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Abstract

Entropic Optimal Transport (EOT), also referred to as the Schrödinger problem, seeks to find a random processes with prescribed initial/final marginals and with minimal relative entropy with respect to a reference measure. The relative entropy forces the two measures to share the same support and only the drift of the controlled process can be adjusted, the diffusion being imposed by the reference measure. Therefore, at first sight, Semi-Martingale Optimal Transport (SMOT) problems (see [1]) seem out of the scope of applications of Entropic regularization techniques, which are otherwise very attractive from a computational point of view.

However, when the process is observed only at discrete times, and become therefore a Markov chain, its relative entropy can remain finite even with variable diffusion coefficients, and discrete semi-martingales can be obtained as solutions of (multi-marginal) EOT problems.

Given a (smooth) semi-martingale, the limit of the relative entropy of its time discretisations, scaled by the time step converges to the so-called "specific relative entropy", a convex functional of its variance process, similar to those used in SMOT.

In this paper we use this observation to build an *entropic time discretization* of continuous SMOT problems. This allows to compute discrete approximations

of solutions to continuous SMOT problems by a multi-marginal Sinkhorn algorithm, without the need of solving the non-linear Hamilton-Jacobi-Bellman pde's associated to the dual problem, as done for example in [1, 2].
We prove a convergence result of the time discrete entropic problem to the continuous time problem, we propose an implementation and provide numerical experiments supporting the theoretical convergence.

Keywords: Smie-Martingale Optimal Transport, Multi-Marginal Optimal Transport, Entropic Penalisation, Specific Relative Entropy, Sinkhorn Algorithm

Notations

Time is denoted by $t \in [0,1]$ and space by $x \in \mathbb{R}^d$. The space of continuous paths $t \to \omega_t$ is denoted $\Omega = C([0,1];\mathbb{R}^d)$ and $\mathcal{P}(\Omega)$ is the space of probability measures on Ω.

The evaluation map at time t is denoted by $e_t : \omega \in \Omega \to \omega_t$, also called the "position at time t" map.

The push forward of a measure μ by a map T is $T_\# \mu$, i.e. $T_\# \mu(B) = \mu(T^{-1}(B))$ for all borel sets B. For $\mathbb{P} \in \mathcal{P}(\Omega)$, $\mathbb{P}_t = (e_t)_\# \mathbb{P}$ is the marginal at time t, and likewise for $(t_1, t_2, ...)$ marginals: $\mathbb{P}_{t_1,t_2,...} = (e_{t_1,t_2,...})_\# \mathbb{P}$. If the time marginals $\mathbb{P}_t$ have densities with respect to the Lebesgue measure, these densities are denoted by ρ_t.

Let $(t_i)_{i=0,N}$ be a regular time discretisation of $[0,1]$ associated with the time step denoted by h, i.e. $h := \frac{1}{N}$. We use a subscript $.^h$ when dealing with time discrete quantities. For instance $\mathbb{P}^h = (e_{t_0,t_1,...,t_N})_\# \mathbb{P}$. Given $\mathbb{P}^h$ we use the simplified marginal notations $\mathbb{P}^h_i = (e_{t_i})_\# \mathbb{P}^h$, $\mathbb{P}^h_{i,i+1} = (e_{t_i,t_{i+1}})_\# \mathbb{P}^h$, etc.

We sometimes abuse notations using the same symbol for a measure and its density with respect to the Lebesgue measure, i.e. $x_i \in \mathbb{R}^d \to \mathbb{P}^h_i(x_i)$ a function or $\mathbb{P}^h_i(B) = \int_B d\mathbb{P}^h_i(x_i) = \int_B \mathbb{P}^h_i(x_i)\, dx_i$ for any Borel $B \in \mathbb{R}^d$ (and the same for multivariate measures). More generally an i or $i, i+1$ subscript always indicates that the variable is a measure or function over $\mathbb{R}^d_{t_i}$ or $\mathbb{R}^d_{t_i} \times \mathbb{R}^d_{t_{i+1}}$.

Througout the paper, $|\cdot|$ denotes the Euclidean norm, aboslute value for a scalar, Euclidian norm for a vector, Frobenius norm for a matrix.

The characteristic function of a set will be denoted ι, hence

$$\iota_{[\lambda,\Lambda]}(x) := \begin{cases} 0 \text{ if } x \in [\lambda, \Lambda], \\ +\infty \text{ otherwise,} \end{cases}$$

and by extension, for a symmetric matrix a,

$$\iota_{[\lambda,\Lambda]}(a) := \begin{cases} 0 \text{ if } \lambda Id \leq a \leq \Lambda Id, \\ +\infty \text{ otherwise.} \end{cases}$$

A classic notation for the transition probability from (s,x) to (t,y) is $\Pi(s,x,t,y) = \mathbb{P}(X_t = y | X_s = x)$ but when dealing with time discrete measures we will also use the more compact non standard notation and decomposition:

$$\mathbb{P}^h_{i\to i+1}(x_i, dx_{i+1}) := \frac{\mathbb{P}^h_{i,i+1}(x_i, dx_{i+1})}{\mathbb{P}^h_i(x_i)} \tag{1}$$

(or simply $\mathbb{P}^h_{i\to i+1}$ when there is no ambiguity). Hence for all x_i, $\mathbb{P}^h_{i\to i+1}(x_i,\cdot)$ is a probability in $\mathcal{P}(\mathbb{R}^d_{t_{i+1}})$, while $\mathbb{P}^h_{i,i+1}$ is a joint probability defined on $\mathbb{R}^d_{t_i}\times\mathbb{R}^d_{t_{i+1}}$.

The expectation with respect to $\mathbb{P}$ is denoted $\mathbb{E}_{\mathbb{P}}(\cdot)$ or $\mathbb{E}(\cdot)$ if there is no ambiguity. Finally, we denote by $(\rho_0,\rho_1)\in\mathcal{P}(\mathbb{R}^d)\times\mathcal{P}(\mathbb{R}^d)$ two probability densities with finite second moments, that will play the role of initial and final distributions for the optimal transport problem. The relative entropy of a measure $\mathbb{P}$ with respect to $\overline{\mathbb{P}}$ is given as:

$$\mathcal{H}(\mathbb{P}|\overline{\mathbb{P}}) := \begin{cases} \mathbb{E}_{\mathbb{P}}\log\left(\dfrac{d\mathbb{P}}{d\overline{\mathbb{P}}}\right) & if\,\mathbb{P}\ll\overline{\mathbb{P}} \\ +\infty & \text{otherwise.}\end{cases}$$

$\mathcal{W}_2(\rho_0,\rho_1)$ is the classical quadratic Wasserstein distance.

Contents

1 Introduction

We propose an *entropic time discretization* method for semi-martingales solutions of Semi-Martingale Optimal Transport (SMOT) problems of the form (5). Such problems are of particular interest in finance, for model calibration (see for instance [1–6]). The numerical solution methods proposed thus far rely on the dual formulation of the problem, which implies performing a gradient descent on the solutions of an associated fully non-linear Hamilton-Jacobi-Bellman (HJB) equation. The approach proposed here avoids this step, by introducing an entropic time discretization of the problem and relying only on the well-known Sinkhorn algorithm for numerical resolution, in particular opening the way to applications in higher dimensions.

Entropic Optimal Transport (EOT) is a well-studied problem. Given $\bar{\mathbb{P}}$ the Wiener measure with variance $\bar{\sigma}$ and initial law ρ_0, $\bar{\mathbb{P}}_{0,1}$ its joint (intial,final) law, the EOT problem can be formulated in a *static form* as

$$\inf_{\mathbb{P}\in\mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d),\mathbb{P}_0=\rho_0,\mathbb{P}_1=\rho_1} \mathcal{H}(\mathbb{P}|\bar{\mathbb{P}}_{0,1}).$$

It also enjoys the equivalent time continuous formulation, known as the Schrödinger problem, defined by (3), (see [7] for a survey). The EOT formulation is extremely useful, since it can be seen, for small parameter $\bar{\sigma}$, as a relaxation of the classical OT problem

$$\inf_{\mathbb{P}\in\mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d),\mathbb{P}_0=\rho_0,\mathbb{P}_1=\rho_1} \mathbb{E}(|X_0-X_1|^2).$$

Observe that the quadratic cost above is *linear* in $\mathbb{P}$, whereas the relative entropy is stricly convex The EOT problem is therefore easier to solve than the classical OT problem, through the celebrated Sinkhorn algorithm (see [8] and [9] for further development of this method).

The minimizers of the Shrödinger problem are semi-martingales with a local drift converging to the OT map (as $\overline{\sigma}\to 0$) and a martingale part with *fixed* volatility $\overline{\sigma}$ imposed by the finite relative entropy w.r.t. the reference measure $\overline{\mathbb{P}}$.

For problems like (5), this is a big limitation, as one one wants to control the process mainly through its diffusion. In financial applications the drift is constrained. The approach we propose here enables to optimize on the diffusion coefficient while still using entropic OT.

Multi-marginal EOT. For this we first perform a time discretisation, i.e. consider $\mathbb{P}^h = (e_{t_0,t_1,\dots,t_N})_\#\mathbb{P}$ and $\overline{\mathbb{P}}^h = (e_{t_0,t_1,\dots,t_N})_\#\overline{\mathbb{P}}$ ($h = 1/N$ is the timestep). We are dealing here with $N+1$ marginal measures and solutions of a *muti-marginal* OT problem. Multi-marginal relative entropic regularization, i.e extra penalization with the relative entropy of a minimization problem, is again helpful numerically. We will now try to minimize the discrete functional (16). As in the two-marginal case, thanks to the entropy, the problem becomes stricly convex, and the dual problem becomes a smooth concave maximisation problem, that can be solved efficiently with extensions of the Sinkhorn algorithm. The size of the dual variable matches the size of the support and is therefore optimal. Depending on the structure of the cost and the reference measure, regularization also preserves the Markovianity of the solution. Separability properties in time yield a linear cost with respect to time discretization. Depending on the tree structure of the Multi-Marginal cost, these properties have been used for instance in [10, 11].

Letting $\mathbb{P}$ be the law of a continuous semi-martingale, and $\mathbb{P}^h$ its time discretization, (and using a similar notation for $\overline{\mathbb{P}}$) one has $\lim_{h\searrow 0}\mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h) = \mathcal{H}(\mathbb{P}|\overline{\mathbb{P}})$. Therefore, even using a time discretization, minimizing $\mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h)$ over $\mathbb{P}^h$ has the strong consequence that in the limit $h\searrow 0$, $\mathbb{P}^h$ must have *asymptotically fixed* volatility $\overline{\sigma}$. Entropic regularisation of a minimization problem over $\mathbb{P}^h$ therefore also adds a hard diffusive constraint to the limit process. It was used in [11] or [12] for example.

Specific Relative Entropy (SRE) regularisation. In order to allow for a variable diffusion, our main idea is to replace the relative entropy with the "specific" relative entropy (introduced in [13] see also [14, 15]). The specific relative entropy is defined as

$$\mathcal{S}^{\mathcal{H}}(\mathbb{P}|\overline{\mathbb{P}}) = \lim_{h\searrow 0} h\,\mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h).$$

For smooth diffusion processes with quadratic variation $a_t^{\mathbb{P}}$ one can show (see Proposition 2), *thanks to the h scaling*, that the limit is well defined for a general $a^{\mathbb{P}}$ and given by

$$\mathcal{S}^{\mathcal{H}}(\mathbb{P}|\overline{\mathbb{P}}) = \tfrac{1}{2}\mathbb{E}_{\mathbb{P}}\left(\int_0^1 \mathcal{S}^{\mathcal{I}}(a_t^{\mathbb{P}}|\overline{\sigma}^2)\,dt\right),$$

see (8) for the definition of $\mathcal{S}^{\mathcal{I}}$. Adding $h\,\mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h)$ to (5) should therefore yield (15) in the limit $h\to 0$ and hence achieve our objective: It yields a time discretization ($\mathcal{V}^h$) of any SMOT problem of the form ($\mathcal{V}^0$), hence provides a "standard" relative entropic regularisation problem solvable by efficient algorithms.

(Note that the cost function in ($\mathcal{V}^0$) is of the form $F+\mathcal{S}^{\mathcal{I}}(a_t^{\mathbb{P}}|\overline{\sigma}^2)$, therefore if F is sufficiently convex, changing F into $F-\mathcal{S}^{\mathcal{I}}$ we can approximate (5) itself).

Theoretical Contribution. Our main result is Theorem 5: Given $\mathbb{P}^0$ a minimizer of $\mathcal{V}^0$ and $(\mathbb{P}^h)_h$ a sequence of minimizers of ($\mathcal{V}^h$), $\mathcal{I}^h(\mathbb{P}^h)$ converges to $\mathcal{I}^0(\mathbb{P}^0)$. In the course of the proof we construct from $(\mathbb{P}^h)_h$ a family of smooth measures on Markov chains $(\mathbb{P}^h_\varepsilon)_{h,\varepsilon}$, depending on a small regularisation parameter ε, such that

i) $\mathcal{I}^h(\mathbb{P}^h_\varepsilon)$ is close to $\mathcal{I}^h(\mathbb{P}^h)$ depending on ε,
ii) for all $\varepsilon>0$, $\mathbb{P}^h_\varepsilon$ converges as $h\to 0$ (in a sense to be defined) to $\mathbb{P}_\varepsilon$, a smooth measure on Ω,
iii) $\mathcal{I}^0(\mathbb{P}_\varepsilon)$ is close to $\mathcal{I}^0(\mathbb{P}^0)$ depending on ε.

Hence we "almost" prove that $\mathbb{P}^h$ converges to $\mathbb{P}^0$, see remark 2). The full proof of this result is an ongoing work. As is, our convergence result relies on additional boundedness assumptions on the coefficients (hypothesis H2). We do not know if that can be relaxed but the bounds can be chosen a priori depending on the marginal data so that the constraint for the discrete problem is not active. It also seems necessary to recover a bona fide diffusion process as the limit of the discrete Markov chain to add an extra constraint on a moment of order larger than 4 (kurtosis) for the discrete process $\mathbb{P}^h$.

Numerical Contribution. Our numerical solution is based on the dual formulation (27-29-30) of the problem. We present the duality in a simplified setting, in particular, we removed additional constraints on the coefficients and the kurtosis as it is dropped for our numerical experiments. The application of Fenchel Rockaffelar convex duality is allowed here, as often in OT, using a linear change of variable involving moments instead of conditional moments and the interpretation of the integrand as a convex lower semicontinuous perspective function (46). The implementation is based on the Sinkhorn algorithm. The convergence analysis of Sinkhorn both in the continuous and discrete case is yet to be studied. Let us point out that existing results for classical multi-marginal Sinkhorn [16, 17] do not cover our continuous non-compact support. We deal here with a multi-marginal "weak" optimal transport formulation for which Sinkhorn is currently being developed but only in the compact case.

Letting (a, b) be respectively the quadratic variation and drift of the process, we use as a cost function $G(b) = \gamma \|b\|^2$ with a large γ, which means that in practice we are only seeking to enforce a martingale constraint, the penalization on a is done through $\mathcal{S}^{\mathcal{I}}$ which is convex in a. We illustrate the convergence as $h \searrow 0$ on a series of test cases including the situation (some are not covered by the theory) of a time-dependent reference diffusion coefficient $\overline{a}_t$. Space discretization is performed using the parabolic scaling $dx^2 = h$. The domain is truncated based on the expected tail behaviors of the diffusion so that the size of the time space grid is $O(1/h^{1.5})$. The implementation trades memory for speed using intermediate recursive variables (36-37) and also uses a coarse to fine in h warm restart strategy. The experiments confirm the expected $O(1/h^{1.5})$ time cost of our implementation.

