Assignment 2
Advanced Image Processing (CS754)

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Contents

1 Question 2

a) Given that the oracular solution is x_s.

Now

$$y = \phi_s \tilde{x} + \eta$$

where ϕ_s is the submatrix with columns corresponding to indices of x_s. Now, through pre-multiplication of ϕ_s^T, we get,

$$\phi_s^T y = \phi_s^T \phi_s \tilde{x} + \phi_s^T \eta$$

Rearranging the terms,

$$\phi_s^T (y - \eta) = \phi_s^T \phi_s \tilde{x}$$

Pre-multiply by $(\phi_s^T \phi_s)^{-1}$

$$\tilde{x} = (\phi_s^T \phi_s)^{-1} \phi_s^T (y - \eta)$$

Here, $\phi_s^T \phi_s)^{-1} \phi_s^T$ is the pseudo inverse of ϕ_s

b) From (a), we know that,

$$\tilde{x} = (\phi_s^T \phi_s)^{-1} \phi_s^T (y - \eta_1)$$

$$\tilde{x} = \phi_s^\dagger (y - \eta_1)$$

where $\phi_s^\dagger$ is the pseudo-inverse of matrix ϕ_s Now, we also know that the true solution is obtained by estimating x from the forward model

$$y = \phi x + \eta$$

Since, x_s and x have the same indices of nonzero values, thus x is also s sparse. The difference between x_s and x is of the magnitude of coefficients. Thus, for the same y

$$y = \phi x + \eta_2$$

Since, x is also s-sparse with same indices as $\tilde{x}$, thus we can write the above equation as

$$y = \phi_s x + \eta_2$$

Thus, we can get x through the steps followed in (a)

$$x = \phi_s^\dagger (y - \eta_2)$$

Now,

$$\tilde{x} - x = \phi_s^\dagger (\eta_2 - \eta_1)$$

let $\eta = \eta_2 - \eta_1$, then

$$\tilde{x} - x = \phi_s^\dagger \eta$$

thus,

$$||\tilde{x} - x||_2 = ||\phi_s^\dagger \eta||_2$$

Now, using the Cautchy Shwartz inequality, we know that for two matrices **v** and **u**

$$|v.u| \leq ||v||_2 ||u||_2$$

Using the same on $||\phi_s^\dagger \eta||_2$, we get

$$||\phi_s^\dagger \eta||_2 \leq ||\phi_s||_2 ||\eta||_2$$

c) Let us say that, $\tilde{x}$ = oracular solution (**s** sparse) and x = real solution (**s** sparse)

Now, since $\tilde{x}$ and x are both s sparse, thus $\tilde{x} - x$ is atmost 2s sparse signal. Thus, let's say if ϕ follows the RIP of order 2S for 2S sparse vector $\tilde{x} - x$ if,

$$(1 - \delta_{2s})||\tilde{x} - x||_2^2 \leq ||\phi(\tilde{x} - x)||_2^2 \leq (1 + \delta_{2s})||\tilde{x} - x||_2^2 \tag{1}$$

Now, consider the two equations, corresponding to the true solution and the oracular solution respectively

$$y = \phi x + \eta_2 \tag{2}$$

$$y = \phi \tilde{x} + \eta_1 \tag{3}$$

Taking difference between the equations (2) and (3), we get the following expression for $\phi(\tilde{x} - x)$

$$\phi(\tilde{x} - x) = \eta_2 - \eta_1$$

let $\eta = \eta_2 - \eta_1$, then

$$\phi(\tilde{x} - x) = \eta$$

Thus, replacing the expression for $||\tilde{x} - x||_2^2$ proved in (b), and substituting the expression of inequality in $\phi(\tilde{x} - x) = \eta$ in (1), the equation changes to the following:

$$(1 - \delta_{2s})||\phi^\dagger \eta||_2^2 \leq ||\eta||_2^2 \leq (1 + \delta_{2s})||\phi^\dagger \eta||_2^2 \tag{4}$$

Considering left side of the inequality (4)

$$(1 - \delta_{2s})||\phi^\dagger \eta||_2^2 \leq ||\eta||_2^2$$

From (b), we know that $||\phi_s^\dagger \eta||_2 \leq ||\phi_s||_2 ||\eta||_2$. Thus, for the condition to hold true, $||\eta||_2^2$ should be greater than the largest value of $||\phi^\dagger \eta||_2^2$. Thus, the inequality changes to:

$$(1 - \delta_{2s})||\phi^\dagger||_2^2 ||\eta||_2^2 \leq ||\eta||_2^2$$

On rearranging the terms, we get

$$||\phi^\dagger||_2 \leq \frac{1}{\sqrt{1 - \delta_{2s}}} \tag{5}$$

Similarly, considering the right side of the inequality (4),

$$||\eta||_2^2 \leq (1 + \delta_{2s})||\phi^\dagger \eta||_2^2$$

Using $||\phi_s^\dagger \eta||_2 \leq ||\phi_s||_2 ||\eta||_2$ from (b), the inequality changes to:

$$||\eta||_2^2 \leq (1 + \delta_{2s})||\phi^\dagger||_2^2 ||\eta||_2^2$$

On rearranging the terms, we get

$$||\phi^\dagger||_2 \geq \frac{1}{\sqrt{1 + \delta_{2s}}} \tag{6}$$

On combining both results, we get the bounds on $||\phi^\dagger||_2$ as

$$\boxed{\frac{1}{\sqrt{1 + \delta_{2s}}} \leq ||\phi^\dagger||_2 \leq \frac{1}{\sqrt{1 - \delta_{2s}}}}$$

d) Given that, the bounds obtained in (c), help in providing bounds for $||\tilde{x} - x||_2^2$, which are as follows:

$$\frac{\epsilon}{\sqrt{1 + \delta_{2s}}} \leq ||\tilde{x} - x||_2^2 \leq \frac{\epsilon}{\sqrt{1 - \delta_{2s}}} \tag{7}$$

Since, x is the solution using basis pursuit, and $\tilde{x}$ is the oracular solution thus, the difference is the two solutions i.e. the error between the two is given by $||\tilde{x} - x||_2$. Since, the error term is always bounded between two constant values as shown in (7). Thus, the actual solution provided by Theorem 3, is only a constant factor worse than the oracular solution.

