LINEAR ALGEBRA AND ITS APPLICATIONS

Renna Sultana
Department of Science and Humanities

LINEAR ALGEBRA AND ITS APPLICATIONS

MATRICES AND
GAUSSIAN ELIMINATION

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Course Content: Prerequisites

What?

Why?

PREREQUISITES:

- **What** is Linear Algebra?

- Linear Algebra is the study of systems of Linear equations.
- It is concerned with vector spaces and linear mappings between such spaces.
- It includes the study of lines, planes and subspaces and is also concerned with properties common to all vector spaces.
- Linear Algebra is the geometry of n-dimensional space and its linear transformations. Thus Linear Algebra helps us to develop our geometric instinct to visualize the concepts in higher dimensions.

PREREQUISITES:

Why do we need to study Linear Algebra?

- Linear Algebra is used in the everyday world to solve problems in Mathematics, Physics, Biology, Chemistry, Engineering, Statistics, Economics, Finance, Psycology, Sociology, etc.,
- Applications that use Linear Algebra include the transmission of information, the development of special effects in film and video, recording of sound, Web search engines on the Internet and economic analyses.
- Linear Algebra is a study of Linear Transformations. It is used in Graph Theory, Networks, Signal Processing, Probability Theory, Real Analysis, Communication.
- Linear Algebra is used in coding theory, Artificial Intelligence, Machine Learning, Image Processing, Computer Graphics, Numerical Analysis, Control Systems, Networking, Ordinary/Partial Differential Equations.

INTRODUCTION:

Course Content of Linear Algebra:

- **Module 1:** Matrices and Gaussian Elimination
- **Module 2:** Vector Spaces
- **Module 3:** Linear Transformations and Orthogonality
- **Module 4:** Orthogonalization , Eigen Values and Eigen Vectors
- **Module 5:** Singular Value Decomposition

MATRICES AND GAUSSIAN ELIMINATION:

Course Content of Module 1:

❖Introduction

❖The Geometry of Linear Equations

❖Gaussian Elimination

❖Singular Cases

❖Elimination Matrices

❖Triangular Factors and Row Exchanges

❖Inverses and Transposes

❖Inverse by Gauss Jordan Method

INTRODUCTION:

Course Content: Introduction

What is a linear Equation?

- A linear equation in n variables is of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

where $x_1, x_2, \cdots, x_n$ are **unknowns or variables,** $a_i \in \square$ $(i = 1, 2, \cdots, n)$

are known as the **Coefficients** of the variables x_i $(i = 1, 2, \cdots, n)$

and $b \in \square$ are constants.

Examples of **Linear equations** are: $2x_1 - 3x_2 = 7$; $(\sqrt{2})x_1 - (\sqrt{5})x_2 = 1$

Examples of **Non-Linear equations** are: $x_1x_2 - 3x_3 = 2; y = \log x + \sin x;$

$$2x + 3y^{1/3} = 5$$

INTRODUCTION:

What is a system of linear Equations?

- A **system of linear equation** is a set of linear equations involving the same **unknowns or variables .**

- A system of **two** equations with **two** unknowns is of the form

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

- A system of **three** equations with **three** unknowns is of the form

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

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INTRODUCTION:

Matrix Notation:

- A system of **m** equations with **n** unknowns is given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

This can be represented in matrix form as **Ax=b** where

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix}_{mxn} \quad ; \; x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{nx1} \quad ; \; b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{mx1}$$

Here **A** is a matrix of order mxn known as **The Co-efficient Matrix**,

x_{nx1} is a nx1 column matrix(or vector) known as **The Vector of unknowns or variables**

b_{mx1} is a column vector of order mx1 known as **The Constant Vector**

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INTRODUCTION:

- a_{ij} is the component in the i^{th} **row** and j^{th} **column** of A.
- **If m=n,** A will be a **Square matrix of order n** and the system will have **n** equations with **n** variables.
- **If all b_i 's are zero,** the system is known as **Homogeneous system of Equations.** If at least one of **b_i 's is not zero** then the system is said to be **Non-Homogeneous system of Equations.**
- **Matrix** $[A:b] = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} & : b_1 \\ a_{21} & a_{22} \cdots & a_{2n} & : b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} & : b_m \end{pmatrix}_{mx(n+1)}$ Is called the **Augmented Matrix.**

INTRODUCTION:

- **Solution to the System of Equations:** A set of values or numbers assigned for variables $x_1, x_2, \cdots, x_n$ which satisfy all the equations simultaneously is defined as the **Solution** to the system of equations.

- The set of all possible solutions is called the **Solution Set** or **General Solution** of Linear system of equations.

- **Consistency:** A linear system of equations is said to be **Consistent** if it has a solution (unique solution or infinitely many solutions).

- A linear system of equations is said to be **Inconsistent** if it has no solution.

INTRODUCTION:

- **Elementary Row Operations:** In order to solve the system of linear equations using Matrix notation we define three basic operations called **Elementary Row Operations** or **Elementary Row Transformations** which are as follows:

- Multiply the entries of a row by a non-zero scalar k i.e $R_i \longrightarrow kR_i \,(k \neq 0)$
- Replace one row by sum of itself and a non-zero scalar multiple k of another row i.e $R_i \longrightarrow R_i + kR_j \,(k \neq 0)$.
- Interchange of any two rows i.e $R_i \leftrightarrow R_j$.

- **Elementary Matrix:** A square matrix of order n is called an **Elementary Matrix** E_n if it can be obtained from an Identity matrix I_n using a single row operation. i.e $I_n \to E_n$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = E_2$$

INTRODUCTION:

- **Equivalent Matrices:** If two matrices A and B are such that each of them can be obtained from the other by a definite number of Elementary transformations then they are said to be **Equivalent Matrices** represented by $A \sim B$.

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & -3 & 4 \end{pmatrix} \xrightarrow{R_2 + \left(\frac{1}{2}\right) R_1} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & 1 \\ 0 & -3 & 4 \end{pmatrix} \xrightarrow{R_3 + 2R_2}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 6 \end{pmatrix} \sim B \quad \therefore A \sim B$$

INTRODUCTION:

- **Echelon Form Of A Matrix:** A rectangular matrix A of order mxn is said to be in **Echelon Form U** if it satisfies the following conditions:

- The first non-zero element in every row is known as the **pivot.**
- Below each pivot is a column of **zeros** obtained by Elementary row operations.
- Each pivot lies to the right of the pivot in the row above. This produces a **Staircase pattern** as shown below.
- Zero rows(if exist) appear at the bottom of the Matrix.

$$A \rightarrow \begin{pmatrix} a & b & c & d & e \\ 0 & 0 & f & g & h \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{4x5} = U$$

INTRODUCTION:

- **Row Reduced Echelon Form Of A Matrix(RREF):** A rectangular matrix A in Echelon Form must further undergo the following operations to reduce to RREF :

- Every row in Echelon form must be divided by its pivot so that the first non-zero entry in every row is 1.

- Using the pivot rows produce zeros above the first non-zero entry (i.e 1).

This RREF is denoted by R.

$$A \rightarrow U = \begin{pmatrix} a & b & c & d & e \\ 0 & 0 & f & g & h \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & b/a & 0 & 0 & e/a \\ 0 & 0 & 1 & 0 & h/f \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R$$

INTRODUCTION:

Example:

$$A \to \begin{pmatrix} 1 & 1 & -2 & 3: & 4 \\ 2 & 3 & 3 & -1: & 3 \\ 5 & 7 & 4 & -1: & 5 \end{pmatrix}$$

References/Links:

https://en.wikipedia.org/wiki

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INTRODUCTION:

Example:

$$A \rightarrow \begin{pmatrix} 1 & 1 & -2 & 3: & 4 \\ 2 & 3 & 3 & -1: & 3 \\ 5 & 7 & 4 & -1: & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 & 3: & 4 \\ 0 & 1 & 7 & -7: & -5 \\ 0 & 0 & 0 & -2: & -5 \end{pmatrix} = U \rightarrow \begin{pmatrix} 1 & 0 & -9 & 0: & -16 \\ 0 & 1 & 7 & 0: & -25/2 \\ 0 & 0 & 0 & 1: & 5/2 \end{pmatrix} = R$$

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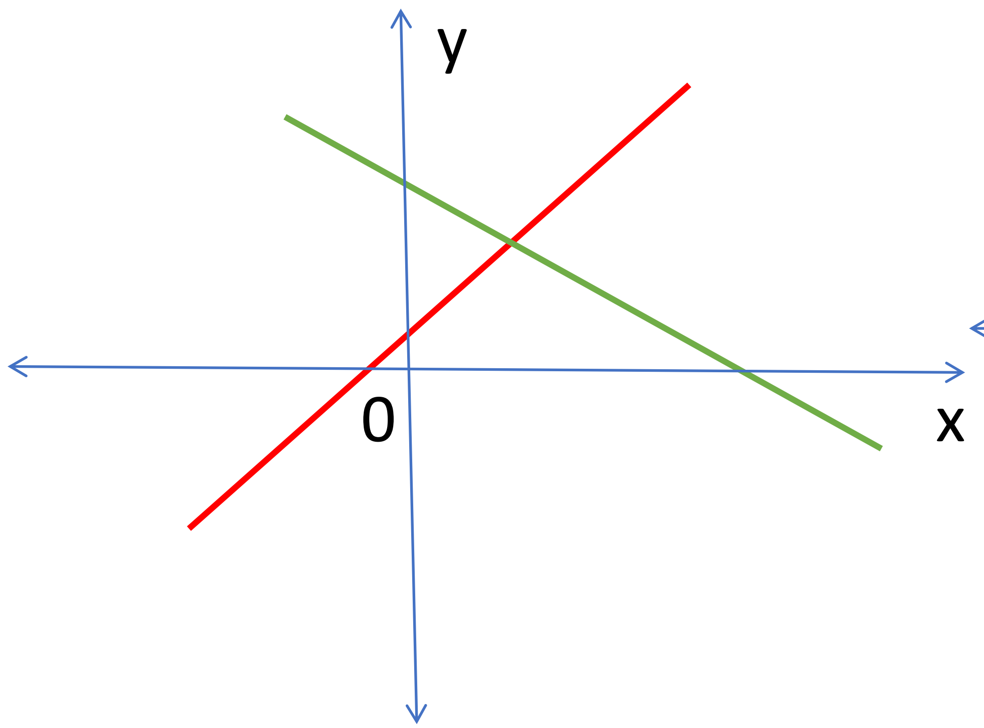
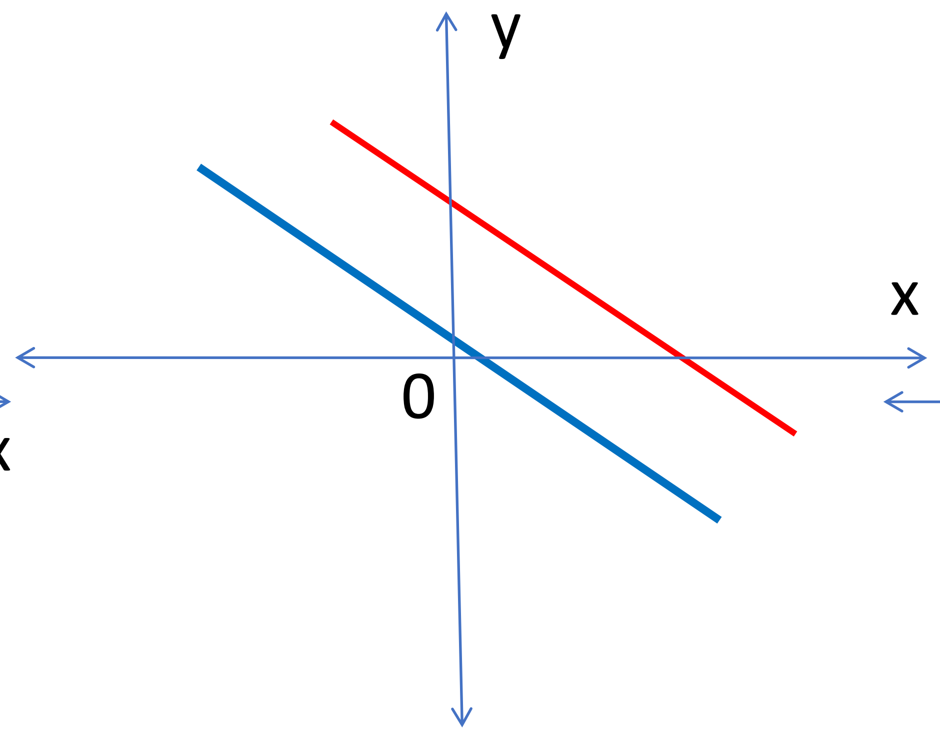
Course Content: The Geometry of Linear Equations

- SYSTEM OF 2 EQUATIONS WITH 2 VARIABLES:

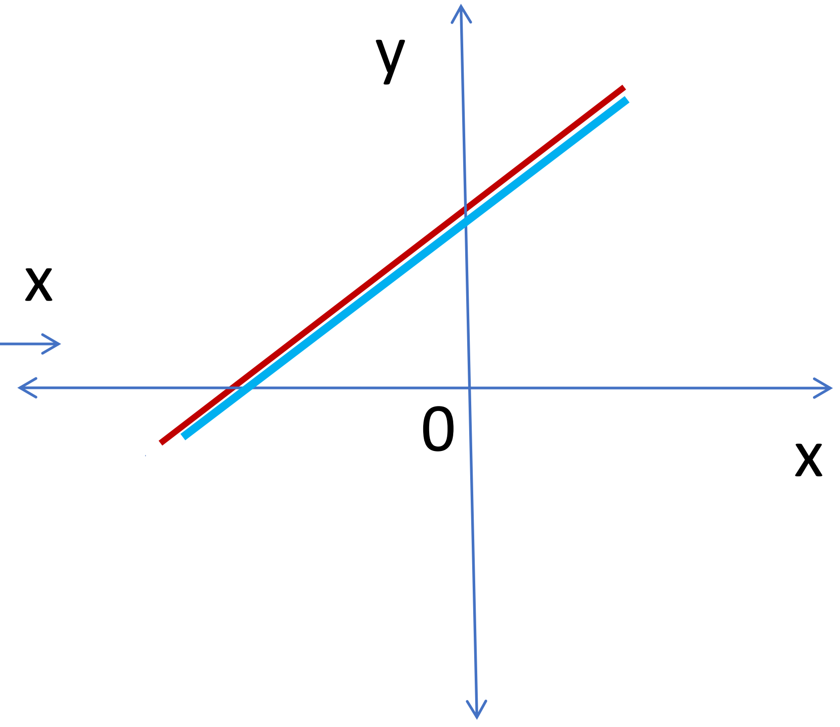
$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$



Intersecting lines
Unique solution



Parallel lines
No Solution



Overlapping lines
Infinite no. of solutions

System of m linear equations with n unknowns have either a unique solution or infinite solutions or no solution.

THE GEOMETRY OF LINEAR EQUATIONS:

- (i) Consider a system of 2 equations with 2 variables

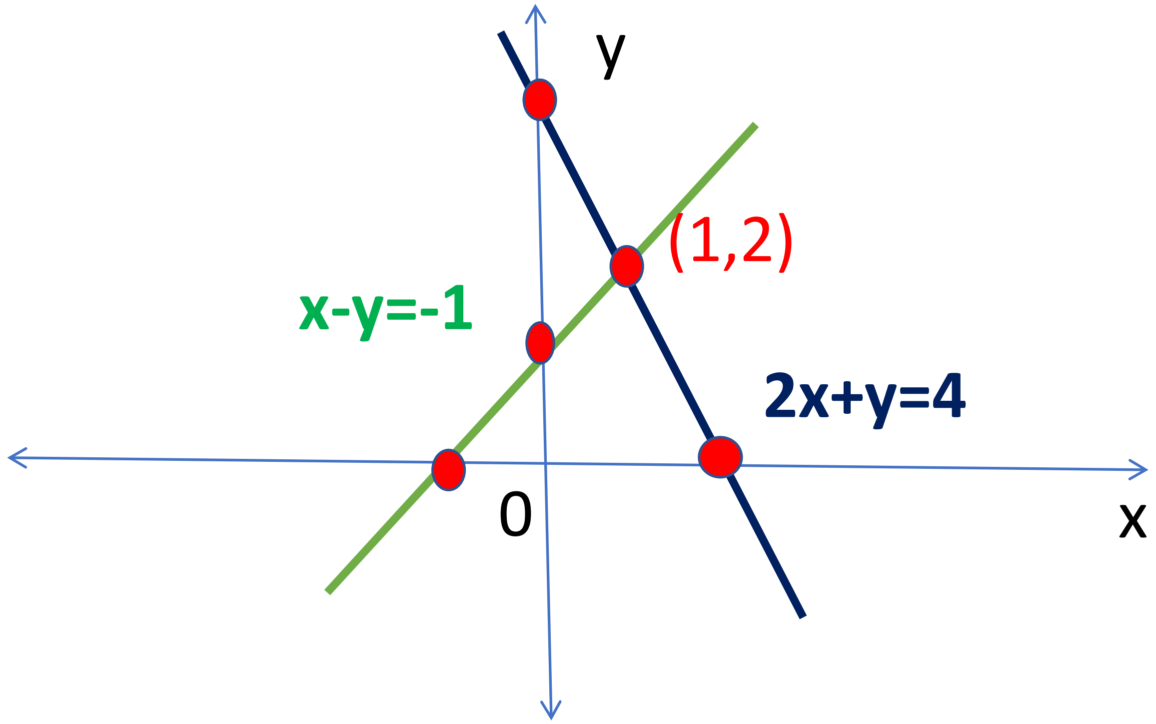
$$x - y = -1$$
$$2x + y = 4$$

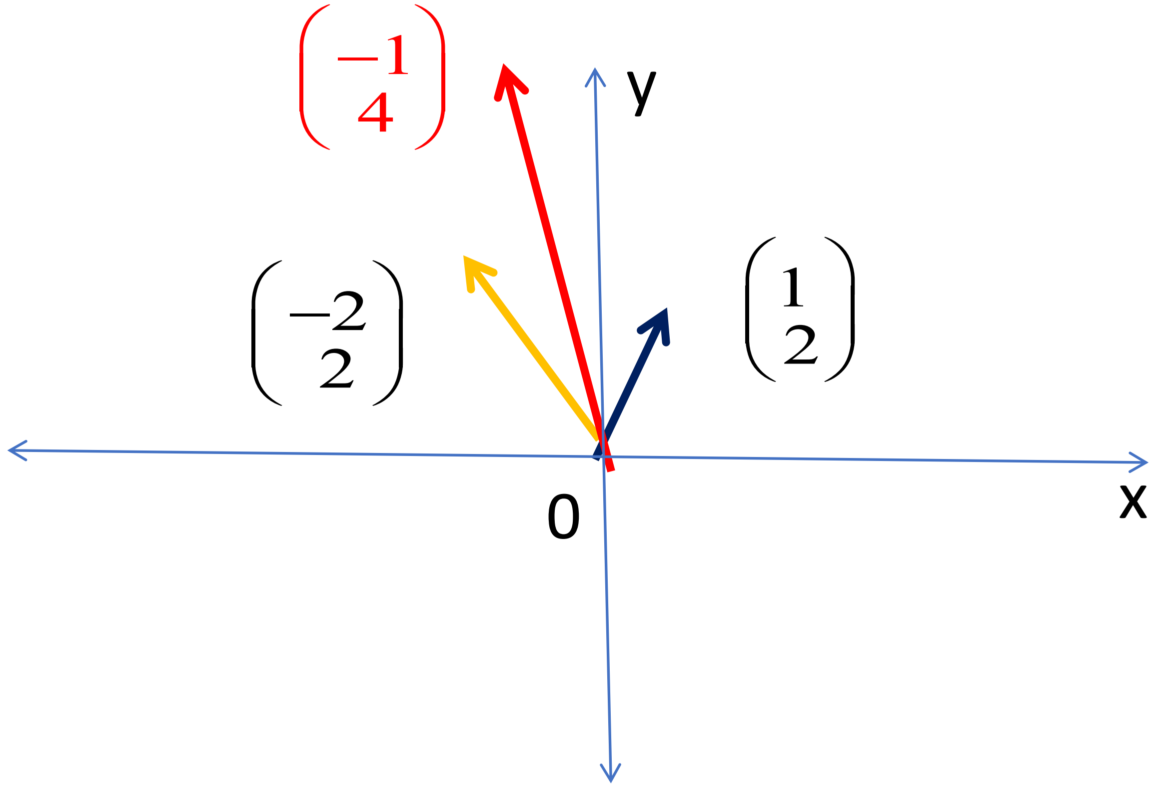
- **Row Picture:** Solving these 2 equations we get (1,2) as the point of intersection of the 2 lines. Hence this system has **a unique solution**.

