Inference & Causality

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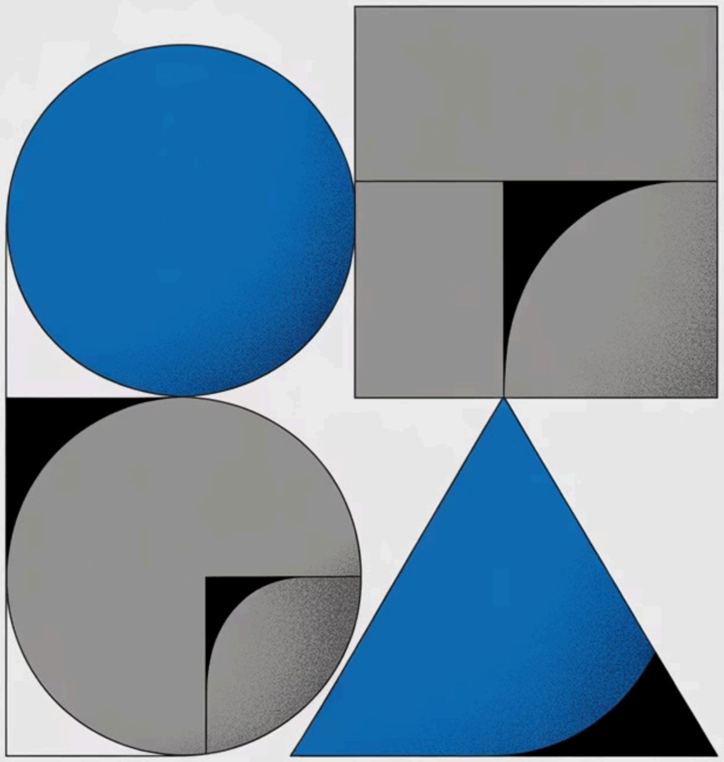
The Three Rules of Do-Calculus

Do-calculus provides a systematic framework for transforming causal expressions involving interventions into observable quantities. These three foundational rules enable rigorous causal inference when simple adjustment formulas are insufficient.



Do-Calculus: The Three Rules and Identifiability

Learning Goals

- Formalize intervention logic through the powerful framework of **do-calculus**
- Master the **three fundamental rules** for systematically transforming causal expressions
- Develop deep understanding of **identifiability** of causal effects from observational data

This session builds the mathematical foundation for rigorous causal inference, extending beyond simple adjustment formulas to handle complex causal structures.

From Adjustment Formulas to General Rules

1

Back-Door Criterion

Special case for blocking confounding paths through conditioning

2

Front-Door Criterion

Special case using mediator variables when confounders are unobserved

3

Do-Calculus

General framework encompassing all adjustment methods

The Central Question

> How can we derive $P(Y \mid do(X))$ from observables when no simple criterion applies?

Consider the classic smoking example: When we cannot directly adjust for confounders like genetic predisposition, do-calculus provides systematic rules to derive causal effects by reasoning about the graph structure. The framework tells us when we can safely replace interventions with observations.

Motivation for Do-Calculus

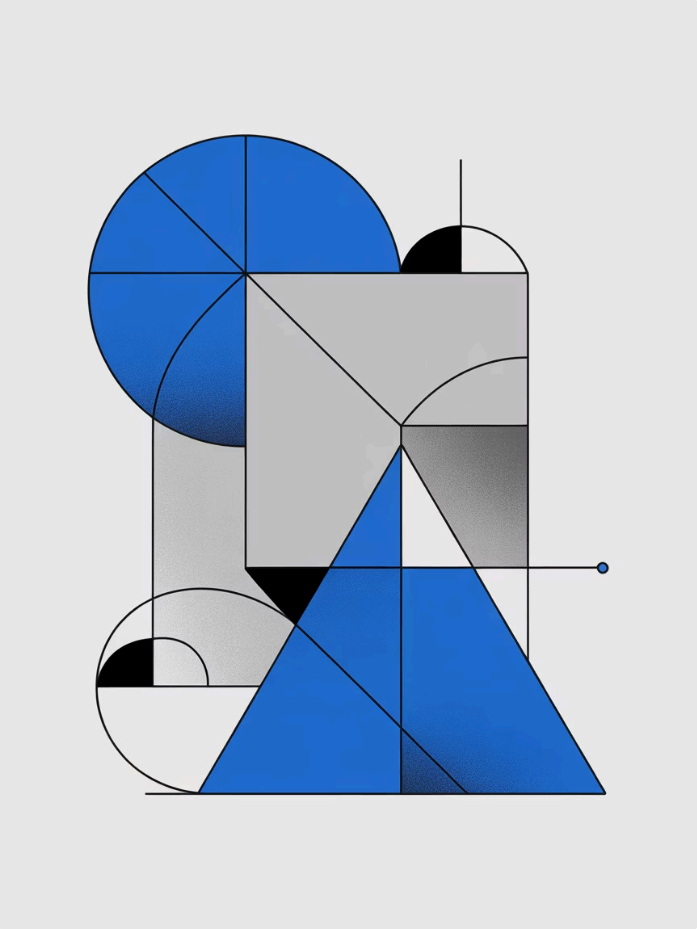
The Challenge

We need a systematic way to replace do(·) operators with regular conditioning wherever the causal structure permits. Without such rules, we're limited to special cases and cannot handle complex causal scenarios.