2 Question 3

According to restricted isometry property (RIP), for integer S={1,2.......,n}, the smallest isometry constant (RIC) δ_s of a matrix $A = \phi\psi(*)$ of size $m * n$ is the smallest number such that for any S-sparse vector, the following is true

$$(1 - \delta_s)||\theta||^2 \leq ||A\theta||^2 \leq (1 + \delta_s)||\theta||^2$$

To prove:
If $s < t$, we need to prove that $\delta_s \leq \delta_t$.
Proof:
Let θ_s be **s** sparse vector and θ_t be **t** sparse vector. So if $t > s$, then

$$||\theta_t|| \geq ||\theta_s||$$

Using RIP of order **s** on a matrix ϕ, we get

$$(1 - \delta_s)||\theta_s||^2 \leq ||\phi\theta_s||^2 \leq (1 + \delta_s)||\theta_s||^2$$

similarly using RIP of order **t** on ϕ

$$(1 - \delta_t)||\theta_t||^2 \leq ||\phi\theta_t||^2 \leq (1 + \delta_t)||\theta_t||^2$$

Now since $t \geq s$, thus,

$$||\theta_t|| \geq ||\theta_s||$$

Now multiplying by a non-singular matrix, and squaring it up we get

$$||\phi\theta_t||^2 \geq ||\phi\theta_s^2||$$

For this to satisfy, the upper bound of $||\phi\theta_t||$ has to be greater than the upper bound on $||\phi\theta_s||$.i.e.

$$(1 + \delta_t)||\theta_t||^2 > (1 + \delta_s)||\theta_s||^2$$

$$\frac{(1 + \delta_t)}{(1 + \delta_s)} \geq \frac{||\theta_s||^2}{||\theta_t||^2}$$

Since maximum value of $\frac{||\theta_s||^2}{||\theta_t||^2}$ is 1 as $||\theta_t|| \geq ||\theta_s||$, So

$$\frac{(1 + \delta_t)}{(1 + \delta_s)} \geq 1$$

Hence,

$$\boxed{\delta_t \geq \delta_s}$$

3 Question 4

a). Title: A novel sensing matrix design for compressed sensing via mutual coherence minimization.
Venue: IEEE 8^{th} International workshop on computational advances in multisensor adaptive processing (CAMSAP).
Authors: Khaled Ardah, Marius Pesavento, Martin Haardt.
Publication year: 2019.
Link: https://ieeexplore.ieee.org/document/9022467

b). In this paper, no imaging system has been demonstrated and it's more inclined towards the mathematical foundations of the SMCM (Sequential mutual coherence minimization). They outlined that direct MCM design problem of sensing matrix **A** in the equation **y=Ax** where $y \in \mathbf{C}^N$, $A \in \mathbf{C}^{N*K}$, $x \in \mathbf{C}^K$, can be solved by dividing original non-convex problem into K convex subproblems.
But for the reconstruction they had pointed to BP (Basis pursuit) and OMP (Orthogonal Matching Pursuit) so the imaging systems discussed in the class follows here as well.

c). The problem to find the sparsest solution x^* can be formulated as

$$x^* = argmin_x ||x||_0 \quad st \quad y = Ax \tag{8}$$

However, to guarantee perfect reconstruction (9) needs to be satisfied.

$$||x||_0 < \frac{1}{2}(1 + \mu^{-1}(A)) \tag{9}$$

Here, $\mu(A)$ denotes the mutual coherence of sensing matrix A. i.e.

$$\mu(A) = \max_{j \neq k} \frac{|a_j^H a_k|}{||a_j||||a_k||} \tag{10}$$

Now the problem of sensing matrix design based on MCM turns out to be

$$\min_{A \ \in \mathbf{C}^{N*K}} \mu(A) = \min_{A \ \in \mathbf{C}^{N*K}} f(|G - I_k|^2) \tag{11}$$

Here, **G** is the gram matrix and **A** is column-wise normalized matrix of **A**. The problem formulated in (11) is non-convex and NP hard so the authors first expanded $|G|^2 = |A^H A|^2 \in \mathbf{C}^{K*K}$. This turns $|G|^2$ to be symmetric matrix with all ones in it's main diagonal.
Now solving (11) in alternating fashion by iterating over the K sub-problems where K^{th} vector a_k is given as

$$find \quad a_k \quad s.t \quad |a_j^H a_k|^2 \leq \beta \quad \forall j \neq k, ||a_k|| = 1 \tag{12}$$

But $||a_k|| = 1$ may result in problem infeasibilty for poor initialized vectors $a_j \forall j \neq k$. So dropping the unit norm constraint.

$$find \quad a_k \quad s.t \quad |a_j^H a_k|^2 \leq \beta \quad \forall j \neq k \tag{13}$$

to obtain the solution (13), one possible approach is given as

$$a_k = \max_{v \in \mathbf{C}^N} |a_k^H v|^2 \quad s.t \quad |a_j^H v|^2 \leq \beta \quad \forall j \neq k$$

Here **v** is the beamforcing vector that we wish to design.

d). Optimization algorithm illustrated in the paper is listed below:

Inputs: initial $A_{(0)}$, lower-bound $\beta = \frac{K-N}{N(K-1)}$, convergence threshold ϵ_{th}, index-set ϕ=1,2,......,K and let t=1.
while not converged **do**
for k=1 to K **do**
1. Obtain v_k using SOP (Sequential Orthogonal Projection)
2. Update kth column of $A^{(t)}$ by $a_k^{(t)} = v_k^{(t)}/||v_k^{(t)}||_2$
end for
If $\epsilon = |\mu(A^t) - \mu(A^{t-1})|^2 \leq \epsilon_{th}$; **break**
end while
Output: designed sensing matrix **A**.

e). SMCM yields best performance by having monotonic and faster convergence rate, within 100 to 200 iterations. It is able to solve nonconvex and NP-hard mutual coherence problem by dividing into K convex sub-problems, where each sub-problem is solved optimally using existing techniques like the SOP (sequential orthogonal projection) method. In the results, author has concluded that SMCM is capable of obtaining a sensing matrix with a mutual coherence close to known **lower bound** which outperforms all the existing methods in terms of performance.