Related works. The method studied in this paper has been applied to volatility calibration for derivatives pricing in the companion paper [18].

With the same motivation in finance, [19] points out the issue of using the relative entropy between singular measures and already mentions using the expectation $\mathcal{S}^{\mathcal{I}}$ to regularize the volatility. The setting however remains purely discrete and the link with the SRE concept and discrete standard relative entropy is not done.

The characterization of specific relative entropy for smooth Ito diffusions can be found in [14]. Our Proposition 2 also assumes smoothness but generalizes to dimensions $d > 1$.

In [20], the authors apply time discretization and scaling ideas replacing the relative entropy by p-Wasserstein distance, and study the properties of this new divergence between martingales.

The regularisation of a stochastic optimal control is proposed using the relative entropy in [21] at the time-continuous level. Only the drift is controlled and the relative entropy is used to relax (doubling the space of the state variable and minimizing also on the reference measure) and define an easier-to-solve alternate minimization algorithm.

2 Background

2.1 Dynamic Entropic Optimal Transport (EOT)

The dynamic formulation of the Classical OT optimisation problem [22] is now well-established and studied. It can be reformulated in a Lagrangian framework:

$$\inf_{\begin{cases} (X_t, b_t) \quad s.t. \\ dX_t = b_t(X_t)\, dt, \\ X_i \sim \rho_i, \quad i \in \{0,1\} \end{cases}} \int_0^1 \frac{1}{2}\mathbb{E}\left(|b_t(X_t)|^2\right)\, dt\,. \tag{2}$$

The usual initial/final probabilities ρ_0 and ρ_1 are given. The optimal speed b_t will be a deterministic function depending on time and space, although it is initally only requested to be an adapted process. We use probabilistic notations even in this deterministic setting, to prepare for the stochastic generalisations to come. Let us start with

$$\inf_{\begin{cases} (X_t, b_t), \quad s.t. \\ dX_t = b_t(X_t)\, dt + \overline{\sigma}\, dW_t, \\ X_i \sim \rho_i, \quad i \in \{0,1\} \end{cases}} \int_0^1 \frac{1}{2}\mathbb{E}\left(|b_t(X_t)|^2\right)\, dt\,. \tag{3}$$

The new ingredients are the nonnegative scalar volatility $\overline{\sigma}$ and W_t the Wiener process.

Setting $\overline{\mathbb{P}}$ the measure of a diffusion process with volatility $\overline{\sigma}$ and initial law ρ_0 (i.e under $\overline{\mathbb{P}}$ the canonical process satisfies $X_0 \sim \rho_0$ and $dX_t = \overline{\sigma}\, dW_t$), (3) can be re-interpreted as the Schroedinger bridge problem (see [7] for a review and also its link with OT when $\overline{\sigma} \searrow 0$):

$$\inf_{\begin{cases} \mathbb{P} \in \mathcal{P}(\Omega) \quad s.t. \\ \mathbb{P}_0 = \rho_0 \quad \mathbb{P}_1 = \rho_1, \end{cases}} \mathcal{H}(\mathbb{P}|\overline{\mathbb{P}}). \tag{4}$$

Remarkably, the solutions of (3) and (4) are the same (the optimal $\mathbb{P}$ in the latter will be the law of the optimal X in the former). Indeed, this is a direct consequence of Girsanov's Theorem that the integrand in (3) is the relative entropy. Moreover, the problem can also be formulated under a static form (i.e. only optimizing on joint law between $t = 0, t = 1$): The additive property of the relative entropy applied to the disintegration of the measures $\mathbb{P}$ and $\overline{\mathbb{P}}$ yields

$$\mathcal{H}(\mathbb{P}|\overline{\mathbb{P}}) = \mathcal{H}(\mathbb{P}_{0,1}|\overline{\mathbb{P}}_{0,1}) + \mathbb{E}_{\mathbb{P}_{0,1}}\left(\mathcal{H}(\mathbb{P}(.|X_0, X_1)|\overline{\mathbb{P}}(.|X_0, X_1))\right),$$

which shows that the reference measure $\overline{\mathbb{P}}$ will enforce $\mathbb{P}$ to share the same point to point Brownian bridges and therefore the same volatility $\overline{\sigma}$. (In fact in order to minimize the left hand side, one will minimize the first term of the right hand side, and cancel to second term).

The control is left to the drift to steer the distribution from ρ_0 to ρ_1. The relative entropy term has been useful in [10, 11, 23] as a regularisation or modelling term or both, [11] for example extends the entropic regularisation method to

$$
\inf_{\begin{cases} \mathbb{P} \in \mathcal{P}(\Omega) \quad s.t. \\ \mathbb{P}_0 = \rho_0, \, \mathbb{P}_1 = \rho_1, \end{cases}} \mathbb{E}_{\mathbb{P}} \left(\int_0^1 \int_{\mathbb{R}^d} F(t, x, \mathbb{P}_t(x)) \, dx \, dt \right) + \overline{\sigma} \, \mathcal{H}(\mathbb{P}|\overline{\mathbb{P}}) \, ,
$$

F being a "nice" convex functional, adding an local interaction term in the objective function. Again, this formulation leads to an optimisation of the drift of a diffusion process with prescribed volatility $\overline{\sigma}$. After time discretisation it becomes a multi-marginal entropic transport problem, and using the specific form of the dual of the relative entropy, a separable Sinkhorn algorithm can be applied formally with an optimal linear memory and operation cost. This will be detailed on our specific problem in Section 4.

2.2 Stochastic OT problem

We follow closely the setup proposed by [1] and reused in [24]. Let $(\Omega, \mathcal{F})$ be the set of continuous paths $C([0,T];\mathbb{R})$ and its canonical filtration. We let $\mathcal{P}$ be the set of Borel probability measures on $(\Omega, \mathcal{F})$. Let $\mathcal{P}^0 \subset \mathcal{P}$ be a subset of measures such that, for each $\mathbb{P} \in \mathcal{P}^0$, $X \in \Omega$ is an $(\mathcal{F}, \mathbb{P})$-semimartingale on $[0,1]$ given by

$$
X_t = X_0 + B_t^{\mathbb{P}} + A_t^{\mathbb{P}}, \quad \langle X \rangle_t = \langle M^{\mathbb{P}} \rangle_t = A_t^{\mathbb{P}}, \quad \mathbb{P}\text{-}as, \quad t \in [0,1],
$$

where $M^{\mathbb{P}}$ is an $(\mathcal{F}, \mathbb{P})$-martingale on $[s,1]$ and $(B^{\mathbb{P}}, A^{\mathbb{P}})$ is $\mathcal{F}$-adapted and $\mathbb{P}$-as absolutely continuous with respect to time. In particular, $\mathbb{P}$ is said to have characteristics $(a^{\mathbb{P}}, b^{\mathbb{P}})$, which are defined in the following way,

$$
a_t^{\mathbb{P}} = \frac{dA_t^{\mathbb{P}}}{dt}, \quad b_t^{\mathbb{P}} = \frac{dB_t^{\mathbb{P}}}{dt} \cdot
$$

Note that $(a^{\mathbb{P}}, b^{\mathbb{P}})$ is $\mathcal{F}$-adapted and determined up to $d\mathbb{P} \times dt$, almost everywhere. We might use a, b instead of $a^{\mathbb{P}}, b^{\mathbb{P}}$ for simplicity of notations sometimes throughout the paper, but it is fundamental to note that the object we are optimising is the measure $\mathbb{P}$, from which a, b are derived. Letting $\mathcal{S}^d$ be the set of symmetric matrices and $\mathcal{S}_+^d$ be the set of positive semidefinite matrices, in general, $(a^{\mathbb{P}}, b^{\mathbb{P}})$ takes values in the space $\mathcal{S}_+^d \times \mathbb{R}^d$. For any $a_1, a_2 \in \mathcal{S}^d$, let us define $a_1 : a_2 := \mathrm{trace}(a_1^\intercal a_2)$.

Denote by $\mathcal{P}^1 \subset \mathcal{P}^0$ the set of probability measures $\mathbb{P}$ whose characteristics $(a^{\mathbb{P}}, b^{\mathbb{P}})$ are $\mathbb{P}$-integrable on the interval $[0,1]$. In other words,

$$
\mathbb{E}\left(\int_0^1 |a^{\mathbb{P}}| + |b^{\mathbb{P}}| \, dt \right) < +\infty,
$$

where $|\cdot|$ denotes the Euclidean norm. We also let $\mathbb{P}_t = X_{t\,\#}\mathbb{P}$, i.e. the marginal of $\mathbb{P}$ at time t.

In this paper, our goal is to extend the benefits of Entropic regularisation to stochastic OT problems (see [1, 3, 4, 24]) of the form

$$\inf_{\mathbb{P}\in\mathcal{P}^1} \mathbb{E}_{\mathbb{P}}\left(\int_0^1 F(t, X_t, b_t^{\mathbb{P}}, a_t^{\mathbb{P}})\, dt\right) + \mathcal{D}(\mathbb{P}_0, \rho_0) + \mathcal{D}(\mathbb{P}_1, \rho_1) \tag{5}$$

where F has to be convex in (a, b) at every (t, x) and also enforces the non-negativity of a the diffusion coefficient, $\mathcal{D}(\mathbb{P}_t, \rho)$ is convex in $\mathbb{P}_t$, for a given ρ. Typically $\mathcal{D}$ will be a penalty function to enforce that $\mathbb{P}_t$ is either equal or close to ρ at initial and final times. We will use hard constraints as $\iota_{\rho_t}(\mathbb{P}_t)$, as in the usual optimal transport problem in the numerical section. Still, the convergence theorem relies on a soft constraint for instance $\mathcal{W}_2(\rho_t, \mathbb{P}_t)$ the quadratic Wasserstein distance. We will assume that there exists C such that for every pair ρ_1, ρ_2

$$\begin{aligned}
&\mathcal{D} \text{ is strictly convex and there exists } C > 0 \text{ such that}\\
&\forall \rho_1, \rho_2,\ 0 \le \mathcal{D}(\rho_1, \rho_2) \le C\mathcal{W}_2^2(\rho_1, \rho_2)\\
&\mathcal{D}(\rho_1, \rho_2) < +\infty \implies \int |x|^4 d\rho_1(x) < \infty\,.
\end{aligned} \tag{H1}$$

In particular, if η is a space convolution kernel with variance v_η, then

$$\mathcal{D}(\eta \star \rho, \rho) \le C v_\eta.$$

Usual coercivity assumptions on F are

$$F \ge C(1 + |a|^p + |b|^p) \text{ for } C > 0, p > 1. \tag{6}$$

We will restrict to a strongly coercive case:

$$F = G + \iota_{[\lambda,\Lambda]}(a) + \iota_{[0,B]}(|b|) \tag{H2}$$

with λ, Λ, B given positive parameters, and G satisfying

$$G \text{ is strictly convex and Lipschitz on } [0, B] \times [\lambda, \Lambda]. \tag{H3}$$

The bounds in (H2) are technical assumptions that allow to handle the convergence of the discrete entropy to the specific entropy.

As explained in [4], when F is a function of (t, X, b, a) the optimal a and b will automatically be functions of (t, X), even if one does not restrict such local processes in the minimisation procedure. This is due to the convexity of F which favors local functions. We state here a general and standard existence result for (5), that can be found in [24].

Theorem 1. *Under conditions (H1, H2, H3), if there exists an admissible solution (i.e. a $\mathbb{P}$ with finite energy in (5)) there exists a minimiser to (5).*

This follows by standard arguments on the compactness of any minimizing sequence, and the lower semi-continuity of the energy functional.

Remark 1. • *Assuming ρ_0 and ρ_1 are in convex order, time-dependent martingale constrained processes (see for example [25]) correspond to (5) with*

$$F = F(a) \text{ if } b \equiv 0,$$
$$+\infty \text{ otherwise.}$$

• *In particular, the Bass Martingale Problem (see [6] for example) corresponds to the choice $F = |a - Id|^2$.*

• *The Schrödinger bridge problem (4) corresponds to*

$$F = |b|^2 \text{ if } a \equiv \varepsilon,$$
$$+\infty \text{ otherwise,}$$

for a positive constant ε.

2.3 Specific relative entropy

The concept of "specific relative entropy" between diffusion processes seems to go back to [13], see also [15] and [14].

Let us consider $\mathbb{P} \in \mathcal{P}^1$ and the reference measure $\overline{\mathbb{P}}$ defined in section 2.1. We denote by $\mathbb{P}^h = (e_{t_1,t_2,\dots})_\# \mathbb{P}$ and $\overline{\mathbb{P}}^h = (e_{t_1,t_2,\dots})_\# \overline{\mathbb{P}}$ their time discretisation with time mesh h (see the notation section). Then under regularity conditions on the drift and diffusion coefficients specified in Proposition 2 below, the specific relative entropy between $\mathbb{P}$ and $\overline{\mathbb{P}}$ is defined as the limit:

$$\mathcal{S}^{\mathcal{H}}(\mathbb{P}|\overline{\mathbb{P}}) := \lim_{h \searrow 0} h\, \mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h).$$

Under additional smoothness and boundedness assumptions on the characteristics $b^{\mathbb{P}}, a^P$ can be characterized (see proposition 2) explicitly:

$$\mathcal{S}^{\mathcal{H}}(\mathbb{P}|\overline{\mathbb{P}}) = \lim_{h \searrow 0} h\, \mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h) = \tfrac{1}{2}\mathbb{E}_{\mathbb{P}}\left(\int_0^1 \mathcal{S}^{\mathcal{I}}(a_t(X_t)|\bar{a}))\, dt\right). \tag{7}$$

The integrand $\mathcal{S}^{\mathcal{I}}$, is given (in dimension 1) by

$$\mathcal{S}^{\mathcal{I}}(a|\bar{a}) := \frac{a}{\bar{a}} - 1 - \log\left(\frac{a}{\bar{a}}\right). \tag{8}$$

In the case of a positive definite symmetric matrix in dimension d, the definition becomes

$$\mathcal{S}^{\mathcal{I}}(a|\bar{a}) := \mathrm{Tr}(\bar{a}^{-1}(a - \bar{a})) - \log\left(\frac{\det(a)}{\det \bar{a}}\right).$$

This function is strictly convex with minimum at $\overline{a}$, a barrier for vanishing a and strictly increasing but just sub-linearly as $a \to +\infty$.

Thanks to the discretisation and renormalisation by h, w will see that the limit (7) is well defined even when the diffusions matrices a and $\bar{a}$ differ, in which case the probability measures $\mathbb{P}$ and $\overline{\mathbb{P}}$ are singular as probability measures on the space of continuous functions, which entails that the relative entropy blows up. This is easily understood in the following simplified setting: instead of discretizing a time continuous process, directly assume that $\mathbb{P}^h$ and $\overline{\mathbb{P}}^h$ are a Markov chains with transisitions following normal distribution with space dependent (μ_i^h, v_i^h) coefficients such that:

$$\mathbb{P}^h_{i\to i+1} = \mathcal{N}(x_i + h\,\mu_i^h(x_i), h\,v_i^h(x_i))$$

and likewise $\overline{\mathbb{P}}^h_{i\to i+1} = \mathcal{N}(x_i, h\,\overline{a})$. A direct computation expresses the discrete relative entropy as

$$h\,\mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h) = h\,\mathcal{H}(\mathbb{P}^h_0|\rho_0) + \frac{h}{2}\sum_{i=0}^{N-1}\mathbb{E}_{\mathbb{P}^h_i}(\mathcal{S}^{\mathcal{I}}(v_i^h(X_i^h)|\overline{a}) + \frac{h}{\overline{a}}\,\|\mu_i^h(X_i^h)\|^2)\,, \tag{9}$$

where $((X_i^h)_{i=0..N}$ is the canonical discrete time process associated to $\mathbb{P}^h)$). Assuming convergence as $h \searrow 0$ of the piecewise constant in time interpolation of the coefficients and *without the* h *scaling* in (9) the zero-order term in (9) blows up consistently with the definition of relative entropy between singular continuous-time diffusion processes. *With the scaling*, we recover Definition 7.

