LINEAR ALGEBRA AND ITS APPLICATIONS
UE19MA251

Unit 3. Linear Transformations and Orthogonality

Projections And Least Squares

The failure of Gaussian Elimination is almost certain when we have several equations in one unknown.

$$a_1 x = b_1$$
$$a_2 x = b_2$$
$$----$$
$$a_m x = b_m$$

This system is solvable if $b = (b_1, \ldots, b_m)$ is a multiple of $a = (a_1, \ldots, a_m)$.

Projections And Least Squares

If the system is inconsistent, then we choose that value of a that minimizes an average error E in the m equations. The most convenient average comes from the ***sum of squares***:

Squared Error $$E^2 = \sum_{i=1}^{m} (a_i x - b_i)^2$$

If there is an exact solution the minimum error is E = 0. If not, the minimum error occurs when $\frac{dE^2}{dx} = 0$

Solving for x, the least squares solution is $\hat{x} = \frac{a^T b}{a^T a}$

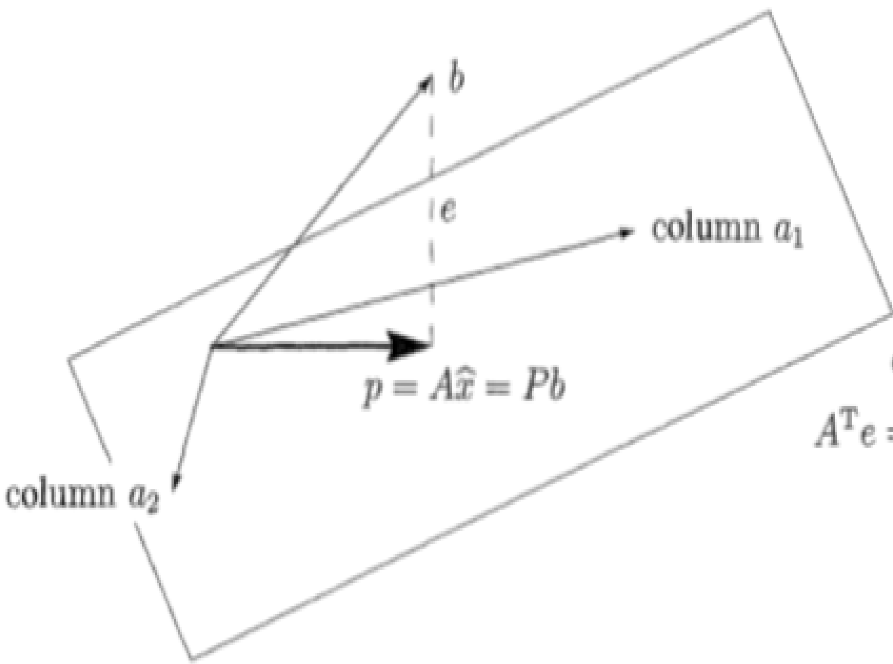
Least Squares Problem With Several Variables

Consider a system of equations Ax = b that is inconsistent.

The vector b lies outside C(A) and we need to project it onto C(A) to get the point p in C(A) that is closest to b. The problem here is the same as to minimize the error $E = \| Ax - b \|$ and this is exactly the distance from b to the point Ax in C(A).

Searching for the least squares solution $\hat{x}$ is the same as locating the point p that is closest to b.

Least Squares Problem With Several Variables



$$a_1^{\mathrm{T}} e = 0$$

$$a_2^{\mathrm{T}} e = 0$$

combine into

$$A^{\mathrm{T}} e = A^{\mathrm{T}}(b - A\hat{x}) = 0$$

Least Squares Problem With Several Variables

The error vector $e = b - A\hat{x}$ must be perpendicular to C(A) and hence can be found in the left null space of A.

Thus, $A^T(b - A\hat{x}) = 0 \; or \; A^T A\hat{x} = A^T b$

These are called the *Normal Equations*.

Solving them, we get the optimal solution $\hat{x}$

Note :

If b is orthogonal to C(A) then its projection is the zero vector.

Problems on Projections and Least squares

Find the projection of $b = (1, 2, 7)$ onto the column space of A spanned by $(1, 1, -2)$ and $(1, -1, 4)$. Split b into $p + q$ with p in $C(A)$ and q in $N(A^T)$.

Solution: Let p be the projection of b onto $C(A)$ which is spanned by $(1, 1, -2)$ and $(1, -1, 4)$

So $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}$ and $p = A\hat{x}$

Problems on Projections and Least squares

Normal equation to find $\hat{x}$ is

$$A^T A \hat{x} = A^T b \Rightarrow \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix} \hat{x} = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$\Rightarrow \hat{x} = \begin{bmatrix} 9/22 \\ 37/22 \end{bmatrix}$$

$$\therefore p = A\hat{x} = \begin{bmatrix} 23/11 \\ -14/11 \\ 65/11 \end{bmatrix}$$

and $p + q = b$

$$\Rightarrow q = b - p$$

$$q = \begin{bmatrix} -12/11 \\ 36/11 \\ 12/11 \end{bmatrix}$$

and $q \in N(A^T)$

Problems on Projections and Least squares

2. Find a basis for the orthogonal complement of the row space of $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$. Split the vector $(3,3,3)$ into a row space component x_r and a null space component x_n.

Solution: x_r and x_n are projections of $x = (3,3,3)$ onto $C(A^T)$ and $N(A)$ respectively.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \underset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow N(A) = \left\{ \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

Problems on Projections and Least squares

Let $a = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

Projection of b onto a line through 'a' is x_n.

$$x_n = \hat{x} a = \frac{a^T b}{a^T a} \cdot a = \frac{-9}{9} \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

We know that $x = x_r + x_n \Rightarrow x_r = x - x_n = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$

THANK YOU