4 Question 5

Given that:

$$P1 : min_x \quad ||x||_1 \quad s.t \quad ||y - \phi x||_2 \leq \epsilon \tag{14}$$

Also the LASSO problem which seeks to minimize J(x),

$$J(x) = ||y - \phi x||_2^2 + \lambda ||x||_1 \qquad \lambda > 0 \tag{15}$$

Proof:
Consider that **z** minimizes the cost function J(.) where $x \neq z$ also.

$$J(x) \geq J(z) \tag{16}$$

Using (14),

$$||y - \phi x||_2 \leq \epsilon$$

Squaring both sides, we get,

$$||y - \phi x||_2^2 \leq \epsilon^2 \tag{17}$$

Now using (15) and (16)

$$||y - \phi x||_2^2 + \lambda ||x||_1 \geq ||y - \phi z||_2^2 + \lambda ||z||_1$$
$$\lambda(||x||_1 - ||z||_1) \geq ||y - \phi z||_2^2 - ||y - \phi x||_2^2$$

Substituting the results from (17), we get

$$\lambda(||x||_1 - ||z||_1) \geq \epsilon^2 - ||y - \phi x||_2^2 \geq 0$$

Therefore, when $\lambda > 0$,

$$||x||_1 - ||z||_1 \geq 0$$
$$||x||_1 \geq ||z||_1$$

Since $||z||_1$ is less than equal to $||x||_1$ so for all **x** for which $||y - \phi x||_2 \leq \epsilon$, **z** would also be the solution of problem P1.

5 Question 6

Dorfman Pooling

Let $P(x)$ denote the probability of event x. Given that there are a total of n individuals out of which k are infected such that $k << n$. The individuals are divided into groups for testing, with each group of size g.

$$P(individual\ getting\ infected) = \frac{k}{n}$$

$$P(individual\ not\ getting\ infected) = 1 - \frac{k}{n}$$

A pool tests positive if there is atleast one postive sample is in a pool of size g.

$$P(atlest\ one\ postive\ in\ pool\ of\ size\ g) = 1 - P(no\ positive\ in\ pool)$$

$$P(atlest\ one\ postive\ in\ pool\ of\ size\ g) = 1 - (1 - \frac{k}{n})^g$$

Number of groups $= \frac{n}{g}$

Now, the concept of Dorfman pooling is that for every pool that tests positive, we need to conduct the test for all the individuals in that pool. Thus, the first step would be to test all the pools if they are positive or negative. Thus, let the number of tests in round 1 be x.

$$x = \frac{n}{g}$$

Now, for a particular pool, the fraction of members that can test positive is

$$Fraction\ of\ members\ that\ can\ test\ positive\ in\ a\ pool = gP(atlest\ one\ positive)$$

$$Fraction\ of\ members\ that\ can\ test\ positive\ in\ a\ pool = g(1 - (1 - \frac{k}{n})^g)$$

Doing this for all the groups we have made, if the pool tests positive. Let the number of tests in this stage be y.

$$y = \frac{n}{g}(g(1 - (1 - \frac{k}{n})^g))$$

Thus, total average number of tests, $T = y + x$,

$$T = \frac{n}{g} + \frac{n}{g}(g(1 - (1 - \frac{k}{n})^g))$$

$$\boxed{T = \frac{n}{g}(1 + g(1 - (1 - \frac{k}{n})^g))}$$

Worst Case

The worst case is when all the infected individuals are present in different groups. In this case, in round 1, we need to test all the groups that we have created. Since, number of groups with a group size of g for n people is equal to $\frac{n}{g}$. Thus, number of tests in round one (say x) , $x = \frac{n}{g}$. Now in the second round, we need to test all the pools having the infected people in them. Since, there are k infected people, one in each group, thus the number of infected groups would be k. Thus, testing every person in these groups having group size g, we get the the total number of tests required in the second round (say y), $y = kg$.

Thus total number of tests required in the worst case is, $T = x + y$.

$$\boxed{T = \frac{n}{g} + kg}$$

In order to obtain the optimal value of the groupsize g in the worst case, we have to minimize the total number of test cases in the worst case (T) with respect to g. Thus,

$$\frac{\partial T}{\partial g} = \frac{-n}{g^2} + k$$

Setting $\frac{\partial T}{\partial g} = 0$, we get

$$\frac{n}{g^2} = k$$

$$\boxed{g_{optimal} = \sqrt{\frac{k}{n}}}$$

6 Question 1

Part 1
To prove : $\delta_{2s} = 1$ could imply that 2s columns of Φ may be linearly dependent. **Proof:** Assume, two s sparse vectors x_1 and x_2. Assuming Φ follows Restricted Isometry Property of order $2s$. Thus, we can write the following inequality,

$$(1 - \delta_{2s})||x_1 - x_2||_2^2 \leq ||\Phi(x_1 - x_2)||_2^2 \leq (1 + \delta_{2s})||x_1 - x_2||_2^2$$

Now if $\delta_{2s} = 1$, then the LHS of the inequality changes to the following

$$0 \leq ||\Phi(x_1 - x_2)||_2^2$$

This means that there is a possibility of $||\Phi(x_1 - x_2)||_2^2 = 0$. Now let $\tilde{x} = x_1 - x_2$. Since,x_1 and $x2$ are both s sparse thus, $\tilde{x}$ is 2s sparse. Now the equation changes to $||\Phi\tilde{x}||_2^2 = 0$. This is possible only if $\Phi\tilde{x} = 0$. Since $\tilde{x}$ is $2s$ sparse, thus linear $\Phi\tilde{x} = 0$, can be written of as a linear combination of 2s columns of Φ summing up to zero. This means that, 2s columns of Φ are linearly dependent.