Considering now an arbitrary Markov chain $\mathbb{P}^h$ we define

$$\begin{cases} b_i^h(x_i) := \dfrac{1}{h}\mathbb{E}_{\mathbb{P}^h_{i\to i+1}}\left[X^h_{i+1} - x_i\right], \\ a_i^h(x_i) := \dfrac{1}{h}\mathbb{E}_{\mathbb{P}^h_{i\to i+1}}\left[(X^h_{i+1} - x_i)(X^h_{i+1} - x_i)^\star\right]. \end{cases} \tag{10}$$

Remark that compared to the previous Gaussian example $b = \mu$ but $a = v + hb^2$. This choice of characteristics is only equivalent when $h \to 0$, but has the fundamental property that $\mathbb{P}^h_i a^h_i, \mathbb{P}^h_i b^h_i$ are linear quantities with respect to $\mathbb{P}^h$. Using this definition we can show in particular that the discrete relative entropy controls the discrete version of the specific relative entropy, we show (Proposition 2 ii)) that

$$h\,\mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h) \geq h\,\mathcal{H}(\mathbb{P}^h_0|\rho_0) + \frac{h}{2}\sum_{i=0}^{N-1}\mathbb{E}_{\mathbb{P}^h_i}(\mathcal{S}^{\mathcal{I}}(a_i^h(X_i^h)|\overline{a})\,. \tag{11}$$

We gather these results in:

Proposition 2 (Specific Relative Entropy)**.** *The following statements hold:*

i) Let $\mathbb{P} \in \mathcal{P}_1$, see (5), assume that both functions $(t,x) \mapsto b_t(x)$ and $(t,x) \mapsto a_t(x)$ are twice continuously differentiable, and that the matrix $a_t(x)$ is invertible for every $(t,x) \in [0,1] \times \mathbb{R}^d$. Assume further that (b,a) are bounded above and a below by a positive constant (in the sense of strong ellipticity for a). Then (7) holds.

ii) Let $\mathbb{P}^h$ be the law of a discrete-time Markov chain and (b_i^h, a_i^h) given by (10). We assume further that $d\mathbb{P}^h = \rho . d\overline{\mathbb{P}}^h$ with a continous density ρ, then (11) holds.

Proof. The proof is given in Annex 6.1 □

2.4 Convergence of Markov chains to time continuous diffusions

This is a generalisation of [26, Th. 11.2.3] to inhomogeneous in time Markov chain.

We consider an inhomogeneous (family in h) of transition probabilities $\mathbb{P}^h_{i \to i+1}(x, dy)$ on $\mathbb{R}^d$ for $i = 0, \dots, N-1$. We associate to it a discrete time Markov chain $X_0, \dots, X_{N-1}$ that is turned itself into a continuous time process $x(t)$ with $x(ih) = X_i$ and linear interpolation for $t \in [ih, (i+1)h)$ for $i = 0, \dots, N-1$ with law $\widetilde{\mathbb{P}}^h \in \mathcal{P}(\Omega)$. The discrete sampling $x(0), x(h), \dots, x((N-1)h)$ is a Markov sequence with inhomogeneous transition

$$\widetilde{\mathbb{P}}^h(x((i+1)h) \in dy \,|\, x(ih) = x) = \mathbb{P}^h_{i \to i+1}(x, dy)\,.$$

We define the piecewise constant in time drift and diffusion coefficients :

$$\begin{cases} \widetilde{b}_t^h(x) := \frac{1}{h} \mathbb{E}^h_{\mathbb{P}_{\lfloor t\,h^{-1} \rfloor \to \lfloor t\,h^{-1} \rfloor + 1}} \big[(X_{i+1} - x) 1_{\|X_{i+1} - x\| \le 1} \big], \\ \widetilde{a}_t^h(x) := \frac{1}{h} \mathbb{E}^h_{\mathbb{P}_{\lfloor t\,h^{-1} \rfloor \to \lfloor t\,h^{-1} \rfloor + 1}} \big[(X_{i+1} - x)(X_{i+1} - x)^\star 1_{\|X_{i+1} - x\| \le 1} \big]\,. \end{cases} \tag{12}$$

We also need a conditional moment of order larger than 4 ($\alpha > 0$) referred to abusively as "kurtosis" (kurtosis corresponds to $\alpha = 0$)

$$c_i^h(x_i) := h^{-(2+\alpha)} \mathbb{E}_{\mathbb{P}^h_{i \to i+1}} (\|X_{i+1} - x_i\|^{4+2\,\alpha})\,. \tag{13}$$

Note that when X_i^h is the discrete sampling of a continuous diffusion with bounded characteristics, $c_i^h(x_i)$ is controlled uniformly by the bound on the characteristics. (see Lemma 11).

Theorem 3. *Assuming $(\widetilde{b}_t^h, \widetilde{a}_t^h)$ defined in (12) and (c_i^h) defined in (13) satisfy*

$$\begin{cases} i) \quad \lim_{h \searrow 0} \sup_{|x| \le R, t \in [0,1]} \|\widetilde{b}_t^h(x) - \widetilde{b}_t^0(x)\| = 0\,, \\ ii) \quad \lim_{h \searrow 0} \sup_{|x| \le R, t \in [0,1]} \|\widetilde{a}_t^h(x) - \widetilde{a}_t^0(x)\| = 0\,, \\ iii) \quad \lim_{h \searrow 0} \max_{i=1,\dots,N-1} \sup_{\|x\| \le R} h\, c_i^h(x) = 0\,, \end{cases} \tag{14}$$

for some pair $\widetilde{b}_t^0(x), \widetilde{a}_t^0(x)$ of integrable functions. Then, $\widetilde{\mathbb{P}}^h$ narrowly converges to $\widetilde{\mathbb{P}}^0$ a diffusion process in the weak sense with drift and diffusion coefficients $(\widetilde{b}_t^0, \widetilde{a}_t^0)$.

Proof. The proof is given in Annex 6.2 □

3 Entropic Optimal Martingale Transport (EOMT) problem, discretization and convergence

3.1 The EOMT problem and its discretisation

We introduce the $\mathcal{S}^{\mathcal{I}}$ regularized time continuous functional

$$\mathcal{I}^0(\mathbb{P}) := \mathbb{E}_{\mathbb{P}}\left(\int_0^1 F(b_t^{\mathbb{P}}(X_t), a_t^{\mathbb{P}}(X_t)) + \mathcal{S}^{\mathcal{I}}(a_t^{\mathbb{P}}(X_t)|\overline{a}))\, dt\right) + \mathcal{D}(\mathbb{P}_0, \rho_0) + \mathcal{D}(\mathbb{P}_1, \rho_1) \quad (15)$$

where $\overline{a}$ is a given reference volatility target, $b^{\mathbb{P}}, a^{\mathbb{P}}$ are the characteristic coefficients associated to $\mathbb{P} \in \mathcal{P}^1$ (see section 2.2). Note that there are, formally at least, no obstacles to considering local in time and space $\overline{a}$. The fidelity terms $\mathcal{D}(\mathbb{P}_t, \rho)$ properties are summarized in (H1).

The time continuous SRE regularisation of (5) is:

$$\inf_{\mathbb{P} \in \mathcal{P}^1} \mathcal{I}^0(\mathbb{P})\,. \quad (\mathcal{V}^0)$$

Penalization/regularisation of the volatility is often necessary to deal with underdetermined problems like volatility calibration of pricing models in finance (see [3, 19, 24, 27, 28]). Here, the choice of the SRE regularisation is driven by its discrete-time formulation, which will allow to rely on a Sinkhorn-like algorithm for numerical resolution. Indeed, based on (7) and Theorem 3 we intend to use the natural time discretisation of (15):

$$\mathcal{I}^h(\mathbb{P}^h) := h \sum_{i=0}^{N-1} \mathbb{E}_{\mathbb{P}_i^h}\left(F(b_i^h(X_i^h), a_i^h(X_i^h))\right) + h\, \mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h) + \mathcal{D}(\mathbb{P}_0^h, \rho_0) + \mathcal{D}(\mathbb{P}_1^h, \rho_1), \quad (16)$$

with (b_i^h, a_i^h), the discrete drift and quadratic variation increments defined in (10). We thus consider

$$\inf_{\begin{cases} \mathbb{P}^h \in \mathcal{P}(\bigotimes_{i=0}^N \mathbb{R}_{t_i}) \; s.t. \\ h \sum_{i=0}^{N-1} \mathbb{E}_{\mathbb{P}_i^h}\left(c_i^h(X_i^h)\right) \le M \end{cases}} \mathcal{I}^h(\mathbb{P}^h)\,. \quad (\mathcal{V}^h)$$

In this formulation, we have introduced an additional constraint on a moment of order larger than 4 ($\alpha > 0$) referred to abusively as *kurtosis* (kurtosis is $\alpha = 0$, see also (13):

$$c_i^h(x_i) := h^{-(2+\alpha)} \mathbb{E}_{\mathbb{P}^h_{i \to i+1}}(\|X_{i+1} - x_i\|^{4+2\,\alpha})\,.$$

This additional constraint seems necessary to apply Theorem 3 and ensures that the sequence $(\mathbb{P}^h)_h$ converges in $\mathcal{P}^1$. Controlling the specific entropy through (11) is not sufficient to guarantee (as in [26] Chap.11 or [29]) that the limit of $(\mathbb{P}^h)_h$ is a jump free diffusion process. We start with an existence result for the discretised problem:

Theorem 4. *Under the hypotheses (H1-H2-H3) problem* $(\mathcal{V}^h)$ *has a unique Markovian solution* $\mathbb{P}^h$.

Proof. Annex 6.3 □

Before studying the convergence of the discretized problem to the continuous one let us first comment on the difference between the two formulations: there is no explicit bound on the kurtosis of the continuous process. This bound comes from the boundedness of the drift and diffusion coefficients as an immediate consequence of Lemma 11. Therefore, choosing the constant M large enough in the discretised problem will be sufficient to ensure that the continuous limit of of this constaint is not saturated. However, this constraint is still important to guarantee the convergence to a diffusion process (with no jumps).

Our main result in this section is the convergence of the values of the discretized problems to the continuous one:

Theorem 5. *Let* $\rho_0, \rho_1 \in \mathcal{P}_2(\mathbb{R}^d)$ *be given. Under the hypotheses (H1-H2-H3), let* $\mathbb{P}^0$ *be a minimizer of* $(\mathcal{V}^0)$ *and* $(\mathbb{P}^h)_h$ *a sequence of minimizers of* $(\mathcal{V}^h)$*. Then,*

$$\lim_{h \searrow 0} \mathcal{I}^h(\mathbb{P}^h) = \mathcal{I}^0(\mathbb{P}^0)\,.$$

Proof. See Section 3.2. □

Remark 2. *Theorem 5 does not elaborates on the convergence of the minimizing sequences* $(b^h, a^h, \mathbb{P}^h)$*, but in the course of the proof however we construct from* $(\mathbb{P}^h)_h$ *a family of smooth measures on Markov chains* $(\mathbb{P}^h_\varepsilon)_{h,\varepsilon}$ *, depending on a small regularisation parameter* ε*, such that*

i) $\mathcal{I}^h(\mathbb{P}^h_\varepsilon)$ *is close to* $\mathcal{I}^h(\mathbb{P}^h)$ *depending on* ε*,*
ii) for all $\varepsilon > 0$*,* $\mathbb{P}^h_\varepsilon$ *converges as* $h \to 0$ *(in a sense to be defined) to* $\mathbb{P}_\varepsilon$*, a smooth measure on* Ω*,*
iii) $\mathcal{I}^0(\mathbb{P}_\varepsilon)$ *is close to* $\mathcal{I}^0(\mathbb{P}^0)$ *depending on* ε*.*

Hence we "almost" prove that $\mathbb{P}^h$ *converges to* $\mathbb{P}^0$*. The proof of the full result, relying on fine regularity properties of the minimizers* $\mathbb{P}^h$*, is still ongoing.*

Remark 3. *We point out that this result is not a classic* Γ*-convergence result* $(\mathcal{I}^h \to \mathcal{I}^0)$ *as we deal with discrete time Markov chains on the* I^h *side and continuous diffusion processes as arguments of* $\mathcal{I}^0$*. Our numerical interest is in the behavior of numerical solutions to* $(\mathcal{V}^h)$*.*

Remark 4. *The discrete approximation of* $(\mathcal{V}^0)$ *corresponds to a* relative entropic regularised *problem and will be solved in Sections 4 and 5 using a Sinkhorn algorithm. Note that under stronger convexity assumptions on* F*, we can apply this strategy to any stochastic control problem of the type (5) simply by writing* $F = F - \mathcal{S}^{\mathcal{I}} + \mathcal{S}^{\mathcal{I}}$ *applying (7) only to the last term.*

3.2 Convergence of the time discretisation

The proof shares similarities with $\Gamma-$convergence and relies on well-chosen regularization of the minimizers. Since the proof is long, we highlight the main steps and associated intermediate lemmas. Details are deferred to the Annex 6.

We start with three technical lemmas. The first lemma concerns the scaling of the drift and diffusion coefficients, which is needed to satisfy the corresponding hard constraints.

Lemma 6 (Diffusion coefficients rescaling)**.** *Let X_t have law $\mathbb{P} \in \mathcal{P}^1$ with characteristics b, a satisfying $\lambda Id < a < \Lambda Id$ and $|b| < B$. For all $1 \gg \delta > 0$, let*

$$X_\delta = \sqrt{\alpha}X + \sqrt{\varepsilon}B(t),$$

$\varepsilon = \delta\frac{\lambda+\Lambda}{\Lambda-\lambda}$ *and* $\alpha = 1 - \frac{2\,\varepsilon}{\lambda+\Lambda}$ *depending on δ, and $B(t)$ an independent standard Brownian motion. Then, X_δ has law $\mathbb{P}_\delta \in \mathcal{P}^1$ with characteristics b_δ, a_δ, and we have that:*

i) Letting $T_{\sqrt{\alpha}}(x) = \sqrt{\alpha}x$, and $\mathbb{P}_{t,\delta} := (e_t)_{\#}\mathbb{P}_\delta$,

$$\mathbb{P}_{t,\delta} := \mathcal{N}(0, \varepsilon t) \star [T_{\sqrt{\alpha}}]_\sharp\, \mathbb{P}_t,$$

ii) Scaling of the coefficients:

$$(\lambda + \delta)\, Id < a_\delta < (\Lambda - \delta)\, Id, \text{ and } |b_\delta| \le (1 - \delta)B\,.$$

iii) $\mathbb{P}_\delta$ narrowly converges to $\mathbb{P}$ as $\delta \searrow 0$ and

$$\mathcal{I}^0(\mathbb{P}_\delta) \le \mathcal{I}^0(\mathbb{P}) + O(\delta)\,,$$

where the notation $O(\delta)$ hides a constant that depends on the Lipschitz contact of G and λ, Λ, B.