- **Column Picture:** $x\begin{pmatrix}1\\2\end{pmatrix} + y\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1\\4\end{pmatrix} \Rightarrow 1\begin{pmatrix}1\\2\end{pmatrix} + 2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1\\4\end{pmatrix}$

  x=1 and y=2 will satisfy this equation.

  Hence the linear combination of the column vectors on LHS produces the vector on RHS.





THE GEOMETRY OF LINEAR EQUATIONS:

- (ii) Consider the system

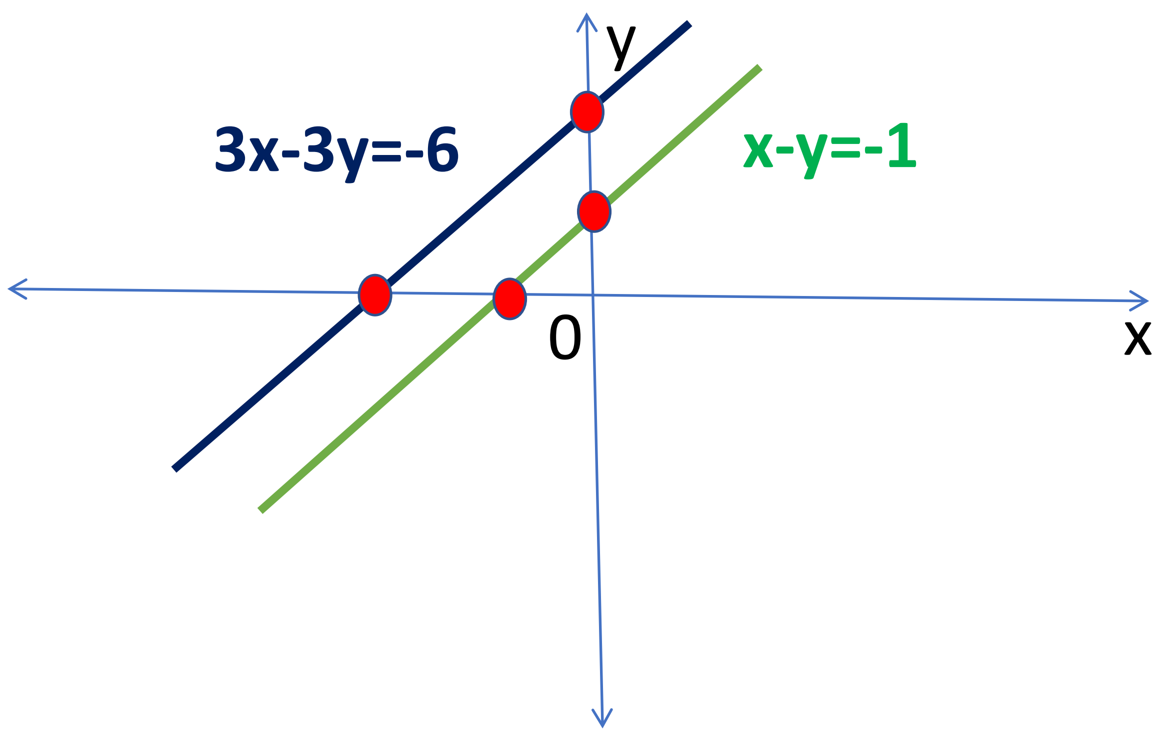
$$x - y = -1$$
$$3x - 3y = -6$$

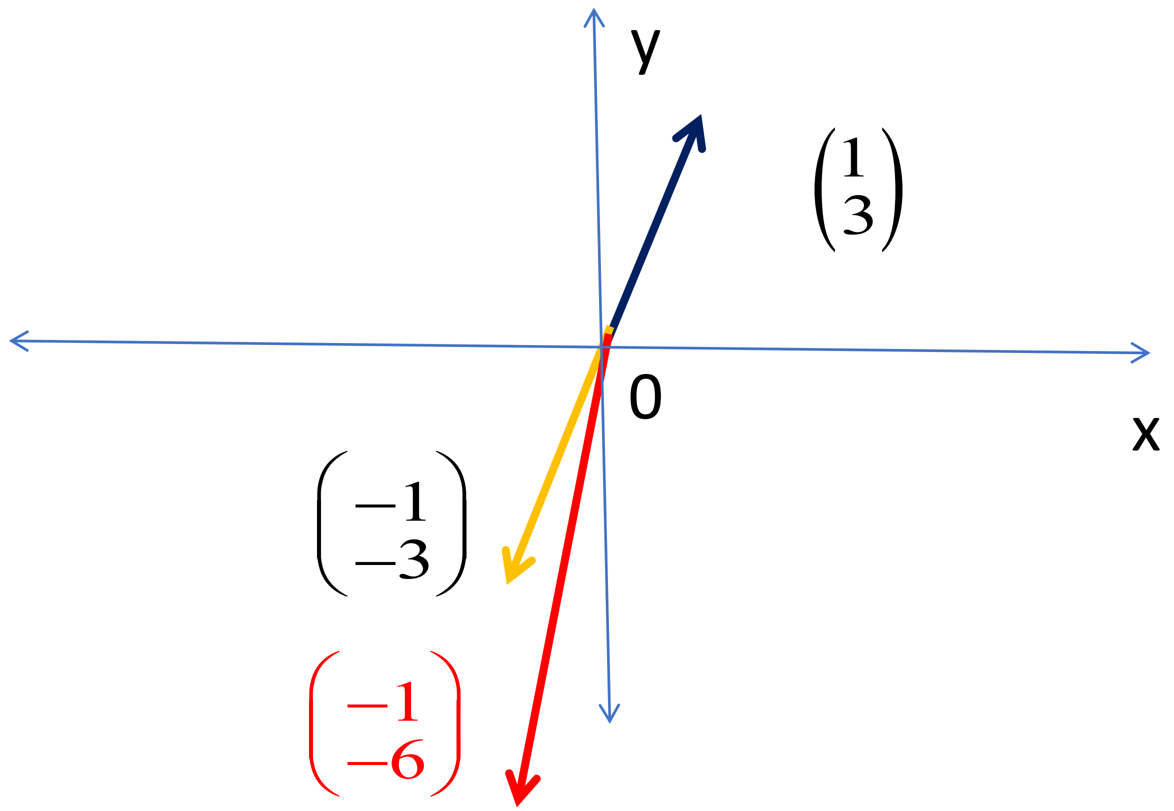
- **Row Picture:** These are parallel lines and so has no point of intersection. Hence this system has **no solution**.

$$x\begin{pmatrix}1\\3\end{pmatrix} + y\begin{pmatrix}-1\\-3\end{pmatrix} = \begin{pmatrix}-1\\-6\end{pmatrix}$$

- **Column Picture:**

There is no linear combination of the column vectors on LHS which produces the vector on RHS.





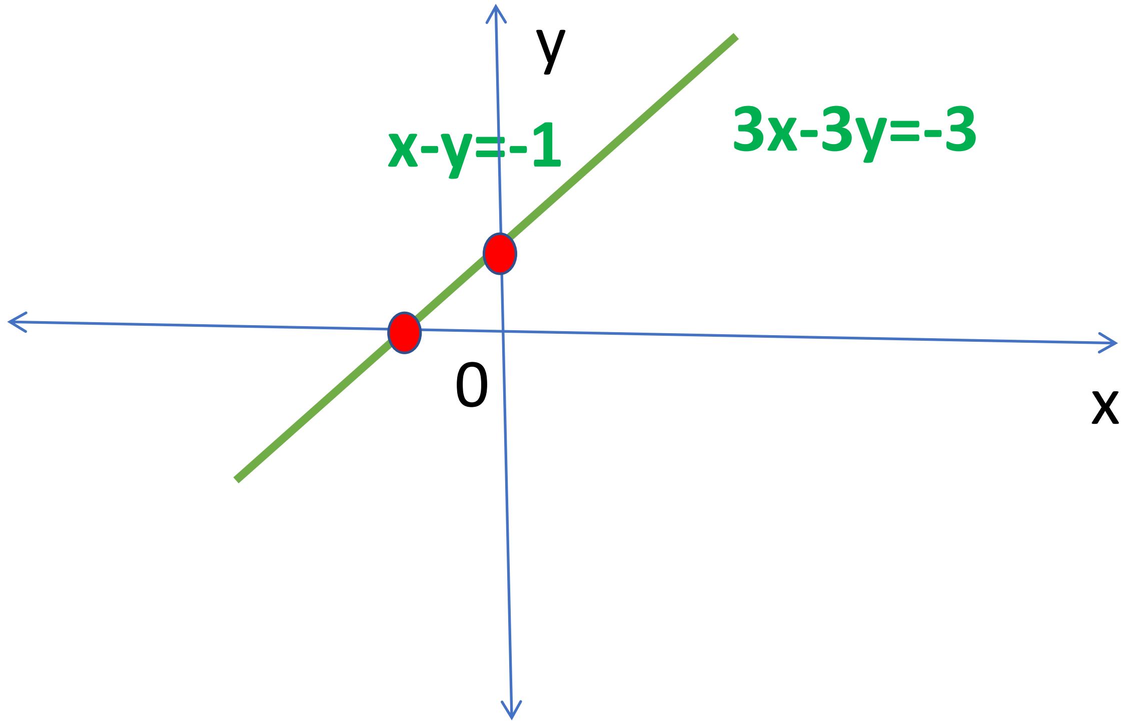
THE GEOMETRY OF LINEAR EQUATIONS:

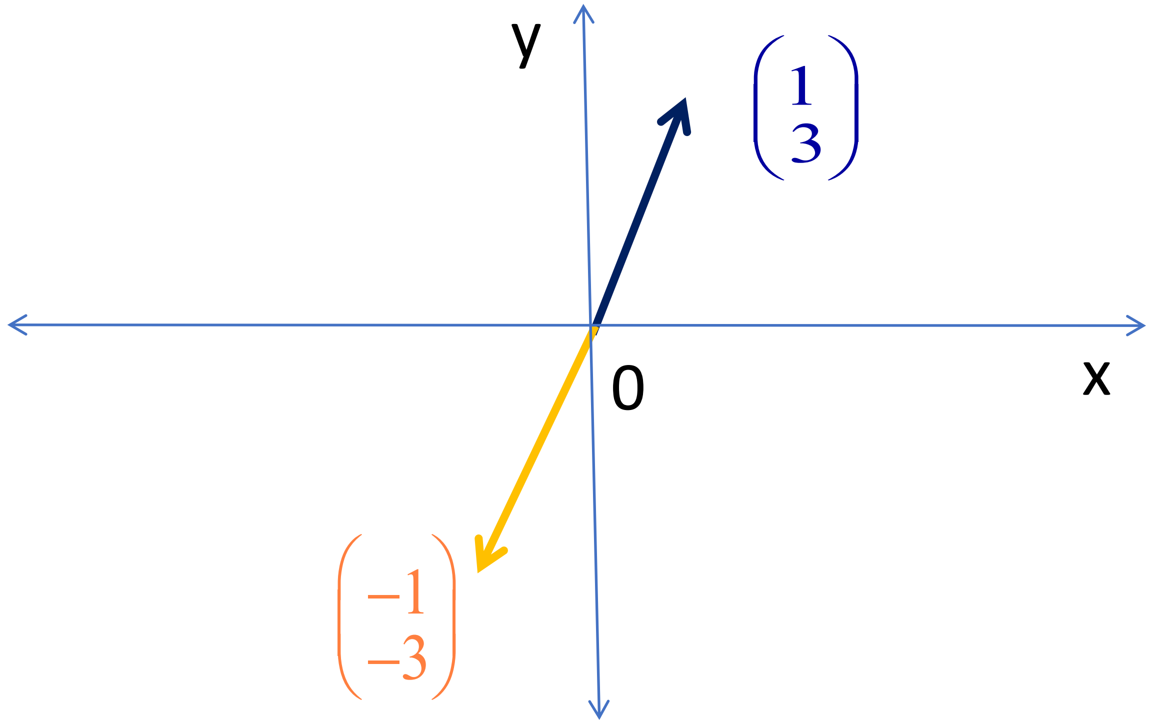
- (iii)Consider the system $\begin{aligned} x-y&=-1 \\ 3x-3y&=-3 \end{aligned}$

- **Row Picture:** These are coincident(overlapping) lines and so has infinite number of solutions. Hence this system has **infinitely many solutions**.

- **Column Picture:** $x\begin{pmatrix}1\\3\end{pmatrix}+y\begin{pmatrix}-1\\-3\end{pmatrix}=\begin{pmatrix}-1\\-3\end{pmatrix}$

  There are infinite number of linear combinations of the column vectors on LHS which produces the vector on RHS.



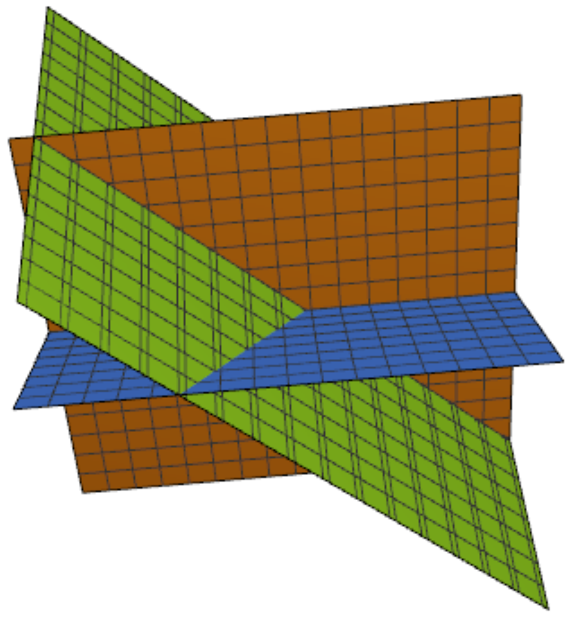


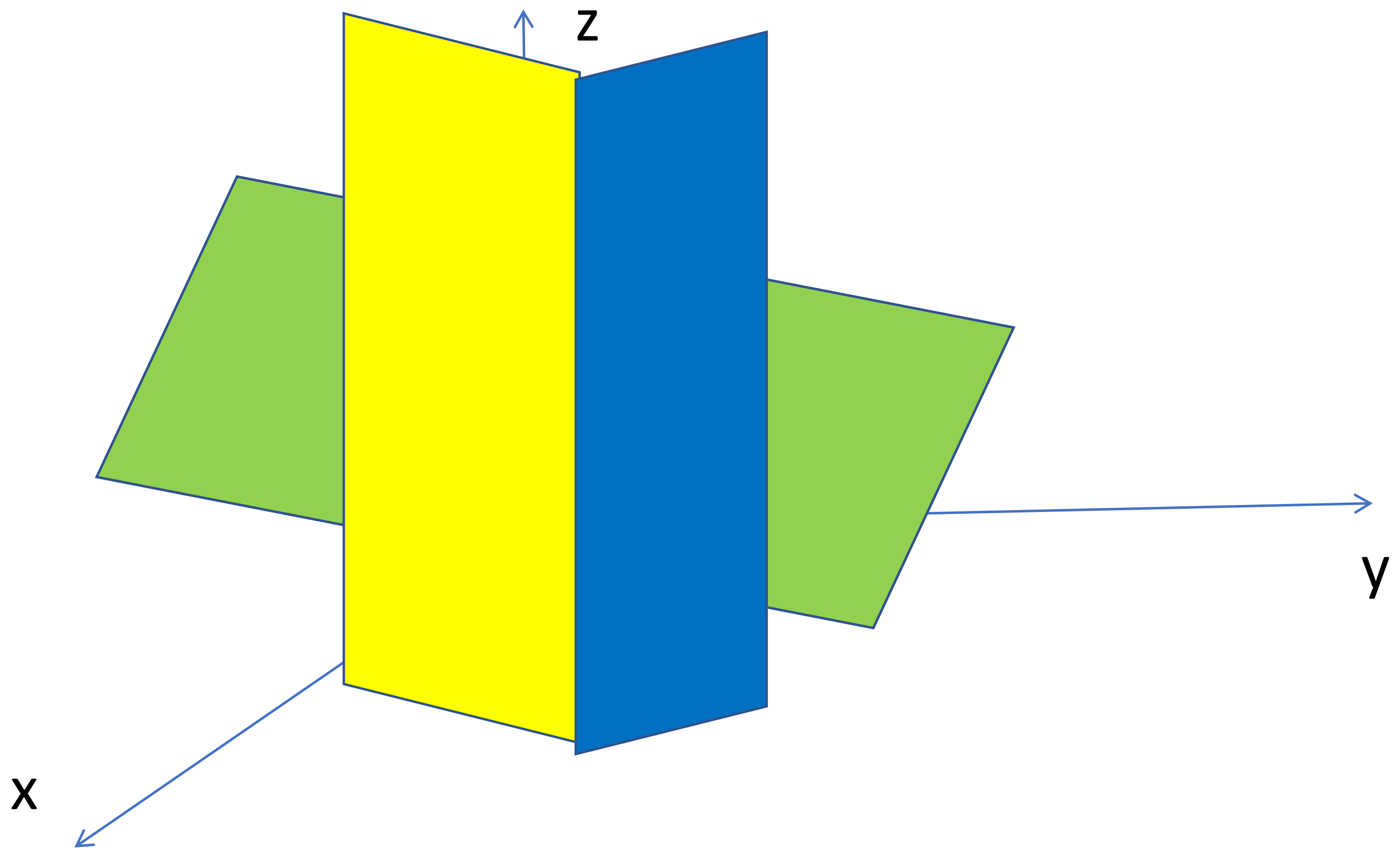
THE GEOMETRY OF LINEAR EQUATIONS:

- **SYSTEM OF 3 EQUATIONS WITH 3 VARIABLES:** Consider the system

$$x + y + 2z = 1$$
$$x + 2y - z = -2$$
$$x + 3y + z = 5$$

- **Row Picture:** Each equation describes a 2-dimensional plane in R^3. The first 2 planes intersect along a line and this line intersects with the third plane to produce a point (-6. 3, 2) which is the unique solution (point of intersection of the three planes) to the system of equations.