Part 2
To Prove: $||\Phi(x^* - x)||_{l_2} \leq ||\Phi x^* - y||_{l_2} + ||y - \Phi x||_{l_2} \leq 2\epsilon$
Proof:
We can rewrite $||\Phi(x^* - x)||_{l_2} = ||\Phi x^* - \Phi x)||_{l_2}$, as $||\Phi x^* + y - y - \Phi x)||_{l_2}$. Now, using the triangle inequality, i.e

$$||v + u||_{l^2} \leq ||v||_{l^2}||u||_{l^2}$$

Upon the vectors $\Phi x^* + y$ and $y - \Phi x)||_{l_2}$, we get,

$$||\Phi x^* + y - y - \Phi x)||_{l^2} \leq ||\Phi x^* - y||_{l^2} + ||y - \Phi x||_{l^2}$$

Thus, **LHS** of the inequality is proved.

Now, since the both x^* and x are solutions to the constrained optimization problem

$$\min_{\tilde{x} \in R^n} ||\tilde{x}||_{l^1} \ subject \ to \ ||y - \Phi\tilde{x}||_{l2} \leq \epsilon$$

Thus, both have the constraint, $||y - \Phi x^*||_{l2} \leq \epsilon$, and $||y - \Phi x||_{l2} \leq \epsilon$. Thus, the upper limit of $||\Phi x^* - y||_{l_2} + ||y - \Phi x||_{l_2}$ is 2ϵ.
Thus, the **RHS** of the inequality is proved.

Part 3
To Prove: $||h_{T_j}||_{l_2} \leq s^{1/2}||h_{T_j}||_{l_\infty} \leq s^{-1/2}||h_{T_{j-1}}||_{l_1}||$
Proof :
For some vector $\mathbf{v}$, that is s sparse, from the relationships of various norms, we know that,

$$||v||_2 \leq ||v||_1 \leq \sqrt{s}||v||_2$$

Now, $||\mathbf{v}||_2 = \sqrt{v_1^2 + v_2^2 + v_3^2 + ... + v_s^2}$. Also, $||\mathbf{v}||_\infty = \max v_1, v_2, ..., v_s$. Now, let $p = v_1^2 + v_2^2 + v_3^2 + ... + v_s^2$. Since, $v_i \leq v_\infty \forall i \in s$. Thus

$$v_1^2 + v_2^2 + v_3^2 + ... + v_s^2 \leq v_\infty^2 + v_\infty^2 + v_\infty^2 + ... + v_\infty^2$$

$$v_1^2 + v_2^2 + v_3^2 + ... + v_s^2 \leq s v_\infty^2$$

Taking square root on both sides, we get,

$$\sqrt{v_1^2 + v_2^2 + v_3^2 + ... + v_s^2} \leq \sqrt{s} v_\infty$$

Now, the above result is valid for any s sparse vector. Since, $x^* - x = h$, and h is decomposed into $h_{T_0}, h_{T_1}, h_{T_2}, h_{T_3}, ...$ Such that each h_{T_i} is s sparse.
Thus, from here we can say that,

$$||h_{T_j}||_{l_2} \leq s^{1/2}||h_{T_j}||_{l_\infty}$$

Thus, the **LHS** of the inequality is proved.
Now, since, T_0 corresponds to the locations of the s largest coefficients of x. So automatically, T_1 will contain the s largest coefficients of x which do not belong to the set T_0. This means that $||h_{T_0}||_1 \geq ||h_{T_1}||_1$. Extending this concept for any arbitrary $j \geq 2$, we get

$$||h_{T_{j-1}}||_1 \geq ||h_{T_j}||_1$$

Now, for any s sparse vector $\mathbf{v}$, we know that

$$||v||_1 \leq s||v||_\infty$$

Using the same inequality above for h_{T_j}, we get the following

$$||h_{T_j}||_1 \leq s||h_{T_j}||_\infty$$

Since, $||h_{T_{j-1}}||_1 \geq ||h_{T_j}||_1$ should hold, thus it should be greater than the maximum value of $||h_{T_j}||_1$. Thus

$$||h_{T_{j-1}}||_1 \geq s||h_{T_j}||_\infty$$

Multiplication with $s^{-1/2}$, we get

$$s^{-1/2}||h_{T_{j-1}}||_1 \geq s^{1/2}||h_{T_j}||_\infty$$

Thus, the **RHS** side of the inequality is also proved.

Part 4

To Prove : $\sum_{j\geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...) \leq s^{-1/2}||h_{T_0^c}||_{l^1}$
Proof :
From part(3), we know that

$$||h_{T_j}||_{l_2} \leq s^{-1/2}||h_{T_{j-1}}||_{l_1}$$

Adding $\sum_{j\geq 2}$ to both sides of the inequality,

$$\sum_{j\geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...)$$

Thus, **LHS** of the inequality is proved.

Now, from the definition of decomposition of h, we know that, $h_{T_0^c} = h - h_{T_0}$. Since h_{T_1} is obtained through the s largest indices of the $h_{T_0^c}$. Thus, $||h_{T_0^c}||_1 \geq ||h_{T_1}||_1 \geq ||h_{T_2}||_1$...... Thus,

$$||h_{T_0^c}||_{l_1} \geq ||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...$$

Multiplication with $s^{-1/2}$, we get

$$s^{-1/2}(||h_{T_2}||_{l_1} + ...) \leq s^{-1/2}||h_{T_0^c}||_{l^1}$$

Thus, **RHS** of the inequality is proved.