What Do-Calculus Provides

- Precise conditions for **adding or removing** observations from conditioning sets
- Rules for **swapping** between interventions and observations
- Methods to **eliminate** unnecessary intervention operators

> **Think of it as algebra for causality:** Just as algebraic rules let us manipulate equations while preserving equality, do-calculus rules let us transform causal expressions while preserving causal meaning.



The ultimate outcome: Express $P(Y \mid do(X))$ purely in terms of the observable joint distribution $P(Y,X,Z)$, enabling causal inference from observational data when the graph structure supports it.

Graph Notation for Do-Calculus

Do-calculus operates on **modified graphs** that reflect different intervention scenarios. Understanding this notation is essential for applying the three rules correctly.

$$G_{\bar{X}}$$

Remove Incoming Arrows

Delete all edges pointing *into* X. This represents intervening to set X to a specific value, breaking the influence of X's causes.

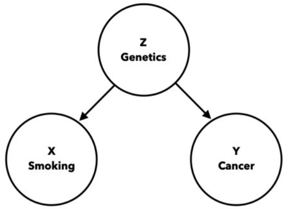
Interpretation: X is no longer determined by its parents; we control it directly.

$$G_{\underline{X}}$$

Remove Outgoing Arrows

Delete all edges pointing *out from* X. This represents scenarios where X's causal influence on descendants is blocked or irrelevant.

Interpretation: X no longer affects its children in the graph.



Rule 1: Insertion/Deletion of Observations

+=x

The Rule

$P(y \mid do(x),z,w) = P(y \mid do(x),w)$

Condition: Y ⊥ Z | X,W in G_{\bar{X}}

Interpretation

If Z is conditionally independent of Y given X and W in the intervened graph (where incoming edges to X are removed), then Z is **irrelevant** for predicting Y once we've intervened on X and conditioned on W. We can freely add or remove Z from the conditioning set without changing the result.

> **Key Insight:** This rule identifies variables that provide no additional information about Y beyond what X and W already tell us under intervention.

Real-World Example: Fire Alarm System

Consider: **Smoke → Alarm → Evacuation**

If we *intervene* to set off the alarm (do(Alarm=on)), whether there's actual smoke becomes irrelevant for predicting evacuation behavior. The alarm's state screens off the smoke signal.

Therefore: P(Evacuation | do(Alarm), Smoke) = P(Evacuation | do(Alarm))

Rule 1: Detailed Example

01

Check Graph Structure

Verify that Y $\perp$ Z | X,W holds in $G_{\bar{X}}$ by checking d-separation after removing incoming edges to X.

02

Apply Rule 1

If the independence holds, we can safely add or remove Z from the conditioning set without affecting the causal effect estimate.

03

Verify Result

Check that the transformed expression is closer to observable quantities or simpler to work with.

Quick Quiz: What must be true in the graph structure for Rule 1 to apply? Answer: After removing all incoming arrows to X, there must be no unblocked path between Y and Z that isn't already blocked by {X,W}.

Without Rule 1	Must condition on all observed variables, increasing complexity
With Rule 1	Can simplify by removing irrelevant variables, making estimation more efficient

Rule 2: Exchange of Action and Observation

The Rule

$P(y \mid do(x), do(z), w) = P(y \mid do(x), z, w)$

Condition: $Y \perp Z \mid X, W$ in $G_{\bar{X}\underline{Z}}$

Interpretation

This powerful rule allows us to **convert interventions into observations**. When Z is independent of Y given X and W in a graph where we've removed incoming edges to X and outgoing edges from Z, then actively setting Z (intervention) has the same effect as simply observing Z (conditioning).

> **Why It Matters:** Interventions require experiments; observations come from data. This rule tells us when we can learn causal effects from observational data instead of experiments.

When to Use Rule 2

- When Z acts on Y only through X and W
- When Z's causal influence is fully mediated by observed variables
- To transform complex intervention expressions into observable conditional probabilities

This is particularly useful when Z is difficult or unethical to intervene on, but X provides sufficient control.

Rule 2: Step-by-Step Example

Scenario: Treatment, Mediator, Outcome

Consider the causal structure: **Treatment (X) → Mediator (Z) → Outcome (Y)**, where Z is independent of Y given X (all of Z's effect on Y is through X).