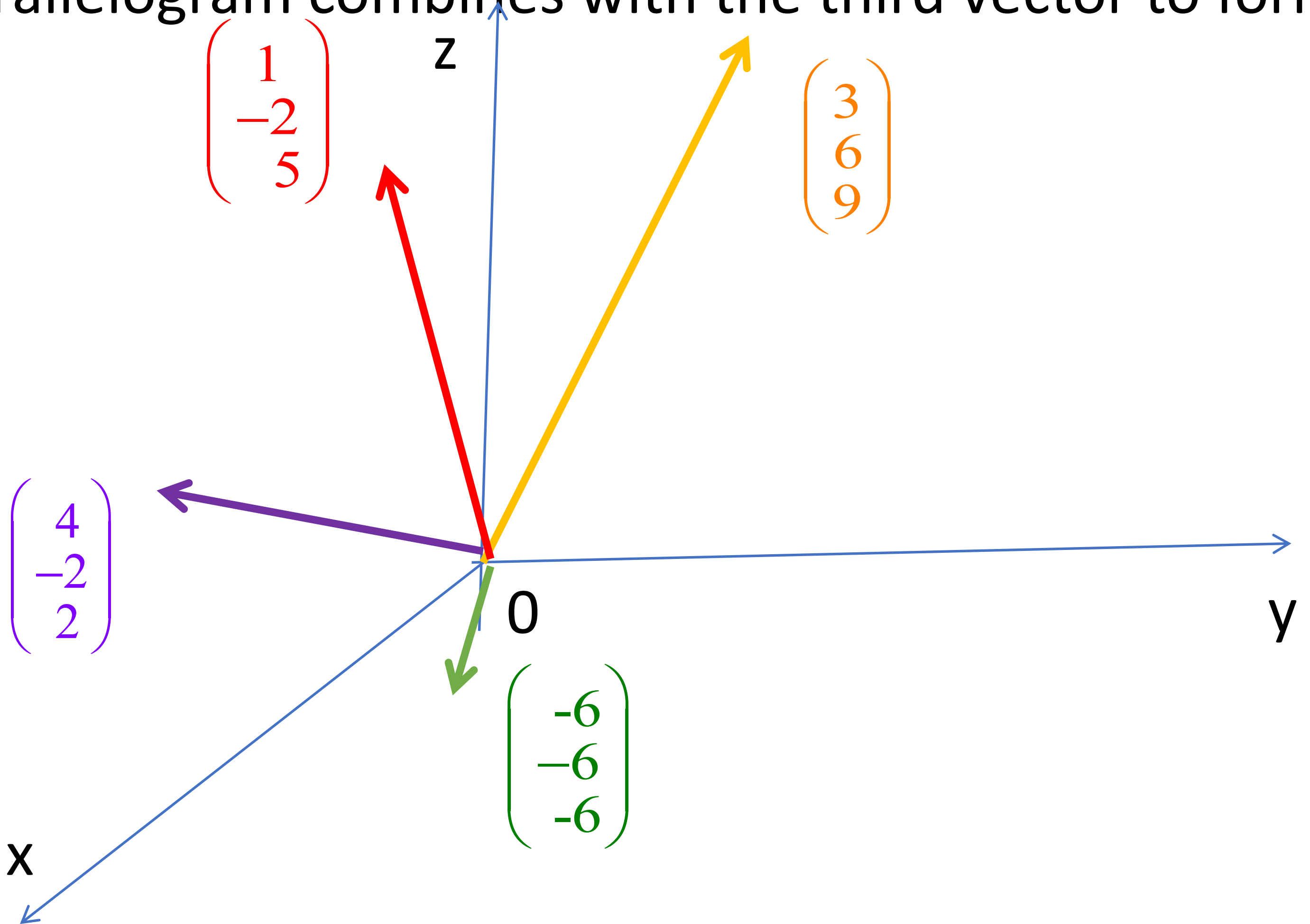


THE GEOMETRY OF LINEAR EQUATIONS:

❖ **Column Picture:** The vector equation for the given system can be written as

$$x\begin{pmatrix}1\\1\\1\end{pmatrix}+y\begin{pmatrix}1\\2\\3\end{pmatrix}+z\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\-2\\5\end{pmatrix} \Rightarrow -6\begin{pmatrix}1\\1\\1\end{pmatrix}+3\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-3\\0\\3\end{pmatrix}+2\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\-2\\5\end{pmatrix}$$

The first two vectors on the LHS combine to form a Parallelogram and the diagonal of this parallelogram combines with the third vector to form a Parallelopiped.



THE GEOMETRY OF LINEAR EQUATIONS:

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- **Row Picture:** Intersection of Lines/Planes.
- **Column Picture:** Combination of Columns.

- **Row Picture:** A Line requires 2 equations in 3-dimensional space. Similarly a Line requires 3 equations in 4-dimensional space. Hence a Line requires (n-1) equations in n-dimensional space.
  The first equation represents a (n-1)dimensional plane in n dimensions. The second plane(equation) intersects it in a smaller set of dimension (n-2). Thus every new plane reduces the dimension by one. At the end when all the 'n' planes are accounted for the intersection has dimension zero. It is a point, which lies on all the planes and its co-ordinates satisfy all 'n' equations. This point is the solution.

References/Links:

https://upload.wikimedia.org/wikipedia/commons/c/c0/Intersecting_Lines.svg

Google search: Graphs of row and column pictiure for a system of linear equations

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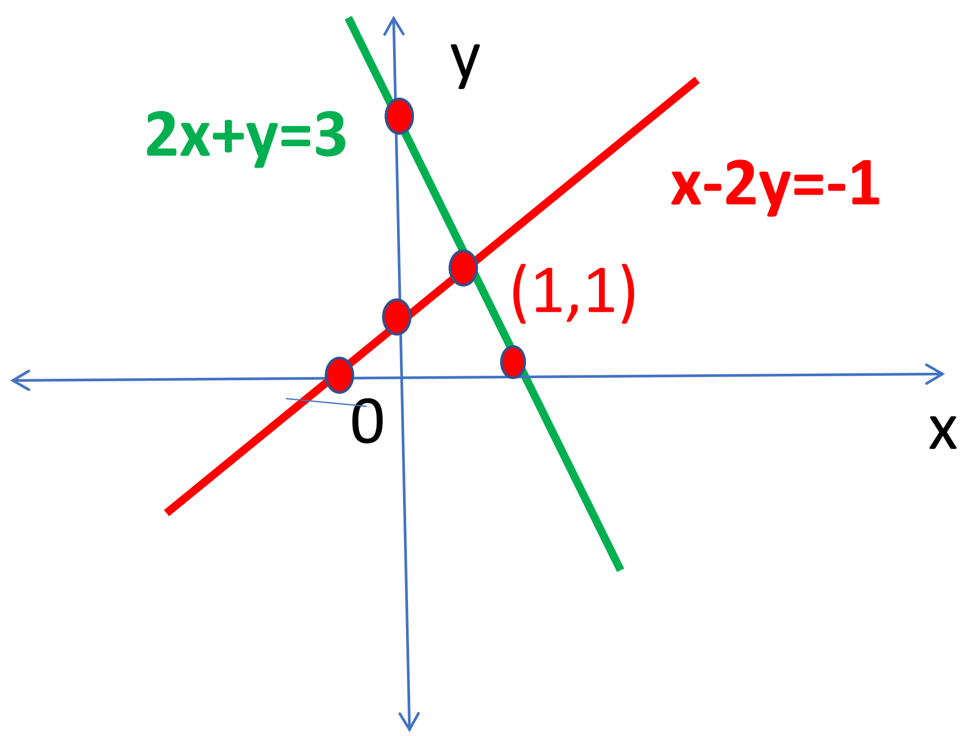
Course Content: The Geometry of Linear Equations

1. Solve the system of equations 2x+y=3; x-2y=-1 and draw the Row picture and Column picture.

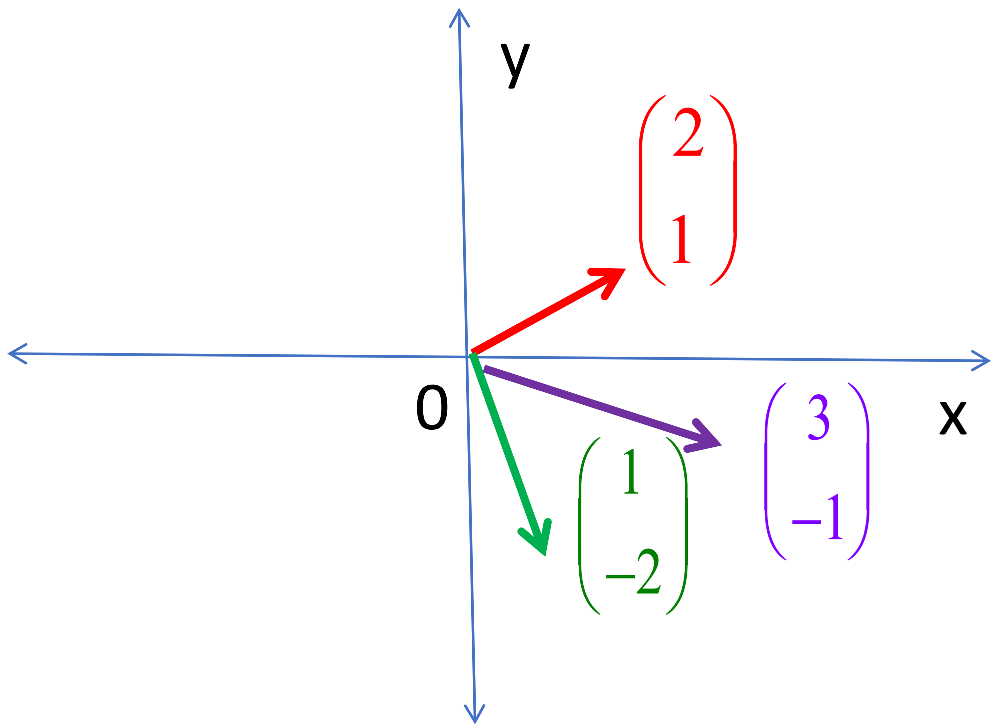
Row picture



Intersecting lines
Unique solution (1,1)

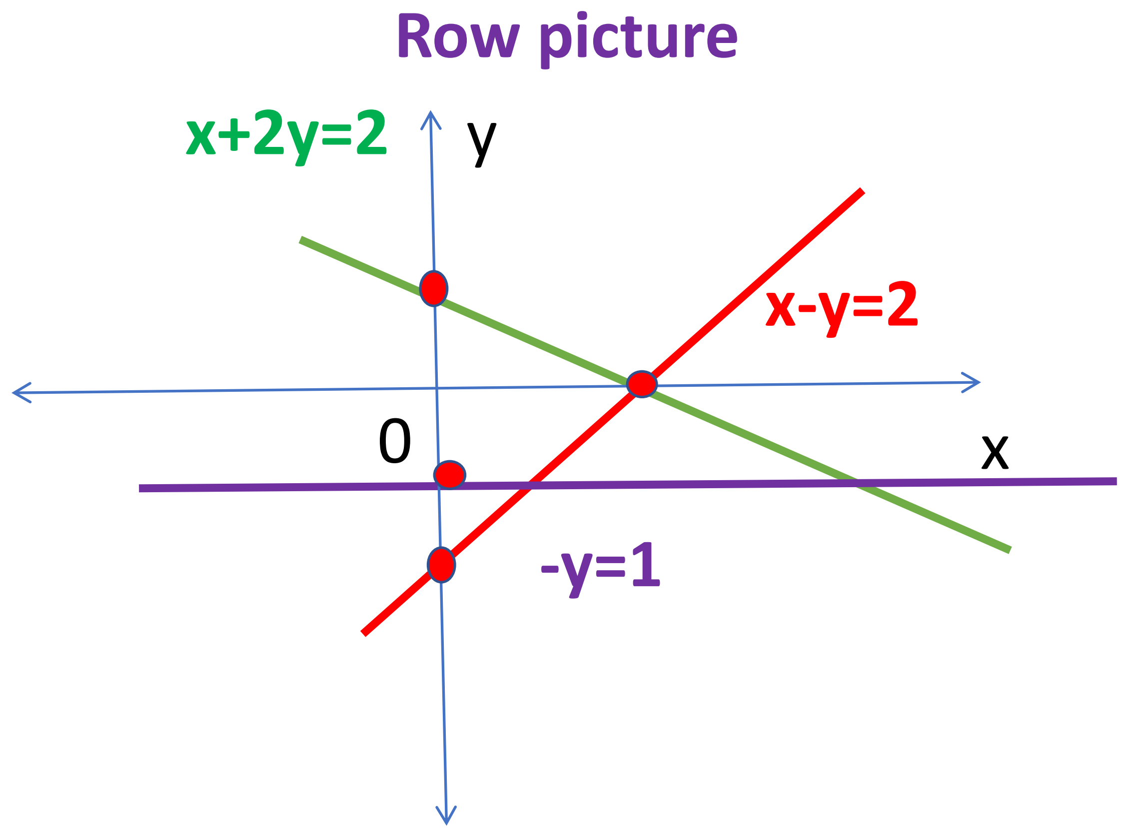
Column picture

$$x\begin{pmatrix}2\\1\end{pmatrix}+y\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}3\\-1\end{pmatrix} \Rightarrow 1\begin{pmatrix}2\\1\end{pmatrix}+1\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}3\\-1\end{pmatrix}$$



THE GEOMETRY OF LINEAR EQUATIONS:

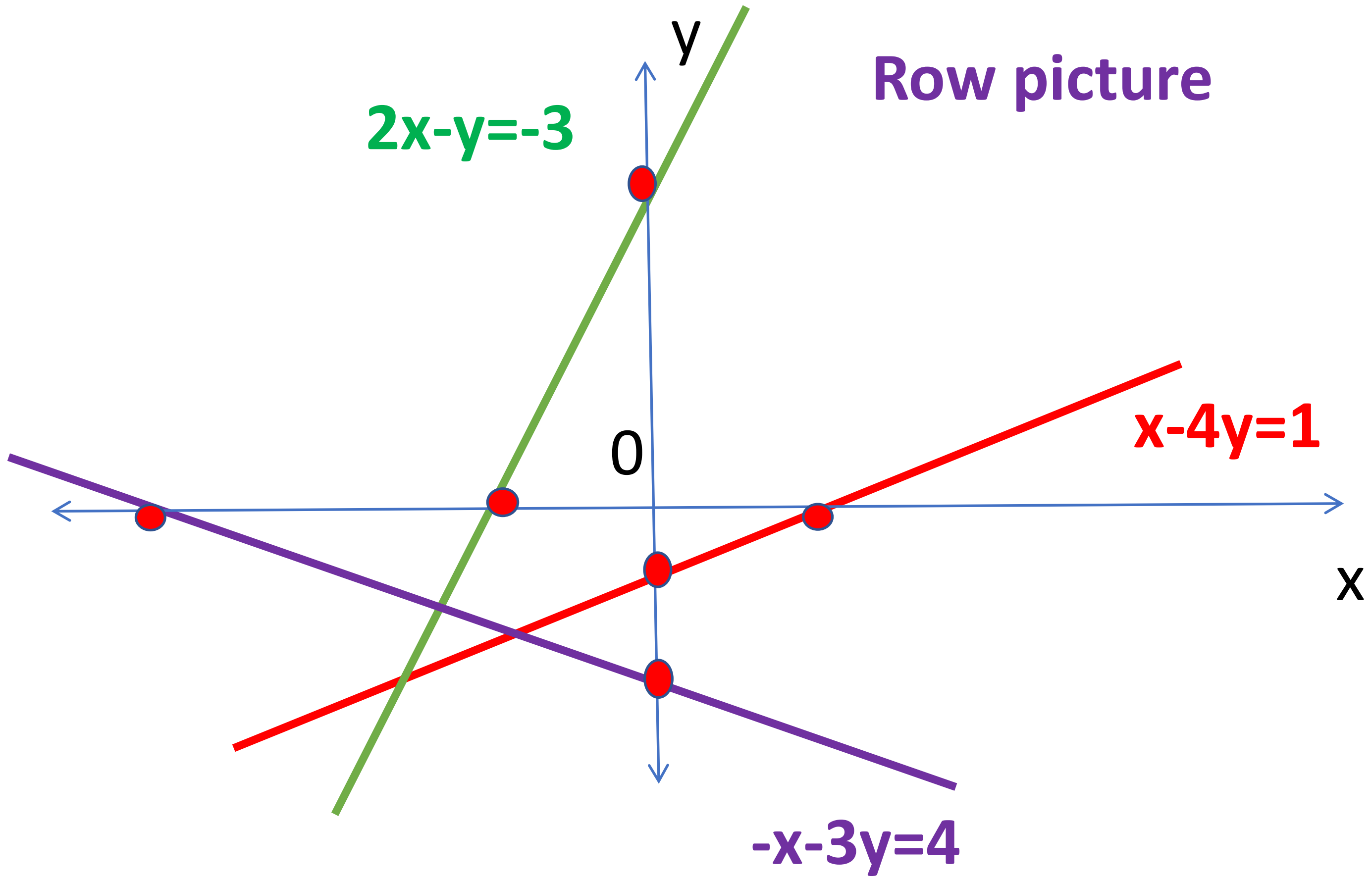
2. Sketch the three lines $x+2y=2$; $x-y=2$; $-y=1$(row picture only) and decide if the three equations are solvable. What happens if all right hand sides are zero? Is there any non-zero choice of right hand sides that allow the three lines to intersect at a common point of intersection?



The given three equations are not solvable as they do not have a common point of intersection. If all right hand sides are zero we get a trivial(zero) solution i.e $x=0$, $y=0$. If RHS of first equation is -1 then the three lines $x+2y=-1$; $x-y=2$; $-y=1$ intersect at a common point of intersection $(1,-1)$

THE GEOMETRY OF LINEAR EQUATIONS:

3. Sketch the three lines $x - 4y = 1$; $2x - y = -3$; $-x - 3y = 4$ and decide if the three equations are solvable. Do they have a common point of intersection? Explain.