Moreover, the same holds for a discrete process X^h with law $\mathbb{P}^h$, considering $\mathbb{P}^h_\delta$ the probability of $X^h_\delta(t) = \sqrt{\alpha}X^h(t) + \sqrt{\varepsilon}B(t)$ as above. We have, for α sufficiently close to 1,

iv) Scaling of the coefficients:

$$(\lambda + \delta)\, Id < a^h_\delta < (\Lambda - \delta)\, Id, \text{ and } |b^h_\delta| \le (1 - \delta)B\,.$$

v) $\mathbb{P}^h_\delta$ narrowly converges to $\mathbb{P}^h$ as $\delta \searrow 0$ and

$$\mathcal{I}^h(\mathbb{P}^h_\delta) \le \mathcal{I}^h(\mathbb{P}^h) + O(\delta)\,. \tag{17}$$

Proof. Annex 6.4. ☐

The next lemma concerns the regularization technique for the time-continuous formulation.

Lemma 7 (Time (and space) regularization - the continuous case). *Let* $(\mathbb{P}^0, b^0, a^0)$ *representing a Markov process* $X(t)$ *solving, for a Brownian motion* $B(t)$*,*

$$dX(t) = b^0(t, X)dt + a^0(t, X)dB(t) \tag{18}$$

such that $\mathcal{I}(\mathbb{P}^0) < +\infty$*. Fix* $\sigma > 0$ *and let* $\varepsilon > 0$ *such that* $2\varepsilon\Lambda < \sigma$*. We further assume that* $|b^0| \leq M/(1+2\varepsilon)$ *and* $a^0 \leq \frac{\Lambda}{(1+2\varepsilon)^2}\,\mathrm{Id}$*. Then, there exists another Markov process* $X_{\varepsilon,\sigma}(t)$*, solving the SDE as above for the quantities* $b_{\varepsilon,\sigma}(t,x), a_{\varepsilon,\sigma}(t,x)$ *which are smooth in* t, x *such that*

$$\mathcal{I}(\mathbb{P}_{\varepsilon,\sigma}) \leq \mathcal{I}(\mathbb{P}^0) + O(\varepsilon) + O(\sigma)\,. \tag{19}$$

Proof. See Annex 6.5. □

Main steps of the proof: Up to step 4, we prove that the value of the continuous problem is lower bounded by the lim sup of the values of the discrete problems. For that, we consider a probability with finite value for the continuous problem, rescale and regularize it, and then discretize it to obtain a candidate for the discrete problem.

* **Step 0:** Recall that $(\mathcal{V}^h)$ is well posed and $\mathbb{P}^h$ is markovian (see Theorem 4). We also have well-posedness and existence of a solution $\mathbb{P}^0$ of the time continuous problem $(\mathcal{V}^0)$ is known (see Theorem 1).
* **Step 1:** Given $\mathbb{P}^0 \in \mathcal{P}^1$ a minimizer of $(\mathcal{V}^0)$ we construct a regularised version $\mathbb{P}^0_{\varepsilon,\sigma}$ by applying lemma 7 (σ depends on δ) . We get

$$\mathcal{I}^0(\mathbb{P}_{\varepsilon,\sigma}) \leq \mathcal{I}^0(\mathbb{P}) + O(\sigma) + O(\varepsilon).$$

* **Step 2:** Set $\mathbb{P}^h_{\varepsilon,\sigma} = (e_{t_0,t_1,\dots,t_N})_{\#}\mathbb{P}_{\varepsilon,\sigma}$ the time discretisation of $\mathbb{P}_{\varepsilon,\sigma}$, the output of **step 1**. We want to compare $\mathcal{I}^h(\mathbb{P}^h_{\varepsilon,\sigma})$ and $\mathcal{I}^0(\mathbb{P}_{\varepsilon,\sigma})$.

Lemma 8. *For all* h *sufficiently small w.r.t.* δ, ε*, we have*

i) The family $(\mathbb{P}^h_{\delta,\varepsilon})_h$ *satisfies the hypothesis of proposition 2 i) and in particular (see (41)) we have:*

$$h\,\mathcal{H}(\mathbb{P}^h_{\varepsilon,\sigma}|\overline{\mathbb{P}}^h) = \frac{1}{2}\sum_{i=1}^{N} h\,\mathbb{E}_{\mathbb{P}_{\delta,\varepsilon}}\left[\mathcal{S}^{\mathcal{I}}(a^{\mathbb{P}_{\varepsilon,\sigma}}(ih, X_{ih})|\overline{a}\right] + O(h^{1/4})\,.$$

ii) It holds

$$\mathcal{I}^h(\mathbb{P}^h_{\varepsilon,\sigma}) \leq \mathcal{I}^0(\mathbb{P}_{\varepsilon,\sigma}) + O(h^{\frac{1}{4}})\,.$$

Proof. Annex 6.6 □

* **Step 3:** For any minimizer of $(\mathcal{V}^h)$, denoted $\mathbb{P}^h$, and gathering the results in steps 1 and 2 we get

$$\mathcal{I}^h(\mathbb{P}^h) \leq \mathcal{I}^h(\mathbb{P}^h_{\varepsilon,\sigma}) \leq \mathcal{I}^0(\mathbb{P}^0) + O(\sigma) + O(\varepsilon) + O(h^{\frac{1}{4}})\,.$$

We pass to the limit first in h and then in ε, σ to obtain

$$\limsup_{h \searrow 0} \mathcal{I}^h(\mathbb{P}^h) \le \mathcal{I}^0(\mathbb{P}^0)\,.$$

The next steps show that the lim inf of the values of the discrete problems is lower bounded by the value of the continuous problem.

*** Step 4:** We need the following definitions.

For $\mathbb{P}^h \in \mathcal{P}(\bigotimes_{i=0}^N \mathbb{R}_{t_i})$ Markovian, we define piecewise constant in time interpolants

$$\widetilde{\rho}_t^h = \sum_{i=0}^N \mathbf{1}_{[t_i,t_{i+1})}\mathbb{P}_i^h, \quad \widetilde{b}_t^h = \sum_{i=0}^N \mathbf{1}_{[t_i,t_{i+1})} b_i^h, \quad \widetilde{a}_t^h = \sum_{i=0}^N \mathbf{1}_{[t_i,t_{i+1})} a_i^h \tag{20}$$

(b_i^h, a_i^h) defined in (12). We use the moment notations

$$\widetilde{m}_t^h = \widetilde{\rho}_t^h\, \widetilde{a}_t^h, \quad \widetilde{n}_t^h = \widetilde{\rho}_t^h\, \widetilde{b}_t^h. \tag{21}$$

We define the proxy functional for $\mathcal{I}^0$

$$\mathcal{J}(\rho, m, n) = \int_{t,x} F(m/\rho, n/\rho) + \mathcal{S}^{\mathcal{I}}(n/\rho)\, d\rho_t(x) dt\,. \tag{22}$$

In particular we have for $\mathbb{P} \in \mathcal{P}^1$ of finite energy $\mathcal{I}^0$

$$\mathcal{I}^0(\mathbb{P}) = \mathcal{J}(\mathbb{P}_t, \mathbb{P}_t\, b^{\mathbb{P}}, \mathbb{P}_t\, a^{\mathbb{P}})\,,$$

and using Proposition 2 ii) :

$$\mathcal{J}(\widetilde{\rho}^h, \widetilde{m}^h, \widetilde{n}^h) \le \mathcal{I}^h(\mathbb{P}^h)$$

and apply the following lemma:

Lemma 9 (Time (and space) regularization - the discrete case)**.** *Let $(\mathbb{P}^h)_h$ be sequence of Markov Chains such that $\sup_h \mathcal{I}^h(\mathbb{P}^h) < +\infty$ and such that the fourth moment of $\mathbb{P}_0^h$ is uniformly bounded in h. Let $\delta, \varepsilon > 0$. Then, there exist $\mathbb{P}_{\delta,\varepsilon}^h \in \mathcal{P}(\mathbb{R}^{dN})$ and $\mathbb{P}_{\delta,\varepsilon} \in \mathcal{P}^1$ such that, up to a subsequence in h,*

i) we have

$$\mathcal{I}^h(\mathbb{P}_{\delta,\varepsilon}^h) \le \mathcal{I}^h(\mathbb{P}^h) + O(\varepsilon) + O(h) + O(\delta) \tag{23}$$

and

$$\mathcal{I}^h(\mathbb{P}_{\delta,\varepsilon}^h) \ge \mathcal{J}(\widetilde{\rho}_{\delta,\varepsilon}^h, \widetilde{m}_{\delta,\varepsilon}^h, \widetilde{n}_{\delta,\varepsilon}^h)\,. \tag{24}$$

where $(\widetilde{\rho}_{\delta,\varepsilon}^h, \widetilde{m}_{\delta,\varepsilon}^h, \widetilde{n}_{\delta,\varepsilon}^h)$ is the change of variable (20-21) associated to $\mathbb{P}_{\delta,\varepsilon}^h$

ii) $\widetilde{\mathbb{P}}_{\delta,\varepsilon}^h$ narrowly converges to $\mathbb{P}_{\delta,\varepsilon} \in \mathcal{P}^1$ with characteristics $a_t^{\mathbb{P}_{\delta,\varepsilon}}, b_t^{\mathbb{P}_{\delta,\varepsilon}}$.

iii) In addition, we have

$$\mathcal{I}^0(\mathbb{P}_{\delta,\varepsilon}) = \mathcal{J}(\mathbb{P}_{\delta,\varepsilon,t}, \widetilde{\mathbb{P}}_{\delta,\varepsilon,t} b_t^{\widetilde{\mathbb{P}}_{\delta,\varepsilon}}, \widetilde{\mathbb{P}}_{\delta,\varepsilon,t} a_t^{\widetilde{\mathbb{P}}_{\delta,\varepsilon}}) \leq \liminf_{h\searrow 0} \mathcal{J}(\widetilde{\rho}^h_{\delta,\varepsilon}, \widetilde{m}^h_{\delta,\varepsilon}, \widetilde{n}^h_{\delta,\varepsilon})\,.$$

Proof. Annex 6.7. Note that it is possible to apply this lemma due to Hypothesis (H1) which ensures that the initial density's fourth moment is finite. □

*** Step 5:** Lemma 9 *i*) and *ii*) gives

$$\mathcal{J}(\widetilde{\rho}^h_{\delta,\varepsilon}, \widetilde{m}^h_{\delta,\varepsilon}, \widetilde{n}^h_{\delta,\varepsilon}) \leq \mathcal{I}^h(\mathbb{P}^h) + O(\varepsilon) + O(\delta) + O(h)$$

and passing to the limit in h using Lemma 9 *iii*) and for a minimizer $\mathbb{P}^0$ of ($\mathcal{V}^0$) we have

$$\mathcal{I}^0(\mathbb{P}^0) \leq \mathcal{I}^0(\mathbb{P}_{\delta,\varepsilon}) \leq \liminf_{h\searrow 0} \mathcal{I}^h(\mathbb{P}^h) + O(\varepsilon) + O(\delta)\,. \tag{25}$$

*** Last Step:** Applying **Step 5** to a sequence of minimizers $(\mathbb{P}^h)_h$ of ($\mathcal{V}^h$) and using **Step 3**, we get

$$\mathcal{I}^0(\mathbb{P}^0) \leq \lim_{\delta,\varepsilon\searrow 0} \mathcal{I}^0(\mathbb{P}_{\delta,\varepsilon}) \leq \liminf_{h\searrow 0} \mathcal{I}^h(\mathbb{P}^h) \leq \limsup_{h\searrow 0} \mathcal{I}^h(\mathbb{P}^h) \leq \mathcal{I}^0(\mathbb{P}^0)$$

which concludes the proof of Theorem 5 *i*).

4 Dual formulation and Sinkhorn algorithm

In this section the support of $\mathbb{P}^h$ is a finite product of compact subsets of $\mathbb{R}^d$: $\Omega^h := \bigotimes_{i=0}^N \mathcal{X}_i$. This restriction is consistent with the space discretisation and truncation of our implementation (Section 5). In order to simplify the presentation and the notations we restrict to dimension $d = 1$ and consider a dependence of the payoff F just on the drift b, generalisations to dependence in a follow the same lines. We also drop all additional constraints on the coefficients and the Kurtosis constraints since it does not seems necessary to obtain convergence of the Sinkhorn algorithm in the next section and also to let h go to 0 together with the space discretisation step. We present the duality without rigorous proofs that are left for further studies. Let us just mentioned that Multi-Marginal OT on a product of compact space is usually easier to deal with as in [17] or [16] for instance.

The simplified discrete primal problem becomes:

$$\inf_{\mathbb{P}^h \in \mathcal{P}(\Omega^h)} \mathbb{E}_{\mathbb{P}^h}\left(h \sum_{i=0}^{N-1} F(b_i^h(X_i^h))\right) + \mathcal{D}(\rho_0, \mathbb{P}^h_0) + \mathcal{D}(\rho_1, \mathbb{P}^h_1) + h\,\mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}) \tag{26}$$

where $b_i^h(x_i) := \frac{1}{h} E_{\mathbb{P}^h_{i\to i+1}}\left(X^h_{i+1} - x_i\right)$ is given as a function of $\mathbb{P}^h$ as in (10).

4.1 Fenchel-Rockafellar duality

We again remark that the drifts b_i^h can be written using local linear functions of $\mathbb{P}^h$ (abusing again notations and using $\mathbb{P}^h$ for the distribution and their densities):

$$b_i^h(x_i) = \frac{1}{h} \frac{\mathbb{E}_{\mathbb{P}^h_{i,i+1}(x_i,.)}(X^h_{i+1} - x_i)}{\mathbb{P}^h_i(x_i)}$$

where $\mathbb{P}^h_{i,i+1}(x_i,.) := \mathbb{P}^h_i \, \mathbb{P}^h_{i\to i+1}$ is to be understood as the measure on $\mathbb{R}_{t_{i+1}}$ obtained by freezing the first variable x_i in the joint $\mathbb{R}_{t_i} \times \mathbb{R}_{t_{i+1}}$ probability $\mathbb{P}^h_{i,i+1}$.

The primal problem (26) can be rewritten

$$\inf_{\left\{\mathbb{P}^h \in \mathcal{P}(\Omega^h)\right\}} \mathcal{F}(\Delta^\dagger \mathbb{P}^h) + \, h \, \mathcal{H}(\mathbb{P}^h | \overline{\mathbb{P}}^h)$$

using the linear change of variable

$$\Delta^\dagger \mathbb{P}^h := \{(\mathbb{P}^h_i)_{i=0}^N, (x_i \to \frac{1}{h} \, \mathbb{E}_{\mathbb{P}^h_{i,i+1}(x_i,.)}(X^h_{i+1} - x_i))_{i=0}^{N-1}\},$$

$$\mathcal{F}((m_{0,i})_{i=0}^N, (m_{1,i})_{i=0}^{N-1}) := \mathcal{D}(\rho_0, m_{0,0}) + \mathcal{D}(\rho_1, m_{0,N}) + h \, \textstyle\sum_{i=0}^{N-1} \mathbb{E}_{m_{0,i}}(F(\frac{m_{1,i}}{m_{0,i}}))\,.$$

We recall that the reference measure is defined by $\overline{\mathbb{P}}^h_{i\to i+1} = \mathcal{N}(x_i + hb_i^h(x_i), h\overline{a})$ and $\overline{\mathbb{P}}^h_0 = \rho_0$.

