LINEAR ALGEBRA AND ITS APPLICATIONS

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MATRICES AND
GAUSSIAN ELIMINATION

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GAUSSIAN ELIMINATION:

Course Content: Gaussian Elimination

❖ **Rank Of A Matrix:** A Square matrix A of order n is said to have rank r if

➢ At least one minor of order r does not vanish $(\neq 0)$.

➢ Every minor of order (r+1) vanishes $(= 0)$.

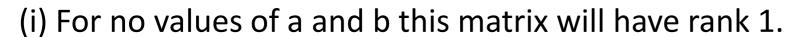**Rank of matrix A is denoted by r i.e.rank(A)=r.**

❖ If $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{mxn}$ is a rectangular matrix, then **Rank of the Matrix** is defined as

the number of non-zero rows in the Echelon form of A. It is also defined as the

maximum number of Linearly Independent Rows or Columns of the Matrix A.

GAUSSIAN ELIMINATION:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 2 & 2 & -3 \\ -2 & -2 & 3 \\ 4 & 4 & 6 \end{pmatrix}; \begin{pmatrix} 1 & 5 \\ -1 & 3 \\ 2 & 3 \end{pmatrix}; \begin{pmatrix} 1 & 5 \\ -1 & 3 \\ 2 & 10 \end{pmatrix}; \begin{pmatrix} 2 & -1 & 3 & 5 \\ 3 & 2 & 1 & 2 \end{pmatrix}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$2 \qquad\qquad 1 \qquad\qquad 2 \qquad\qquad 2 \qquad\qquad 2$$

Ex: Find the conditions on a and b so that the Matrix $\begin{pmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 1 & 3 & 6 \end{pmatrix}$ has rank 1, 2, 3.

$$\begin{pmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 1 & 3 & 6 \end{pmatrix} \xrightarrow{R_3 - \left(\frac{1}{a}\right)R_1} \begin{pmatrix} a & 1 & 2 \\ 0 & 2 & b \\ 0 & 3-(1/a) & 6-(2/a) \end{pmatrix}$$

(i) For no values of a and b this matrix will have rank 1.
(ii) If a=1/3 and b=4, rank of the matrix is 2.
(iii) If $a \neq 1/3$ and $b \neq 4$, rank of the matrix is 3.

GAUSSIAN ELIMINATION:

❖ **Relation between Rank, Consistency and Solution:**

❖ If rank(A)=r, then the following hold good:

(i) If **rank(A)=rank[A:b}=r**, system Ax=b is **consistent** and has **a solution**.

(ii) If **rank(A)=rank[A:b}=r=n**, system Ax=b is **consistent** and has **a unique solution**.

(iii) If **rank(A)=rank[A:b}=r<n**, system Ax=b is **consistent** and has **infinite number of**

solutions.

(iv) If **rank(A) ≠ rank[A:b]** , system Ax=b is **inconsistent** and has **no solution**.

GAUSSIAN ELIMINATION:

Gaussian Elimination

Gaussian Elimination is used to check for Consistency and solve a System of linear equations.

For a given system of equations Ax=b apply Elementary row transformations to the Augmented Matrix [A:b] and reduce it to [U:c] where U is an **Upper Triangular Matrix** so that we get an equivalent system Ux=c which can be solved by **Backward Substitution**.

Here A and U are Equivalent Matrices and hence solution of Ax=b is same as Ux=c.

❖ **The following steps are to be followed while performing Elementary Row Transformations in Gaussian Elimination:**

➤ **No exchange** of rows.

➤ First row should be **retained** as it is (not altered).

➤ The first non-zero element in every non-zero row is called **Pivot**.

➤ The original system Ax=b and new system obtained Ux=c have the **same solution**.

GAUSSIAN ELIMINATION:

Gaussian Elimination is illustrated below for a system of 3 equations with 3 variables.

Consider a system of 3 equations in 3 variables

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$[A:b] = \begin{pmatrix} a_{11} & a_{12} & a_{13} : b_1 \\ a_{21} & a_{22} & a_{23} : b_2 \\ a_{31} & a_{32} & a_{33} : b_3 \end{pmatrix} \xrightarrow[R_3 - \left(\frac{a_{31}}{a_{11}}\right)R_1]{R_2 - \left(\frac{a_{21}}{a_{11}}\right)R_1} \begin{pmatrix} a_{11} & a_{12} & a_{13} : b_1 \\ 0 & d_{22} & d_{23} : c_2 \\ 0 & d_{32} & d_{33} : c_3 \end{pmatrix}$$

$$\xrightarrow{R_3 - \left(\frac{d_{32}}{d_{22}}\right)R_2} \begin{pmatrix} a_{11} & a_{12} & a_{13} : b_1 \\ 0 & d_{22} & d_{23} : c_2 \\ 0 & 0 & e_{33} : c_4 \end{pmatrix} = [U:c]$$

$$\Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ \qquad d_{22}x_2 + d_{23}x_3 = c_2 \\ \qquad\qquad e_{33}x_3 = c_4 \end{cases}$$

GAUSSIAN ELIMINATION:

Check for consistency and solve the following system of equations if consistent:

(i)
$$x_1 + x_2 - 2x_3 + 4x_4 = 5$$
$$2x_1 + 2x_2 - 3x_3 + x_4 = 3$$
$$3x_1 + 3x_2 - 4x_3 - 2x_4 = 1$$

$$[A:b] = \begin{pmatrix} 1 & 1 & -2 & 4:5 \\ 2 & 2 & -3 & 1:3 \\ 3 & 3 & -4 & -2:1 \end{pmatrix}$$

$$\xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & 1 & -2 & 4:5 \\ 0 & 0 & 1 & -7:-7 \\ 0 & 0 & 2 & -14:-14 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 1 & -2 & 4:5 \\ 0 & 0 & 1 & -7:-7 \\ 0 & 0 & 0 & 0:0 \end{bmatrix}$$

r(A)=2 = r[A:b]<n(=4)

System is **consistent** and has **infinitely many solutions**.

$$\Rightarrow \begin{cases} x_1 + x_2 - 2x_3 + 4x_4 = 5 \\ x_3 - 7x_4 = -7 \end{cases}$$

Solution is $(x_1, x_2, x_3, x_4) = (10k_1 - k_2 - 9,\ k_2,\ 7k_1 - 7,\ k_1)$

Depending upon values of k_1 and k_2 we get infinity of solutions.

THANK YOU

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