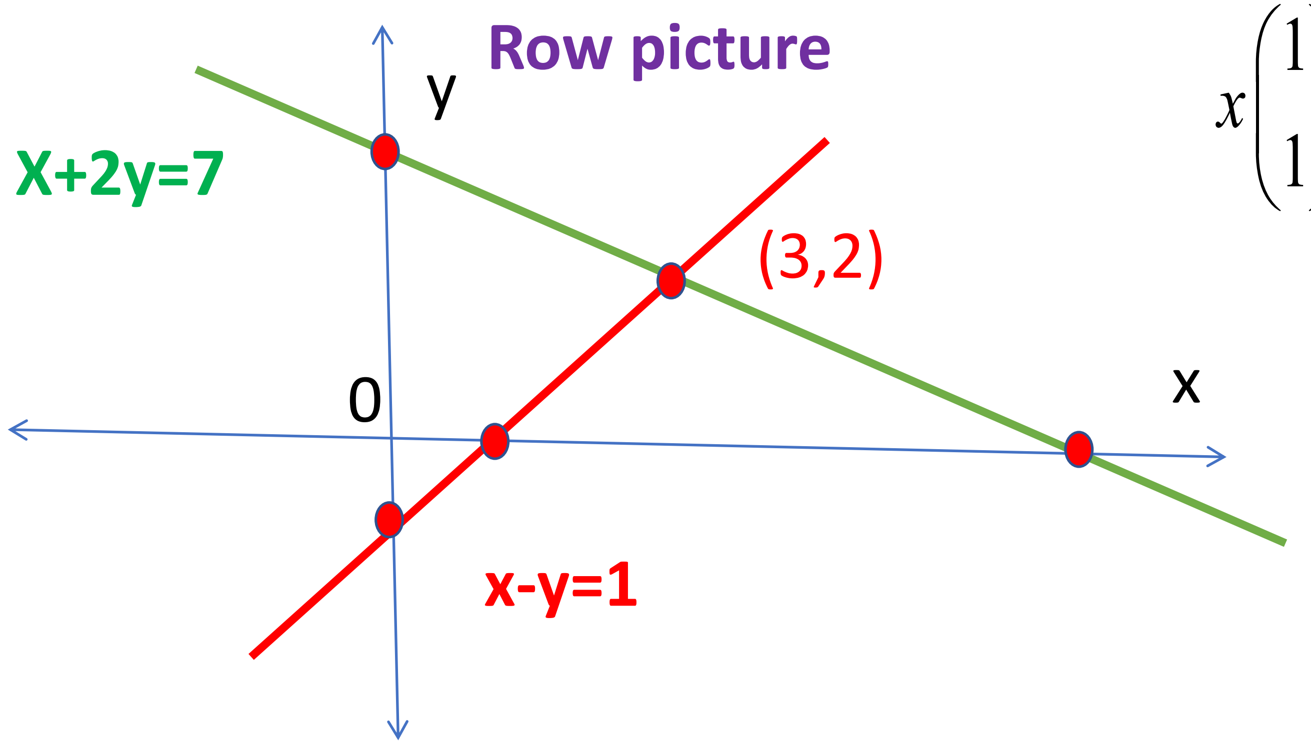
Given three equations are not solvable as they have no common point of intersection.

THE GEOMETRY OF LINEAR EQUATIONS:

4. Draw the row picture and column picture for the following system of equations and discuss its consistency, singularity, and existence of the solution:

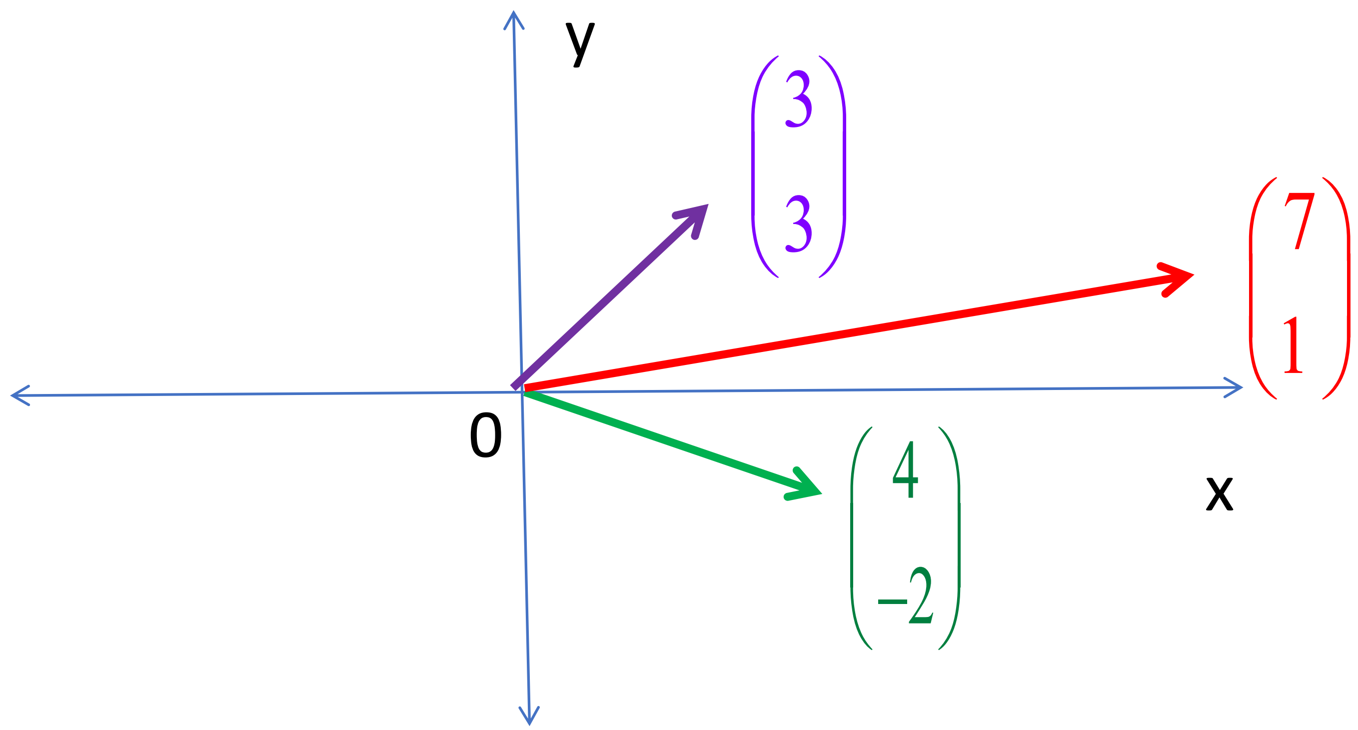
(i) x + 2y = 7; x - y = 1

Row picture



Column picture

$$x\begin{pmatrix}1\\1\end{pmatrix}+y\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}7\\1\end{pmatrix}\Rightarrow 3\begin{pmatrix}1\\1\end{pmatrix}+2\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}7\\1\end{pmatrix}$$



(ii) 2x + 3y = 6; 4x + 6y = 12 (iii) x - 3y = 5; x - 3y = -5

INTRODUCTION:

Example:

$$A \rightarrow \begin{pmatrix} 1 & 1 & -2 & 3: & 4 \\ 2 & 3 & 3 & -1: & 3 \\ 5 & 7 & 4 & -1: & 5 \end{pmatrix} \xrightarrow[R_3-5R_1]{R_2-2R_1} \begin{pmatrix} 1 & 1 & -2 & 3: & 4 \\ 0 & 1 & 7 & -7: & -5 \\ 0 & 2 & 14 & -16: & -15 \end{pmatrix} \xrightarrow{R_3-2R_2}$$

$$\begin{pmatrix} 1 & 1 & -2 & 3: & 4 \\ 0 & 1 & 7 & -7: & -5 \\ 0 & 0 & 0 & -2: & -5 \end{pmatrix} = U \xrightarrow{R_3/(-2)} \begin{pmatrix} 1 & 1 & -2 & 3: & 4 \\ 0 & 1 & 7 & -7: & -5 \\ 0 & 0 & 0 & 1: & 5/2 \end{pmatrix} \xrightarrow[R_2+7R_3]{R_1-3R_3}$$

$$\begin{pmatrix} 1 & 1 & -2 & 0: & -7/2 \\ 0 & 1 & 7 & 0: & 25/2 \\ 0 & 0 & 0 & 1: & 5/2 \end{pmatrix} \xrightarrow{R_1-R_2} \begin{pmatrix} 1 & 0 & -9 & 0: & -16 \\ 0 & 1 & 7 & 0: & 25/2 \\ 0 & 0 & 0 & 1: & 5/2 \end{pmatrix} = R$$

References/Links:

https://upload.wikimedia.org/wikipedia/commons/c/c0/Intersecting_Lines.svg

Google search: Graphs of row and column pictiure for a system of linear equations

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THE GEOMETRY OF LINEAR EQUATIONS:

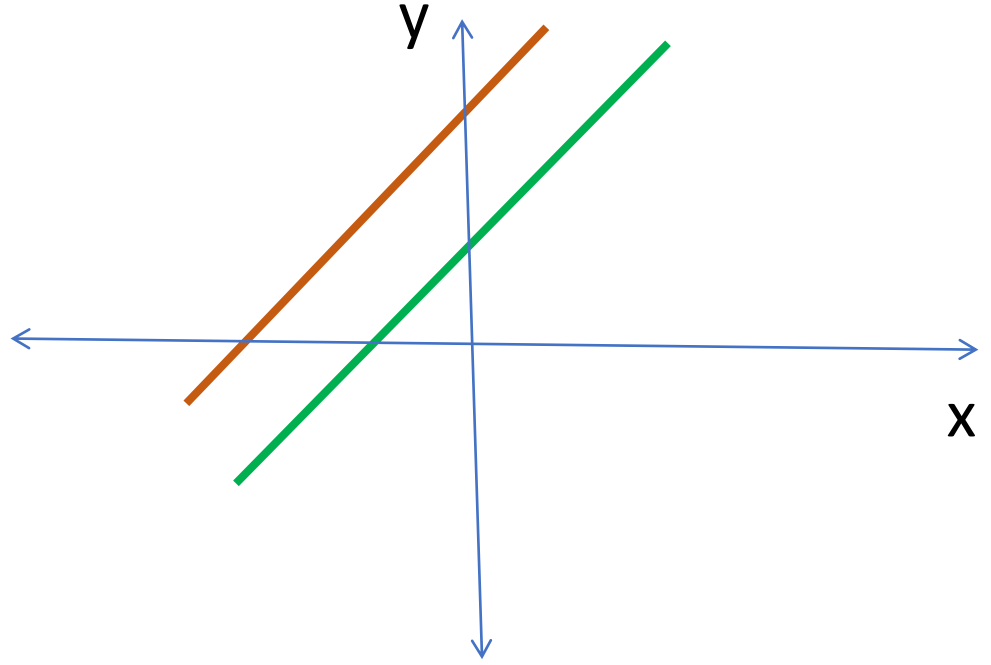
Course Content: Singular Cases

- **SINGULAR CASES in Two dimensions:** A system of linear equations is said to be singular $\left(|A|=0\right)$ if it has no solution or has infinite number of solutions.

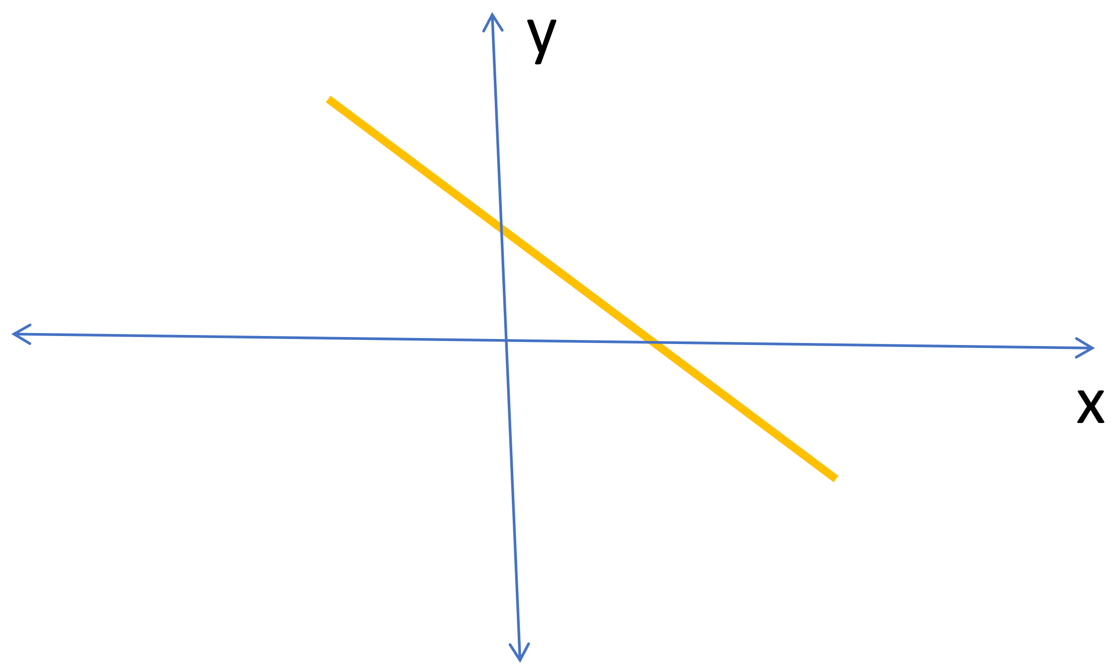
- **(i) ROW PICTURE (Two dimensions):** In two dimensions the lines are parallel if they have no solution and coincident if they ha e infinite number of solutions. In such a case the matrix A will have dependent row /column and det (A)=0. Such a matrix is called Singular Matrix.

(2 lines):

Lines parallel (no solution)



Lines coincident (infinite no. of solutions)



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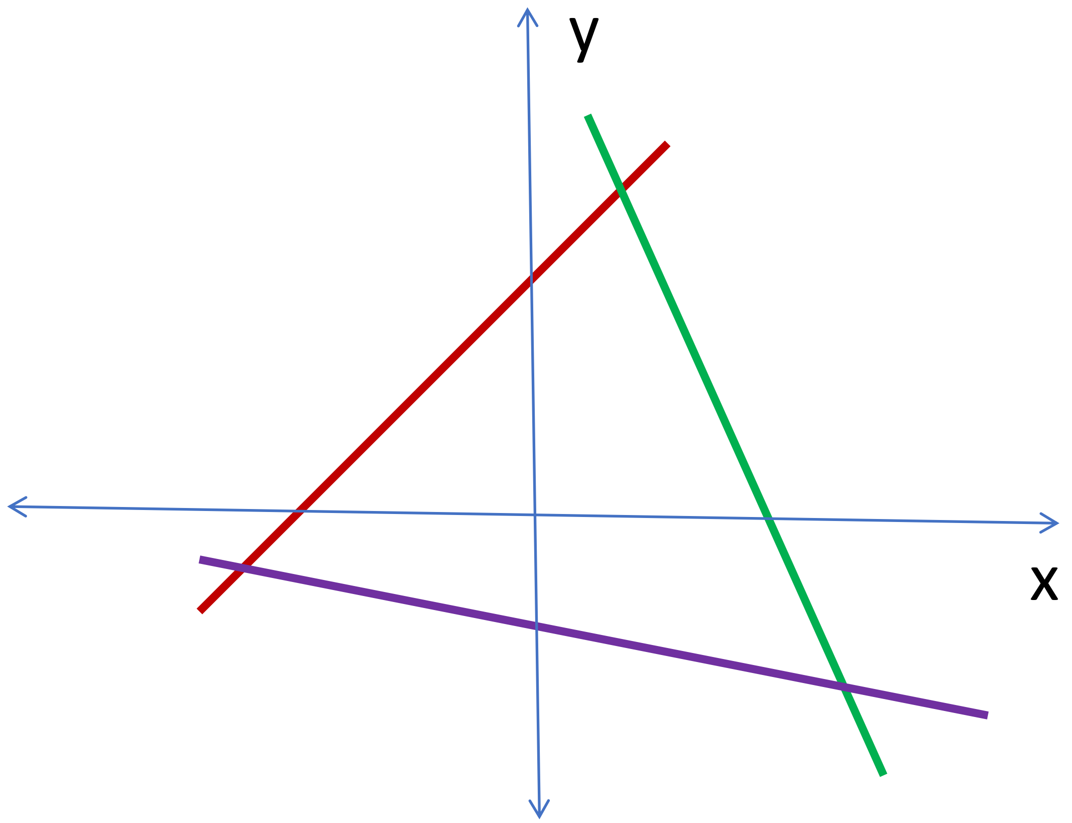
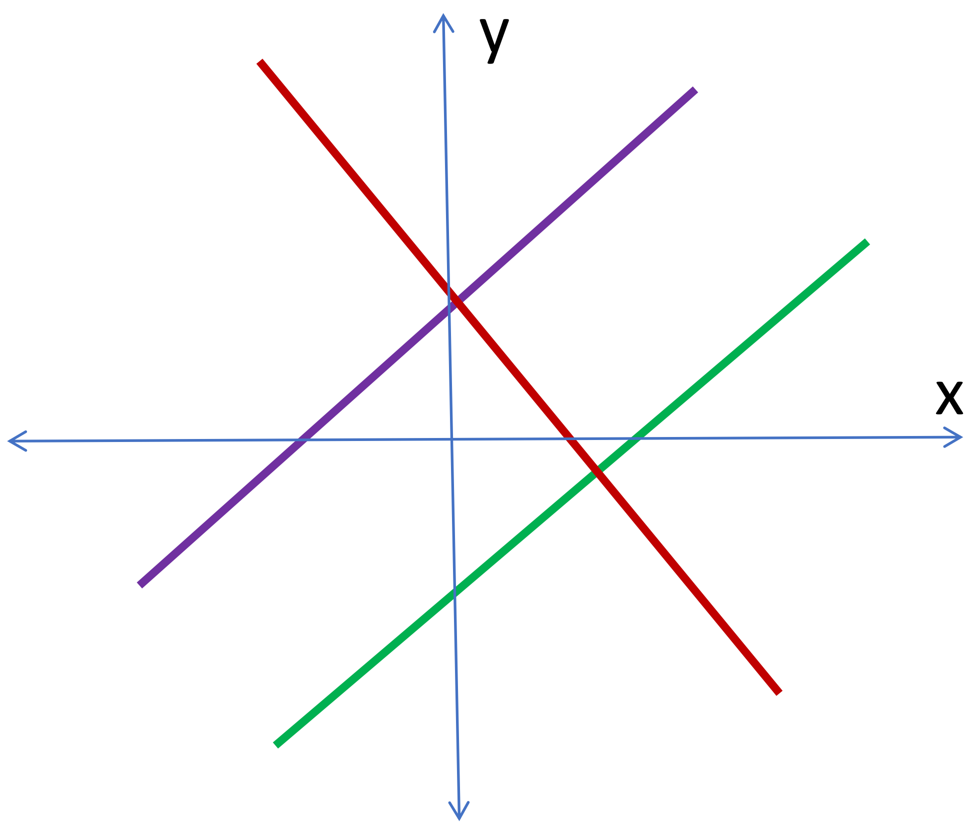
THE GEOMETRY OF LINEAR EQUATIONS:

(3 lines):