Part 5
To Prove :$|| \sum_{j\geq 2} h_{T_j}||_{l_2} \leq \sum_{j\geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}||h_{T_0^c}||_{l_1}$ **Proof :**
Using vector summation inequality, we can say that

$$||h_{T_1} + h_{T_2} + h_{T_3} + ...||_{l_2} \leq ||h_{T_1}||_{l2} + ||h_{T_2}||_{l2} + ||h_{T_3}||_{l2} + ...$$

Thus, **LHS** of the inequality has been proved.
From part(4), we know that

$$\sum_{j\geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}||h_{T_0^c}||_{l^1}$$

Thus, **RHS** of the inequality has been proved.

Part 6
To Prove : $\sum_{i\in T_0} |x_i + h_i| + \sum_{i\in T_0^c} |x_i + h_i| \geq ||x_{T_0}||_{l_1} - ||h_{T_0}||_{l_1} + ||h_{T_0^c}||_{l_1} - ||x_{T_0^c}||_{l_1}$
Proof :
We know that, the reverse triangle inequality is as follows:

$$|||v||_{l2} - ||w||_{l2}| \leq ||v + u||_{l2}$$

Also, through relation of norms, we know that $||v||_2 \leq ||v||_1$. Thus

$$||v + u||_{l2} \leq ||v + u||_{l1}$$

Since, $|||v||_{l2} - ||w||_{l2}| \leq ||v + u||_{l2}$, thus

$$|||v||_{l2} - ||w||_{l2}| \leq ||v + u||_{l1}$$

Now, $||v||_{l2} \leq ||v||_{l1}$ and $|w||_{l2} \leq ||w||_{l1}$. Thus, the above inequality transforms to the following:

$$|||v||_{l1} - ||w||_{l1}| \leq ||v + u||_{l1}$$

Using the above expression in $\sum_{i\in T_0}|x_i+h_i|+\sum_{i\in T_0^c}|x_i+h_i|$, we get the following,

$$\sum_{i\in T_0}|x_i+h_i|+\sum_{i\in T_0^c}|x_i+h_i| \leq |||x_{T_0}||_{l1}-||h_{T_0}||_{l1}|+|||x_{T_0^c}||_{l1}-||h_{T_0^c}||_{l1}|$$

In order to open the modulus, we need to know which magnitude is higher for both the components on the left side. From our definition of h, we know that h_{T_0} is composed of the s largest indices of x. Thus, $||x_{T_0}||_{l_1} \geq ||h_{T_0}||_{l_1}$. Similarly, $||x_{T_0^c}||_{l1} \geq ||h_{T_0^c}||_{l1}$. Thus the above inequality, after opening the modulus changes to the following:

$$\sum_{i\in T_0}|x_i+h_i|+\sum_{i\in T_0^c}|x_i+h_i| \geq ||x_{T_0}||_{l_1}-||h_{T_0}||_{l_1}+||h_{T_0^c}||_{l_1}-||x_{T_0^c}||_{l_1}$$

Thus, proved.

Part 7
To Prove: $||h_{T_0^c}||_{l_1} \leq ||h_{T_0}||_{l_1}+2||x_{T_0^c}||_{l_1}$
Proof:
From part (6), we know that,

$$||x||_{l_1} \geq ||x_{T_0}||_{l^1}-||h_{T_0}||_{l^1}+||h_{T_0^c}||_{l^1}-||x_{T_0^c}||_{l^1}$$

Now, we can say that

$$x = x_{T_0^c}+x_{T_0}$$
$$||x||_{l_1} \geq ||x_{T_0^c}||_{l_1}+||x_{T_0}||_{l_1}$$

Thus, the inequality proved in part(5) changes to the following:

$$||x||_{l_1}+||h_{T_0}||_{l^1}+||x_{T_0^c}||_{l^1}-||x_{T_0}||_{l^1} \geq ||h_{T_0^c}||_{l^1}$$

Replacing the inequality $||x||_{l_1} \geq ||x_{T_0^c}||_{l_1}+||x_{T_0}||_{l_1}$ above, we get,

$$||h_{T_0}||_{l^1}+2||x_{T_0^c}||_{l^1} \geq ||h_{T_0^c}||_{l^1}$$

Thus, proved.

Part 8
To prove: $||h_{(T_0\cup h_{T_1})^c}||_{l_2} \leq ||h_{T_0}||_{l^2}+2e_0, e_0 = s^{-1/2}||x-x_s||_{l_1}$
Proof :
Since we know that $x_{T_0^c} = ||x-x_s||_{l_1}$ (by definition). Also, from part (4), we know that

$$h_{(T_0\cup T_1)^c}||_{l_2} \leq s^{-1/2}||h_{T_0^c}||_{l_1}$$

We also know from part(7), that

$$||h_{T_0^c}||_{l_1} \leq ||h_{T_0}||_{l_1}+2||x_{T_0^c}||_{l_1}$$

Now, multiplying with $s^{-1/2}$, we get the following

$$s^{-1/2}||h_{T_0^c}||_{l_1} \leq s^{-1/2}||h_{T_0}||_{l_1}+2s^{-1/2}||x_{T_0^c}||_{l_1}$$

Now, replacing $h_{(T_0\cup T_1)^c}||_{l_2}$ from the inequality $h_{(T_0\cup T_1)^c}||_{l_2} \leq s^{-1/2}||h_{T_0^c}||_{l_1}$, in the above equation.

$$||h_{(T_0\cup T_1)^c}||_{l_2} \leq ||h_{T_0}||_{l^2}+2e_0$$

Thus, proved

Part 9
To Prove : $|\langle \Phi h_{T_0 \cup T1}, \Phi h\rangle| \leq ||\Phi h_{T_0 \cup T1}||_{l_2}||\Phi h||_{l_2} \leq 2\epsilon\sqrt{1+\delta_{2s}}||h_{(T_0 \cup T_1)}||_{l_2}$
Proof
From Cauchy-Schwartz inequality for two vectors **v** and **u**, we know that,

$$||v.u||_2 \leq ||v||_{l_2}||u||_{l_2}$$

thus, for $|\langle \Phi h_{T_0 \cup T1}, \Phi h\rangle|$, we can write the following,

$$|\langle \Phi h_{T_0 \cup T1}, \Phi h\rangle| \leq ||\Phi h_{T_0 \cup T1}||_{l_2}||\Phi h||_{l_2}$$

Thus, LHS of the inequality is proved.

