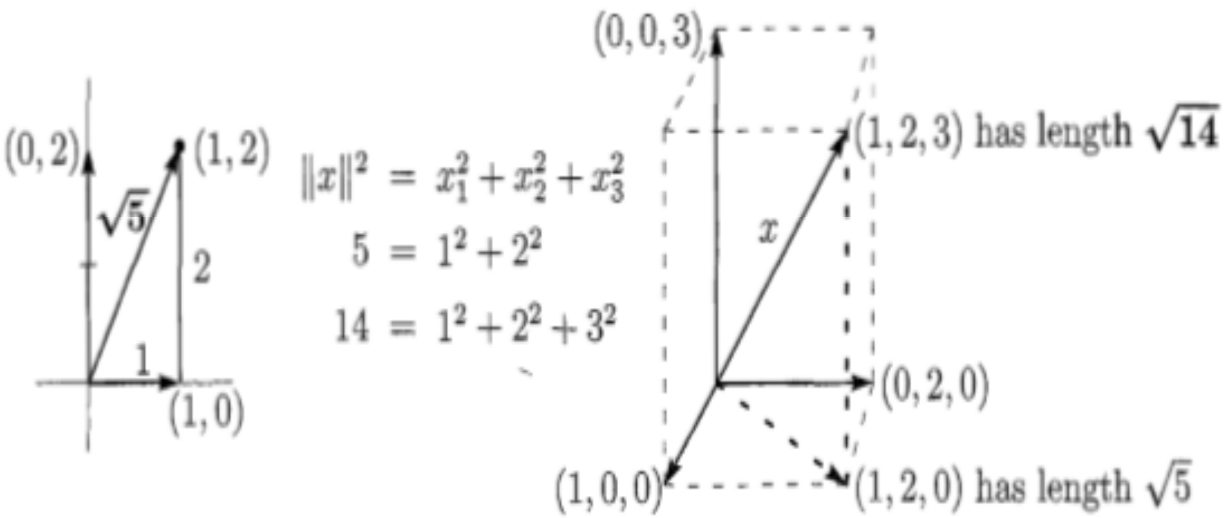
The ***norm or length*** of a n-dimensional vector $x = (x_1, x_2, \ldots, x_n)$ is written as $\| x \|$ and is defined as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

We can also write $$\|x\|^2 = x^T x$$

Note : Zero is the only vector whose norm is 0.

Orthogonal Vectors & Subspaces



(a) (b)

Orthogonal Vectors & Subspaces

Definition:

The ***inner product*** or dot product or scalar product of two vectors x = (x_1, x_2, …., x_n) and y = (y_1, y_2, …., y_n) is denoted by

$$x^T y \ \ or \ \ x \circ y \ \ or \ \ \langle x , y \rangle$$

and is defined by

$$x^T y = x_1 y_1 + x_2 y_2 + \ldots.. + x_n y_n$$

Note that $x^T y = y^T x$

Orthogonal Vectors & Subspaces

Definition :

Two vectors $x = (x_1, x_2, \ldots., x_n)$ and $y = (y_1, y_2, \ldots., y_n)$ are said to be ***orthogonal*** if

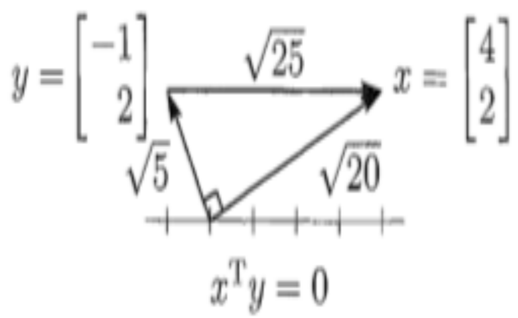
$$x^T y = y^T x = 0$$

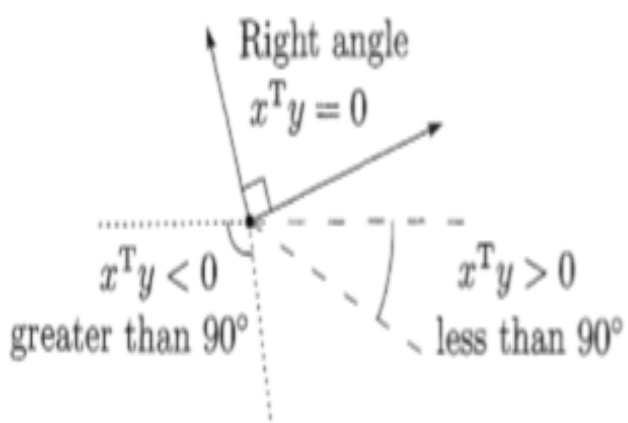
Note :

1. Zero is the only vector that is orthogonal to itself.
2. Zero is the only vector that is orthogonal to every other vector.

Unit 3. Linear Transformations and Orthogonality

Orthogonal Vectors & Subspaces





Unit 3. Linear Transformations and Orthogonality

Orthogonal Vectors & Subspaces

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Examples

1.The coordinate vectors (1, 0, …., 0) , (0, 1, 0,…, 0) , ….., (0, 0, …., 0, 1) are mutually orthogonal in R^n.

2. The vectors (c, s) , (-s, c) are orthogonal in R^2.

3. The vectors (2, 1, 0) , (-1, 2, 0) are orthogonal in R^3.

Orthogonal Vectors & Subspaces

Theorem : If the non-zero vectors $v_1, v_2, \ldots\ ..v_k$ are mutually orthogonal then these vectors are linearly independent but convex need not be true.

Example : Vectors (2, 1) and (1, 2) are linearly independent but they are not mutually orthogonal

Orthogonal Vectors & Subspaces

Definition :

Two subspaces S and T of a vector space V are ***orthogonal*** if every vector x in S is orthogonal to every vector y in T. Thus,

$$x^T y = 0$$

for all x ∈ S and y ∈ T.

Orthogonal Vectors & Subspaces

Examples

1. $Z = \{0\}$ is orthogonal to all subspaces.