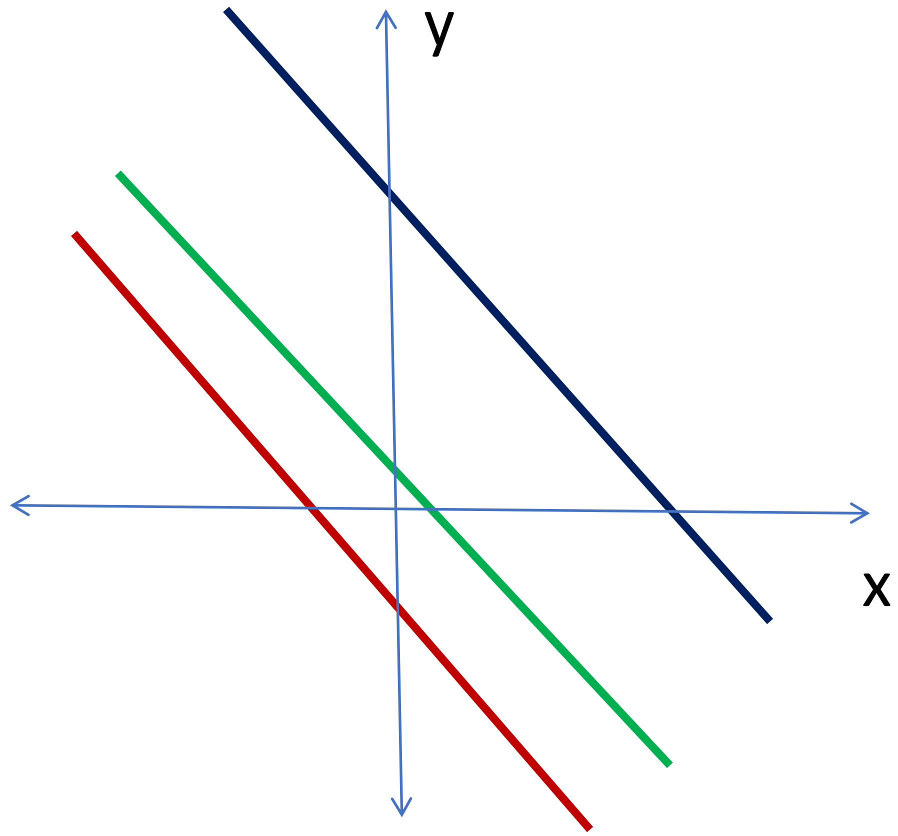
Lines intersecting in pairs (no solution)



2 Lines parallel & one intersecting(no solution)



All 3 parallel lines (no solution)

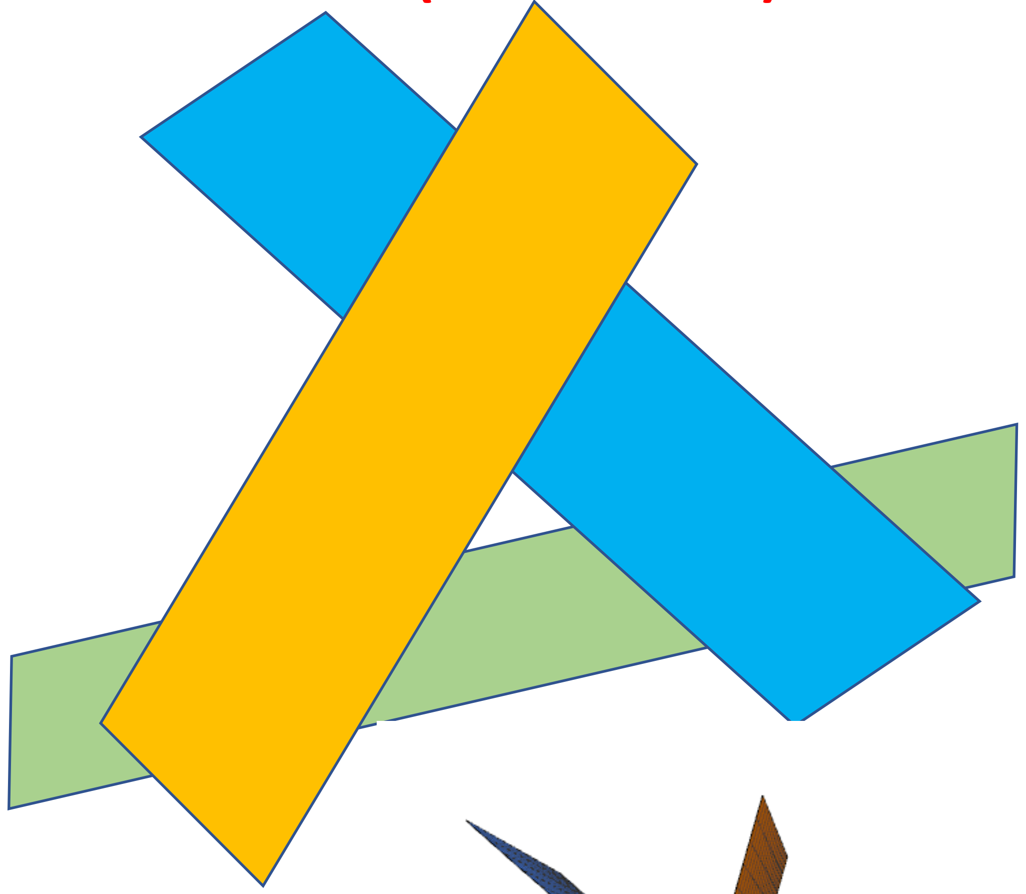


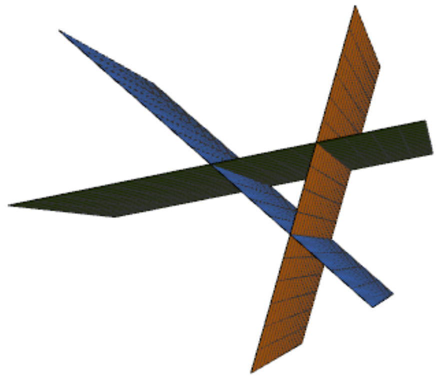
THE GEOMETRY OF LINEAR EQUATIONS:

- **ROW PICTURE(Three dimensions):** In three dimensions if the 3 Planes do not intersect then we have the following cases :

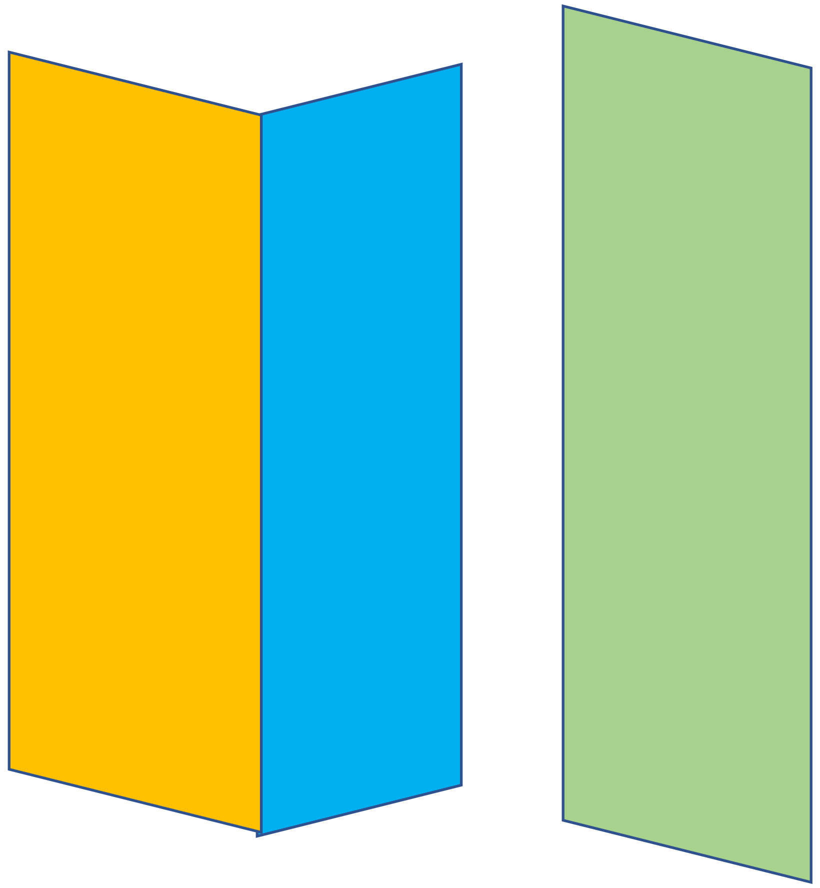
(3 planes):

Every pair of planes intersecting in a line (no solution)

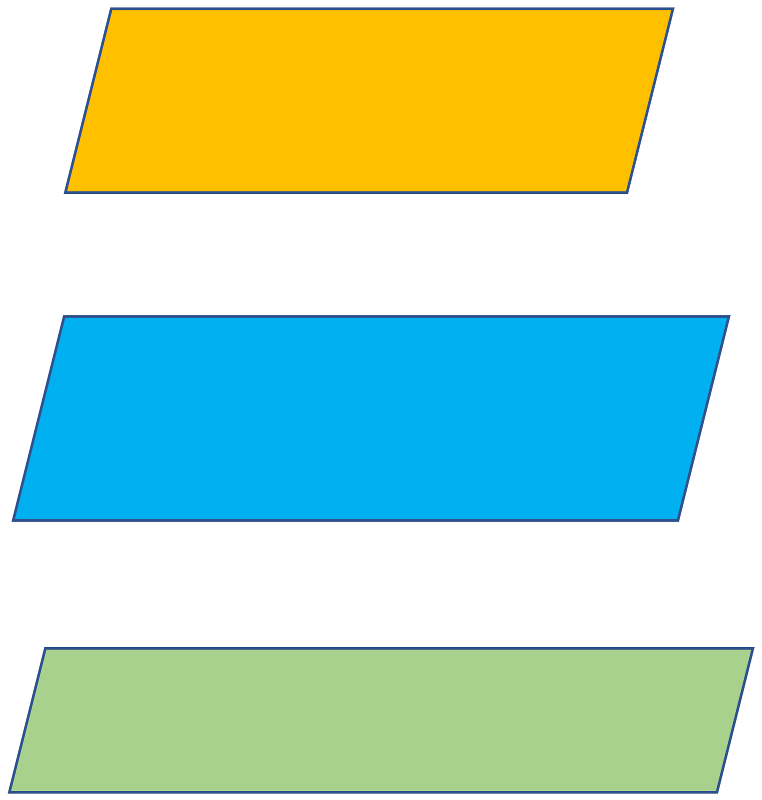




2 planes intersecting in a line and 3rd is parallel to this line (no solution)



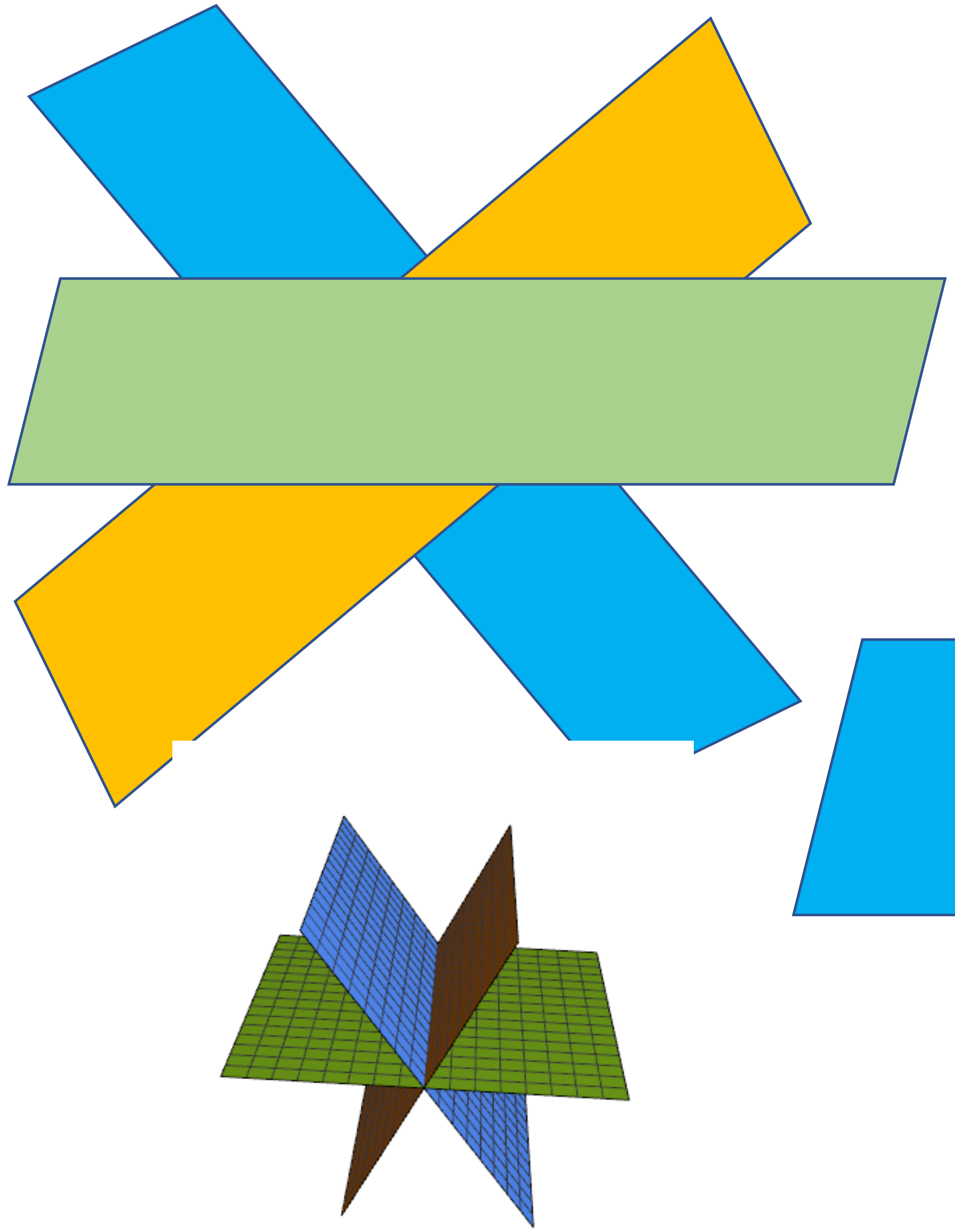
All 3 parallel planes (no solution)



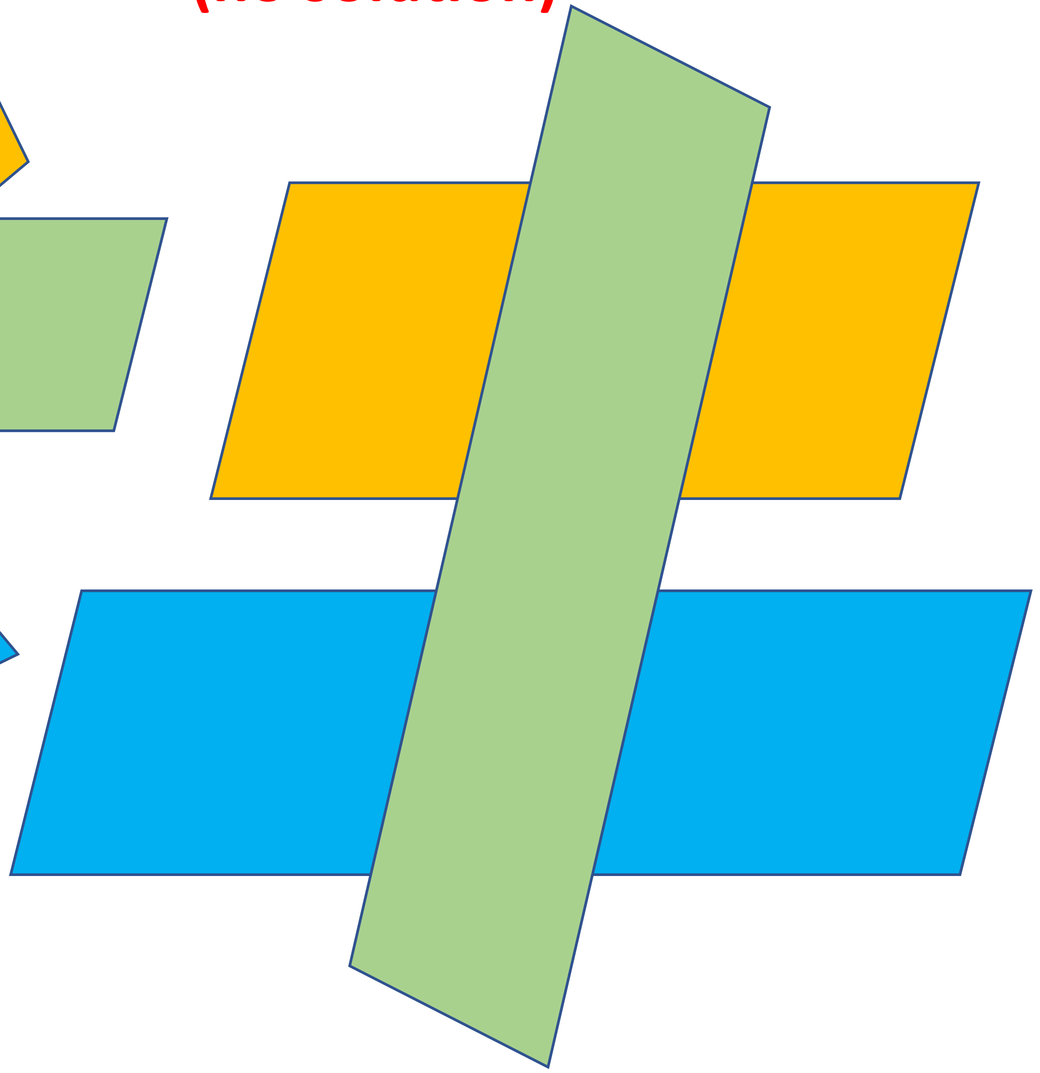
THE GEOMETRY OF LINEAR EQUATIONS:

All 3 overlapping planes (infinite no. of solutions)

All 3 planes intersecting in a line (infinite no solutions)



2 parallel planes and third plane intersecting them (no solution)

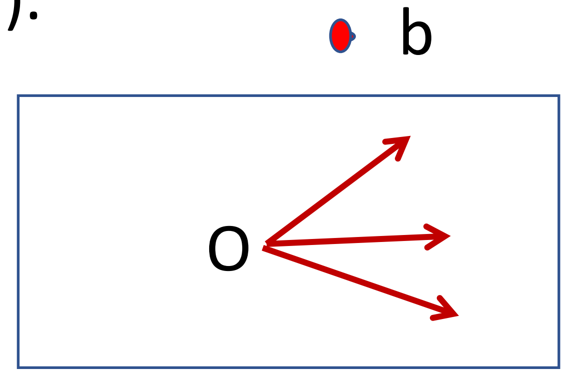


THE GEOMETRY OF LINEAR EQUATIONS:

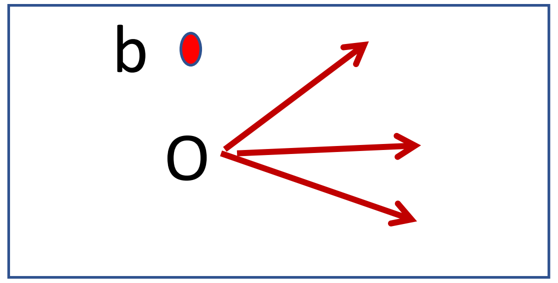
- **(ii) COLUMN PICTURE (Three dimensions):** Consider the column picture for a system of 3equations in 3 variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \Rightarrow x\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + y\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + z\begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Each of these column vectors is a position vector with origin as a point, These column vectors lie in a plane (as they pass through the origin). Then every combination of these vectors on LHS lie in the same plane (3 vectors are coplanar).