In this section (again for simplicity) we use hard constraints for the loss function on initial/final marginals. It corresponds to the characteristic function $\mathcal{D}(\mu, \rho) := 0$ if $\mu = \rho$ and $+\infty$ otherwise. The (jointly convex) perspective functions $(m_0, m_1) \to h \, \mathbb{E}_{m_0}(F(m_1/m_0))$ are naturally extended to $+\infty$ if $m_0 \leq 0$ and 0 if $m_1 = m_0 = 0$. The $.^\dagger$ notation denotes that $\Delta^\dagger$ is the adjoint of the linear operator defined in (28).

Fenchel Rockafellar duality yields the equivalent dual problem:

$$\sup_{\Phi^h := (\phi^h_{0,i})_{i=0}^N, (\phi^h_{1,i})_{i=0}^{N-1}} -\mathcal{F}^*(-\Phi) - h \, \mathbb{E}_{\overline{\mathbb{P}}^h}(\exp\left(\frac{\Delta \, \Phi^h}{h}\right) - 1)\,. \tag{27}$$

The dual variable is a vector of continuous functions on the spaces $(\mathcal{X}_i)_{i=0}^N$ $\Phi^h = (\phi^h_{0,i})_{i=0}^N, (\phi^h_{1,i})_{i=0}^{N-1}$. The second term in (27) is the Legendre Fenchel transform of the relative entropy and

$$\Delta \, \Phi = \bigoplus_{i=0}^{N-1} \phi_{0,i} + \frac{1}{h}(x_{i+1} - x_i) \, \phi_{1,i} + \phi_{0,N}\,. \tag{28}$$

The Legendre Fenchel transform $\mathcal{F}^*$ of $\mathcal{F}$ is explicitly given, using the separability in space and in "times" i by:

$$\mathcal{F}^*(\Phi) = \mathbb{E}_{\rho_0}(\phi_{0,0} + h\,F^*(\frac{\phi_{1,0}}{h})) + \sum_{i=1}^{N-1} \chi_0(\phi_{0,i} + h\,F^*(\frac{\phi_{1,i}}{h})) + \mathbb{E}_{\rho_1}(\phi_{0,N})$$

where $\chi_0(f) := 0$ if f is the null function in $C(\mathcal{X})$ and $+\infty$ else. It is convenient to eliminate the $(\phi_{0,i})_{i=1}^{N-1}$ variables from Φ (we keep the same notations) and simplify problem (27) using

$$\Delta\,\Phi = \phi_{0,0} + \frac{1}{h}(x_1 - x_0)\,\phi_{1,0} + \bigoplus_{i=1}^{N-1} h\,F^*(-\frac{1}{h}\phi_{1,i}) + \frac{1}{h}(x_{i+1} - x_i)\,\phi_{1,i} + \phi_{0,N} \tag{29}$$

and

$$\mathcal{F}^*(\Phi) = \mathbb{E}_{\rho_0}(\phi_{0,0} + h\,F^*(\frac{\phi_{1,0}}{h})) + \mathbb{E}_{\rho_1}(\phi_{0,N})\,. \tag{30}$$

Applying classical Fenchel-Rockafellar duality ([30] theorem 6.3)

Proposition 10 (Solutions of (27))**.** *We have the following statements*

i) Problem (27) has a unique Markovian *solution which can be expressed, in terms of $\Phi^h = (\phi^h_{0,i=0}, \phi^h_{0,i=N}), (\phi^h_{1,i})_{i=0}^{N-1}$ a vector of continuous functions on the spaces $(\mathbb{R}_{t_i})$, the solution of the dual problem (27)-(29)-(30):*

$$\mathbb{P}^h = \exp\frac{1}{h}\Delta\,\Phi^h\,\overline{\mathbb{P}}\,,$$

$$\Delta\,\Phi = \phi_{0,0} + \frac{1}{h}(x_1 - x_0)\,\phi_{1,0} + \bigoplus_{i=1}^{N-1} h\,F^*(-\frac{1}{h}\phi_{1,i}) + \frac{1}{h}(x_{i+1} - x_i)\,\phi_{1,i} + \phi_{0,N}\,.$$

ii) $\mathbb{P}^h$ satisfies, for all i, the factorisation $\mathbb{P}^h$ can be tensorized with densities:

$$\mathbb{P}^h = \rho_0\,\Pi_{i=0}^{N-1}\mathbb{P}^h_{i\to i+1} \qquad \mathbb{P}^h_{i\to i+1} = \frac{\mathbb{P}^h_{i,i+1}}{\mathbb{P}^h_i}$$

and $\forall\, i$

$$\mathbb{P}^h_{i,i+1} = \mathcal{U}_i\;exp\left(\frac{1}{h^2}(x_{i+1} - x_i)\,\phi_{1,i}\right)\,\overline{\mathbb{P}}^h_{i,i+1}\,\mathcal{D}_{i+1}\,. \tag{31}$$

Each $(\mathcal{U}_i)$ and $(\mathcal{D}_i)$ are functions defined recursively $\mathcal{U}$pward and $\mathcal{D}$ownward using Φ^h and $\overline{\mathbb{P}}^h$ by

$$\begin{cases} \mathcal{U}_0 = exp\left(\frac{1}{h}\,\phi_{0,0}\right), \\ \mathcal{U}_{i+1} = exp\left(F^*(-\frac{1}{h}\phi_{1,i+1})\right)\,\int \mathcal{U}_i\;exp\left(\frac{1}{h^2}(x_{i+1} - x_i)\,\phi_{1,i}\right)\overline{\mathbb{P}}^h_{i,i+1}\,dx_i\,, \\ \;for\ i = 0, N-2. \end{cases} \tag{32}$$

$$
\begin{cases}
\mathcal{D}_N = exp\left(\frac{1}{h}\,\phi_{0,N}\right), \\
\mathcal{D}_i = exp\left(F^*(-\frac{1}{h}\phi_{1,i})\right) \int \overline{\mathbb{P}}^h_{i,i+1}\, exp\left(\frac{1}{h^2}(x_{i+1}-x_i)\,\phi_{1,i}\right) \mathcal{D}_{i+1}\,dx_{i+1}, \\
for\ i = N-1, 1.
\end{cases} \tag{33}
$$

Please recall that all variable subscripted by i are to be understood as function of x_i.

Remark 5. *The modifications to (29) and (30) in the more general case of time and diffusion dependent functions* $(t_i, b_i, a_i) \to F_i(b_i, a_i)$ *are simply* $\Phi^h = (\phi^h_{0,i=0}, \phi^h_{0,i=N}), (\phi^h_{1,i})_{i=0}^{N-1}, (\phi^h_{2,i})_{i=0}^{N-1}$ *the additional potentials* $\phi^h_{2,i}$ *are dual to* $\mathbb{P}^h_i\, a^h_i$ *and*

$$
\begin{aligned}
\Delta\,\Phi =& \phi_{0,0} + \frac{1}{h}(x_1 - x_0)\,\phi_{1,0} + \frac{1}{h}(x_1 - x_0)^2\,\phi_{2,0} + \\
& \bigoplus_{i=1}^{N-1} h\,F^*_i(-\frac{1}{h}\phi_{1,i}, -\frac{1}{h}\phi_{2,i}) + \frac{1}{h}(x_{i+1}-x_i)\,\phi_{1,i} + \frac{1}{h}(x_{i+1}-x_i)^2\,\phi_{2,i} \\
& + \phi_{0,N}
\end{aligned}
$$

and

$$
\mathcal{F}^*(\Phi) = \mathcal{D}^*_{\rho_0}(\phi_{0,0} + h\,F^*_0(\frac{\phi_{1,0}}{h}, \frac{\phi_{2,0}}{h})) + \mathcal{D}^*_{\rho_1}(\phi_{0,N})
$$

where $\mathcal{D}^*_{\rho_i}$ *is the Legendre Fenchel Transform of the (convex) marginal fidelity term* $\rho \to \mathcal{D}(\rho, \rho_i)$.

4.2 Multi-Marginal Sinkhorn Algorithm

We are now working with a fixed h and drop the dependence in the notations for clarity.

The simplest interpretation of Sinkhorn Algorithm (see [9]) is to perform an iterative coordinate-wise (in the components of Φ) ascent to maximize the concave dual problem (27-29-30):

$$
\sup_{\Phi := (\phi_{0,0}, (\phi_{1,i})_{i=0}^{N-1}, \phi_{0,N})} -\mathcal{F}^*(-\Phi) - h\,\mathbb{E}_{\overline{\mathbb{P}}}(\exp\left(\frac{1}{h}\Delta\,\Phi\right) - 1).
$$

One cycle of dual variable optimisation will be indexed by k and the potentials updated “à la Gauss Seidel” in the inner loop over the $N+2$ functions (Kantorovich potentials) in Φ (the order is not important). Notations are a little bit more involved than in the classical two marginals problem, to ease the presentation we set:

$$\begin{cases} \Phi^{k,0} = (\phi_{0,0}, (\phi^k_{1,i})_{i=0}^{N-1}, \phi^k_{0,N}), \\ \Phi^{k,i+1} = (\phi^{k+1}_{0,0}, (\phi^{k+1}_{1,0} \dots \phi^{k+1}_{1,i-1}, \phi_{1,i}, \phi^k_{1,i+1} \dots \phi^k_{1,N}), \phi^k_{0,N}), \quad i = 0 \dots N-1, \\ \Phi^{k,N+2} = (\phi^{k+1}_{0,0}, (\phi^{k+1}_{1,i})_{i=0}^{N-1}, \phi_{0,N}). \end{cases} \tag{34}$$

One Sinkhorn cycle $k \to k+1$, the updates from $\Phi^{k,0}$ to $\Phi^{k+1,0}$ can be written in compact form as a loop on its components:

$$\begin{cases} \phi^{k+1}_{0,0} = \arg\sup_{\phi_{0,0}} -\mathcal{F}^*(-\Phi^{k,0}) - h\,\mathbb{E}_{\overline{\mathbb{P}}}(\exp\left(\frac{1}{h}\Delta\,\Phi^{k,0}\right) - 1), \\ \phi^{k+1}_{1,j+1} = \arg\sup_{\phi_{1,j}} -\mathcal{F}^*(-\Phi^{k,i+1}) - h\,\mathbb{E}_{\overline{\mathbb{P}}}(\exp\left(\frac{1}{h}\Delta\,\Phi^{k,i+1}\right) - 1), \quad i = 0 \dots N-1, \\ \phi^{k+1}_{0,N} = \arg\sup_{\phi_{0,N}} -\mathcal{F}^*(-\Phi^{k,N+1}) - h\,\mathbb{E}_{\overline{\mathbb{P}}}(\exp\left(\frac{1}{h}\Delta\,\Phi^{k,N+2}\right) - 1). \end{cases} \tag{35}$$

Each of these maximization problems is stricly concave and sufficiently smooth (depending on F^*). They are also separable in space and (35) amounts to solve in sequence the following set of equations (∂F^* is the Frechet derivative, a gradient in finite dimention):

$$\begin{cases} \rho_0 = \exp\left(\frac{1}{h}\phi^{k+1}_{0,0}(x_0)\right) \int \exp\left(\frac{1}{h^2}(x_1 - x_0)\,\phi^k_{1,0}\right) \overline{\mathbb{P}}^h_{0,1}\,\mathcal{D}^k_1\,dx_1 \\ \rho_0\,\partial F^*(-\frac{1}{h}\phi^{k+1}_{1,0}) = \exp\left(\frac{1}{h}\phi^{k+1}_{0,0}\right) \int \frac{1}{h}(x_1 - x_0) \exp\left(\frac{1}{h^2}(x_1 - x_0)\,\phi^{k+1}_{1,0}\right) \overline{\mathbb{P}}^h_{0,1}\,\mathcal{D}^k_1\,dx_1 \\ 0 = \mathcal{U}^{k+1}_i \int \left(\partial F^*(-\frac{1}{h}\phi^{k+1}_{1,i}) + \frac{1}{h}(x_{i+1} - x_i)\right) \exp\left(F^*(-\frac{1}{h}\phi^{k+1}_{1,i}) + \frac{1}{h^2}(x_{i+1} - x_i)\,\phi^{k+1}_{1,0}\right) \overline{\mathbb{P}}^h_{i,i+1}\,\mathcal{D}^k_{i+1}\,dx_{i+} \\ \text{for all } i = 1, N-1 \\ \rho_1 = \mathcal{U}^{k+1}_{N-1} \exp\left(\frac{1}{h}\phi^{k+1}_{0,N}\right) \end{cases}$$

Recall that all variable subscripted by i are to be understood as function of x_i. Each line is to be understood as point-wise in space. The functions $(\mathcal{U}^k_i, \mathcal{D}^k_i)$ are defined by the recursions (32-33) applied to the iterative Gauss-Seidel update of the potentials: (34)

$$\begin{cases} \mathcal{U}^k_0 = \exp\left(\frac{1}{h}\,\phi^k_{0,0}\right) \\ \mathcal{U}^k_{i+1} = \exp\left(F^*(-\frac{1}{h}\phi^k_{1,i+1})\right) \int \mathcal{U}^k_i \exp\left(\frac{1}{h^2}(x_{i+1} - x_i)\,\phi^k_{1,i}\right) \overline{\mathbb{P}}^h_{i,i+1}\,dx_i \\ \text{for } i = 0, N-2 \end{cases} \tag{36}$$

$$\begin{cases} \mathcal{D}_N^k = \exp\left(\dfrac{1}{h}\,\phi_{0,N}^k\right) \\ \mathcal{D}_i^k = \exp\left(F^*(-\dfrac{1}{h}\phi_{1,i}^k)\right)\displaystyle\int \overline{\mathbb{P}}_{i,i+1}^h \exp\left(\dfrac{1}{h^2}(x_{i+1}-x_i)\,\phi_{1,i}^k\right)\mathcal{D}_{i+1}^k\,dx_{i+1} \\ \text{for } i = N-1, 1 \end{cases} \tag{37}$$

The numerical study in Section 5 gives experimental convergence curves. In the case of finite dimension in space, i.e, $\mathbb{P}^h$ as support on a truncated grid in space. The analysis of the convergence (in k) of this algorithm relies on a multi-marginal extension of [31], see also [16] or [17] and the references therein. Note however that they apply to "standard" multi-marginal constraints while we use a convex payoff of the moments of $\mathbb{P}^h$. This problem is closer in spirit to the "weak" OT [32] setting or rather the entropic regularization of a weak OT problem.