Since, $h_{T_0 \cup T1} = h - \sum_{j \geq 2} h_{T_j}$. As h and each of the h_{T_j} are s sparse by design. Thus, we can say that $h_{T_0 \cup T1}$ is $2s$ sparse. Now, assuming Φ follows RIP of order $2s$, we get the following equation.

$$(1-\delta_{2s})||h_{T_0 \cup T_1}||_2^2 \leq ||\Phi(h_{T_0 \cup T_1})||_2^2 \leq (1+\delta_{2s})||h_{T_0 \cup T_1}||_2^2$$

thus,

$$||\Phi(h_{T_0 \cup T_1})||_2^2 \leq (1+\delta_{2s})||h_{T_0 \cup T_1}||_2^2$$

Also, from part(2), we know that $||\Phi(x^* - x)||_{l_2} \leq 2\epsilon$. Since, $x^* = x + h$ thus, $||\Phi(h)||_{l_2} \leq 2\epsilon$. Now using this result and the above inequality we get the following result.

$$||\Phi h_{T_0 \cup T1}||_{l_2}||\Phi h||_{l_2} \leq 2\epsilon\sqrt{1+\delta_{2s}}||h_{(T_0 \cup T_1)}||_{l_2}$$

Thus, RHS of the inequality is also proved.

Part 10
To Prove : $|\langle \Phi h_{T_0}, \Phi h_{T_j}\rangle| \leq \delta_{2s}||h_{T_0}||_{l_2}||h_{T_j}||_{l_2}$
Proof
We can write the inner product $\langle \Phi h_{T_0}, \Phi h_{T_j}\rangle$ as

$$\langle \Phi h_{T_0}, \Phi h_{T_j}\rangle = \frac{1}{4}\Bigg| ||\Phi(h_{T_0} + h_{T_j})||_2^2 - ||\Phi(h_{T_0} - h_{T_j})||_2^2 \Bigg|$$

From RIP of order $2s$ on i we know that,

$$||\Phi(h_{T_0} + h_{T_j})||_2^2 \leq (1+\delta_{2s})||h_{T_0} + h_{T_j})||_2^2$$

and

$$(1-\delta_{2s})||h_{T_0} - h_{T_j})||_2^2 \leq ||\Phi(h_{T_0} - h_{T_j})||_2^2$$

Now the maximum value for $\langle \Phi h_{T_0}, \Phi h_{T_j}\rangle$, will happen when $||\Phi(h_{T_0} + h_{T_j})||_2^2$ will attain its maximum value while $||\Phi(h_{T_0} - h_{T_j})||_2^2$. Thus, the inequality changes to the following:

$$\langle \Phi h_{T_0}, \Phi h_{T_j}\rangle \leq \frac{1}{4}\Bigg|(1+\delta_{2s})||h_{T_0} + h_{T_j}||_2^2 - (1-\delta_{2s})||h_{T_0} - h_{T_j}||_2^2\Bigg|$$

The above expression simplifies to

$$\langle \Phi h_{T_0}, \Phi h_{T_j}\rangle \leq \frac{1}{4}\Bigg|4\delta_{2s}\langle h_{T_0}, h_{T_j}\rangle\Bigg|$$

Now, we know from Cauchy-Schwartz inequality that,

$$\langle h_{T_0}, h_{T_j} \rangle \leq ||h_{T_0}||_{l_2} ||h_{T_j}||_{l_2}$$

Thus, replacing $\langle h_{T_0}, h_{T_j} \rangle$ in $\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle \leq \frac{1}{4} \left| 4\delta_{2s} \langle h_{T_0}, h_{T_j} \rangle \right|$. We get the following

$$|\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| \leq \delta_{2s} ||h_{T_0}||_{l_2} ||h_{T_j}||_{l_2}$$

Thus proved.

Part II

To prove : $\|h_{T_0}\|_{\ell_2} + \|h_{T_1}\|_{\ell_2} \leq \sqrt{2}\, \|h_{T_0 \cup T_1}\|_{\ell_2}$

Proof:

$\cancel{h_{T_0}}$

We know,

$$h_{T_0 \cup T_1} = h - \sum_{j \geq 2} h_{T_j}$$

$$h_{T_0} = h - \sum_{j \neq 0, \forall j} h_{T_j}$$

$$h_{T_1} = h - \sum_{j \neq 1, \forall j} h_{T_j}$$

Now,

$$h_{T_0} + h_{T_1} = 2h - \sum_{j \neq 0, T_1} h_{T_j} - \sum_{j \neq 1, T_1} h_{T_j}.$$

$$\Rightarrow h_{T_0} + h_{T_1} = h - 2\sum_{j \geq 2} h_{T_j}$$

& $\therefore$

$$h_{T_0} + h_{T_1} = h - \sum_{j \geq 2} h_{T_j} - \sum_{j \geq 2} h_{T_j}$$

$$\Rightarrow h_{T_0} + h_{T_1} = h_{T_0 \cup T_1} - \sum_{j \geq 2} h_{T_j}$$

Now,

$$\sum_{j \geq 2} h_{T_j} = h_{T_0 \cup T_1} - h.$$

thus,

$$h_{T_0} + h_{T_1} = 2h_{T_0 \cup T_1} - h$$

thus,

$$2h_{T_0 \cup T_1} = \cancel{2T_0}\, h_{T_0} + h_{T_1} + h$$

thus,

$$2h_{T_0 \cup T_1} \geq h_{T_0} + h_{T_1}.$$

Taking ℓ_2 norm on either side

$$\sqrt{2}\,\|h_{T_0 \cup T_1}\|_2 \geq \|h_{T_0} + h_{T_1}\|_2$$

$\because$ this should hold for all possible values of $\|h_{T_0}+h_{T_1}\|_2$

$\therefore$ Using triangle inequality.