If the vector b is not in that plane, then solution is impossible. This system is singular and has no solution.



If the vector b lies in the plane, (i.e b is also coplanar), then there are too many solutions. The 3 columns combine in infinitely many ways to produce b. This system is singular and has infinite no. of solutions.

References/Links:

https://upload.wikimedia.org/wikipedia/commons/c/c0/Intersecting_Lines.svg

Google search: Graphs of row and column pictiure for a system of linear equations

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MATRICES AND GAUSSIAN ELIMINATION

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GAUSSIAN ELIMINATION:

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Course Content: Gaussian Elimination

- **Rank Of A Matrix:** A Square matrix A of order n is said to have rank r if
  - At least one minor of order r does not vanish $(\neq 0)$.
  - Every minor of order (r+1) vanishes $(=0)$.

  Rank of matrix A is denoted by r i.e.rank(A)=r.

- If $A = \left[a_{ij} \right]_{mxn}$ is a rectangular matrix, then **Rank of the Matrix** is defined as the number of non-zero rows in the Echelon form of A. It is also defined as the maximum number of Linearly Independent Rows or Columns of the Matrix A.

GAUSSIAN ELIMINATION:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 2 & 2 & -3 \\ -2 & -2 & 3 \\ 4 & 4 & 6 \end{pmatrix}; \begin{pmatrix} 1 & 5 \\ -1 & 3 \\ 2 & 3 \end{pmatrix}; \begin{pmatrix} 1 & 5 \\ -1 & 3 \\ 2 & 10 \end{pmatrix}; \begin{pmatrix} 2 & -1 & 3 & 5 \\ 3 & 2 & 1 & 2 \end{pmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$2 \qquad 1 \qquad 2 \qquad 2 \qquad 2$$

Ex: Find the conditions on a and b so that the Matrix $\begin{pmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 1 & 3 & 6 \end{pmatrix}$ has rank 1, 2, 3.

$$\begin{pmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 1 & 3 & 6 \end{pmatrix} \xrightarrow{R_3-\left(\frac{1}{a}\right)R_1} \begin{pmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 0 & 3-(1/a) & 6-(2/a) \end{pmatrix}$$

(i) For no values of a and b this matrix will have rank 1.

(ii) If a=1/3 and b=4, rank of the matrix is 2.

(iii) If $a \neq 1/3$ and $b \neq 4$, rank of the matrix is 3.

GAUSSIAN ELIMINATION:

- **Relation between Rank, Consistency and Solution:**
- If rank(A)=r, then the following hold good:

(i) If **rank(A)=rank[A:b}=r**, system Ax=b is **consistent** and has **a solution**.

(ii) If **rank(A)=rank[A:b}=r=n**, system Ax=b is **consistent** and has **a unique solution**.

(iii) If **rank(A)=rank[A:b}=r<n**, system Ax=b is **consistent** and has **infinite number of solutions**.

(iv) If **rank(A) $\neq$ rank[A:b]** , system Ax=b is **inconsistent** and has **no solution**.

Gaussian Elimination

Gaussian Elimination is used to check for Consistency and solve a System of linear equations.

For a given system of equations Ax=b apply Elementary row transformations to the Augmented Matrix [A:b] and reduce it to [U:c] where U is an **Upper Triangular Matrix** so that we get an equivalent system Ux=c which can be solved by **Backward Substitution**.

Here A and U are Equivalent Matrices and hence solution of Ax=b is same as Ux=c.

- **The following steps are to be followed while performing Elementary Row Transformations in Gaussian Elimination:**
  - **No exchange** of rows.
  - First row should be **retained** as it is (not altered).
  - The first non-zero element in every non-zero row is called **Pivot**.
  - The original system Ax=b and new system obtained Ux=c have the **same solution**.

GAUSSIAN ELIMINATION:

Gaussian Elimination is illustrated below for a system of 3 equations with 3 variables.

Consider a system of 3 equations in 3 variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

$$[A:b] = \begin{pmatrix} a_{11} & a_{12} & a_{13} & : b_1 \\ a_{21} & a_{22} & a_{23} & : b_2 \\ a_{31} & a_{32} & a_{33} & : b_3 \end{pmatrix} \xrightarrow[R_3 - \left(\frac{a_{31}}{a_{11}}\right)R_1]{R_2 - \left(\frac{a_{21}}{a_{11}}\right)R_1} \begin{pmatrix} a_{11} & a_{12} & a_{13} & : b_1 \\ 0 & d_{22} & d_{23} & : c_2 \\ 0 & d_{32} & d_{33} & : c_3 \end{pmatrix}$$

$$\xrightarrow{R_3 - \left(\frac{d_{32}}{d_{22}}\right)R_2} \begin{pmatrix} a_{11} & a_{12} & a_{13} & : b_1 \\ 0 & d_{22} & d_{23} & : c_2 \\ 0 & 0 & e_{33} & : c_4 \end{pmatrix} = [U:c]$$

$$\Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ \qquad\quad d_{22}x_2 + d_{23}x_3 = c_2 \\ \qquad\qquad\qquad e_{33}x_3 = c_4 \end{cases}$$

GAUSSIAN ELIMINATION:

Check for consistency and solve the following system of equations if consistent:

(i)
$$\begin{aligned} x_1 + x_2 - 2x_3 + 4x_4 &= 5 \\ 2x_1 + 2x_2 - 3x_3 + x_4 &= 3 \\ 3x_1 + 3x_2 - 4x_3 - 2x_4 &= 1 \end{aligned} \qquad [A:b] = \begin{pmatrix} 1 & 1 & -2 & 4:5 \\ 2 & 2 & -3 & 1:3 \\ 3 & 3 & -4 & -2:1 \end{pmatrix}$$

$$\xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & 1 & -2 & 4:5 \\ 0 & 0 & 1 & -7:-7 \\ 0 & 0 & 2 & -14:-14 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 1 & -2 & 4:5 \\ 0 & 0 & 1 & -7:-7 \\ 0 & 0 & 0 & 0:0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_2 - 2x_3 + 4x_4 = 5 \\ \qquad\qquad x_3 - 7x_4 = -7 \end{cases}$$

r(A)=2 = r[A:b]<n(=4)

System is **consistent** and has **infinitely many solutions**.

Solution is $(x_1, x_2, x_3, x_4) = (10k_1 - k_2 - 9,\ k_2,\ 7k_1 - 7,\ k_1)$

Depending upon values of k_1 and k_2 we get infinity of solutions.

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GAUSSIAN ELIMINATION:

1. Check for consistency and solve the following system of equations if consistent:

(i)
$$\begin{aligned} x_1 + x_2 - 2x_3 + 3x_4 &= 4 \\ 2x_1 + 3x_2 + 3x_3 - x_4 &= 3 \\ 5x_1 + 7x_2 + 4x_3 + x_4 &= 5 \end{aligned}$$

$$[A:b] = \begin{pmatrix} 1 & 1 & -2 & 3:4 \\ 2 & 3 & 3 & -1:3 \\ 5 & 7 & 4 & 1:5 \end{pmatrix} \xrightarrow{R_2-2R_1,\ R_3-5R_1}$$

$$\begin{bmatrix} 1 & 1 & -2 & 3 & : 4 \\ 0 & 1 & 7 & -7 & : -5 \\ 0 & 2 & 14 & -14 & : -15 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 1 & -2 & 3 & : 4 \\ 0 & 1 & 7 & -7 & : -5 \\ 0 & 0 & 0 & 0 & : -5 \end{bmatrix}$$

This gives 0=-5 which is not possible.

Also r(A)=2 and r[A:b]=3

System is **inconsistent** and has **no solution**

GAUSSIAN ELIMINATION:

(ii)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + 5x_2 - x_3 &= -4 \\ 3x_1 - 2x_2 - x_3 &= 5 \end{aligned}$$

$$[A:b] = \left(\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{array}\right)$$

$$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{array}\right] \xrightarrow{R_3 + 8R_2} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 + x_3 = 3 \\ x_2 - 3x_3 = -10 \\ -28x_3 = -84 \end{cases}$$

r(A)=r[A:b]=3=n. System is **consistent** and has **a unique solution**.

$$(x_1, x_2, x_3) = (2, -1, 3)$$

GAUSSIAN ELIMINATION:

(iii)

$$\begin{aligned} 2x - 3y + 2z &= 1 \\ 5x - 8y + 7z &= 1 \\ y - 4z &= 3 \end{aligned}$$

$$\left(\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \\ 0 & 1 & -4 & 3 \end{array}\right) \xrightarrow{R_2 - \left(\frac{5}{2}\right)R_1} \left(\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & -1/2 & 2 & -3/2 \\ 0 & 1 & -4 & 3 \end{array}\right) \xrightarrow{R_3 + 2R_2} \left(\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & -1/2 & 2 & -3/2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} 2x - 3y + 2z = 1 \\ -(1/2)\,y + 2z = -3/2 \end{cases}$$

r(A)=r(A:b)=2<n(=3) hence system is **consistent** and has **infinite number of solutions**.

i.e **(x, y, z)=(5k+5, 4k-+3, k)**

GAUSSIAN ELIMINATION:

2. Find all values of a for which the resulting linear system has (a)no solution (b)a unique solution and (c) infinitely many solutions:

$$x + y - z = 2$$
$$x + 2y + z = 3$$
$$x + y + (a^2 - 5)z = a$$

$$[A:b] = \begin{pmatrix} 1 & 1 & -1 : 2 \\ 1 & 2 & 1 : -3 \\ 1 & 1 & a^2 - 5 : a \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & -1 : 2 \\ 0 & 1 & 2 : -5 \\ 0 & 0 & a^2 - 4 : a - 2 \end{pmatrix}$$

(a) System has **no solution** if $a = -2$ (when r(A) $\neq$r(A:b))

(b) System has **a unique solution** if $a \neq \pm 2$(when r(A) = r(A:b)=3=n)

(c) System has **infinitely many solutions** if $a = 2$ (when r(A) = r(A:b)=2<n)

GAUSSIAN ELIMINATION:

3. Find an equation relating a, b and c so that the linear system
is consistent for any values of a, b and c that satisfy that equation. When (a,b,c)=(2,3,9), then what is the solution of the system.

$$x + 2y - 3z = a$$
$$2x + 3y + 3z = b$$
$$5x + 9y - 6z = c$$

$$\begin{pmatrix} 1 & 2 & -3 : a \\ 2 & 3 & 3 : b \\ 5 & 9 & -6 : c \end{pmatrix} \xrightarrow[R_3 - 5R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 2 & -3 : a \\ 0 & -1 & 9 : b-2a \\ 0 & -1 & 9 : c-5a \end{pmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 2 & -3 : a \\ 0 & -1 & 9 : b-2a \\ 0 & 0 & 0 : c-b-3a \end{pmatrix}$$

The given linear system will be **consistent** if a,b,c satisfy the relation c-b-3a=0.

When (a, b, c)=(2, 3, 9), then the solution of the system is **(x, y, z)=(-15k, 9k+1, k)**

.

GAUSSIAN ELIMINATION:

Linear System of Equations

Consistent

Solution Exists (rank(A)=rank[A:b}=r)

Unique solution
(rank(A)=rank[A:b}=r=n)
(Non-Singular Matrix
i.e $|A| \neq 0$)
$(A^{-1} exists)$

Infinite number of solutions.
(rank(A)=rank[A:b}=r<n)
(Singular Matrix i.e $|A| = 0$)

Inconsistent

No Solution (rank(A) $\neq$rank[A:b})
(Singular Matrix i.e $|A| = 0$)

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MATRICES AND GAUSSIAN ELIMINATION

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BREAKDOWN OF GAUSSIAN ELIMINATION:

Course Content: Breakdown of Gaussian Elimination

- If a zero appears in a pivot position, elimination has to stop –either temporarily or permanently. In this case the system may or may not be singular.

  In many cases this problem can be cured and elimination can proceed. Such a system is non-singular and has a full set of pivots.

  In other cases, when the breakdown is unavoidable (permanent). These systems are singular and have no solution or have infinitely many solutions. For such systems a full set of pivots cannot be found.

- Non-Singular & Curable $\left(|A| \neq 0\right)$
- Singular & InCurable $\left(|A| = 0\right)$
- Singular $\left(|A| = 0\right)$

BREAKDOWN OF GAUSSIAN ELIMINATION:

❖(i) Non-Singular & Curable $\left(|A| \neq 0\right)$:

❖ Consider the system

$$\begin{aligned} x + y + z &= 6 \\ x + y + 3z &= 10 \\ x + 2y + 4z &= 12 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 1 : 6 \\ 1 & 1 & 3 : 10 \\ 1 & 2 & 4 : 12 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 : 6 \\ 0 & 0 & 2 : 4 \\ 0 & 1 & 3 : 6 \end{pmatrix}$$

Here there is a zero in the second pivot position which can be avoided by a row exchange. Thus breakdown is **temporary** and **curable**.

Hence this system reduces to an upper triangular system which can be solved by Back Substitution and so system will become **consistent** and will have **a unique solution**.

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 1 : 6 \\ 0 & 1 & 3 : 6 \\ 0 & 0 & 2 : 4 \end{pmatrix} \Rightarrow \left.\begin{aligned} x + y + z &= 6 \\ y + 3z &= 6 \\ 2z &= 4 \end{aligned}\right\} \Rightarrow (x, y, z) = (4, 0, 2)$$

BREAKDOWN OF GAUSSIAN ELIMINATION:

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- **(ii) Singular & InCurable** $\left(|A| = 0\right)$:
- Consider the system

$$x + y + z = 6$$
$$x + y + 3z = 10$$
$$x + y + 4z = 13$$

$$\begin{pmatrix} 1 & 1 & 1 : 6 \\ 1 & 1 & 3 : 10 \\ 1 & 1 & 4 : 13 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 : 6 \\ 0 & 0 & 2 : 4 \\ 0 & 0 & 3 : 7 \end{pmatrix}$$

Here there is a zero in the second pivot position which cannot be avoided by any row exchange.

Hence breakdown cannot be avoided which is **incurable**.

The system is **singular** and has **no solution.**

$$\Rightarrow \left.\begin{aligned} x + y + z &= 6 \\ 2z &= 4 \\ 3z &= 7 \end{aligned}\right\}$$

Here we get z=2 and z=7/3 which is not possible.

BREAKDOWN OF GAUSSIAN ELIMINATION:

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- (iii) Singular $\left(|A| = 0\right)$:
- Consider the system

$$x + y + z = 6$$
$$x + y + 3z = 10$$
$$x + y + 4z = 12$$

$$\begin{pmatrix} 1 & 1 & 1:6 \\ 1 & 1 & 3:10 \\ 1 & 1 & 4:12 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1:6 \\ 0 & 0 & 2:4 \\ 0 & 0 & 3:6 \end{pmatrix}$$

Here there is a zero in the second pivot position which cannot be avoided. But since last two equations are **consistent** and we get **infinite number of solutions**.

$$\Rightarrow \left.\begin{aligned} x + y + z &= 6 \\ 2z &= 4 \\ 3z &= 6 \end{aligned}\right\} \Rightarrow z = 2 \text{ and } x + y = 4$$

$$y = k \Rightarrow x = 4 - k$$

$$\Rightarrow (x, y, z) = (4 - k, k, 2)$$

BREAKDOWN OF GAUSSIAN ELIMINATION:

(1) Apply Gaussian elimination to the system of equations

$$u + v + w = -2$$
$$3u + 3v - w = 6$$
$$u - v + w = -1$$

When does elimination fail at which pivot position.
Is the breakdown temporary or permanent, discuss.
What coefficient of v in the third equation, in place of the present -1, would make it impossible to proceed-and force elimination to break down?

$$\begin{pmatrix} 1 & 1 & 1 : -2 \\ 3 & 3 & -1 : 6 \\ 1 & -1 & 1 : -1 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \begin{pmatrix} 1 & 1 & 1 : -2 \\ 0 & 0 & -4 : 12 \\ 0 & -2 & 0 : 1 \end{pmatrix}$$

Elimination fails at second pivot position. This breakdown is **temporary** which can be cured by exchanging 2nd and 3rd row. Then system becomes consistent and we get a unique solution.

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 1 : -2 \\ 0 & -2 & 0 : 1 \\ 0 & 0 & -4 : 12 \end{pmatrix} \implies \left.\begin{aligned} u + v + w &= -2 \\ -2v &= 1 \\ -4w &= 12 \end{aligned}\right\}$$

$$\implies (u, v, w) = \left(\frac{3}{2}, -\frac{1}{2}, -3\right)$$

If the coefficient of v in the third equation is 1 instead of -1, then elimination breaks down **permanently** and it is impossible to proceed.

BREAKDOWN OF GAUSSIAN ELIMINATION:

(2) For which three numbers a will elimination fail to give three pivots?

$$ax + 2y + 3z = b_1$$
$$ax + ay + 4z = b_2$$
$$ax + ay + az = b_3$$

$$\begin{pmatrix} a & 2 & 3 : b_1 \\ a & a & 4 : b_2 \\ a & a & a : b_3 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} a & 2 & 3 : b_1 \\ 0 & a-2 & 1 : b_2 - b_1 \\ 0 & a-2 & a-3 : b_3 - b_1 \end{pmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{pmatrix} a & 2 & 3 : b_1 \\ 0 & a-2 & 1 : b_2 - b_1 \\ 0 & 0 & a-4 : b_3 - b_2 \end{pmatrix}$$

If **a=0, a=2, a=4** will fail to give 3 pivots.

BREAKDOWN OF GAUSSIAN ELIMINATION:

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3. For what values of a and b does the following system have (i)a unique solution (ii) Infinitely many solutions (iii) No solution.

$$x + 2y + 3z = 2$$
$$-x - 2y + az = -2$$
$$2x + by + 6z = 5$$

$$\begin{pmatrix} 1 & 2 & 3:2 \\ -1 & -2 & a:-2 \\ 2 & b & 6:5 \end{pmatrix} \xrightarrow[R_3-2R_1]{R_2+R_1} \begin{pmatrix} 1 & 2 & 3:2 \\ 0 & 0 & a+3:0 \\ 0 & b-4 & 0:1 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 3:2 \\ 0 & b-4 & 0:1 \\ 0 & 0 & a+3:0 \end{pmatrix}$$

(i) If $a \neq -3$ and $b \neq 4$ then r(A)=r(A:b)=3=n hence system will be **consistent** and will have **a unique solution**.

(ii) If $a = -3$ then r(A)=r(A:b)=2<n(=3) hence system will be **consistent** and will have infinitely **many solutions**.

(iii) If $a = -3,\ b = 4\ (\text{or } a \neq -3,\ b = 4)$, then r(A)=1(or 2) and r(A:b)=2(or 3) hence system will be **inconsistent** and will have **no solution**.

