LINEAR ALGEBRA AND ITS APPLICATIONS
UE19MA251

Orthogonal complement:

Definition :

Given a subspace V of R^n, the space of all vectors orthogonal to V is called the ***orthogonal complement*** of V written as $V^{\perp}$ and read as " V perp ".

Note : The orthogonal complement of a subspace V is unique.

Fundamental Theorem of Linear Algebra- Part-II

The null space is the orthogonal complement of the row space in R^n and the column space is the orthogonal complement of the left null space in R^m.

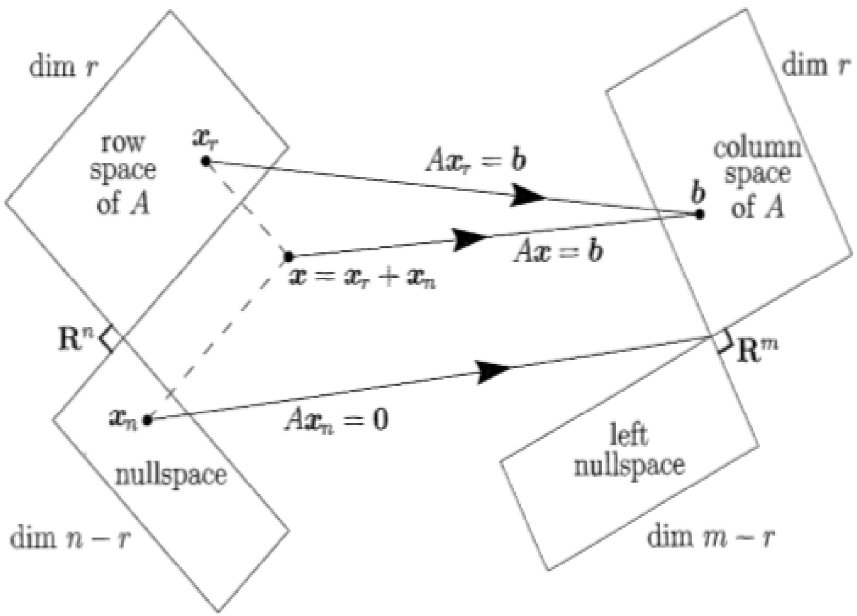
Note :

1. If S and T are orthogonal complements in R^n then it is always true that

$$\dim S + \dim T = n$$

Unit 3. Linear Transformations and Orthogonality

The Matrix And The Subspace



The Matrix And The Subspace

Splitting R^n into orthogonal parts V and W will split every vector into x = v + w.

The vector v is the projection onto the subspace V and the orthogonal component w is the projection of x onto W.

The true effect of matrix multiplication is that every Ax is in C(A). The null space goes to zero. The row space component goes to C(A). Nothing is carried to the left null space.

The Matrix And The Subspace

Note:

- $Ax = b$ is solvable if and only if $y^Tb=0$ whenever $y^TA = 0$.

- From the row space to the column space A is actually invertible, every vector ‘b’ in the column space comes from exactly one vector in the row space.

- Every vector transforms its row space onto its column space.

The Matrix And The Subspace

Problems

Find a vector x orthogonal to the row space of A, a vector y orthogonal to the column space of A and a vector z orthogonal to the null space of A where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}$$

Solution: Null space of A is orthogonal to Row space of A

$\therefore x \in N(A)$ and $z \in C(A^T)$

Left Null space is orthogonal to column space of A

$\therefore y \in N(A^T)$

To find the vectors x, y and z we need to find $C(A^T)$, $N(A)$ and $N(A^T)$.

The Matrix And The Subspace

To find the vectors x, y and z we need to find $C(A^T)$, $N(A)$ and $N(A^T)$.

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 3 & 4 & b_2 \\ 2 & 4 & 6 & b_3 \end{array}\right] \underset{R_3 \to R_3 - 2R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -1 & -2 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 \end{array}\right] = [U:c]$$

$$Ux = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = -2x_2 - 3x_3$$

$$x_2 = -2x_3$$

$$\therefore x_1 = -2(-2x_3) - 3x_3 = x_3.$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The Matrix And The Subspace

$\therefore N(A) = \left\{ x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}, \quad x_3 \in R$

$\therefore \quad x = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$C(A^T) = \left\{ c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}, \quad c_1, c_2 \in R$

$\therefore z = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$Ax = b$ is solvable only if $b_3 - 2b_1 = 0$

[Only then $\rho(A) = \rho(A:b) = 2$]

$\therefore N(A^T) = \left\{ c_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} \quad c_3 \in R$

$\Rightarrow y = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

The Matrix And The Subspace

2. Let P be the plane whose equation is x - 2y +9z = 0. Find a vector perpendicular to P. What matrix has the plane P as its null space and what matrix has P as its row space?

Solution:- Let $P = \{(x, y, z)$ such that $x - 2y + 9z = 0\}$

$$\Rightarrow \begin{pmatrix} 1 & -2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$\therefore$ The vector perpendicular to P is $\begin{pmatrix} 1 \\ -2 \\ 9 \end{pmatrix}$

So the matrix $A = \begin{pmatrix} 1 & -2 & 9 \end{pmatrix}$ has the plane P as its null space.

The Matrix And The Subspace

To find B find the Null space of P.

ie $\begin{pmatrix} 1 & -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow x = 2y + 9z$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 9 \\ 0 \\ 1 \end{pmatrix}$$

Null space $= \left\{ y\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 9 \\ 0 \\ 1 \end{pmatrix},\ x, y \in \mathbb{R} \right\}$

$\therefore B = \begin{bmatrix} 2 & 1 & 0 \\ 9 & 0 & 1 \end{bmatrix}$ So that $C(B^T) = P$

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THANK YOU