$$\|h_{T_0} + h_{T_1}\|_2 \leq \|h_{T_0}\|_2 + \|h_{T_1}\|_2$$

thus,

$$\boxed{\sqrt{2}\,\|h_{T_0 \cup T_1}\|_2 \geq \|h_{T_0}\|_2 + \|h_{T_1}\|_2.}$$

Thus, proved

Part 12

To prove: $(1-\delta_{2s})\|h_{T_0 \cup T_1}\|_2^2 \le \|\phi h_{T_0 \cup T_1}\|_2^2 \le \|h_{T_0 \cup T_1}\|_{\ell_2} \left(2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s} \sum_{j\ge 2} \|h_{T_j}\|_{\ell_2}\right)$

Proof:

$\because$ $h_{T_0 \cup T_1}$ is a $2s$ sparse vector, and assuming ϕ obeys RIP of order $2s$, we get the following condition

$$(1-\delta_{2s})\|h_{T_0 \cup T_1}\|_2^2 \le \|\phi h_{T_0 \cup T_1}\|_2^2 \le (1+\delta_{2s})\|h_{T_0 \cup T_1}\|_{\ell_2}^2$$

LHS, proved,

We wish for a harder constraint on the upper bound of $\|\phi h_{T_0 \cup T_1}\|_2^2$. Thus,

$$\because \quad h_{T_0 \cup T_1} = h - \sum_{j\ge 2} h_{T_j}.$$

$$\phi h_{T_0 \cup T_1} = \phi h - \sum_{j\ge 2} \phi h_{T_j}$$

Taking L2 norms on both sides, we get

$$\|\phi h_{T_0 \cup T_1}\|_2^2 \le \langle \phi h_{T_0 \cup T_1}, \phi h\rangle - \langle \phi h_{T_0 \cup T_1}, \sum_{j\ge 2} \phi h_{T_j}\rangle$$

Through lemma (2.1).

from Cauchy Schwartz inequality,

$$\langle \phi h_{T_0 \cup T_1}, \phi h\rangle \le \|\phi h_{T_0 \cup T_1}\|_2 \|\phi h\|_2 .$$

from part (9), we know that,

$$\|\phi h_{T_0 \cup T_1}\|_2 \|\phi h\|_2 \leq 2\epsilon\sqrt{1+\delta_{2s}}\, \|h_{T_0 \cup T_1}\|_2$$

thus,

$$\|\phi(h_{T_0 \cup T_1})\|_2^2 \leq |\langle \phi h_{T_0 \cup T_1}, \phi h\rangle|_2 - \langle \phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \phi h_{T_j}\rangle$$

thus,

$$\|\phi(h_{T_0 \cup T_1})\|_2^2 \leq 2\epsilon\sqrt{1+\delta_{2s}}\, \|h_{T_0 \cup T_1}\|_2 - \sum_{j \geq 2} \langle \phi h_{T_0 \cup T_1}, \phi h_{T_j}\rangle$$

$\because \quad \langle \phi h_{T_0 \cup T_1}, \phi h_{T_j}\rangle \leq \delta_{2s} \|h_{T_0}\|_2 \|h_{T_j}\|_2$

thus,

$$\|\phi(h_{T_0 \cup T_1})\|_2^2 \leq 2\epsilon\sqrt{1+\delta_{2s}}\, \|h_{T_0 \cup T_1}\|_2 - \delta_{2s} \|h_{T_0 \cup T_1}\|_2 \sum_{j \geq 2} \|h_{T_j}\|_2$$

$\because$ Magnitude must be normalized because of treating inner products as unit vectors, thus

$$\Rightarrow \|\phi(h_{T_0 \cup T_1})\|_2^2 \leq \left(2\epsilon\sqrt{1+\delta_{2s}} - \sqrt{2}\,\delta_{2s} \sum_{j \geq 2} \|h_{T_j}\|_2\right) \|h_{T_0 \cup T_1}\|_2$$

thus, RHS of the inequality is also proved.

Part 13

~~To prove: $\|h_{T_0 \cup T_1}\|_{\ell_2} \leq \alpha\epsilon + \rho s^{-1/2} \|h_{T_0^c}\|_{\ell_2}$,~~

P

Part 13

To prove: $\|h_{T_0 \cup T_1}\|_2 \le \alpha\epsilon + \rho s^{-1/2}\|h_{T_0^c}\|_1$,

$\alpha = \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}$, $\rho = \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}$.

From part 12

$$(1-\delta_{2s})\|h_{T_0 \cup T_1}\|_2 \le 2\epsilon\sqrt{1+\delta_{2s}} - \delta_{2s}\sqrt{2}\sum_{j\ge 2}\|h_{T_j}\|_2$$

$$\Rightarrow \|h_{T_0 \cup T_1}\|_2 \le \underbrace{\frac{2\epsilon\sqrt{1+\delta_{2s}}}{(1-\delta_{2s})}}_{\alpha} - \underbrace{\frac{\delta_{2s}\sqrt{2}}{(1-\delta_{2s})}}_{\rho}\sum_{j\ge 2}\|h_{T_j}\|_2$$

$$\Rightarrow \|h_{T_0 \cup T_1}\|_2 \le \alpha\epsilon + \rho\sum_{j\ge 2}\|h_{T_j}\|_2$$

From part (5), we know that

$$\sum_{j\ge 2}\|h_{T_j}\|_2 \le s^{-1/2}\|h_{T_0^c}\|_1,$$

thus,

$$\boxed{\|h_{T_0 \cup T_1}\|_2 \le \alpha\epsilon + \rho s^{-1/2}\|h_{T_0^c}\|_1,}$$

Thus, proved.