Step 1: Initial Expression

Start with P(y | do(x), do(z))

Step 3: Apply Rule 2

Replace do(z) with z: P(y | do(x), z)

1

2

3

4

Step 2: Check Independence

Verify $Y \perp Z \mid X$ in $G_{X\underline{Z}}$

Step 4: Interpret

We've converted an intervention into observable conditioning

Symbolic Walkthrough

Given: X → Z → Y

Original: P(Y | do(X=x), do(Z=z))

Check: In G_X̄Z , is Y Z | X?
Yes — Z's outgoing edge is removed, X blocks all paths

Apply Rule 2: P(Y | do(X=x), do(Z=z)) = P(Y | do(X=x), Z=z)

Result: Intervention on Z converted to observation of Z

Rule 3: Insertion/Deletion of Actions

The Rule

P(y ∣ do(x),do(z),w) = P(y ∣ do(x),w)

Condition: Y ⊥ Z | X,W in $G_{\bar{X},\underline{Z(W)}}$

Interpretation

Rule 3 allows us to **remove unnecessary intervention operators**. If Z does not affect Y once we account for X and W (in a graph where incoming edges to X are removed and outgoing edges from Z except those to W are removed), then intervening on Z provides no additional information. We can drop the do(z) term entirely.

Key Distinction from Rules 1 & 2

- **Rule 1:** Adds/removes observations (conditioning)
- **Rule 2:** Swaps interventions for observations
- **Rule 3:** Eliminates interventions completely

> **Practical Value:** Each intervention we can eliminate reduces the experimental burden. If Rule 3 applies, we don't need to intervene on Z at all, saving resources and potentially avoiding ethical concerns.

The notation Z(W) means we remove outgoing edges from Z except those pointing to variables in W, allowing Z's effect through W to remain while cutting off other causal pathways.

The Complete Do-Calculus Framework

Rule	Operation	Graph Condition	Primary Use Case
1	Add/remove observation	Y ⊥ Z | X,W in $G_{\bar{X}}$	Simplify conditioning sets by removing irrelevant variables
2	Swap do and observe	Y ⊥ Z | X,W in $G_{\bar{X}\underline{Z}}$	Transform interventions into observations
3	Add/remove action	Y ⊥ Z | X,W in $G_{\bar{X},\,\underline{Z(W)}}$	Eliminate unnecessary interventions

These three rules form a **complete calculus** for causal inference. Any causal effect that can be identified from a graph can be derived by repeatedly applying these rules in combination. The art of do-calculus lies in determining which rule to apply at each step.

Worked Example: Deriving the Front-Door Formula

The front-door criterion can be derived systematically using the three rules of do-calculus. This demonstrates how general principles produce specific adjustment formulas.

Setup

Goal: Derive P(Y | do(X)) when X ← U → Y exists but we observe mediator Z where X → Z → Y.

Step 1: Apply Rule 2

Replace do(Z) with Z conditioning: P(Y | do(X)) = Σz P(Y | do(X), z) P(z | do(X))

Step 2: Apply Rule 3

Remove do(X) from Y term when blocked by Z: P(Y | do(X), z) = Σx' P(Y | x', z) P(x' | do(X))

Step 3: Apply Rule 2 Again

Replace remaining do(X) with X: P(x' | do(X)) = P(x')

Step 4: Simplify Z term

Since Z is only affected by X: P(z | do(X)) = P(z | X)

Final Front-Door Formula

$$P(Y|do(X)) = \sum_{z} P(Z|X) \sum_{x'} P(Y|x', z) P(x')$$

Every term is now expressed using only **observable quantities** from the joint distribution P(X,Y,Z). No interventions remain — we've achieved identifiability through systematic application of do-calculus rules.

Identifiability of Causal Effects

Definition

> A causal effect P(Y|do(X)) is **identifiable** if it can be expressed as a function of the observed joint distribution P(V) using only observable quantities.

The Identifiability Question

For any causal query and graph structure, we ask: Can we eliminate all do(·) operators by applying the three rules of do-calculus?

- **YES** → Effect is identifiable; we can estimate it from observational data
- **NO** → Effect requires experimental intervention; observational data insufficient

Examples of Non-Identifiable Structures

Unobserved Confounding

When U is latent: X ← U → Y with no mediator

Cannot eliminate do(X) — requires randomized experiment

Selection Bias

When sample selection depends on both treatment and outcome

Observational distribution is biased; causal effect not recoverable

Understanding identifiability helps researchers determine whether their research question can be answered with available data or requires experimental design.

Summary & Looking Ahead

Three Rules of Do-Calculus

- **Rule 1:** Add/remove observations when conditionally independent
- **Rule 2:** Exchange interventions and observations under d-separation
- **Rule 3:** Remove unnecessary interventions when effects are blocked

Connection to Adjustment Formulas

Back-door and front-door criteria are special cases derivable from the general do-calculus framework. The rules provide a unified theory of causal identification.

Identifiability

Not all causal effects can be identified from observational data. Do-calculus provides a systematic way to determine when identification is possible and derive the identifying formula.

Next Unit Preview

> **Common Fallacies and Misinterpretations in Causal Reasoning**
> We'll explore how causal inference can go wrong, from Simpson's paradox to confusing correlation with causation. Understanding these pitfalls will help you apply do-calculus correctly and interpret results appropriately.

Prepare by reviewing examples where conditioning on the wrong variables leads to biased estimates. Come ready to discuss real-world cases of causal misinterpretation.