2. In R^2, a line can be orthogonal to another line.

3. In R^3, a line can be orthogonal to another line or a plane. But, a plane cannot be orthogonal to another plane.

Note :

If S and T are orthogonal in V then

$$\dim S + \dim T \leq \dim V$$

Orthogonal Vectors & Subspaces

Fundamental theorem of Orthogonality :

Let A be an m x n matrix then row space of A is orthogonal to its null space in R^n and the column space is orthogonal to left null space in R^m .

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Orthogonal complement:

Definition :

Given a subspace V of R^n, the space of all vectors orthogonal to V is called the ***orthogonal complement*** of V written as $V^{\perp}$ and read as " V perp ".

Note : The orthogonal complement of a subspace V is unique.

Fundamental Theorem of Linear Algebra- Part-II

The null space is the orthogonal complement of the row space in R^n and the column space is the orthogonal complement of the left null space in R^m.

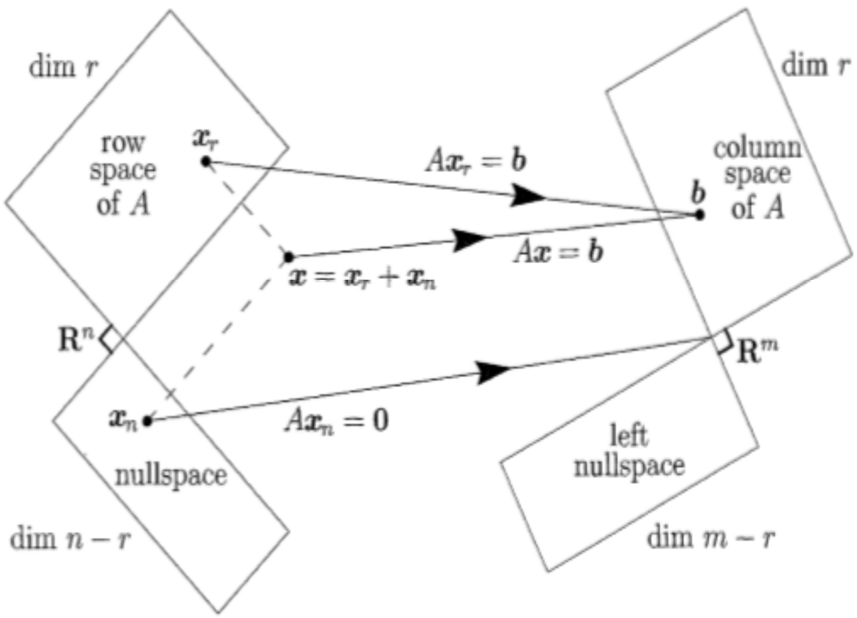
Note :

1. If S and T are orthogonal complements in R^n then it is always true that

$$\dim S + \dim T = n$$

Unit 3. Linear Transformations and Orthogonality

The Matrix And The Subspace



The Matrix And The Subspace

Splitting R^n into orthogonal parts V and W will split every vector into x = v + w.

The vector v is the projection onto the subspace V and the orthogonal component w is the projection of x onto W.

The true effect of matrix multiplication is that every Ax is in C(A). The null space goes to zero. The row space component goes to C(A). Nothing is carried to the left null space.

The Matrix And The Subspace

Note:

- $Ax = b$ is solvable if and only if $y^T b=0$ whenever $y^T A = 0$.

- From the row space to the column space A is actually invertible, every vector 'b' in the column space comes from exactly one vector in the row space.

- Every vector transforms its row space onto its column space.

The Matrix And The Subspace

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Problems

Find a vector x orthogonal to the row space of A, a vector y orthogonal to the column space of A and a vector z orthogonal to the null space of A where
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}$$

Solution: Null space of A is orthogonal to Row space of A

$\therefore$ $x \in N(A)$ and $z \in C(A^T)$

Left Null space is orthogonal to column space of A

$\therefore$ $y \in N(A^T)$

To find the vectors x, y and z we need to find $C(A^T)$, $N(A)$ and $N(A^T)$.

The Matrix And The Subspace

To find the vectors x, y and z we need to find $C(A^T)$, $N(A)$ and $N(A^T)$.

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 3 & 4 & b_2 \\ 2 & 4 & 6 & b_3 \end{array}\right] \underset{R_3 \to R_3 - 2R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -1 & -2 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 \end{array}\right] = [U : c]$$

$$Ux = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = -2x_2 - 3x_3$$

$$x_2 = -2x_3$$

$$\therefore x_1 = -2(-2x_3) - 3x_3 = x_3.$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The Matrix And The Subspace

$\therefore \; N(A) = \left\{ x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}, \; x_3 \in R$

$\therefore \quad x = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$C(A^T) = \left\{ c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}, \; c_1, c_2 \in R$

$\therefore \; z = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$Ax = b$ is solvable only if $b_3 - 2b_1 = 0$

[Only then $\rho(A) = \rho(A:b) = 2$]

$\therefore \; N(A^T) = \left\{ c_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} \; c_3 \in R$

$\Rightarrow y = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

The Matrix And The Subspace

2. Let P be the plane whose equation is x - 2y +9z = 0. Find a vector perpendicular to P. What matrix has the plane P as its null space and what matrix has P as its row space?

Solution:- Let $P = \{(x, y, z) \text{ such that } x - 2y + 9z = 0\}$

$$\Rightarrow \begin{pmatrix} 1 & -2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$\therefore$ The vector perpendicular to P is $\begin{pmatrix} 1 \\ -2 \\ 9 \end{pmatrix}$

So the matrix $A = \begin{pmatrix} 1 & -2 & 9 \end{pmatrix}$ has the plane P as its null space.

The Matrix And The Subspace

To find B find the Null space of P.

ie $\begin{pmatrix} 1 & -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow x = 2y + 9z$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 9 \\ 0 \\ 1 \end{pmatrix}$$

Null space $= \left\{ y\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 9 \\ 0 \\ 1 \end{pmatrix},\ x, y \in R \right\}$

$\therefore\ B = \begin{bmatrix} 2 & 1 & 0 \\ 9 & 0 & 1 \end{bmatrix}$ So that $C(B^T) = P$

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Cosines And projections Onto Lines

Definition :

If $a = (a_1, a_2, \ldots a_n)$, $b = (b_1, b_2, \ldots, b_n)$ include an angle θ between them the ***cosine formula*** states that

$$\cos\theta = \frac{a^T b}{\|a\| \, \|b\|}$$

Projections Onto A Line

To find the projection of b onto the line through a given vector ‘ a ‘, we find the point p on the line that is closest to b.

