VECTOR SPACES

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CLASS 1 : CONTENT

- Definition of Vector Space
- Examples of Vector Space
- Definition of Subspace
- Examples of Subspaces

VECTOR SPACES : DEFINITION

A Real vector space V is a nonempty set of objects called vectors, together with (Scalar multiplication and Vector addition)satisfying the following axioms :

I. If $u, v \in V$, then $u + v \in V \Rightarrow V$ is closed under vector addition.

II. If $\mathrm{c} \in \mathrm{R}\ \&\ \mathrm{u} \in \mathrm{V}$, then $cu \in \mathrm{V} \Rightarrow V$ is closed under scalar multiplication.

These operations satisfy the following properties for $u, v, w \in V$ & c_1, c_1 are scalars

a) $u + v = v + u$ (commutative law)

b) $u + (v + w) = (u + v) + w$(Associative law)

VECTOR SPACES

c) there is a unique zero vector i.e., 0 such that $0 + u = u + 0 = u$

(identity law) Additive identity '0' ∈ V

d) for each u there is a unique vector $(-u)$ such that

$u + (-u) = (-u) + u = 0$ (Inverse law)

e) $c_1(u+v) = c_1 u + c_1 v$

f) $(c_1 + c_2)u = c_1 u + c_2 u$

g) $(c_1 + c_2)u = c_1 u + c_2 u$

h) $1u = u$, Where 1 is a multiplicative identity s.t. 1 £ R

VECTOR SPACES: EXAMPLES

Example 1 The following are examples of vector spaces:

1. The set of all real number $\mathbb{R}$ associated with the addition and scalar multiplication of real numbers.

2. The set of all the complex numbers $\mathbb{C}$ associated with the addition and scalar multiplication of complex numbers.

3. The set of all polynomials $R_n(x)$ with real coefficients associated with the addition

VECTOR SPACES: EXAMPLES

4. The set of all vectors of dimension n written as $\mathbb{R}^n$ associated with the addition and scalar multiplication as defined for 3-d and 2-d vectors for example.

5. The set of all matrices of dimension $m \times n$ associated with the addition and scalar multiplication as defined for matrices.

Example 1 :

Prove that the set of all 2 by 2 matrices associated with the matrix addition and the scalar multiplication of matrices is a vector space.

Solution: Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ s.t. $A, A' \in V$ and $r, s \in R$

Let V be the set of all 2 by 2 matrices.

1) Addition of matrices gives

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$$

Adding any 2 by 2 matrices gives a 2 by 2 matrix and therefore the result of the addition

VECTOR SPACES: EXAMPLES

Scalar multiplication of matrices gives gives

$$r\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$$

Multiply any 2 by 2 matrix by a scalar and the result is a 2 by 2 matrix is an element of V.

3) Commutativity

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$
$$= \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$$
$$= \begin{bmatrix} a'+a & b'+b \\ c'+c & d'+d \end{bmatrix}$$
$$= \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

4) Associativity of vector addition

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}$$
$$= \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}$$
$$= \begin{bmatrix} (a+a')+a'' & (b+b')+b'' \\ (c+c')+c'' & (d+d')+d'' \end{bmatrix}$$
$$= \begin{bmatrix} a+(a'+a'') & b+(b'+b'') \\ c+(c'+c'') & d+(d'+d'') \end{bmatrix}$$
$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}\right)$$

VECTOR SPACES: EXAMPLES

5) Associativity of multiplication

$$r\left(s\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = r\left(\begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}\right)$$
$$= \begin{bmatrix} rsa & rsb \\ rsc & rsd \end{bmatrix}$$
$$= \begin{bmatrix} (rs)a & (rs)b \\ (rs)c & (rs)d \end{bmatrix}$$
$$= (rs)\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

6) Zero vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} a+0 & b+0 \\ c+0 & d+0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

7) Negative vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$
$$= \begin{bmatrix} a+(-a) & b+(-b) \\ c+(-c) & d+(-d) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

VECTOR SPACES: EXAMPLES

8) Distributivity of sums of matrices:

$$r\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right)$$
$$= \begin{bmatrix} r(a+a') & r(b+b') \\ r(c+c') & r(d+d') \end{bmatrix}$$
$$= \begin{bmatrix} ra+ra' & rb+rb \\ rc+rc' & rd+rd \end{bmatrix}$$
$$= r\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + r\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right)$$

9) Distributivity of sums of real numbers:

$$(r+s)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (r+s)a & (r+s)b \\ (r+s)c & (r+s)d \end{bmatrix}$$
$$= \begin{bmatrix} ra+sa & rb+sb \\ rc+sc & rd+sd \end{bmatrix}$$
$$= \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} + \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}$$
$$= r\begin{bmatrix} a & b \\ c & d \end{bmatrix} + s\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

10) Multiplication by 1.

$$1\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a & 1b \\ 1c & 1d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

VECTOR SPACES: EXAMPLES

Example 2 :

Show that the set of all real polynomials with a degree $n \leq 3$ associated with the addition of polynomials and the multiplication of polynomials by a scalar form a vector space.

Solution

The addition of two polynomials of degree less than or equal to 3 is a polynomial of degree lass than or equal to 3.

The multiplication, of a polynomial of degree less than or equal to 3, by a real number results in a polynomial of degree less than or equal to 3

VECTOR SPACES: EXAMPLES

Hence the set of polynomials of degree less than or equal to 3 is closed under addition

and scalar multiplication (the first two conditions above).

The remaining 8 rules are automatically satisfied since the polynomials are real.

VECTOR SPACES: EXAMPLES

Example 3 :

Show that the set of integers associated with addition and multiplication by a real number

IS NOT a vector space

Solution :

The multiplication of an integer by a real number may not be an integer.