5 Numerical experiments

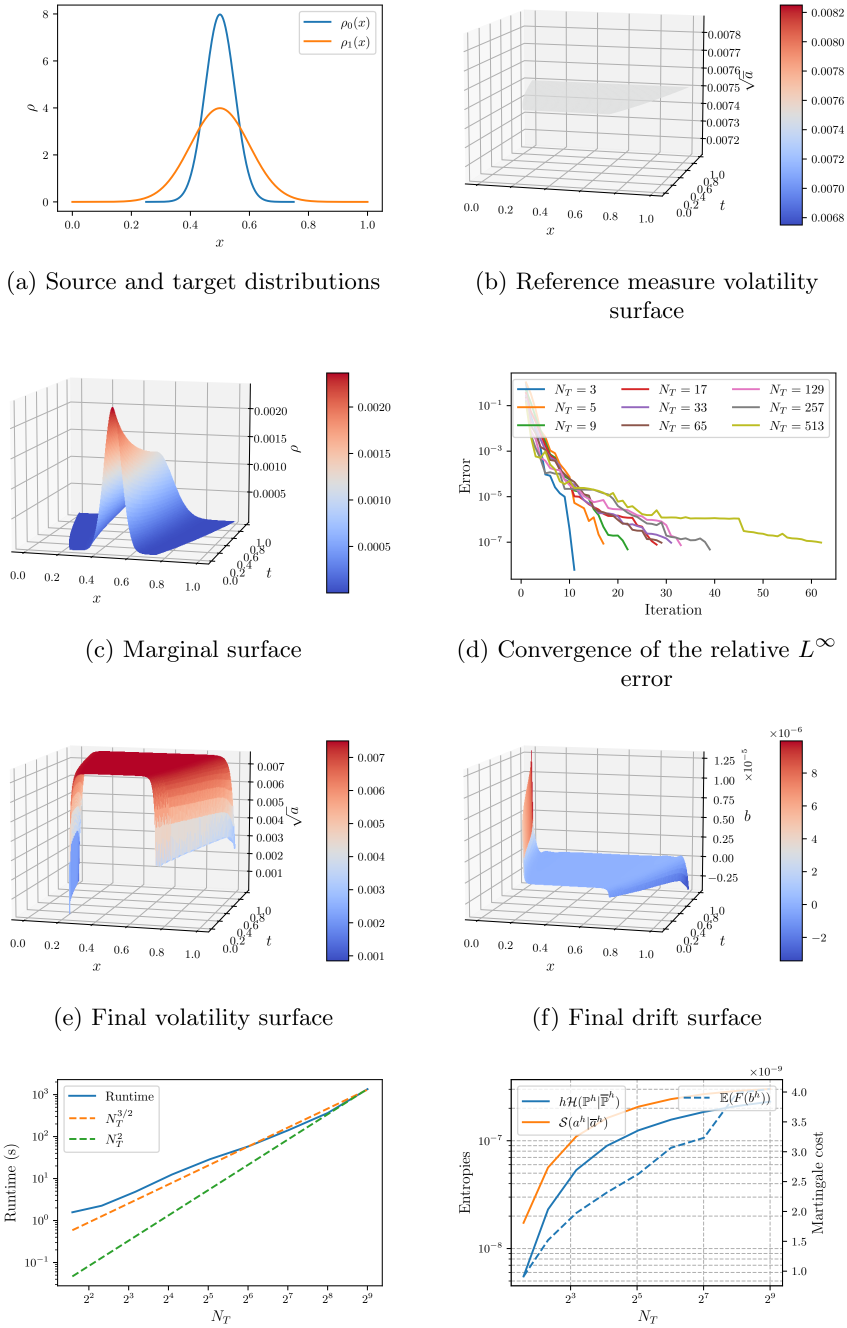
In all experiments we use the function $F = \gamma\|\cdot\|^2$ with a large parameter γ (typically $1.E06$), this is to enforce a soft martingale constraint (no drift). The space discretisation is chosen as $dx = (10\,\overline{\sigma})/K\sqrt{h}$ the parameter K corresponds to the number of "active" points in the reference measure Gaussian kernel (typically $K = 64$). Note that, we extend the test cases to reference measures $\overline{\mathbb{P}}^h$ with variable in time and space volatility $\overline{\sigma}_i = \sqrt{\overline{a}_i}$. The finest computed discretisation in time corresponds to $1/h = N_t = 513$ but we will also provide convergence curves in N_t.

We comment below the different test cases providing figures (1-8) in each case for

(a) Initial and final densities: $\rho_{0,1}$.
(b) The reference measure volatility surface which can be "local" $(t,x) \to \overline{\sigma}(t,x)$.
(c) The surface corresponding to the curve in time of optimal marginals $i \to \mathbb{P}_i^h$, for the finest time step h.
(d) The convergence curves (in the number of interations) of Sinkhorn. It monitors the relative difference $\frac{\|\Phi^{k+1}-\Phi^k\|_\infty}{\|\Phi^k\|_\infty}$ between the dual potentials iterates.
(e) The surface corresponding to the curve in time of optimal volatility $i \to \sqrt{a_i^h}$, for the finest h.
(f) The surface corresponding to the curve in time of optimal drift $i \to b_i^h$, for the finest h. Remember that we are in dimension $d = 1$.
(g) A curve showing the runtime versus $N_t = 1/h$, all experiments confirm the $O(N_t^{3/2})$ optimal cost.
(h) A plot showing 1) the components of the cost function, the kinetic part labelled $\mathbb{E}_{\mathbb{P}^h}(F(b^h))$ and the entropic part labelled $h\,\mathcal{H}(\mathbb{P}^h\,\overline{\mathbb{P}}^h)$ versus N_t, note that the scales are different for these plots; 2) under the same scale as $h\,\mathcal{H}(\mathbb{P}^h\,\overline{\mathbb{P}}^h)$ the curve of SRE $\mathcal{S}(a^h|\overline{a}^h)$ in order to test numerically Theorem 5, i.e. that we are inded penalising the volatility in the limit $h \searrow 0$.

Figure 1: Gaussian to Gaussian with compatible reference volatility

This is a standard diffusion between two Gaussian measures in convex order. The source and target measures are Gaussian measures in convex order, with mean $\mu = 0.5$ and respective standard deviation $\sigma_0 = 0.05$ and $\sigma_1 = 0.1$. The reference measure is the corresponding diffusion starting from ρ_0 and constant diffusion coefficient $\overline{D} = \sigma_1^2 - \sigma_0^2$.



(a) Source and target distributions

(b) Reference measure volatility surface

(c) Marginal surface

(d) Convergence of the relative L^∞ error

(e) Final volatility surface

(f) Final drift surface

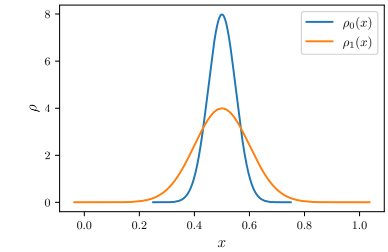
(g) Runtime curve

(h) Convergence of the entropy and specific entropy

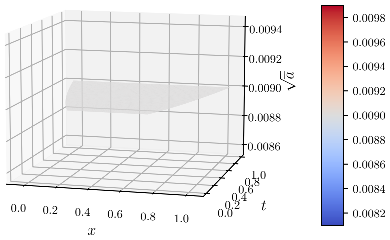
Fig. 1: Gaussian to gaussian with solution reference measure

Figure 2: Gaussian to Gaussian with excess reference volatility

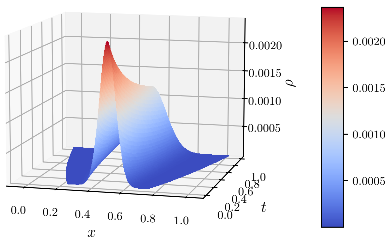
Same setting as figure 1, the volatility of the reference measure is larger, i.e. $\overline{\sigma} = \sqrt{a} > \sqrt{\sigma_1^2 - \sigma_0^2}$.



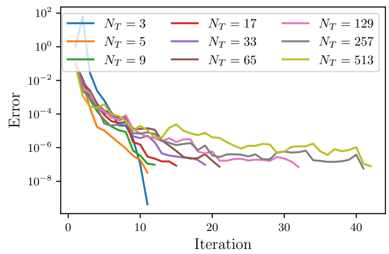
(a) Source and target distributions



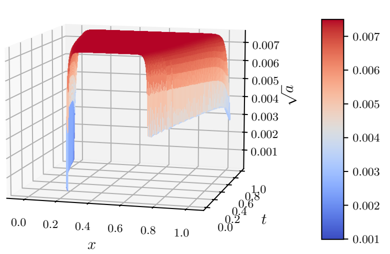
(b) Reference measure volatility surface



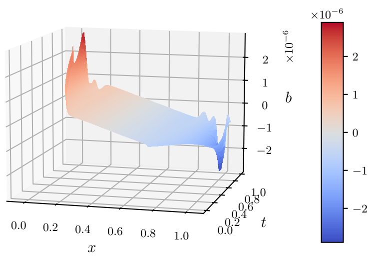
(c) Marginal surface



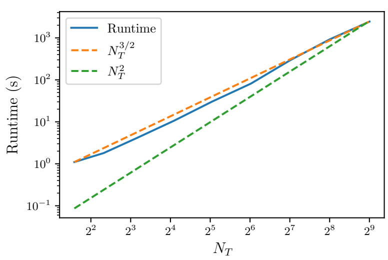
(d) Convergence of the relative L^∞ error



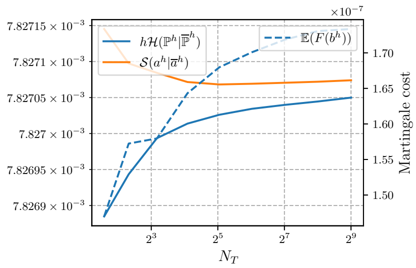
(e) Final volatility surface



(f) Final drift surface



(g) Runtime curve

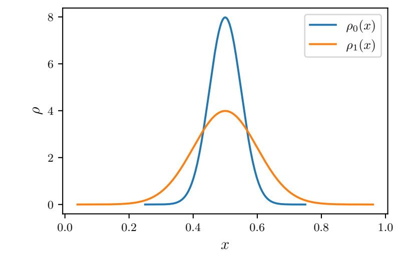


(h) Convergence of the entropy and specific entropy

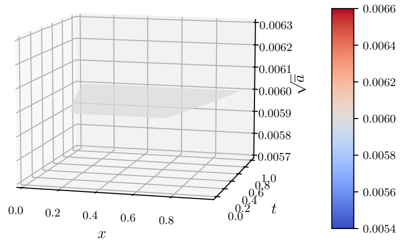
Fig. 2: Gaussian to gaussian with higher reference measure

Figure 3: Gaussian to Gaussian with default volatility

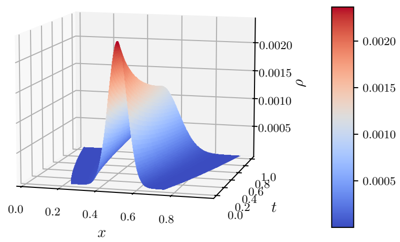
Same setting as figure 1, the volatility of the reference measure is smaller , i.e. $\overline{\sigma} = \sqrt{a} < \sqrt{\sigma_1^2 - \sigma_0^2}$.



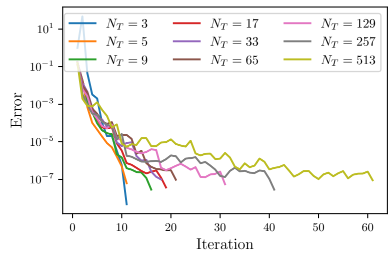
(a) Source and target distributions



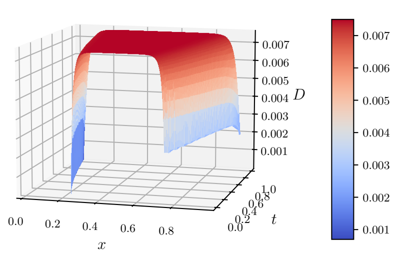
(b) Reference measure volatility surface



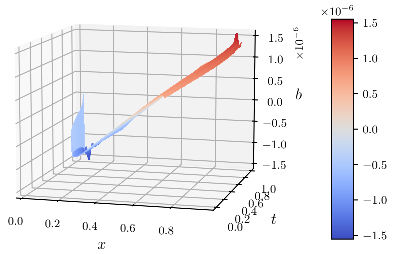
(c) Marginal surface



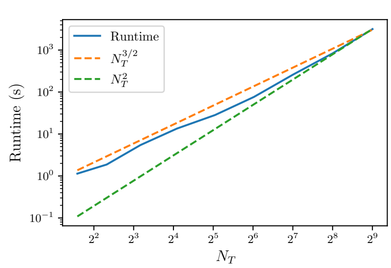
(d) Convergence of the relative L^∞ error



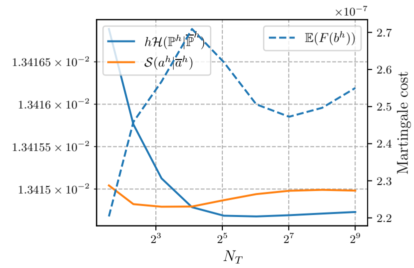
(e) Final volatility surface



(f) Final drift surface



(g) Runtime curve



(h) Convergence of the entropy and specific entropy

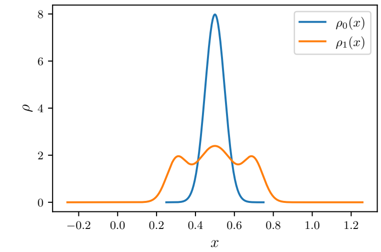
Fig. 3: Gaussian to gaussian with lower reference measure

Figure 4: a Gaussian to a sum of Gaussians

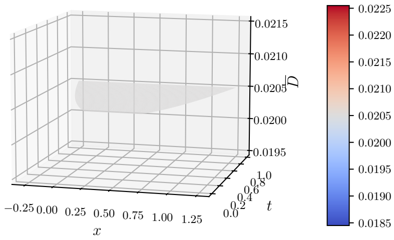
ρ_0 is as in figure 1 and ρ_1 is defined as a sum of three Gaussian measures:

$$\rho_1(x) = p\mathcal{N}(x;\mu_1,\sigma_1) + \frac{1-p}{2}\mathcal{N}(x;\mu_1 - d_1,\sigma_1^{\mathrm{LR}}) + \frac{1-p}{2}\mathcal{N}(x;\mu_1 + d_1,\sigma_1^{\mathrm{LR}})$$

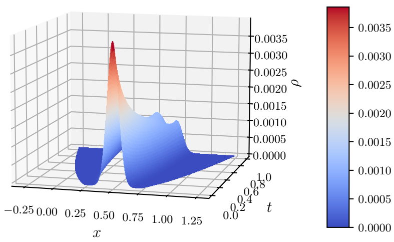
where $p = 0.6$, $\mu_1 = 0.5$, $\sigma_1 = 0.1$, $d_1 = 0.2$ and $\sigma_1^{\mathrm{LR}} = 0.05$. The reference measure is chosen to have a diffusion coefficient $\overline{D} = \mathrm{Var}\,\rho_1 - \mathrm{Var}\,\rho_0$ consistent with the variance increase.