This point must be some multiple of ‘ a ‘ say $p = \hat{x} a$.

Now, the line from b to the closest point p is perpendicular to the vector a and hence

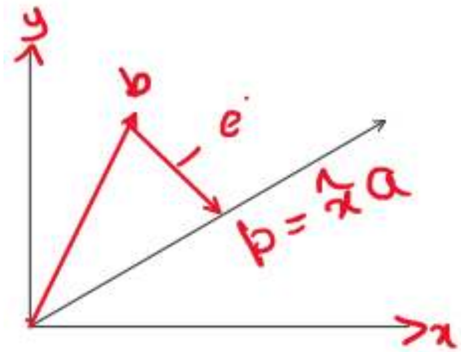
$e = b - p$ Since $a \perp e$

$\Rightarrow a^T e = 0 \Rightarrow a^T(b-p) = 0$

$\Rightarrow a^T b - a^T(\hat{x} a) = 0$

$\Rightarrow \hat{x} = \dfrac{a^T b}{a^T a}$

$\therefore p = \hat{x} a$



Schwarz Inequality

All vectors a and b in R^n satisfy the ***Schwarz Inequality*** which is

$$\left| a^T b \right| \leq \|a\| \, \|b\|$$

Note :

1. Equality holds if and only *if a and b are dependent vectors. The angle is* $\theta = 0^o$ or *180^o*. In this case, b is identical with its projection p and the distance between b and p is zero.

2. Schwarz inequality is also stated as $|\cos\theta| \leq 1$

Projection Matrix of Rank 1

Projections onto a line through a given vector ‘ a ‘ is carried out by a ***Projection Matrix*** given by

$$P = \frac{a\,a^T}{a^T a}$$

This matrix multiplies b and produces p.
That is,

$$Pb = \frac{a\,a^T}{a^T a} b = a \frac{a^T b}{a^T a} = a\,\hat{x} = p$$

Projection Matrix of Rank 1

Note :

1. P is a symmetric matrix.
2. $P^n = P$ for n = 1, 2, 3, ……
3. The rank of P is one.
4. The trace of P is one.
5. If ‘ a’ is a n- dimensional vector then P is a square matrix of order n.
6. If ‘a ‘ is a unit vector then $P = a\,a^T$.

Unit 3. Linear Transformations and Orthogonality

Problems

What multiple of $a = (1,1,1)$ is closest to $b = (2,4,4)$? Find also the point on the line throug 'b' that is closest to a?

Solution: Let 'p' be the point on the line through $a = (1,1,1)$ is closest to $b = (2,4,4)$.

$$\therefore\ p = \hat{x}a = \frac{a^Tb}{a^Ta}.a = \frac{10}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Let p_1 be the point on the line through 'b' is closest to a

$$\text{So } p_1 = \hat{x}_1 b = \frac{a^Tb}{b^Tb}\cdot b = \frac{10}{36}\begin{pmatrix}2\\4\\4\end{pmatrix}$$

Problems

Find the matrix that projects every point in $\mathbb{R}^3$ onto the line of intersection of the planes $x+y+z=0$ and $x-z=0$. What are the column space and row space of this matrix?

Solution:- The line of intersection of these planes is

i.e. $\quad x+y+z=0$
$\quad x - z=0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \underset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} = U$$

$$x = -y - z \quad \text{and} \quad y = -2z$$

Problems

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$\Rightarrow$ The line passing through $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is the line of intersection.

Let $a = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and projection matrix through 'a' is $P = \frac{aa^T}{a^Ta} = \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

P is a symmetric matrix of Rank 1.

Therefore column space and row space are $c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ where $c_1 \in R$.

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Projections And Least Squares

The failure of Gaussian Elimination is almost certain when we have several equations in one unknown.

$$a_1 x = b_1$$
$$a_2 x = b_2$$
$$----$$
$$a_m x = b_m$$

This system is solvable if $b = (b_1, \ldots, b_m)$ is a multiple of $a = (a_1, \ldots, a_m)$.

Projections And Least Squares

If the system is inconsistent, then we choose that value of a that minimizes an average error E in the m equations. The most convenient average comes from the ***sum of squares***:

Squared Error $$E^2 = \sum_{i=1}^{m} (a_i x - b_i)^2$$

If there is an exact solution the minimum error is E = 0. If not, the minimum error occurs when $\frac{dE^2}{dx} = 0$

Solving for x, the least squares solution is $\hat{x} = \frac{a^T b}{a^T a}$

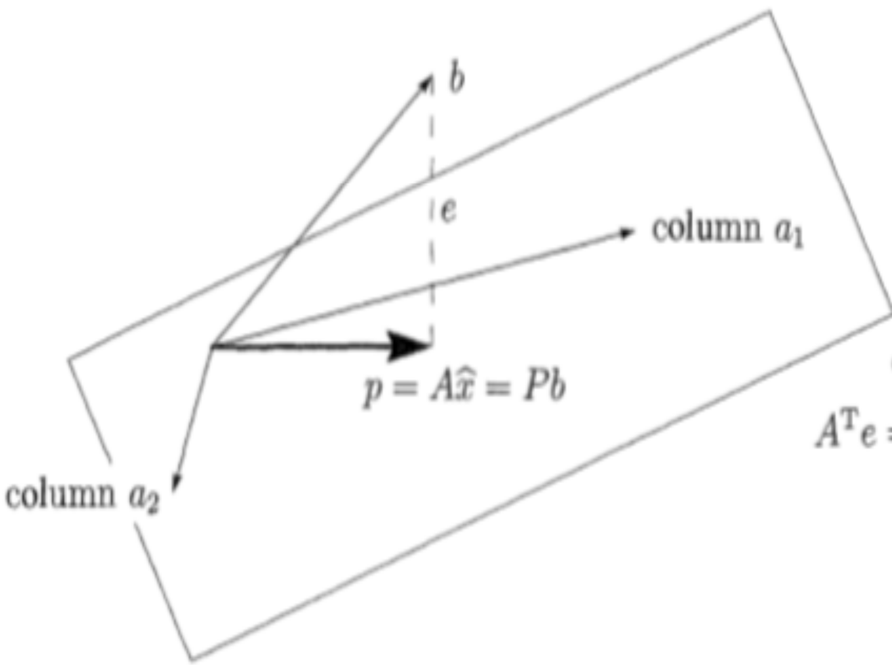
Least Squares Problem With Several Variables

Consider a system of equations Ax = b that is inconsistent.