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MATRICES AND GAUSSIAN ELIMINATION

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Course Content: Elementary Matrices

- Elementary Matrix E_{ij} is obtained from the Identity Matrix I by using transformation $R_i - l_{ij}R_j$ where l_{ij} is the multiplier

  i.e $$I \longrightarrow E_{ij}$$

  Example: E_{32} is obtained as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} = E_{32}$$

MATRICES AND GAUSSIAN ELIMINATION:

Example:

Consider the matrix $A=\begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow[R_3-1/3R_1]{R_2-2/3R_1} \begin{pmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 5/3 & 1/3 \end{pmatrix}$

$$\xrightarrow{R_3+(5/11)R_2} \begin{pmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{pmatrix} = U$$

$$E = E_{21} = R_2 - (2/3)R_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F = E_{31} = R_3 - (1/3)R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}$$

$$G = E_{32} = R_3 + (5/11)R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5/11 & 1 \end{pmatrix}$$

$$E_{32}E_{31}E_{21}A = U$$

MATRICES AND GAUSSIAN ELIMINATION:

Problems:

1. Which elimination matrices put A into upper triangular matrix U?

$$A=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \xrightarrow{R_2+(1/2)R_1} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \xrightarrow{R_3+(2/3)R_2} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow{R_4+(3/4)R_3} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{pmatrix} = U$$

The elimination matrices E_{21}, E_{32} and E_{43} put A into upper triangular matrix i.e.,

$$E_{43}E_{32}E_{21}A = U$$

$$E_{43}E_{32}E_{21}A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A = U$$

MATRICES AND GAUSSIAN ELIMINATION:

2. Which elementary matrices convert $A = \begin{pmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{pmatrix}$ into upper triangular matrix U?

$$A = \begin{pmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{pmatrix} \xrightarrow[R_3 + 1/2R_1]{R_2 + 1/2R_1} \begin{pmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 & 12 \end{pmatrix}$$

$$\xrightarrow{R_3 - 3R_2} \begin{pmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = U$$

Matrices E_{21}, E_{31}, E_{32} convert A into triangular form U.

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix} \quad E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

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MATRICES AND GAUSSIAN ELIMINATION

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Course Content: Triangular Factors

- Consider the system of equations Ax=b. Using elementary row transformations we reduce A to U an Upper triangular matrix. While doing so Ax=b reduces to Ux=c(i.e [b] reduces to [c]) which can be solved by back substitution to obtain x. The steps involved are $Ax = b \Rightarrow E_{32}E_{31}E_{21}Ax = Ux = c\left(\because E_{32}E_{31}E_{21}A = U\right)$

  To undo the steps of elimination we must trace back the steps instead of subtracting we must add . For this we need inverses of Elementary Matrices i.e $E_{21}^{-1}, E_{31}^{-1}, E_{32}^{-1}$

  E_{21}^{-1} can be obtained by changing –l to l in the transformation. Similarly

  E_{31}^{-1} and E_{32}^{-1} can also be obtained. But while going from U to A the inverses should be in reverse direction i.e., $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = A$

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Triangular Factorization A=LU:

- Any square matrix A can be factored as A=LU where
- L is a Lower Triangular Matrix
- With 1's on the diagonal
- Having multiplier's l_{ij} below the diagonal in their respective positions.
- **And**
- U is an Upper Triangular Matrix
- Having pivots on the diagonal (u_{11}, u_{22}, u_{33})

i.e $L=\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} U=\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}=\begin{pmatrix} d_1 & u_{12} & u_{13} \\ 0 & d_2 & u_{23} \\ 0 & 0 & d_3 \end{pmatrix}$

MATRICES AND GAUSSIAN ELIMINATION:

Consider the matrix A= $\begin{pmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow[R_3 - 1/3R_1]{R_2 - 2/3R_1} \begin{pmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 5/3 & 1/3 \end{pmatrix}$

$$\xrightarrow{R_3 + (5/11)R_2} \begin{pmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{pmatrix} = U \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -5/11 & 1 \end{pmatrix}$$

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -5/11 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{pmatrix}$$

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Triangular Factorization A=LDU:

- **A=LU** factorization is Unsymmetric as L has 1's on the diagonal whereas U has pivots on the diagonal. In order to make this factorization symmetric we do **A=LDU** factorization
- **D is a Diagonal Matrix with pivots d_1, d_2, d_3 on the diagonal**
- **L is a Lower Triangular Matrix**
- **With 1's on the diagonal**
- **Having multiplier's l_{ij} below the diagonal in their respective positions.**
- **U is an Upper Triangular Matrix with 1's on the diagonal obtained by dividing each row by its pivot.**

i.e

$$L=\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \quad D=\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \quad U=\begin{pmatrix} 1 & u_{12}/u_{11} & u_{13}/u_{11} \\ 0 & 1 & u_{23}/u_{22} \\ 0 & 0 & 1 \end{pmatrix}$$

MATRICES AND GAUSSIAN ELIMINATION:

Factorise A=LU and hence A=LDU.

$$A=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \xrightarrow{R_2+(1/2)R_1} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \xrightarrow{R_3+(2/3)R_2} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow{R_4+(3/4)R_3} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{pmatrix} = U \quad A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{pmatrix}$$

$$A = LDU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

MATRICES AND GAUSSIAN ELIMINATION:

- To solve a system of equations Ax=b, factorize **A=LU, then Ax=Lux where**

  Let Ux=c, then Lc=b=Ax.

  Solve Lc=b, using Forward elimination and then find x using Ux=c by Backward substitution which gives x.

Ex: Solve as triangular systems without multiplying LU to find A.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

Let $Lc = b \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

$$\Rightarrow \begin{cases} c_1 = 2 \\ c_2 = -2 \\ c_3 = 0 \end{cases} \text{then } Ux = c \Rightarrow \begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2x + 4y + 4z = 2 \\ y + 2z = -2 \\ z = 0 \end{cases}$$

$$(x, y, z) = (5, -2, 0)$$

- Find A=LU and LDU factorization given

$$A=\begin{pmatrix}6 & -2 & -4 & 4\\ 3 & -3 & -6 & 1\\ -12 & 8 & 21 & -8\\ -6 & 0 & -10 & 7\end{pmatrix}\xrightarrow[\substack{R_3+2R_1\\ R_4+R_1}]{R_2-1/2R_1}\begin{pmatrix}6 & -2 & -4 & 4\\ 0 & -2 & -4 & -1\\ 0 & 4 & 13 & 0\\ 0 & -2 & -14 & 11\end{pmatrix}\xrightarrow[R_4-R_2]{R_3+2R_2}\begin{pmatrix}6 & -2 & -4 & 4\\ 0 & -2 & -4 & -1\\ 0 & 0 & 5 & -2\\ 0 & 0 & -10 & 12\end{pmatrix}$$

$$\xrightarrow{R_4+2R_3}\begin{pmatrix}6 & -2 & -4 & 4\\ 0 & -2 & -4 & -1\\ 0 & 0 & 5 & -2\\ 0 & 0 & 0 & 8\end{pmatrix}=U \quad A=LU=\begin{pmatrix}1 & 0 & 0 & 0\\ 1/2 & 1 & 0 & 0\\ -2 & -2 & 1 & 0\\ -1 & 1 & -2 & 1\end{pmatrix}\begin{pmatrix}6 & -2 & -4 & 4\\ 0 & -2 & -4 & -1\\ 0 & 0 & 5 & -2\\ 0 & 0 & 0 & 8\end{pmatrix}$$

$$A=LDU=\begin{pmatrix}1 & 0 & 0 & 0\\ 1/2 & 1 & 0 & 0\\ -2 & -2 & 1 & 0\\ -1 & 1 & 5/2 & 1\end{pmatrix}\begin{pmatrix}6 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & 5 & 0\\ 0 & 0 & 0 & 8\end{pmatrix}\begin{pmatrix}1 & -1/3 & -2/3 & 2/3\\ 0 & 1 & 2 & 1/2\\ 0 & 0 & 1 & -2/5\\ 0 & 0 & 0 & 1\end{pmatrix}$$

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Course Content: Cholesky Decomposition Or Factorization:

Cholesky decomposition or Cholesky factorization is decomposition of a Hermitian positive definite matrix which is factored into lower triangular matrix L and its conjugate transpose L^T. In this L has real positive diagonal entries.

- A Hermitian positive definite matrix A can be factored as $A=LL^T$ where $L=\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$

$$A=LL^T \rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}$$

Algorithm:

Let $A_{3x3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a symmetric positive definite matrix.

Then A can be factored as LL^T

$A=LL^T \rightarrow$
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}$$

$$= \begin{pmatrix} l_{21}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{31} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{pmatrix}$$

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This gives $l_{11}^2 = a_{11} \Longrightarrow l_{11} = \sqrt{a_{11}}$;

$$l_{21} = \frac{a_{21}}{l_{11}}; \qquad l_{22} = \sqrt{a_{22} - l_{21}^2}$$

$$l_{31} = \frac{a_{31}}{l_{11}} ; \qquad l_{32} = \frac{a_{32} - l_{31}l_{21}}{l_{22}}; \qquad l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2}$$

Example: Factorize A using Cholesky Decomposition given A= $\begin{pmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 22 \end{pmatrix}_3$

$$A=\begin{pmatrix}4 & 2 & 6\\ 2 & 2 & 5\\ 6 & 5 & 22\end{pmatrix} \qquad l_{11}=\sqrt{a_{11}}=2\ ;\ l_{21}=\frac{a_{21}}{l_{11}}=1\ ;\quad l_{22}=\sqrt{a_{22}-l_{21}^2}=1$$

$$l_{31}=\frac{a_{31}}{l_{11}}=3\quad ;\qquad l_{32}=\frac{a_{32}-l_{31}l_{21}}{l_{22}}=2\quad ;\qquad l_{33}=\sqrt{a_{33}-l_{31}^2-l_{32}^2}=3$$

Hence $\quad L=\begin{pmatrix}2 & 0 & 0\\ 1 & 1 & 0\\ 3 & 2 & 3\end{pmatrix} \qquad L^T=\begin{pmatrix}2 & 1 & 3\\ 0 & 1 & 2\\ 0 & 0 & 3\end{pmatrix}$

$$\therefore A=LL^T=\begin{pmatrix}2 & 0 & 0\\ 1 & 1 & 0\\ 3 & 2 & 3\end{pmatrix}\begin{pmatrix}2 & 1 & 3\\ 0 & 1 & 2\\ 0 & 0 & 3\end{pmatrix}$$

Factorize $A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix}$ using Cholesky factorization.

Answer: $L = \begin{pmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{pmatrix}$

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Course Content: Row Exchanges and Permutation Matrices

- Consider the system of equations Ax=b. While solving the system if a zero appears In the pivot position, it calls for a row exchange.

  This row exchange is taken care by **Permutation Matrices P .**

  Here **A ≠ LU** then **PA=LU** where **P is a Permutation Matrix** which is an Identity Matrix with rows in different order.

Ex:Consider the system $y=b_1$; $2x-3y=b_2$

$$Ax = b \Rightarrow \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Here Gaussian elimination fails and so calls for a row exchange i.e., $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_2 \\ b_1 \end{pmatrix}$

This is same as PAX=Pb

$$PA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = U \text{ and } \quad Pb = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_2 \\ b_1 \end{pmatrix}$$

$$\therefore PAx = PB \Rightarrow \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_2 \\ b_1 \end{pmatrix}$$

Permutation Matrices:

- **P is a Permutation Matrix** which is an Identity Matrix with rows in different order.
- **Product of two permutation matrices is also a Permutation Matrix.**
- **Inverse of a permutation matrices is also a Permutation Matrix.**
- **P^{-1} is always same as P^T .**
- **Permutation Matrices of order 2 are 2!=2 in number.**

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

MATRICES AND GAUSSIAN ELIMINATION:

❖ **Permutation Matrices of order 3 are 3!=6 in number.**

$$I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix};\qquad P_{21}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}=P_{21}^{-1};$$

$$P_{31}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}=P_{31}^{-1};\qquad P_{32}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}=P_{32}^{-1};$$

$$P_{21}P_{31}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}=\left(P_{21}P_{32}\right)^{-i};\qquad P_{21}P_{32}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}=\left(P_{21}P_{32}\right)^{-1}$$

MATRICES AND GAUSSIAN ELIMINATION:

Ex: $A=\begin{pmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{pmatrix} \xrightarrow{R_2-3R_1} \begin{pmatrix} 2 & 3 & 3 \\ 0 & 0 & -1 \\ 0 & 5 & 7 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{pmatrix} = U$

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{pmatrix} \neq A$$

$$P_{23}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{pmatrix} \xrightarrow{R_3-3R_1} \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{pmatrix} = U$$

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{pmatrix} = P_{23}A$$

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Problem: Explain why A is not factorizable into LU? How can A be modified so that the new matrix can be factored into LU? Also obtain the factors L,D,U for the new matrix. What is the relation between L and U thus obtained? Explain.

$$A=\begin{pmatrix}1 & -2 & 2\\ 2 & -4 & 5\\ -2 & 5 & -4\end{pmatrix}\xrightarrow[R_3+2R_1]{R_2-2R_1}\begin{pmatrix}1 & -2 & 2\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\xrightarrow{R_2\leftrightarrow R_3}\begin{pmatrix}1 & -2 & 2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

A is not factorizable into LU as Gaussian elimination fails and Row exchange is required.

So A should be multiplied with permutation matrix P_{23} so that PA=LU.

$$P_{23}A=\begin{pmatrix}1 & -2 & 2\\ -2 & 5 & -4\\ 2 & -4 & 5\end{pmatrix}\xrightarrow[R_3-2R_1]{R_2+2R_1}\begin{pmatrix}1 & -2 & 2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=U \qquad L=\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 2 & 0 & 1\end{pmatrix}$$

$$L=\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 2 & 0 & 1\end{pmatrix}\quad D=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\quad U=\begin{pmatrix}1 & -2 & 2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$L=U^T$

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Course Content: Inverses and Transposes

- Let A be a square matrix of order n the **Inverse of A** is the matrix B such that **AB=I=BA**. Here $B=A^{-1}$.

- **Properties:** Inverse of a matrix is unique. i.e., AB=BA=I and AC=CA=I, then B=C
- Inverse of the product is the product of Inverses. $(ABCD)^{-1}=D^{-1}C^{-1}B^{-1}A^{-1}$
- If A=LU then $A^{-1}=U^{-1}L^{-1}$.
- Since $E_{32}E_{31}E_{21}A=U$ we have $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U=A \Rightarrow A^{-1}=U^{-1}E_{32}E_{31}E_{21}$

  $\Rightarrow L=E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$

- A matrix A is invertible if and only if elimination produces n pivots with or without row exchanges. Elimination solves Ax=b without explicitly finding A^{-1} .
- If A is invertible, the one and only one solution to Ax=b is $x=A^{-1}b$

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Gauss Jordan Method of computing A^{-1} :

- The inverse of an invertible matrix is obtained by a set of row operations that transforms A to I and I to A^{-1} . This process is known as **Gauss Jordan Method**.
- Consider the Augmented Matrix **[A:I].** Then perform row operations on it so that A reduces to Echelon form **U** and at the same time **I** reduces to **C.** Further reduce **U to I** using elementary row transformations which reduces **C to A^{-1}** .
- i.e., $\left[A:I\right]\rightarrow\left[U:C\right]\rightarrow\left[I:A^{-1}\right]$

Ex: Compute **A^{-1}** using Gauss Jordan Method given

$$A=\begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -2 & 2 & 2 \end{pmatrix}$$

MATRICES AND GAUSSIAN ELIMINATION:

Ex: $[A:I]=\begin{pmatrix} 2 & 1 & 1:1 & 0 & 0 \\ 4 & 3 & 4:0 & 1 & 0 \\ -2 & 2 & 2:0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{pmatrix} 2 & 1 & 1:1 & 0 & 0 \\ 0 & 1 & 2:-2 & 1 & 0 \\ 0 & 3 & 3:1 & 0 & 1 \end{pmatrix}$

$\xrightarrow{R_3-3R_2} \begin{pmatrix} 2 & 1 & 1:1 & 0 & 0 \\ 0 & 1 & 2:-2 & 1 & 0 \\ 0 & 0 & -3:7 & -3 & 1 \end{pmatrix} = [U:C] \xrightarrow[R_2+2/3R_3]{R_1+1/3R_3} \begin{pmatrix} 2 & 1 & 0:10/3 & -1 & 1/3 \\ 0 & 1 & 0:8/3 & -1 & 2/3 \\ 0 & 0 & -3:\ 7 & -3 & 1 \end{pmatrix}$

$\xrightarrow{R_1-R_2} \begin{pmatrix} 2 & 0 & 0:2/3 & 0 & -1/3 \\ 0 & 1 & 0:8/3 & -1 & 2/3 \\ 0 & 0 & -3:7 & -3 & 1 \end{pmatrix} \xrightarrow[R_3=-1/3R_3]{R_1=1/2R_1} \begin{pmatrix} 1 & 0 & 0:1/3 & 0 & -1/6 \\ 0 & 1 & 0:8/3 & -1 & 2/3 \\ 0 & 0 & 1:-7/3 & 1 & -1/3 \end{pmatrix}$

$B=A^{-1}=\begin{pmatrix} 1/3 & 0 & -1/6 \\ 8/3 & -1 & 2/3 \\ -7/3 & 1 & -1/3 \end{pmatrix}$ $\quad =[I:B]$

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Transpose of a Matrix A^T :

- If $A = \left[a_{ij} \right]_{mxn}$ is an mxn matrix, then its transpose is obtained by interchanging its rows and columns and is denoted by $A^T = \left[a_{ji} \right]_{nxm}$.

Ex: If $A = \begin{pmatrix} 2 & 1 & -3 \\ 4 & 2 & 0 \end{pmatrix}_{2x3}$ then $A^T = \begin{pmatrix} 2 & 4 \\ 1 & 2 \\ -3 & 0 \end{pmatrix}_{3x2}$

- **Properties:**
- The Transpose of a Lower Triangular Matrix is an Upper Triangular Matrix.

$$\left(A^T\right)^T = A \; ; \quad (AB)^T = B^T A^T \; ; \quad \left(A^{-1}\right)^T = \left(A^T\right)^{-1} \; ;$$

$$(A \pm B)^T = A^T \pm B^T \; ; \quad \left(A^{-1}\right)^T A^T = \left(AA^{-1}\right)^T = I$$

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Symmetric Matrices:

- If A is a matrix of order n is said to be symmetric matrix of order if $A^T = A$

Ex: If $A=\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$ then $A^T=\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$

- **Properties:**
- If A is symmetric then A^{-1} may or may not exist.
- If for a symmetric matrix A^{-1} exists then A^{-1} is also symmetric.
- For a symmetric matrix A, we have $\left(A^{-1}\right)^T = (A)^{-1} \left(\because A^T = A\right)$

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Symmetric Products AA^T, A^TA, LDL^T:

- If A is a matrix of order mxn, then AA^T *and* A^TA are both symmetric

Ex: If $A=\begin{pmatrix} 2 & 1 & -3 \\ 4 & 2 & 0 \end{pmatrix}_{2x3}$ then $AA^T=\begin{pmatrix} 14 & 10 \\ 10 & 20 \end{pmatrix}$ $A^TA=\begin{pmatrix} 20 & 10 & -6 \\ 10 & 5 & -3 \\ -6 & -3 & 9 \end{pmatrix}$

- If A is symmetric and if $A = LDU$ then,

$$A = A^T = LDL^T \left(\because U = L^T \,\&\, L = U^T \right)$$

$$A=\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 2 & 3 \end{pmatrix} \xrightarrow{R_3+2R_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 7 \end{pmatrix} = U$$

$$LDU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \qquad U = L^T$$

MATRICES AND GAUSSIAN ELIMINATION:

- For which three numbers “c” is this matrix not invertible, and why not?

$$A=\begin{pmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{pmatrix} \xrightarrow[R_3-4R_1]{R_2-c/2R_1} \begin{pmatrix} 2 & c & c \\ 0 & c-\left(c^2/2\right) & c-\left(c^2/2\right) \\ 0 & 7-4c & -3c \end{pmatrix}$$

$$\xrightarrow{R_3-\left(\frac{7-4c}{2c-c^2}\right)2R_2} \begin{pmatrix} 2 & c & c \\ 0 & c-\left(c^2/2\right) & c-\left(c^2/2\right) \\ 0 & 0 & c-7 \end{pmatrix}$$

- Matrix A is not invertible for c=0,2,7
- For c=0.2,7 elimination gives one zero row ,hence A will be singular and so A will not be invertible.