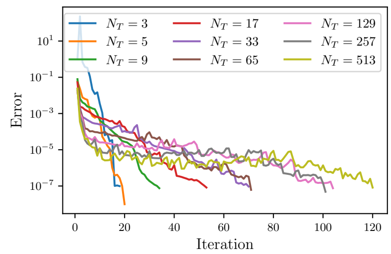
(a) Source and target distributions



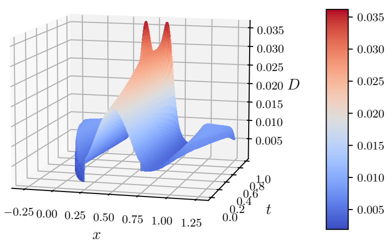
(b) Reference measure volatility surface



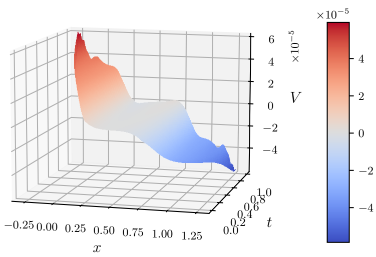
(c) Marginal surface



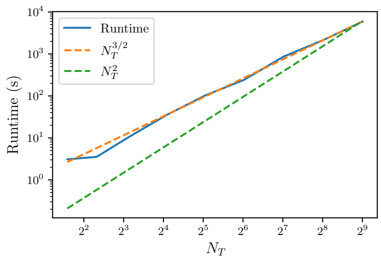
(d) Convergence of the relative L^∞ error



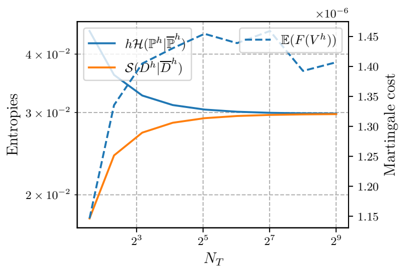
(e) Final volatility surface



(f) Final drift surface



(g) Runtime curve



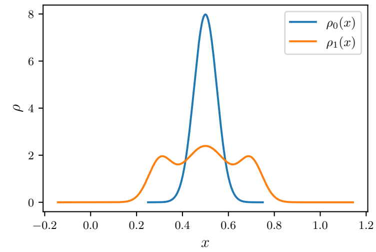
(h) Convergence of the entropy and specific entropy

Fig. 4: a Gaussian to sum of gaussian

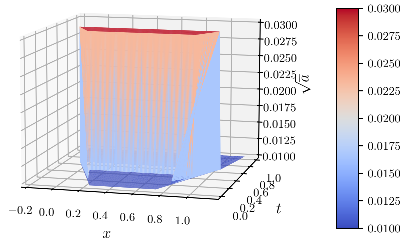
Figure 5: a Gaussian to a sum of Gaussians with a discontinuous in time volatility

The setting is similar to figure 4 except for $\overline{a}$

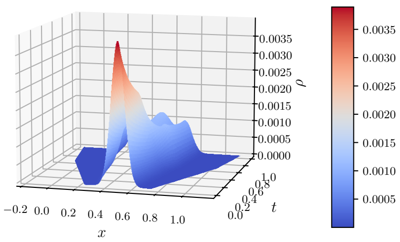
$$\overline{a} = \begin{cases} 0.03 & \text{if } |t - 0.5| \leq 0.2 \\ 0.01 & \text{otherwise.} \end{cases}$$



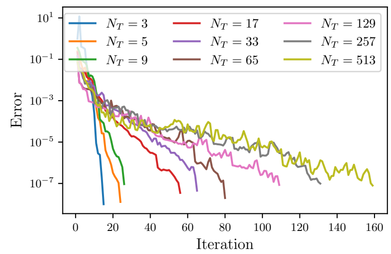
(a) Source and target distributions



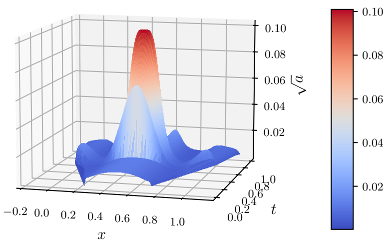
(b) Reference measure volatility surface



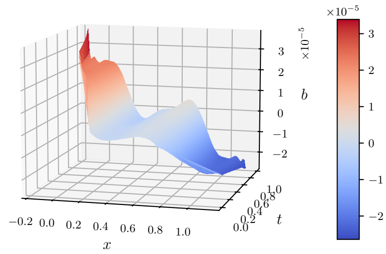
(c) Marginal surface



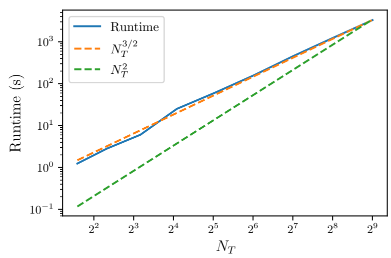
(d) Convergence of the relative L^∞ error



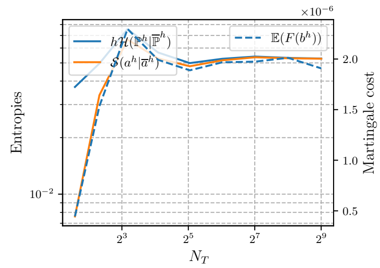
(e) Final volatility surface



(f) Final drift surface



(g) Runtime curve



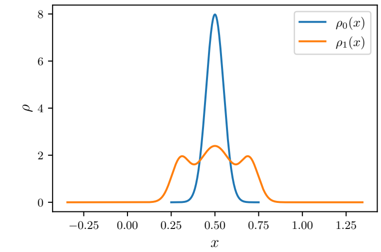
(h) Convergence of the entropy and specific entropy

Fig. 5: a Gaussian to sum of gaussians with a discontinuous in time volatility

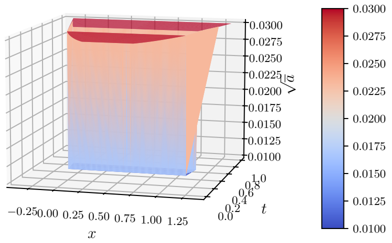
Figure 6: a Gaussian to a sum of Gaussians with a discontinuous in time volatility

The setting is similar to figure 4 except for $\overline{a}$"

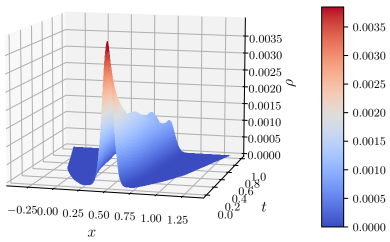
$$\overline{a} = \begin{cases} 0.03 & \text{if } |t - 0.5| \leq 0.2 \\ 0.01 & \text{otherwise.} \end{cases}$$



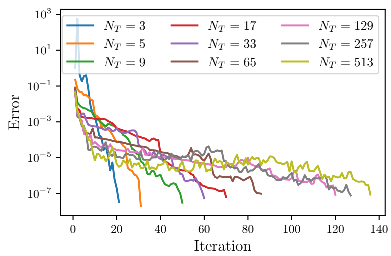
(a) Source and target distributions



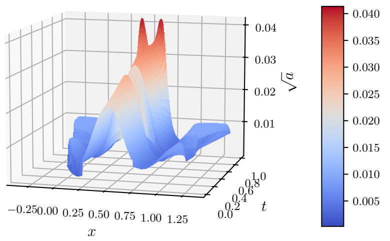
(b) Reference measure volatility surface



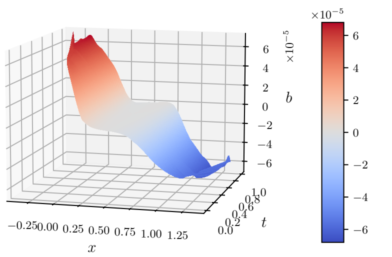
(c) Marginal surface



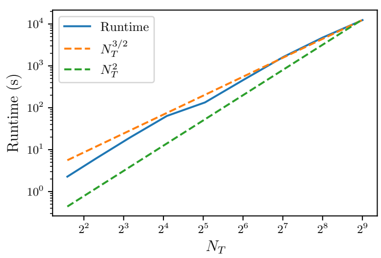
(d) Convergence of the relative L^∞ error



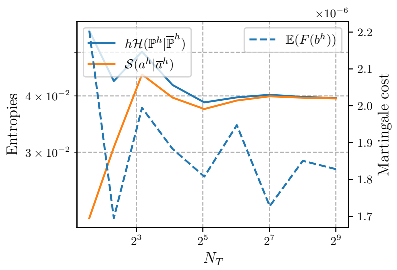
(e) Final volatility surface



(f) Final drift surface



(g) Runtime curve



(h) Convergence of the entropy and specific entropy

Fig. 6: Gaussian to sum of gaussian with discontinuous in time reference volatility

Figure 7: A sum of Gaussians to a sum of Gaussians

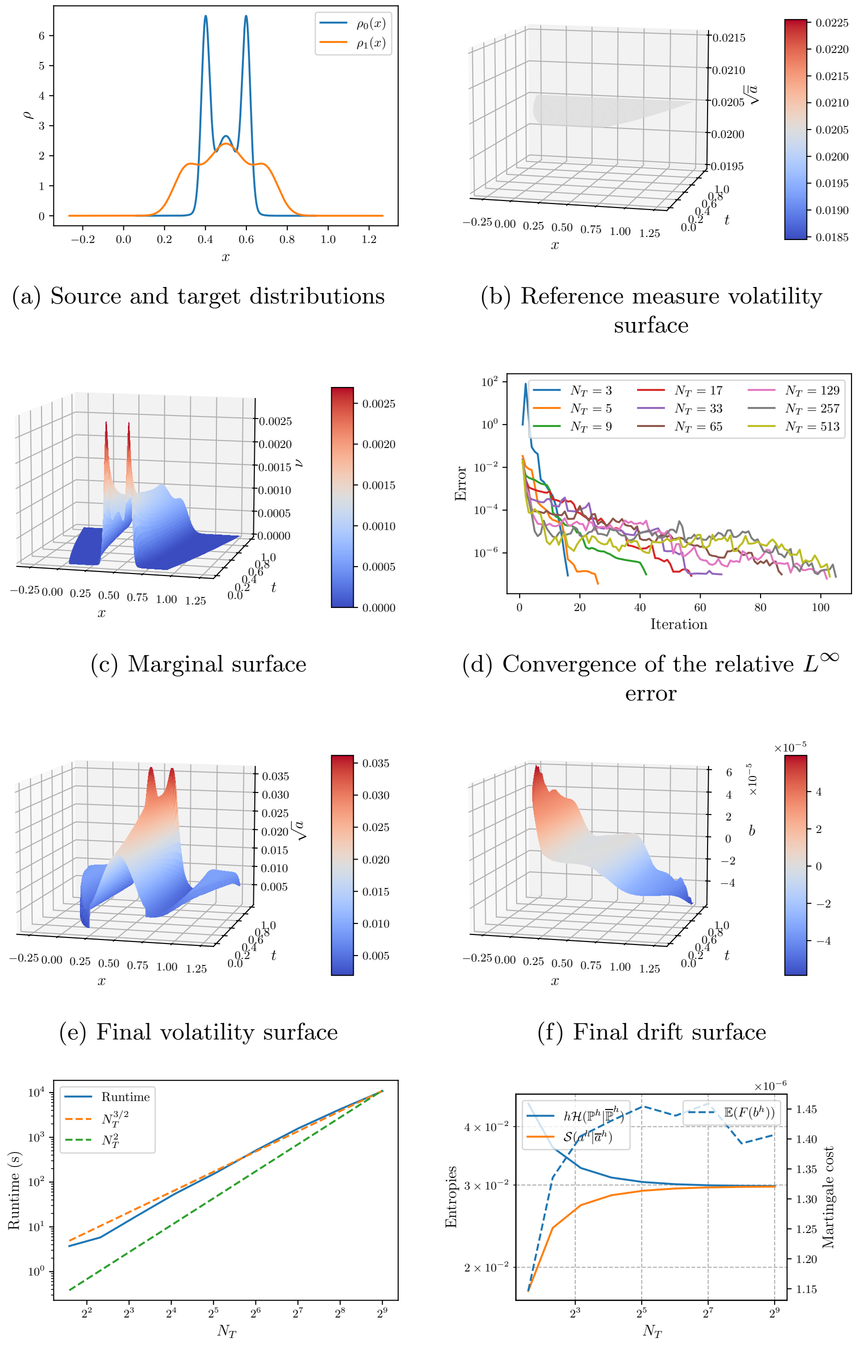
The source and target measures are sums of Gaussians, defined as :

$$\rho_0(x) = p_1\mathcal{N}(x;\mu_0,\sigma_0) + \frac{1-p_1}{2}\mathcal{N}(x;\mu_0-d_0,\sigma_0^{\mathrm{LR}}) + \frac{1-p_1}{2}\mathcal{N}(x;\mu_0+d_0,\sigma_0^{\mathrm{LR}})$$

where $p_1 = 0.4$, $\mu_0 = 0.5$, $\sigma_0 = 0.06$, $d_0 = 0.1$ and $\sigma_1^{\mathrm{LR}} = 0.05$, and

$$\rho_1(x) = p_2\mathcal{N}(x;\mu_1,\sigma_1) + \frac{1-p_2}{2}\mathcal{N}(x;\mu_1-d_1,\sigma_1^{\mathrm{LR}}) + \frac{1-p_2}{2}\mathcal{N}(x;\mu_1+d_1,\sigma_1^{\mathrm{LR}})$$

where $p_2 = 0.6$, $\mu_1 = 0.5$, $\sigma_1 = 0.1$, $d_1 = 0.2$ and $\sigma_1^{\mathrm{LR}} = 0.06$.



(a) Source and target distributions

(b) Reference measure volatility surface

(c) Marginal surface

(d) Convergence of the relative L^∞ error

(e) Final volatility surface

(f) Final drift surface

(g) Runtime curve

(h) Convergence of the entropy and specific entropy

Fig. 7: Sum of gaussian to sum of gaussian

Figure 8: Discontinuous marginals, not in convex order

The source density is defined as :

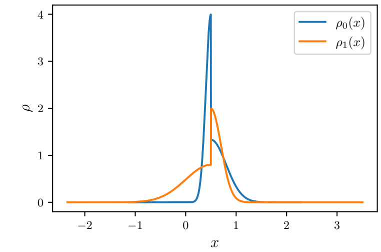
$$\rho_0(x) = \begin{cases} \mathcal{N}(x; \mu, \sigma_{0,1}) & \text{if } x \leq \mu \\ \mathcal{N}(x; \mu, \sigma_{0,2}) & \text{if } x > \mu \end{cases}$$

where $\sigma_{0,1} = 0.1$ and $\sigma_{0,2} = 0.3$, and the density of the final measure is defined as:

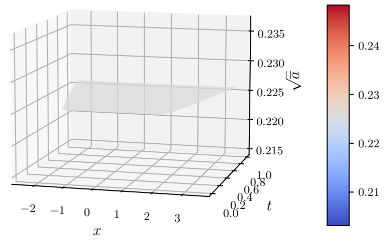
$$\rho_1(x) = \begin{cases} \mathcal{N}(x; \mu, \sigma_{1,1}) & \text{if } x \leq \mu \\ \mathcal{N}(x; \mu, \sigma_{1,2}) & \text{if } x > \mu \end{cases}$$

where $\sigma_{1,1} = 0.5$ and $\sigma_{1,2} = 0.2$, $\mu = 0.5$.

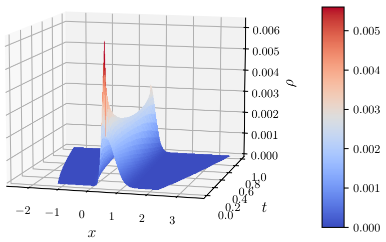
We observe that the penalization of the drift is damped in the domain contributes to the default in convex order making the volatility penalization large.