The vector b lies outside C(A) and we need to project it onto C(A) to get the point p in C(A) that is closest to b. The problem here is the same as to minimize the error $E = \| Ax - b \|$ and this is exactly the distance from b to the point Ax in C(A).

Searching for the least squares solution $\hat{x}$ is the same as locating the point p that is closest to b.

Least Squares Problem With Several Variables



$a_1^T e = 0$

$a_2^T e = 0$

combine into

$A^T e = A^T(b - A\hat{x}) = 0$

Least Squares Problem With Several Variables

The error vector $e = b - A\hat{x}$ must be perpendicular to C(A) and hence can be found in the left null space of A.

Thus, $A^T(b - A\hat{x}) = 0 \;\; or \;\; A^T A\hat{x} = A^T b$

These are called the *Normal Equations*.

Solving them, we get the optimal solution $\hat{x}$

Note :

If b is orthogonal to C(A) then its projection is the zero vector.

Problems on Projections and Least squares

Find the projection of $b = (1, 2, 7)$ onto the column space of A spanned by $(1, 1, -2)$ and $(1, -1, 4)$. Split b into $p + q$ with p in $C(A)$ and q in $N(A^T)$.

Solution! Let p be the projection of b onto $C(A)$ which is spanned by $(1, 1, -2)$ and $(1, -1, 4)$

So $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}$ and $p = A\hat{x}$

Problems on Projections and Least squares

Normal equation to find $\hat{x}$ is

$A^TA\hat{x} = A^Tb \Rightarrow \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix} \hat{x} = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$

$\Rightarrow \hat{x} = \begin{bmatrix} 9/22 \\ 37/22 \end{bmatrix}$

$\therefore p = A\hat{x} = \begin{bmatrix} 23/11 \\ -14/11 \\ 65/11 \end{bmatrix}$ and $p + q = b$

$\Rightarrow q = b - p$

$q = \begin{bmatrix} -12/11 \\ 36/11 \\ 12/11 \end{bmatrix}$

and $q \in N(A^T)$

Unit 3. Linear Transformations and Orthogonality

Problems on Projections and Least squares

2. Find a basis for the orthogonal complement of the row space of $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$. Split the vector $(3,3,3)$ into a row space component x_r and a null space component x_n.

Solution: x_r and x_n are projections of $x = (3,3,3)$ onto $C(A^T)$ and $N(A)$ respectively.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \underset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow N(A) = \left\{ \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

Problems on Projections and Least squares

Let $a = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

Projection of b onto a line through 'a' is x_n.

$$x_n = \hat{x} a = \frac{a^T b}{a^T a} \cdot a = \frac{-9}{9} \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

We know that $x = x_r + x_n \Rightarrow x_r = x - x_n = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$

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Projection Matrices

The matrix P that projects onto C(A) is given by

$$\textbf{Projection matrix} \qquad P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}.$$

Also , if P and Q are the matrices that project onto orthogonal subspaces then it is always true that PQ = 0 and P + Q = I

Least Squares Fitting Of Data

Suppose we do a series of experiments and expect the output b to be a linear function of the input t. We look for a straight line

$$b = C + Dt$$

If there is no experimental error then two measurements of b will determine the line. But, if there is error, we minimize it by the method of least squares and find the optimal straight line.

Least Squares Fitting Of Data

Consider the following system of equations:

$$C + Dt_1 = b_1$$
$$C + Dt_2 = b_2 .....$$
$$C + Dt_m = b_m$$

In matrix form, $\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ - & - \\ 1 & t_m \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ - \\ b_m \end{bmatrix}$ or $Ax = b$

The best solution $\hat{x}$ can be obtained by solving the normal equations.

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Problems on Least Squares Fitting Of Data

Use the method of least square to fit the best line to the data $b = 4, 3, 1, 0$ at $t = -2, -1, 0, 2$ respectively. Find the projection of $b = (4, 3, 1, 0)$ onto the column space of A. Calculate the error vector 'e' and check that 'e' is orthogonal to the columns of A.

Solution: Let $C + Dt = b$ be the best fit straight line for the given data.

Problems on Least Squares Fitting Of Data

Given that $b = 4, 3, 1, 0$ at $t = -2, -1, 0, 2$

$$\Rightarrow \left.\begin{aligned} C + D(-2) &= 4 \\ C + D(-1) &= 3 \\ C + D(0) &= 1 \\ C + D(2) &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow Ax = b$$

The system is inconsistent. To find least square solution $\hat{x}$, we have to solve normal equation

ie $\quad A^T A \hat{x} = A^T b$

$$\Rightarrow \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix} \hat{x} = \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$$\hat{x} = \begin{pmatrix} \hat{C} \\ \hat{D} \end{pmatrix} = \begin{pmatrix} 61/35 \\ -36/35 \end{pmatrix}$$

$\therefore$ The best straight line fit for the given data is

$$b = \frac{61}{35} - \frac{36}{35} t$$

Let p be the projection of $b = (4, 3, 1, 0)$ onto $C(A)$

$$\therefore \quad p = A\hat{x} = \frac{1}{35} \begin{pmatrix} 133 \\ 97 \\ 61 \\ -11 \end{pmatrix}$$

Problems on Least Squares Fitting Of Data

The error vector

$$e = b - p = \begin{pmatrix} 1/5 \\ 8/35 \\ -26/35 \\ 11/35 \end{pmatrix}.$$

The error vector e is orthogonal to both the columns of A

i.e $e^T \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$ and $e^T \begin{pmatrix} -2 \\ -1 \\ 0 \\ 2 \end{pmatrix} = 0$

Therefore the vector 'e' is orthogonal to column space of A.