~~Part 14~~

Part 14

To prove: $||h_{T_0 \cup T_1}||_2 \leq \alpha\epsilon + \rho||h_{T_0 \cup T_1}||_2 + 2\rho e_0$

from part (7), we know that,

$$||h_{T_0^c}||_{\ell_1} \leq ||h_{T_0}||_{\ell_1} + 2||x_{T_0^c}||_{\ell_1}$$

Multiplication by $s^{-1/2}$

$$s^{-1/2}||h_{T_0^c}||_{\ell_1} \leq s^{-1/2}||h_{T_0}||_{\ell_2} + 2s^{-1/2}||x_{T_0^c}||_{\ell_1}$$

$$\Rightarrow s^{-1/2}||h_{T_0^c}||_{\ell_1} \leq ||h_{T_0 \cup T_1}|| + 2e_0.$$

thus, from part (13) we know,

$$||h_{T_0 \cup T_1}||_2 \leq \alpha\epsilon + \rho s^{-1/2}||h_{T_0^c}||_{\ell_1}$$

$$\Rightarrow \boxed{||h_{T_0 \cup T_1}||_2 \leq \alpha\epsilon + \rho(h_{T_0 \cup T_1} + 2e_0).}$$

Hence, proved.

Part 15

To prove: $||h||_{\ell_2} \leq ||h_{T_0 \cup T_1}||_{\ell_2} + ||h_{(T_0 \cup T_1)^c}||_{\ell_2} \leq 2||h_{T_0 \cup T_1}||_{\ell_2} + 2e_0 \leq 2(1-\rho)^{-1}(\alpha\epsilon + (1+\rho)e_0)$

Proof:
We can write h as follows.

$$h = h_{T_0 \cup T_1} + h_{(T_0 \cup T_1)^c}$$

$||h||_{\ell_2} \leq ||h_{T_0 \cup T_1}||_2 + ||h_{(T_0 \cup T_1)^c}||_2$ (Triangle Inequality).

Now,

$$\|h_{(T_0 \cup T_1)^c}\|_{\ell_2} \leq \|h_{T_0}\|_{\ell_2} + 2\varepsilon_0 \quad \text{(from part (2))}$$

thus,

$$\|h\|_{\ell_2} \leq \|h_{T_0 \cup T_1}\|_2 + \|h_{T_0}\|_{\ell_2} + 2e_0.$$

$\because$ $\|h_{T_0}\|_{\ell_2} \leq \|h_{T_0 \cup T_1}\|_{\ell_2}$

thus,

$$\|h\|_{\ell_2} \leq 2\|h_{T_0 \cup T_1}\|_{\ell_2} + 2e_0.$$

$$\Rightarrow \boxed{\|h\|_{\ell_2} \leq 2(1-p)^{-1}(\alpha e + (1+p)e_0)}$$

$\because$ from part 14

$$\|h_{T_0 \cup T_1}\|_2 \leq \alpha e + s\|h_{T_0 \cup T_1}\|_2 + 2se_0$$

$$\Rightarrow \boxed{\|h_{T_0 \cup T_1}\|_2 \leq (\alpha e + 2se_0)(1+s)^{-1}}$$

thus, proved.

Part 16

To prove: $\|h_{\ell_1}\| = \|h_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1} \le 2(1+\rho)(1-\rho)^{-1}\|x_{T_0^c}\|_{\ell_1}$

Proof.

Using the Lemma (2.2), we know,

$$\|h_{T_0}\|_{\ell_1} \le \rho\|h_{T_0^c}\|_{\ell_1}$$

Now,

$$\cancel{\|h\|_{\ell_1} = \|h_{T_0}\|_{\ell_1}}$$

$$\|h\|_{\ell_1} \le \rho\|h_{T_0^c}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1}$$

$$\Rightarrow \|h\|_{\ell_1} \le (1+\rho)\|h_{T_0^c}\|_{\ell_1} \quad \text{—— (i)}$$

Now, from part (7), we know that

$$\|h_{T_0^c}\|_{\ell_1} \le \|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1} \quad \text{—— (ii)}$$

~~thus, replacing value of $\|h_{T_0^c}\|_{\ell_1}$ in (i), we get~~

$$\cancel{\Rightarrow \|h\|_{\ell_1} \le (1+\rho)[\|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1}]}$$

$\Rightarrow$ from, lemma, 2.2.

$$\|h_{T_0}\|_{\ell_1} \le \rho(\|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1})$$

$\Rightarrow$

$$(1-\rho)\|h_{T_0}\|_{\ell_1} \le 2\rho\|x_{T_0^c}\|_{\ell_1}$$

$$\Rightarrow \|h_{T_0}\|_{\ell_1} \le 2\rho(1-\rho)^{-1}\|x_{T_0^c}\|_{\ell_1}$$

Replacing in equation (ii).

$\Rightarrow \|h_{T_0^c}\|_{\ell_1} \leq (2\delta(1-\delta)^{-1}+2)\|x_{T_0^c}\|_{\ell_1}$

thus,

~~$\|h\|_{\ell_1} \leq (1+\delta)$~~

$\|h_{\ell_1}\| \leq (1+\delta)(2\delta(1-\delta)^{-1}+2)\|x_{T_0^c}\|_{\ell_1}$

$\Rightarrow \boxed{\|h_{\ell_1}\| \leq 2(1+\delta)(1-\delta)^{-1}\|x_{T_0^c}\|_{\ell_1}}$

Thus, proved.