THANK YOU

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VECTOR SPACES

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CLASS 10 : CONTENT

- Problems on Four Fundamental Subspaces

FUNDAMENTAL SUBSPACES

Problem 1: Find the Basis and Dimension for the four fundamental subspaces.

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix}$$

Solution :- StepI :- Apply Gauss Elimination and reduce the matrix to Echelon form.

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - R_1} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

FUNDAMENTAL SUBSPACES

Column space : $C(A)$

$$C(A) = \left\{ c_1 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \,\middle/\, c_1, c_2 \in R \right\}$$

$C(A)$ is a 2 Dimensional space spanned by $(1, 1, 3)$ and $(2, 3, 7)$ in R^3

$$\text{Basis of } C(A) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \right\}$$

Dim of $C(A) = 2$.

Note: Select columns of A and not from 'U' to span $C(A)$
$\because$ Columns are not Preserved in elementary Row operations.

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Row space : $C(A^T)$

$$C(A^T) = \left\{ c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} \middle/ c_1, c_2 \in R \right\}$$

or

$$C(A^T) = \left\{ c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \middle/ c_1, c_2 \in R \right\}$$

$C(A^T)$ is a 2 Dim Plane spanned by the linear combination of vectors $(1,2,1,2)$ and $(0,0,0,1)$ in R^4

Basis $C(A^T) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} \right\}$ Dimension of $C(A^T) = 2$

Note: Either non zero rows of A or U can be selected to span $C(A^T)$

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Null Space : N(A)

Special solutions to $Ax=0 \Rightarrow Ux=0$ forms the basis of N(A).

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x+2y+y+2t=0 \Rightarrow x=-2y-z \;,\; t=0$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \sim \begin{pmatrix} -2y-z \\ y \\ z \\ 0 \end{pmatrix} \sim y\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

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N(A) is a 2 Dim Plane spanned by linear combination of vectors $(-2, 1, 0, 0)$ and $(-1, 0, 1, 0)$ in R^4.

Basis of N(A) : $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ Dimension = 2

$$N(A) = \left\{ y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} ;\ y, z \in R \text{ (as } y, z \text{ are free variables)} \right\}$$

Left Null space : $N(A^T)$:-

To find $N(A^T)$ Solve for $A^T y = 0 \Rightarrow y^T A = 0$.

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$$A = \begin{array}{c} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \qquad \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 1 & 2 & 1 & 3 & b_2 \\ 3 & 6 & 3 & 7 & b_3 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_2 - R_1$, $R_3 - 3R_1$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 1 & b_3 - 3b_1 \end{array}\right)$$

$R_3 - R_2$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & (b_3 - 3b_1) - (b_2 - b_1) \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right)$$

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$$N(A^T) = \left\{ c_1 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} ; c_1 \in R \right\}$$

$N(A^T)$ is a line spanned by $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ in R^3.

Basis for $N(A^T) = \left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$

Dimension for $N(A^T) = 1$

Note :- To find the left Null space, find the combination of the rows of A which produces zero rows. (instead of finding A^T)

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Problem 2 : Describe the Column space and the Null space for the following matrices.

1. $\begin{bmatrix} 0 \end{bmatrix}$

$C(A) = Z$; $N(A) = R$

2. $\begin{bmatrix} 0 , -3 \end{bmatrix}$

$C(A) = R$; $N(A) = \{ x \text{ axis in } R^2 \}$

3. $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$C(A) = \{ y \text{ axis in } R^2 \}$; $N(A) = Z$

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4. $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$C(A) = \{x \text{ axis in } R^2\}$; $N(A) = \{\text{line } x=y \text{ in } R^2\}$

5. $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$C(A) = \{x \text{ axis in } R^2\}$; $N(A) = \left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} ; x \in R \right\}$

$N(A)$ is line spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ in R^3

6. $\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$

$C(A) = \{\text{whole of } R^2\}$; $N(A) = \{\text{origin}(0,0) \text{ in } R^2\}$

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Problem 3: Find the Column Space and the Null space of $A = \begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 5 & 3 \end{pmatrix}$. Give an example of a matrix whoose C(A) is same as that of A but the Null space is different.

Solution:- C(A) is 2 Dim Plane in R^3

N(A) is origin in R^2.

$A' = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 7 & 9 \\ 5 & 3 & 8 \end{pmatrix}$ has same C(A') as C(A)

N(A') is line spanned by $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ in R^3.

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Problem 4: Let $V = \{(a,b,c,d) / b+c+d=0\}$ and $W = \{(a,b,c,d) / a+b=0$ and $c=2d\}$ be subspaces of R^4. Find the dimension of $V \cap W$.

Solution: $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$; $b+c+d=0$, $\sim \begin{pmatrix} a \\ -c-d \\ c \\ d \end{pmatrix} \sim a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

V is a 3 Dim Plane in R^4

$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$; $a+b=0$ and $c=2d$ $\begin{pmatrix} -b \\ b \\ 2d \\ d \end{pmatrix} \sim b\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$

W is a 2 Dim Plane in R^4.

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$V \cap W$

$b+c+d=0$ and $a+b=0$, $c=2d$

$b+3d=0 \Rightarrow b=-3d$, $\Rightarrow a=3d$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \sim \begin{pmatrix} 3d \\ -3d \\ 2d \\ d \end{pmatrix} \sim d \begin{pmatrix} 3 \\ -3 \\ 2 \\ 1 \end{pmatrix}$$

$V \cap W$ is $\therefore$ a line spanned by $(3,-3,2,1)$ in R^4.

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