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Unit-4

Orthogonalization , Eigen Values and Eigen Vectors

Topics in the Module:

- Orthogonal Bases, orthogonal matrices and properties
- Rectangular matrices with orthonormal columns
- The Gram- Schmidt Orthogonalization
- A=QR factorization
- Introduction to Eigenvalues and Eigenvectors
- Properties of Eigenvalues and Eigenvectors and C-H theorem
- Problems on Eigenvalues and Eigenvectors and C-H theorem
- Symmetric Matrices and Diagonalization of a Matrix
- Problems on diagonalization of a matrix
- Powers and products of matrices

CLASS-1

ORTHOGONAL BASES, ORTHOGONAL MATRICES AND PROPERTIES

In an **orthogonal basis**, every vector is perpendicular to every other vector. The coordinate axes are mutually orthogonal.

Mutually perpendicular unit vectors are called **Orthonormal** vectors.

For the vector space R^2,

1. The set (2, 0) , (0, 2) is an orthogonal basis.
2. The set (1, -2) , (2 , 1) is an orthogonal basis.
3. The set (1, 0), (0, 1) is an orthonormal basis.

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- A matrix with Orthonormal columns will be called Q.
- A square matrix with Orthonormal columns is called an ***Orthogonal matrix*** denoted by Q.

Ex: Rotation matrix , any permutation matrix .

Properties of Q

- If Q (square or rectangular) has orthonormal columns, then $Q^T Q = I$.
- An orthogonal matrix is a square matrix with orthonormal columns. Then Q^T is Q^{-1}.
- If Q is rectangular then Q^T is left inverse of Q.
- Multiplication by any Q preserves length. The norms of x and Qx are equal.

Properties of Q (Continued.....)

- Also, Q preserves inner products and angles, since $(Q x)^T (Qy) = x^TQ^TQy = x^Ty$.

If $q_1, q_2 \ldots . q_n$ are orthonormal basis of R^n then any vector b from R^n can be expressed as

$$b = x_1 q_1 + x_2 q_2 + \ldots . + x_n q_n \qquad \ldots . \text{Eqn } (1)$$

Multiply both sides by q_1^T. Then $q_1^T b = x_1$.

Similarly, $x_2 = q_2^T b, \ldots \ldots, x_n = q_n^T b$.

Hence, $b = (q_1^T b) q_1 + (q_2^T b) q_2 + \ldots . + (q_n^T b) q_n$

= sum of one dimensional projections on to q_i's.

The matrix form of equation (1) is $Qx = b$ and the solution of this system of equations is

$$x = Q^{-1} b = Q^T b$$

Properties of Q (Continued…..)

- The rows of a square matrix are orthonormal whenever the columns are

Orthonormal columns
Orthonormal rows

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}.$$

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CLASS-2

RECTANGULAR MATRICES WITH ORTHONORMAL COLUMNS

Rectangular matrices with orthonormal columns

- If Q has orthonormal columns, the least-squares problem becomes easy.
- $Q^T Qx = Q^T b$ are the normal equations for the best solution -in which $Q^T Q = I$.
- $x = Q^T b$
- $p=Qx$ the projection of b is $(q_1^T b)q_1+\ldots+(q_n^T b)q_n$
- $p = QQ^T b$, the projection matrix is $P = Q Q^T$.

Special cases:-

1. If Q is a square matrix, $\hat{x} = Q^T b$

$\hat{x} = Q^{-1} b \quad [Q^T = Q^{-1}]$

2. If Q is a square matrix, $P = Q\hat{x}$

$= QQ^{-1} b$

$P = b$

Problems:

1. Project $b=(0,3,0)$ onto each of the orthonormal vectors $a_1=\left(\frac{2}{3},\frac{2}{3},-\frac{1}{3}\right)$, $a_2=\left(-\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$ & then find its projection p onto the plane of a_1 & a_2.

Sol^n The Projection of b onto a_1 is given by

$$p_1=\left(\frac{a_1^Tb}{a_1^Ta_1}\right)a_1 \qquad \text{but } a_1^Ta_1=||a_1||=\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}}=1$$

$$p_1=\left(\begin{bmatrix}\frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}\begin{bmatrix}0\\3\\0\end{bmatrix}\right)\begin{bmatrix}2/3\\2/3\\-1/3\end{bmatrix} \Rightarrow p_1=\begin{bmatrix}4/3\\4/3\\-2/3\end{bmatrix}$$

Problems:

Similarly, $P_2 = \frac{(a_2^T b) a_2}{a_2^T a_2} = \begin{bmatrix} -2/3 \\ 4/3 \\ 4/3 \end{bmatrix}$

$\therefore$ P is the projection onto the plane containing a_1 & a_2 i.e. $P = P_1 + P_2$

$$= \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix}$$

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Problem 2:- Find a third column so that the matrix

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & - \\ 1/\sqrt{3} & 2/\sqrt{14} & - \\ 1/\sqrt{3} & -3/\sqrt{14} & - \end{bmatrix}$$

is orthogonal. verify that the rows automatically become orthonormal at the same time.

Ans: By the definition of orthogonal matrix, first and second columns are orthogonal to the required third column. Let the third column be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

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$$B = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & x \\ 1/\sqrt{3} & 2/\sqrt{14} & y \\ 1/\sqrt{3} & -3/\sqrt{14} & z \end{bmatrix} \Rightarrow a^T c = 0 \Rightarrow \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = 0 \Rightarrow x + y + z = 0 \longrightarrow (1)$$

Similarly, $b^T c = 0 \Rightarrow \begin{bmatrix} 1/\sqrt{14} & 2/\sqrt{14} & -3/\sqrt{14} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$$\Rightarrow x + 2y - 3z = 0 \longrightarrow (2)$$

Solving ① & ②, $(x, y, z) = (-5, 4, 1)$

$\therefore \quad Q = (a \quad b \quad c)$

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & -5/\sqrt{42} \\ 1/\sqrt{3} & 2/\sqrt{14} & 4/\sqrt{42} \\ 1/\sqrt{3} & -3/\sqrt{14} & 1/\sqrt{42} \end{bmatrix}$$

For the rows to be orthonormal, the norm must be equal to 1.

i.e. $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{5}{\sqrt{42}}\right)^2 = 1$

$$\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{4}{\sqrt{42}}\right)^2 = 1$$

& $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{-3}{\sqrt{14}}\right)^2 + \left(\frac{1}{\sqrt{42}}\right)^2 = 1$

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Unit-4

Orthogonalization , Eigen Values and Eigen Vectors

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CLASS-3

THE GRAM-SCHMIDT ORTHOGONALIZATION

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The Gram-Schmidt process:

- It is a process of converting linearly independent vectors into orthonormal vectors.