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THANK YOU

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LINEAR ALGEBRA AND ITS APPLICATIONS

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MATRICES AND
GAUSSIAN ELIMINATION

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Course Content: Supplementary problems

1. Find PA=LU and PA=LDU for A= $\begin{pmatrix} 3 & -1 & 0 \\ 6 & -2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

$$A=\begin{pmatrix} 3 & -1 & 0 \\ 6 & -2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \xrightarrow[R_3-1/3R_1]{R_2-2R_1} \begin{pmatrix} 3 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 4/3 & 2 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{pmatrix} 3 & -1 & 0 \\ 0 & 4/3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$PA=\begin{pmatrix} 3 & -1 & 0 \\ 1 & 1 & 2 \\ 6 & -2 & 0 \end{pmatrix} \xrightarrow[R_3-2R_1]{R_2-1/3R_1} \begin{pmatrix} 3 & -1 & 0 \\ 0 & 4/3 & 2 \\ 0 & 0 & 0 \end{pmatrix} = U$$

$$PA=\begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 & 0 \\ 0 & 4/3 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{pmatrix}$$

MATRICES AND GAUSSIAN ELIMINATION:

2. For what values of a and b does the following system have (i)a trivial solution

(ii) Infinitely many solutions.

$$x+2y+3z=0$$
$$-x-2y+az=0$$
$$2x+by+6z=0$$

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & a \\ 2 & b & 6 \end{pmatrix} \xrightarrow[R_3-2R_1]{R_2+R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & a+3 \\ 0 & b-4 & 0 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & b-4 & 0 \\ 0 & 0 & a+3 \end{pmatrix}$$

This system will have only either a trivial solution or infinitely many solutions. It will have

(i) **a trivial solution** if $a \neq -3 \ \& \ b \neq 4$ then r(A)=3=n

(ii) **Infinitely many solutions** if $a=-3$ or $b=4$ *or both* then r(A) will be 2 or 1 respectively.

GAUSSIAN ELIMINATION:

3. Check for consistency and solve the following system of equations if consistent. Also discuss its rank:

$$3x+y+2z=3$$
$$2x-3y-z=-3$$
$$x+2y+z=4$$

$$\begin{pmatrix} 3 & 1 & 2:3 \\ 2 & -3 & -1:-3 \\ 1 & 2 & 1:4 \end{pmatrix} \xrightarrow[R_3-\left(\frac{1}{3}\right)R_1]{R_2-\left(\frac{2}{3}\right)R_1} \begin{pmatrix} 3 & 1 & 2:3 \\ 0 & -11/3 & -7/3:-5 \\ 0 & 5/3 & -1/3:3 \end{pmatrix}$$

$$\xrightarrow{R_3+\left(\frac{5}{11}\right)R_2} \begin{pmatrix} 3 & 1 & 2:3 \\ 0 & -11/3 & -7/3:-5 \\ 0 & 0 & -24/33:8/11 \end{pmatrix} \Rightarrow \begin{cases} 3x+y+2z=3 \\ (-11/3)y-7/3z=-5 \\ (-24/33)z=8/11 \end{cases}$$

r(A)=r(A:b)=3=n hence system is **consistent** and has **a unique solution**.

i.e **(x, y, z)=(1, 2,-1). Its rank is 3.**

4. Find an LU and LDU factorization for A. What is the rank of A?

$$A=\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix} \xrightarrow[\substack{R_3-R_1 \\ R_4+3R_1}]{R_2+2R_1} \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{pmatrix} \xrightarrow[R_4-4R_2]{R_3+3R_2} \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{pmatrix}$$

$$\xrightarrow{R_4-2R_3} \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \quad A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

$$A = LDU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1/2 & 5/2 & -1 \\ 0 & 1 & 1/3 & 2/3 & -1 \\ 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Rank of A is 4.

MATRICES AND GAUSSIAN ELIMINATION:

5. Solve Ax=b for x if $A=\begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 3 \\ 4 & 2 & 5 \end{pmatrix}$ and b= $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

$$[A:I]=\begin{pmatrix} 1 & 0 & -2:1 & 0 & 0 \\ 2 & 1 & 3:0 & 1 & 0 \\ 4 & 2 & 5:0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3-4R_1]{R_2-2R_1} \begin{pmatrix} 1 & 0 & -2: & 1 & 0 & 0 \\ 0 & 1 & 7: & -2 & 1 & 0 \\ 0 & 2 & 13: & -4 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3-2R_2} \begin{pmatrix} 1 & 0 & -2: & 1 & 0 & 0 \\ 0 & 1 & 7: & -2 & 1 & 0 \\ 0 & 0 & -1: & 0 & -2 & 1 \end{pmatrix} \xrightarrow[R_2+7R_3]{R_1-2R_3} \begin{pmatrix} 1 & 0 & 0: & 1 & 4 & -2 \\ 0 & 1 & 0: & -2 & -13 & 7 \\ 0 & 0 & -1: & 0 & -2 & 1 \end{pmatrix}$$

$$\xrightarrow{-R_3} \begin{pmatrix} 1 & 0 & 0: & 1 & 4 & -2 \\ 0 & 1 & 0: & -2 & -13 & 7 \\ 0 & 0 & 1: & 0 & 2 & -1 \end{pmatrix} \quad A^{-1}=\begin{pmatrix} 1 & 4 & -2 \\ -2 & -13 & 7 \\ 0 & 2 & -1 \end{pmatrix} \quad x=\begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$$

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Sum of Subspaces

The Sum of two subspaces U and W of a vector space V is defined as

$$U + W = \{u \in U, w \in W\}$$

Definition: Let U, W be subspaces of V . Then V is said to be the direct sum of U and W, and we write $V = U \oplus W$,
if $V = U + W$ and $U \cap W = \{0\}$.
Let U, W be subspaces of V . Then $V = U \oplus W$ if and only if for every $v \in V$ there exist unique vectors $u \in U$ and $w \in W$ such that $v = u + w$.

Properties :

1. The zero vector ‘0’ of V is in U +W .
2. For any $u, w \in U + W$, we have $u + v \in U + W$.
3. For any $v \in U + W$ and $\alpha \in R$, we have $\alpha v \in V \in U + W$
4. $v = u + w$ must be unique .

Sum of subspaces

Example : Consider $U = \{ (a, 0, 0) / a \in R \}$
$W = \{ (0, b, c) / b, c \in R \}$

Thus $V = U + W = \{ (a, b, c) / a, b, c \in R \}$

Hence the direct sum of subspaces U and W results into vector space R^3

VECTOR SPACES

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CLASS 1 : CONTENT

- Definition of Vector Space
- Examples of Vector Space
- Definition of Subspace
- Examples of Subspaces

VECTOR SPACES : DEFINITION

A Real vector space V is a nonempty set of objects called vectors, together with (Scalar multiplication and Vector addition)satisfying the following axioms :

I. If $u, v \in V$, then $u + v \in V$ $\Rightarrow V$ is closed under vector addition.

II. If $\mathrm{c} \in \mathrm{R}$ & $\mathrm{u} \in \mathrm{V}$, then $cu \in \mathrm{V}$ $\Rightarrow V$ is closed under scalar multiplication.

These operations satisfy the following properties for $u, v, w \in V$ & c_1, c_1 are scalars

a) $u + v = v + u$ (commutative law)

b) $u + (v + w) = (u + v) + w$(Associative law)

VECTOR SPACES

c) there is a unique zero vector i.e., 0 such that $0 + u = u + 0 = u$

(identity law) Additive identity '0' $\in V$

d) for each u there is a unique vector $(-u)$ such that

$u + (-u) = (-u) + u = 0$ (Inverse law)

e) $c_1(u+v) = c_1u + c_1v$

f) $(c_1 + c_2)u = c_1u + c_2u$

g) $(c_1 + c_2)u = c_1u + c_2u$

h) $1u = u$, Where 1 is a multiplicative identity s.t. 1 £ R

VECTOR SPACES: EXAMPLES

Example 1 The following are examples of vector spaces:

1. The set of all real number $\mathbb{R}$ associated with the addition and scalar multiplication of real numbers.
2. The set of all the complex numbers $\mathbb{C}$ associated with the addition and scalar multiplication of complex numbers.
3. The set of all polynomials $R_n(x)$ with real coefficients associated with the addition

VECTOR SPACES: EXAMPLES

4. The set of all vectors of dimension n written as $\mathbb{R}^n$ associated with the addition and scalar multiplication as defined for 3-d and 2-d vectors for example.

5. The set of all matrices of dimension $m \times n$ associated with the addition and scalar multiplication as defined for matrices.

VECTOR SPACES: EXAMPLES

Example 1 :

Prove that the set of all 2 by 2 matrices associated with the matrix addition and the scalar multiplication of matrices is a vector space.

Solution: Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ s.t. $A, A' \in V$ and $r, s \in R$

Let V be the set of all 2 by 2 matrices.

1) Addition of matrices gives

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a + a' & b + b' \\ c + c' & d + d' \end{bmatrix}$$

Adding any 2 by 2 matrices gives a 2 by 2 matrix and therefore the result of the addition

VECTOR SPACES: EXAMPLES

Scalar multiplication of matrices gives gives

$$r\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$$

Multiply any 2 by 2 matrix by a scalar and the result is a 2 by 2 matrix is an element of V.

3) Commutativity

$$\begin{aligned} &\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \\ &= \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} \\ &= \begin{bmatrix} a'+a & b'+b \\ c'+c & d'+d \end{bmatrix} \\ &= \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{aligned}$$

4) Associativity of vector addition

$$\begin{aligned} &\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} \\ &= \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} \\ &= \begin{bmatrix} (a+a')+a'' & (b+b')+b'' \\ (c+c')+c'' & (d+d')+d'' \end{bmatrix} \\ &= \begin{bmatrix} a+(a'+a'') & b+(b'+b'') \\ c+(c'+c'') & d+(d'+d'') \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}\right) \end{aligned}$$

VECTOR SPACES: EXAMPLES

5) Associativity of multiplication

$$r\left(s\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = r\left(\begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}\right)$$
$$= \begin{bmatrix} rsa & rsb \\ rsc & rsd \end{bmatrix}$$
$$= \begin{bmatrix} (rs)a & (rs)b \\ (rs)c & (rs)d \end{bmatrix}$$
$$= (rs)\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

6) Zero vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} a+0 & b+0 \\ c+0 & d+0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

7) Negative vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$
$$= \begin{bmatrix} a+(-a) & b+(-b) \\ c+(-c) & d+(-d) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

VECTOR SPACES: EXAMPLES

8) Distributivity of sums of matrices:

$$r\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right)$$
$$= \begin{bmatrix} r(a+a') & r(b+b') \\ r(c+c') & r(d+d') \end{bmatrix}$$
$$= \begin{bmatrix} ra+ra' & rb+rb \\ rc+rc' & rd+rd \end{bmatrix}$$
$$= r\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + r\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right)$$

9) Distributivity of sums of real numbers:

$$(r+s)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (r+s)a & (r+s)b \\ (r+s)c & (r+s)d \end{bmatrix}$$
$$= \begin{bmatrix} ra+sa & rb+sb \\ rc+sc & rd+sd \end{bmatrix}$$
$$= \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} + \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}$$
$$= r\begin{bmatrix} a & b \\ c & d \end{bmatrix} + s\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

10) Multiplication by 1.

$$1\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a & 1b \\ 1c & 1d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

VECTOR SPACES: EXAMPLES

Example 2 :

Show that the set of all real polynomials with a degree $n \leq 3$ associated with the addition of polynomials and the multiplication of polynomials by a scalar form a vector space.

Solution

The addition of two polynomials of degree less than or equal to 3 is a polynomial of degree lass than or equal to 3.

The multiplication, of a polynomial of degree less than or equal to 3, by a real number results in a polynomial of degree less than or equal to 3

VECTOR SPACES: EXAMPLES

Hence the set of polynomials of degree less than or equal to 3 is closed under addition

and scalar multiplication (the first two conditions above).

The remaining 8 rules are automatically satisfied since the polynomials are real.

VECTOR SPACES: EXAMPLES

Example 3 :

Show that the set of integers associated with addition and multiplication by a real number

IS NOT a vector space

Solution :

The multiplication of an integer by a real number may not be an integer.

Example: Let $x = -2$

If you multiply x by the real number $\sqrt{3}$ the result is NOT an integer.

VECTOR SPACE

Few examples :

1. R = the set of all real numbers

2. R^2 = { (x, y) / x, y Є R }

3. R^3 = { (x, y , z) / x, y, z Є R }

4. R^n = { (x1 , x2 , ..., xn) / xi Є R }

5. R^∞ = { (x1 , x2 , ...,) / xi Є R }

VECTOR SPACES: EXAMPLES

Problem 1:

Verify whether the following

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} ; x \geq 0, y \geq 0, x, y \in r \right\}$$

Is a vector space ……

under usual vector addition and scalar multiplication.

VECTOR SPACES

- Closure property holds good.
- Associative property holds
- $\exists \ 0 \in V \ \exists \ u + 0 = u = 0 + u$,
- $\forall u \in V \ \exists \ -u \notin V \quad \left[Eg: u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in V, \ -u = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \notin V \right]$

Therefore Inverse law doesn't hold

Hence V is not a vector space.

Problem 2 : $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 / x + y = 0 \right\} u, v \in V$

All the properties holds good

Hence V is a vector space

VECTOR SPACES

Precisely ,

We can add any two vectors and we can multiply all vectors by scalars. In other words, we can take linear combinations.

SUBSPACES

A nonempty subset of a vector space is called a subspace of V , if it is itself a vector space under the same operations of vector addition and scalar multiplication as defined in vector space .

The following are the properties satisfied by a subspace of V

i) $0 \in W$ (zero vector always belongs to a subspace)

ii) $if \quad u, v \in W$ Then $u + v \in W$

iii) If 'c' is a scalar and $u \in W$ then $cu \in W$

SUBSPACES : DEFINITION

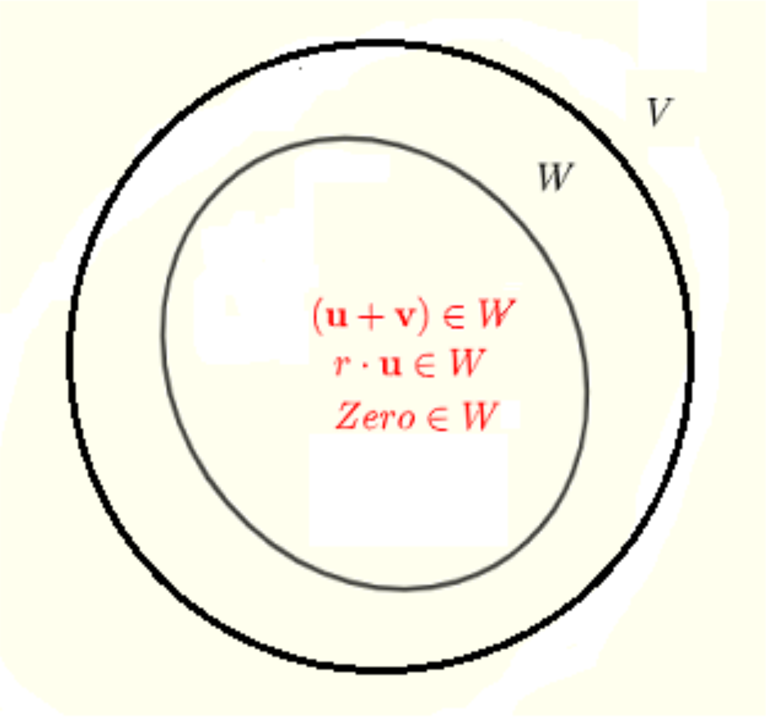
If W is a subset of a vector space V and if W is itself a vector space under the inherited

operations of addition and scalar multiplication from V, then W is called a subspace

To show that the W is a subspace of V, it is enough to show that

1. W is a subset of V
2. The zero vector of V is in W
3. For any vectors and in W, $\mathbf{u}+\mathbf{v}$ is in W. (closure under additon)
4. For any vector $\mathbf{u}$ and scalar r, $r\cdot\mathbf{u}$ is in W. (closure under scalar multiplication).

SUBSPACES : DEFINITION



SUBSPACES : EXAMPLES

Example 1

The set W of vectors of the form $(x, 0)$ where $x \in \mathbb{R}$ is a subspace of $\mathbb{R}^2$ because:

W is a subset of $\mathbb{R}^2$ whose vectors are of the form (x, y) where $x \in \mathbb{R}$ and $y \in \mathbb{R}$

The zero vector $(0, 0)$ is in W

$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$, closure under addition

$r \cdot (x, 0) = (rx, 0)$, closure under scalar multiplication

SUBSPACES : EXAMPLES

Example 2

The set W of vectors of the form (x, y) such that $x \geq 0$ and $y \geq 0$ is not a subspace of

$\mathbb{R}^2$ because it is not closed under scalar multiplication.

Vector $\mathbf{u} = (2, 2)$ is in W but its negative $-1(2, 2) = (-2, -2)$ is not in W.

SUBSPACES :

- Note : If U and W are two subspaces of a vector space V, intersection $U \cap W$ is also a subspace of V .

  $0 \in U$ and $0 \in W$ since U and W are subspaces they must contain '0' . $0 \in U \cap W$

- The intersection of any number of subspaces of a vector space V is a subspace of V

SUBSPACES

Subspace of $\mathbb{R}^3$

i. $\mathbb{R}^3$ itself

ii. zero vector i.e., $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$

iii. line passing through origin

iv. plane passing through origin

v. In general, if V = R^n , the possible subspaces are , lines through origin, 2-d planes through origin, 3-d planes through origin, ….. , (n-1)- d planes through origin and the space itself .

THANK YOU

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