Example: Let $x = -2$

If you multiply x by the real number $\sqrt{3}$ the result is NOT an integer.

VECTOR SPACE

Few examples :

1. R = the set of all real numbers

2. R^2 = { (x, y) / x, y Є R }

3. R^3 = { (x, y , z) / x, y, z Є R }

4. R^n = { (x1 , x2 , …, xn) / xi Є R }

5. R^∞ = { (x1 , x2 , …,) / xi Є R }

VECTOR SPACES: EXAMPLES

Problem 1:

Verify whether the following

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}; x \geq 0, y \geq 0, x, y \in r \right\}$$

Is a vector space ……

under usual vector addition and scalar multiplication.

VECTOR SPACES

- Closure property holds good.
- Associative property holds
- $\exists \ 0 \in V \ \exists \ u + 0 = u = 0 + u$,
- $\forall u \in V \ \exists -u \notin V \quad \left[Eg: u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in V, \ -u = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \notin V \right]$

Therefore Inverse law doesn't hold

Hence V is not a vector space.

Problem 2 : $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 / x + y = 0 \right\} u, v \in V$

All the properties holds good

Hence V is a vector space

VECTOR SPACES

Precisely ,

We can add any two vectors and we can multiply all vectors by scalars. In other words, we can take linear combinations.

SUBSPACES : DEFINITION

SUBSPACES

A nonempty subset of a vector space is called a subspace of V , if it is itself a vector space under the same operations of vector addition and scalar multiplication as defined in vector space .

The following are the properties satisfied by a subspace of V

i) $0 \in W$ (zero vector always belongs to a subspace)

ii) $if \quad u, v \in W$ Then $u + v \in W$

iii) If 'c' is a scalar and $u \in W$ then $cu \in W$

SUBSPACES : DEFINITION

If W is a subset of a vector space V and if W is itself a vector space under the inherited

operations of addition and scalar multiplication from V, then W is called a subspace

To show that the W is a subspace of V, it is enough to show that

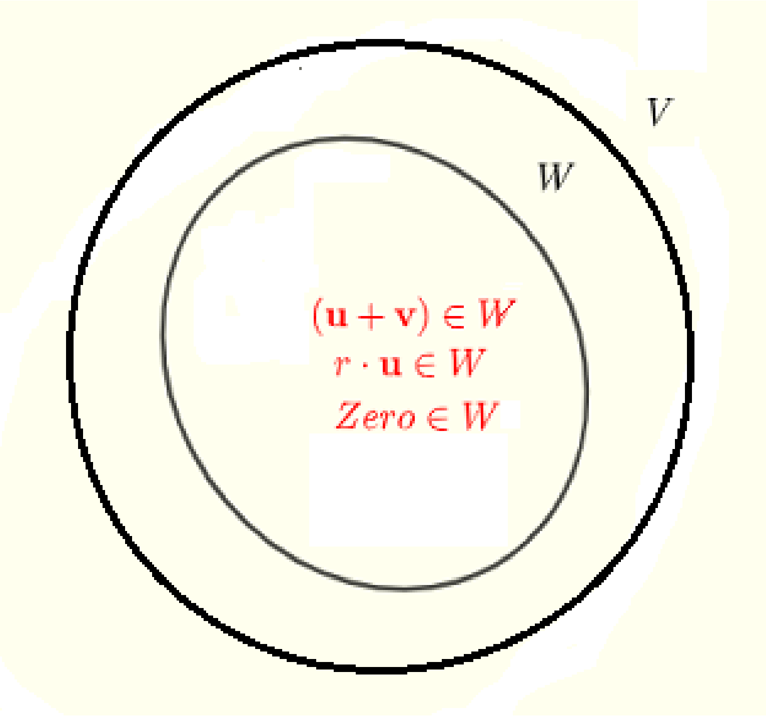
1. W is a subset of V

2. The zero vector of V is in W

3. For any vectors and in W, $\mathbf{u} + \mathbf{v}$ is in W. (closure under additon)

4. For any vector $\mathbf{u}$ and scalar r, $r \cdot \mathbf{u}$ is in W. (closure under scalar multiplication).

SUBSPACES : DEFINITION



SUBSPACES : EXAMPLES

Example 1

The set W of vectors of the form $(x, 0)$ where $x \in \mathbb{R}$ is a subspace of $\mathbb{R}^2$ because:

W is a subset of $\mathbb{R}^2$ whose vectors are of the form (x, y) where $x \in \mathbb{R}$ and $y \in \mathbb{R}$

The zero vector $(0, 0)$ is in W

$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$, closure under addition

$r \cdot (x, 0) = (rx, 0)$, closure under scalar multiplication

SUBSPACES : EXAMPLES

Example 2

The set W of vectors of the form (x, y) such that $x \geq 0$ and $y \geq 0$ is not a subspace of

$\mathbb{R}^2$ because it is not closed under scalar multiplication.

Vector $\mathbf{u} = (2, 2)$ is in W but its negative $-1(2, 2) = (-2, -2)$ is not in W.

SUBSPACES :

- Note : If U and W are two subspaces of a vector space V, intersection $U \bigcap W$ is also a subspace of V .

  $0 \in U$ and $0 \in W$ since U and W are subspaces they must contain '0' . $0 \in U \bigcap W$

- The intersection of any number of subspaces of a vector space V is a subspace of V

SUBSPACES

Subspace of $\mathbb{R}^3$

i. $\mathbb{R}^3$ itself

ii. zero vector i.e., $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$

iii. line passing through origin

iv. plane passing through origin

v. In general, if V = R^n , the possible subspaces are , lines through origin, 2-d planes through origin, 3-d planes through origin, ….. , (n-1)- d planes through origin and the space itself .

THANK YOU

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