- Consider any 3 independent vectors a, b, c. Then the first orthonormal q_1= a/norm(a).

- If ‘b’ is perpendicular to the vector ‘a’ then q_2=b/norm(b) otherwise B=b-(q_1^Tb)q_1 and q_2=B/norm(B).

The Gram-Schmidt process (Continued.....)

- If 'c' is perpendicular to the plane spanned by the vectors a and b then $q_3=c/\text{norm}(c)$ otherwise $C=c-(q_1^Tc)q_1 - (q_2^Tc)q_2$ and $q_3=C/\text{norm}(C)$.

This is the one idea of the whole Gram-Schmidt process, to subtract from every new vector its components in the directions that are already settled. That idea is used over and over again. When there is a fourth vector, we subtract away its components in the directions of q_1, q_2, q_3.

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Problems:-

1. From the vectors a, b, c find orthonormal vectors q_1, q_2 & q_3 given $a = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, c = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Ans: From Gram-Schmidt Process $q_1 = \frac{a}{\|a\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$q_2 = \frac{e_2}{\|e_2\|}$ where $e_2 = b - (q_1^T b) q_1$

$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left[\left(1/\sqrt{2} \;\; 1/\sqrt{2} \;\; 0 \right) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right] \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$

$\therefore \; q_2 = \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right)$

$\cdot\ q_3 = \dfrac{e_3}{\|e_3\|}$, $e_3 = c - (q_1^T c)q_1 - (q_2^T c)q_2$

$$e_3 = \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix} , \quad \|e_3\| = \frac{2\sqrt{3}}{3}$$

$$\therefore q_3 = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

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CLASS-4

A=QR FACTORIZATION

The factorization A=QR

- Let A be a matrix whose columns are a, b, c
- Let Q be the matrix whose columns are q_1, q_2 and q_3 which are determined using Gram $-$ Schmidt process.
- Then to find the third matrix which connects A and Q, express a, b, c as a linear combination of q_1, q_2, q_3.

The whole factorization is

$$A = \begin{bmatrix} & & \\ a & b & c \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ q_1 & q_2 & q_3 \\ & & \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{bmatrix}$$

$$A = QR$$

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Note: Consider the normal equation for $Ax=b$,

$$A^T A\hat{x} = A^T b \quad \text{But } A = QR$$

$$(QR)^T QR\hat{x} = (QR)^T b$$

$$R^T Q^T Q R\hat{x} = R^T Q^T b$$

$$R^T R\hat{x} = R^T Q^T b$$

$$\therefore \boxed{R\hat{x} = Q^T b}$$

$\Rightarrow$ When $Ax=b$ is not solvable, we consider $R\hat{x} = Q^T b$ and solve.

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⇒ If A is a square matrix of order 'n' then Q & R are also square matrix of order 'n'. But if A is a matrix of order $m \times n$ then Q is also a matrix of order $m \times n$ but R is a square matrix of order 'n'.

$$\begin{bmatrix} \uparrow & \uparrow \\ a & b \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow \\ q_1 & q_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b \\ 0 & q_2^T b \end{bmatrix}$$

Problem 1:- Factor $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix}$ into QR,

recognizing that first column is already a unit vector.

Ans: $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix}$, $Q = [q_1 \; q_2]$

$\downarrow a \qquad \downarrow b$

$q_1 = \frac{a}{\|a\|} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, $e_2 = b - (q_1^T b)\, q_1$

$e_2 = \begin{pmatrix} \sin\theta(1-\cos^2\theta) \\ -\cos\theta \sin^2\theta \end{pmatrix}$

$$q_2 = \frac{c_2}{\|c_2\|} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

$$R = \begin{pmatrix} q_1^T a & q_1^T b \\ 0 & q_2^T b \end{pmatrix} = \begin{pmatrix} 1 & \cos\theta\sin\theta \\ 0 & \sin^2\theta \end{pmatrix}$$

$$A = QR \Rightarrow \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} 1 & \cos\theta\sin\theta \\ 0 & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \cos^2\theta\sin\theta + \sin^3\theta \\ \sin\theta & \cos\theta\sin^2\theta - \cos\theta\sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix}$$

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Problem: Find an orthonormal set q_1, q_2, q_3 for which q_1, q_2 span the column space of $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$. Which fundamental subspace contains q_3. What is the least squares solution of $Ax = b$ if $b = [1 \ 2 \ 7]^T$?

Ans: $a = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, b = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \Rightarrow q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{9}} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$q_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$, $q_2 = \frac{e_2}{\|e_2\|}$ where $e_2 = b - (q_1^T b) q_1$

$e_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$\therefore q_2 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$

Let $e_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. To find q_3, let the vector e_3 be assumed as orthogonal to the plane spanned by q_1 & q_2 such that $q_1^T e_3 = 0$ & $q_2^T e_3 = 0$.

$$\Rightarrow \begin{bmatrix} 1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \; , \; \begin{bmatrix} 2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow x + 2y - 2z = 0 \; , \; 2x + y + 2z = 0$$

On solving, $e_3 = (-2, 2, 1)$

$$\therefore q_3 = \frac{e_3}{\|e_3\|} = \left(-2/3 , 2/3 , 1/3\right).$$

Consider $Ax = b \Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

The above system is inconsistent. i.e $Ax = b$ is not solvable & hence we consider $R\hat{x} = Q^Tb$ where

$$R = \begin{bmatrix} q_1^T a & q_1^T b \\ 0 & q_2^T b \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}$$

$$\therefore R\hat{x} = Q^Tb \Rightarrow \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}\hat{x} = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}\hat{x} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} \Rightarrow \hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

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CLASS-5

INTRODUCTION TO EIGEN VALUES AND EIGEN VECTORS

Definition :

Let A be a square matrix of order n. If there exists a real or complex number λ and a non zero vector x such that $Ax = \lambda x$ then x is called the ***Eigen vector of A*** and λ is its corresponding ***Eigen value***.

Note :

- The vector x is in the null space of A- λ I.
- The number λ is chosen so that A- λ I has a null space.
- A- λ I must be singular.
- Det(A- λ I)=0 is called the *characteristic equation of A* and roots of this equation are called *characteristic roots or Eigen values or Latent roots*.

Corresponding to 'n' distinct Eigen values we get 'n' independent Eigen vectors. But when 2 or more eigen values are equal, it may or may not be possible to get linearly independent Eigen vectors corresponding to repeated roots.

Procedure to find eigenvalues and eigenvectors

- Compute the determinant of $A - \lambda I$. With a λ subtracted along the diagonal, this determinant is a polynomial of degree n. It starts with $(-\lambda)^n$.
- Find the roots of this polynomial. The n roots are the eigenvalues of A.

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For each eigenvalue λ, solve the equation $(A - \lambda I)x = 0$. Since the determinant of $A - \lambda I$ is zero, there are solutions other than $x = 0$. Those are the eigenvectors.

Example: Find Eigen values and Eigen vectors

If

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

then the characteristic equation is

$$|\mathbf{A} - \lambda \cdot \mathbf{I}| = \left| \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} \right| = \lambda^2 + 3\lambda + 2 = 0$$

and the two eigenvalues are

$$\lambda_1=-1,\ \lambda_2=-2$$

Let's find the eigenvector, $\mathbf{v}_1$, associated with the eigenvalue, λ_1=-1, first.

$$\mathbf{A}\cdot\mathbf{v}_1 = \lambda_1\cdot\mathbf{v}_1$$

$$\left(\mathbf{A}-\lambda_1\right)\cdot\mathbf{v}_1 = 0$$

$$\begin{bmatrix} -\lambda_1 & 1 \\ -2 & -3-\lambda_1 \end{bmatrix}\cdot\mathbf{v}_1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\cdot\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\cdot\begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = 0$$

so clearly from the top row of the equations we get

$$v_{1,1}+v_{1,2}=0, \quad \text{so}$$
$$v_{1,1} = -v_{1,2}$$

Note that if we took the second row we would get

$$-2\cdot v_{1,1}+-2\cdot v_{1,2}=0, \quad \text{so again}$$
$$v_{1,1} = -v_{1,2}$$

In either case we find that the first eigenvector is any 2 element column vector in which the two elements have equal magnitude and opposite sign.

$$\mathbf{v}_1 = k_1 \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

where k_1 is an arbitrary constant. Note that we didn't have to use +1 and -1, we could have used any two quantities of equal magnitude and opposite sign.

Going through the same procedure for the second eigenvalue:

$$\mathbf{A} \cdot \mathbf{v}_2 = \lambda_2 \cdot \mathbf{v}_2$$

$$(\mathbf{A} - \lambda_2) \cdot \mathbf{v}_2 = \begin{bmatrix} -\lambda_2 & 1 \\ -2 & -3-\lambda_2 \end{bmatrix} \cdot \mathbf{v}_2 = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} = 0 \quad \text{so}$$

$$2 \cdot v_{2,1} + 1 \cdot v_{2,2} = 0 \quad \left(\text{or from bottom line: } -2 \cdot v_{2,1} - 1 \cdot v_{2,2} = 0\right)$$

$$2 \cdot v_{2,1} = -v_{2,2}$$

$$\mathbf{v}_2 = k_2 \begin{bmatrix} +1 \\ -2 \end{bmatrix}$$

Again, the choice of +1 and -2 for the eigenvector was arbitrary; only their ratio is important.

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LINEAR ALGEBRA AND ITS APPLICATIONS

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CLASS-6

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS AND CAYLEY-HAMILTON THEOREM

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS:

- If λ is an Eigen value of A with x as the corresponding Eigen vector then λ^2 is an Eigen value of A^2 with the same Eigen vector x.
- For a given Eigen vector x, there corresponds only one Eigen value λ.
- For a given Eigen value there corresponds infinitely many Eigen vectors.

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS (Contd……..)

- $\lambda = 0$ is an Eigen value of A , if and only if A is singular i.e det(A)=0.
- If λ is an Eigen value of A with x as the Eigen vector then $1/\lambda$ is an Eigen value of A^{-1} provided A^{-1} exists.
- A and its transpose A^T have the same Eigen values.

- The Eigen values of a diagonal matrix are just the diagonal elements of the matrix.
- The Eigen values of an idempotent matrix are either zero or unity.
- The sum of the Eigen values of a matrix is the sum of the elements of the principal diagonal.
- The product of the Eigen values of a matrix A is equal to its determinant.

"Every square matrix satisfies its own characteristic equation"

Proof: Let A be an n-square matrix. Let $D(\lambda)$ be the characteristic polynomial of A given by,

$$D(\lambda) = |\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_1\lambda + c_0 \longrightarrow (1)$$

Let $B(\lambda)$ be the adjoint of $(\lambda I - A)$. The elements of $B(\lambda)$ are cofactors of the matrix $(\lambda I - A)$ and are polynomials in λ of degree not exceeding $n-1$. Thus

$$B(\lambda) = B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + \cdots + B_1\lambda + B_0 \longrightarrow (2)$$

where B_i are n-square matrices whose elements are functions of the elements of A and independent of λ.

We know that $(\lambda I - A) \cdot \operatorname{adj}(\lambda I - A) = |\lambda I - A| I$

$$(\lambda I - A) \cdot B(\lambda) = |\lambda I - A| I$$

From (1) & (2), we have

$$(\lambda I - A)(B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + \cdots + B_1\lambda + B_0)$$

$$= I(\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0) \longrightarrow (3)$$

Equating the like powers of λ on both sides of (3), we get

$$B_{n-1} = I$$

$$B_{n-2} - AB_{n-1} = C_{n-1} I$$

$$B_{n-3} - AB_{n-2} = C_{n-2} I$$

$$\cdots\cdots$$

$$B_0 - AB_1 = C_1 I$$

$$-AB_0 = C_0 I$$

Multiplying both sides of the above matrix equations by $A^n, A^{n-1}, A^{n-2}, \ldots, A, I$, respectively, we have

$$A^n B_{n-1} = A^n$$

$$A^{n-1} B_{n-2} - A^n B_{n-1} = c_{n-1} A^{n-1}$$

$$A^{n-2} B_{n-1} - A^{n-1} B_{n-2} = c_{n-2} A^{n-2}$$

$$- - - - \cdots$$

$$AB_0 - A^2 B_1 = c_1 A$$

$$-AB_0 = c_0 I$$

By adding all the above equations, we get

$$0 = A^n + c_{n-1} A^{n-1} + c_{n-2} A^{n-2} + \cdots + c_1 A + c_0 I \longrightarrow (4)$$

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· Since all the terms on the L.H.S cancel each other. Thus A satisfies its own characteristic equation.

Inverse by Cayley-Hamilton theorem:-

Multiplying (4) by A^{-1},

$$0 = A^{n-1} + c_{n-1}A^{n-2} + c_{n-2}A^{n-3} + \cdots + c_1 I + c_0 A^{-1}$$

Solving for A^{-1}, we get

$$A^{-1} = -\frac{1}{c_0}\left[A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I\right]$$

Note:- A^{-1} exists only iff c_0 = determinant of $A \neq 0$.

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CLASS-7

PROBLEMS ON EIGEN VALUES AND EIGEN VECTORS AND CAYLEY-HAMILTON THEOREM

Problem:-

1. Find the Eigen values & Eigen vectors for the given matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Ans: Characteristic eqn is $|A - \lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (6-\lambda)[(3-\lambda)^2 - 1] + 2[-2(3-\lambda)+2] + 2(2 - 2(3-\lambda)) = 0$$

$$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$\lambda = 8$ is one of the roots.

$$\begin{array}{r|rrrr} 8 & 1 & -12 & 36 & -32 \\ & 0 & 8 & -32 & 32 \\ \hline & 1 & -4 & 4 & 0 \end{array} \Rightarrow \lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow \lambda = 2, 2$$

$\therefore \lambda_1 = 8, \lambda_2 = 2, \lambda_3 = 2.$

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Consider $|A-\lambda I|X=0$

$$\Rightarrow \quad (6-\lambda)x-2y+2z=0$$
$$-2x+(3-\lambda)y-z=0$$
$$2x-y+(3-\lambda)z=0$$

When $\lambda=8$,

$$\left.\begin{aligned} -2x-2y+2z&=0\\ -2x-5y-z&=0\\ 2x-y-5z&=0 \end{aligned}\right\} \text{ On solving } X_1=\begin{bmatrix}2\\-1\\1\end{bmatrix}$$

When $\lambda=2$,

$$\left.\begin{aligned} 4x-2y+2z&=0\\ -2x+y-z&=0\\ 2x-y+z&=0 \end{aligned}\right\} \text{ On solving } X_2=\begin{bmatrix}\frac{k_1}{2}-\frac{k_2}{2}\\ k_1\\ k_2\end{bmatrix}=X_3$$

Where k_1 & k_2 are arbitrary & can take any value.

$\therefore$ Eigen vector is $X=[X_1\ X_2\ X_3]$

2. Find the Eigen values & Eigen vectors for the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

Ans: The characteristic equation is $|A-\lambda I|=0$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & -\lambda & 0 \\ -1 & 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3+\lambda=0$$

$$\Rightarrow \lambda=0,\ \lambda^2+1=0$$

$$\lambda=0, \pm i$$

$\therefore \lambda_1=0,\ \lambda_2=i,\ \lambda_3=-i$ are the required eigen values.

Now consider $|A-\lambda I|X=0 \Rightarrow (1-\lambda)x-y+z=0$

$$x-\lambda y=0$$

$$-x+y+(-1-\lambda)z=0$$

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Case 1:- When $\lambda = 0$

$$\left.\begin{aligned} x - y + z &= 0 \\ x &= 0 \\ -x + y - z &= 0 \end{aligned}\right\}$$ On solving $X_1 = \begin{bmatrix} 0 \\ k_1 \\ k_2 \end{bmatrix}$. Let $k_1 = k_2 = 1$

$$\therefore X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case 2:- When $\lambda = i$ $\Rightarrow$ $(1-i)x - y + z = 0$

$$x - iy + 0.z = 0$$

$$-x + y + (-1-i)z = 0$$

$$X_2 = \begin{bmatrix} i \\ 1 \\ -i \end{bmatrix}$$

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Case 3:- When $\lambda = -i$,

$$\left.\begin{array}{l} (1+i)x - y + z = 0 \\ x + iy + 0z = 0 \\ -x + y + (-1+i)z = 0 \end{array}\right\} \quad X_3 = \begin{bmatrix} -i \\ 1 \\ i \end{bmatrix}$$

$\therefore$ Eigen vectors corresponding to $\lambda = 0, i, -i$ is

$$X = [X_1 \ X_2 \ X_3] = \begin{bmatrix} 0 & i & -i \\ 1 & 1 & 1 \\ 1 & -i & i \end{bmatrix}$$

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Problem:- If $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$. Evaluate A^{-1} & A^{-2} using Cayley-Hamilton theorem or (C-H) theorem.

Ans:- $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

According to C-H theorem $(\lambda \to A)$

$$A^3 - 4A^2 - A + 4I = 0 \longrightarrow (1)$$

Now to find A^{-1} & A^{-2}

To find A^{-1}

Multiplying eqn (1) by A^{-1},

$$\Rightarrow A^2 - 4A - I + 4A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{4}\left[-A^2 + 4A + I\right]$$

$$A^{-1} = \frac{1}{4}\left[-\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} + 4\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}$$

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To find A^{-2}:

Multiplying eqn (1) by A^{-2}

$$A - 4I - A^{-1} + 4A^{-2} = 0$$

$$\Rightarrow A^{-2} = \frac{1}{4}\left[-A + A^{-1} + 4I\right]$$

$$A^{-2} = \frac{1}{16}\begin{bmatrix} 1 & -18 & -18 \\ -5 & 10 & -6 \\ 5 & 6 & 22 \end{bmatrix}$$

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