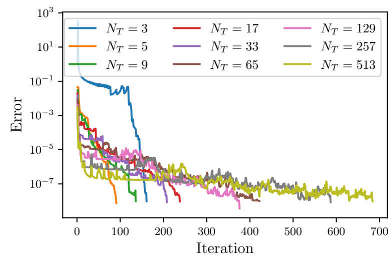
(a) Source and target distributions



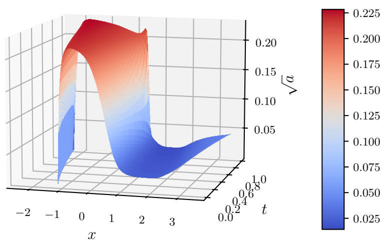
(b) Reference measure volatility surface



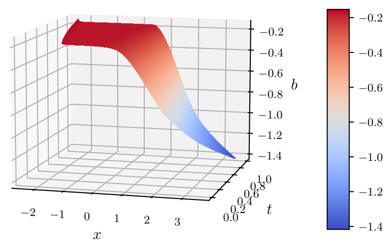
(c) Marginal surface



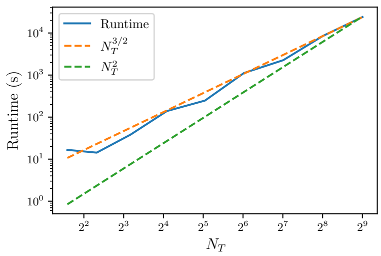
(d) Convergence of the relative L^∞ error



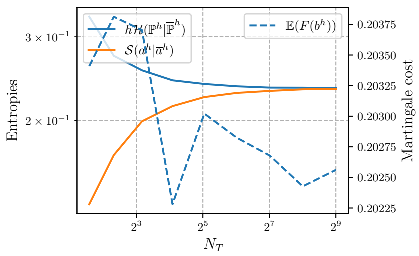
(e) Final volatility surface



(f) Final drift surface



(g) Runtime curve



(h) Convergence of the entropy and specific entropy

Fig. 8: Disontinuous marginals, not in convex order

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6 Annex

6.1 Proof of proposition 2 [Specific Relative Entropy]

Proof of prop. 2 *i)*

We have $X_i^h = X_{ih}$, where $(X_t)_{t\in[0,1]}$ solves $dX_t = b_t(X_t)dt + a_t(X_t)^{1/2}dB_t$.

Step 1: We recall the ingredient we need for small time asymptotics of the heat kernel. Let $p_{a,b}^h(t,x,y)$ denote the transition density associated with the diffusion process

$$dX_t = b(t, X_t)dt + \sigma(t,x)dB_t$$

with values in $\mathbb{R}^d$ for some d-dimensional standard Brownian motion, and $a(t,x) = \sigma(t,x)\sigma(t,x)^\star$. We assume that both functions $(t,x) \mapsto b(t,x)$ and $(t,x) \mapsto a(t,x)$ are twice continuously differentiable, and that the matrix $a(t,x)$ is invertible for every $(t,x) \in [0,1] \times \mathbb{R}^d$. For $x, y \in \mathbb{R}^d$ and $0 \leq t \leq t+h \leq 1$, let

$$p_{a,b}(t,h,x,y)dy = \mathbb{P}(X_{t+h} \in dy \,|\, X_t = x)$$

denote the transition probability associated with the diffusion process with drift b and diffusion matrix a. From [33], the following asymptotic expansion is known: for every compact $U \subset \mathbb{R}^d$, there exists $\tau > 0$ such that for $\|x-y\| \leq \tau$, we have

$$p_{a,b}(t,h,x,y) = \frac{1}{(2\pi h)^{d/2}\det(a(t,x)))^{1/2}} e^{\left(\frac{1}{2h}(y-x)^\star a(t,x)^{-1}(y-x)\right)}\big(1 + hR(t,h,x,y)\big) \tag{38}$$

where

$$\sup_{0\leq t\leq t+h\leq 1, x,y\in U} |R(t,h,x,y)| \leq C$$

and R only depends on $\partial_t^i\partial_x^j\sigma$ and $\partial_t^i\partial_x^j b$, for $0 \leq i+j \leq 2$, computed on x and on f_u, $0 \leq u \leq 1$, where f_u is the unit length geodesic joining x and y for the Riemannian metric associated with the metric tensor $a(t,x)^{-1}$.

Step 2: We separate the basic Kullback-Leibler divergence into a local and a global part. Define

$$q_a(t,h,x,y) = \frac{1}{(2\pi h)^{d/2}\det(a(t,x))^{1/2}}\exp\left(-\frac{1}{2h}(y-x)^\star a(t,x)^{-1}(y-x)\right).$$

From (38), a first-order expansion in h yields

$$\log\left(\frac{p_{a,b}(t,h,x,y)}{p_{\bar{a},0}(t,h,x,y)}\right) = \log\left(\frac{q_a(t,h,x,y)}{q_{\bar{a}}(t,h,x,y)}\right) + h\widetilde{R}(t,h,x,y)$$

where $\widetilde{R}$ has the same properties as R with dependence upon a, b and $\bar{a}$, *i.e.* it is valid for $\|x-y\| \leq \tau$. For $\|x-y\| \geq \tau$, we always have the Aaronson's type estimate [34] that reads

$$c_- h^{-d/2} e^{-C_- h^{-1}\|x-y\|^2} \leq p_{a,b}(t,h,x,y) \leq c_+ h^{-d/2} e^{-C_+ h^{-1}\|x-y\|^2} \tag{39}$$

where $c_\pm, C_\pm$ depend on $\inf_{t,x} \|a(t,x)\| > 0$ and $\sup_{t,x}(\|b(t,x)\|, \|a(t,x)\|)$, thus requiring uniform boundedness. Thus

$$\log\left(\frac{p_{a,b}(t,h,x,y)}{p_{\bar{a},0}(t,h,x,y)}\right) = \left(\log\left(\frac{q_a(t,h,x,y)}{q_{\bar{a}}(t,h,x,y)}\right) + h\widetilde{R}(t,h,x,y)\right)\mathbf{1}_{\|x-y\|\le\eta} + V_\eta(t,h,x,y),$$

where

$$V_\eta(t,h,x,y) = \log\left(\frac{p_{a,b}(t,h,x,y)}{p_{\bar{a},0}(t,h,x,y)}\right)\mathbf{1}_{\|x-y\|\ge\eta}.$$

The estimate (39) yields

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[V_\eta(ih,h,X_{ih},X_{(i+1)h})\right]\\
&\le \frac{c^+_{a,b}}{c^-_{\bar{a},0}}\mathbb{P}(\|X_{(i+1)h}-X_{ih}\|\ge\eta) - (C^+_{a,b}-C^-_{a,0})h^{-1}\mathbb{E}_{\mathbb{P}}[\|X_{(i+1)h}-X_{ih}\|^2\mathbf{1}_{\|X_{(i+1)h}-X_{ih}\|\ge\eta}]\\
&\le \big(\frac{c^+_{a,b}}{c^-_{\bar{a},0}} - (C^+_{a,b}-C^-_{a,0})h^{-1}\mathbb{E}_{\mathbb{P}}[\|X_{(i+1)h}-X_{ih}\|^4]^{1/2}\big)\eta^{-1/2}\mathbb{E}\big[\|X_{(i+1)h}-X_{ih}\|\big]^{1/2}\\
&= O(h^{1/4}).
\end{aligned}$$

Likewise,

$$\mathbb{E}_{\mathbb{P}}\big[h\widetilde{R}(ih,h,X_{ih},X_{(i+1)h})\big)\mathbf{1}_{\|X_{(i+1)h}-X_{ih}\|\le\eta}\big] \le Ch,$$

so that

$$\mathbb{E}_{\mathbb{P}}\big[\log\left(\frac{p_{a,b}(ih,h,X_{ih},X_{(i+1)h})}{p_{\bar{a},0}(ih,h,X_{ih},X_{(i+1)h})}\big]\right)\big] = \mathbb{E}_{\mathbb{P}}\big[\log\left(\frac{q_{a,b}(ih,h,X_{ih},X_{(i+1)h})}{q_{\bar{a},0}(ih,h,X_{(i+1)h},X_{ih})}\right)\big] + O(h^{1/4}) \tag{40}$$

and therefore

$$\begin{aligned}
\mathcal{H}(\mathbb{P}^h\,|\,\overline{\mathbb{P}}^h) &= \mathbb{E}_{\mathbb{P}}\Big[\log\left(\prod_{i=1}^N \frac{p_{a,b}(ih,h,X_{ih},X_{(i+1)h})}{p_{\bar{a},0}(ih,h,X_{ih},X_{(i+1)h})}\right)\Big]\\
&= \sum_{i=1}^N \mathbb{E}_{\mathbb{P}}\Big[\log\left(\frac{q_a(ih,h,X_{ih},X_{(i+1)h})}{q_{\bar{a}}(ih,h,X_{ih},X_{(i+1)h})}\right)\Big] + O(Nh^{1/4})
\end{aligned}$$

according to (40).

Step 3. Control of the entropy of the proxy transitions. We have

$$\begin{aligned}
&\log\left(\frac{q_a(ih,h,X_{ih},X_{(i+1)h})}{q_{\bar{a}}(ih,h,X_{ih},X_{(i+1)h})}\right) = -\tfrac{1}{2}\log\left(\frac{\det(a(ih,X_{ih}))}{\det(\bar{a}(ih,X_{ih}))}\right)\\
&- \tfrac{1}{2}h^{-1}(X_{(i+1)h}-X_{ih})^\star(a(ih,X_{ih})^{-1}-\bar{a}(ih,X_{ih})^{-1})(X_{(i+1)h}-X_{ih}).
\end{aligned}$$

Next, we have

$$
\begin{aligned}
&\tfrac{1}{2}h^{-1}(X_{(i+1)h}-X_{ih})^\star(a(ih,X_{ih})^{-1}-\bar a(ih,X_{ih})^{-1})(X_{(i+1)h}-X_{ih})\\
&=-\tfrac{1}{2}h^{-1}\big(\int_{ih}^{(i+1)h}\sigma(s,X_s)dB_s\big)^\star(a(ih,X_{ih})^{-1}-\bar a(ih,X_{ih})^{-1})\big(\int_{ih}^{(i+1)h}\sigma(s,X_s)dB_s\big)+\zeta_i^h,
\end{aligned}
$$

where

$$
\begin{aligned}
\zeta_i^h&=h^{-1}\big(\int_{ih}^{(i+1)h}b(s,X_s)ds\big)^\star(a(ih,X_{ih})^{-1}-\bar a(ih,X_{ih})^{-1})\big(\int_{ih}^{(i+1)h}b(s,X_s)ds\big)\\
&+h^{-1}\big(\int_{ih}^{(i+1)h}\sigma(s,X_s)dB_s\big)^\star(a(ih,X_{ih})^{-1}-\bar a(ih,X_{ih})^{-1})\big(\int_{ih}^{(i+1)h}b(s,X_s)ds\big)\\
&+h^{-1}\big(\int_{ih}^{(i+1)h}b(s,X_s)ds\big)^\star(a(ih,X_{ih})^{-1}-\bar a(ih,X_{ih})^{-1})\big(\int_{ih}^{(i+1)h}\sigma(s,X_s)dB_s\big)
\end{aligned}
$$

The boundedness of b and σ (and its inverse) readily yields

$$\max_{1\le i\le N}\mathbb{E}_{\mathbb{P}}[\|\zeta_i^h\|]=O(h^{1/2})$$

by applying Cauchy-Schwarz inequality repeatedly. Also, writing

$$h^{-1/2}\int_{ih}^{(i+1)h}\sigma(s,X_s)dB_s=\sigma(ih,X_{ih})+\xi_i^h,$$

where

$$\xi_i^h=h^{-1/2}\int_{ih}^{(i+1)h}(\sigma(s,X_s)-\sigma(ih,X_{ih}))dB_s$$

satisfies

$$\max_{1\le i\le N}\mathbb{E}_{\mathbb{P}}[\|\xi_i^h\|]=O(h^{1/2})$$

thanks to the smoothness of $(t,x)\mapsto\sigma(t,x)$ (at least Lipschitz in both variables) and standard moment estimates, we obtain

$$
\begin{aligned}
&-\tfrac{1}{2}h^{-1}\big(\int_{ih}^{(i+1)h}\sigma(s,X_s)dB_s\big)^\star(a(ih,X_{ih})^{-1}-\bar a(ih,X_{ih})^{-1})\big(\int_{ih}^{(i+1)h}\sigma(s,X_s)dB_s\big)\\
&=\tfrac{1}{2}\mathrm{Tr}\big(\bar a(ih,X_{ih})^{-1}(a(ih,X_{ih})-\bar a(ih,X_{ih})))\big)+\widetilde{\xi}_i^h,
\end{aligned}
$$

where now, in the same way as for the remainder term ξ_i^h, we readily have

$$\max_{1\le i\le N}\mathbb{E}_{\mathbb{P}}[|\widetilde{\xi}_i^h|]=O(h^{1/2}).$$

Putting together all our estimates, we obtain

$$\log\left(\frac{q_a(ih,h,X_{ih},X_{(i+1)h})}{q_{\bar a}(ih,h,X_{ih},X_{(i+1)h})}\right) = -\tfrac{1}{2}\log\left(\frac{\det(a(ih,X_{ih}))}{\det(\bar a(ih,X_{ih}))}\right)$$
$$+\tfrac{1}{2}\mathrm{Tr}\big(\bar a(ih,X_{ih})^{-1}(a(ih,X_{ih})-\bar a(ih,X_{ih})))\big)+\widetilde{\zeta}_i^h,$$

for some stochastic remainder term $\widetilde{\zeta}_i^h$ that satisfies the moment estimate

$$\max_{1\le i\le N}\mathbb{E}_{\mathbb{P}}[|\widetilde{\zeta}_i^h|]=O(h^{1/2}).$$

By definition of the relative entropy $\mathcal{S}^{\mathcal{I}}$, we thus obtain

$$\mathcal{H}(\mathbb{P}^h\,|\,\overline{\mathbb{P}}^h)=\frac{1}{2}\sum_{i=1}^{N}\mathbb{E}_{\mathbb{P}}\big[\mathcal{S}^{\mathcal{I}}(a(ih,X_{ih})|\bar a(ih,X_{ih})\big]+O(Nh^{1/4}).$$

Finally, using that $h\sim N^{-1}$, we obtain

$$h\,\mathcal{H}(\mathbb{P}^h\,|\,\overline{\mathbb{P}}^h)=\frac{1}{2}\sum_{i=1}^{N}h\mathbb{E}_{\mathbb{P}}\big[\mathcal{S}^{\mathcal{I}}(a(ih,X_{ih})|\bar a(ih,X_{ih})\big]+O(h^{1/4}) \tag{41}$$

and

$$\lim_{h\to 0}h\mathcal{H}(\mathbb{P}^h\,|\,\overline{\mathbb{P}}^h)=\mathbb{E}_{\mathbb{P}}\Big(\int_0^1\mathcal{S}^{\mathcal{I}}(a(s,X_s)|\bar a(s,X_s)ds\Big)$$

by Riemann approximation, which is the desired result. More specifically, one readily checks that

$$\int_{ih}^{(i+1)h}\mathcal{S}^{\mathcal{I}}(a(s,X_s)|\bar a(s,X_s)ds=h\mathcal{S}^{\mathcal{I}}(a(ih,X_{ih})|\bar a(ih,X_{ih})+\rho_i^h,$$

where $\max_{1\le i\le N}\mathbb{E}[|\rho_i^h|]=O(h^{1/2})$ using that both $\mathcal{S}^{\mathcal{I}}$ and a, $\bar a$ are smooth functions.

Proof of Proposition 2 *ii)* Since $\mathbb{P}^h$ is Markov we rewrite the relative Entropy using the transition probabilities

$$\mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h)=\mathcal{H}(\mathbb{P}_0|\rho_0)+\sum_{i=0}^{N-1}\mathbb{E}_{\mathbb{P}_i^h}(\mathcal{H}(\mathbb{P}^h_{i\to i+1}|\overline{\mathbb{P}}^h_{i\to i+1}))\,.$$

It is therefore sufficient to focus on one transition and show

$$\mathcal{H}(\mathbb{P}^h_{i\to i+1}|\overline{\mathbb{P}}^h_{i\to i+1})\ge\frac{1}{2}\mathcal{S}^{\mathcal{I}}(a_i^h(X_i^h)|\bar a)\,.$$

Let us simplify the notations and look at $\mathcal{H}(\mathbb{P}_x|\overline{\mathbb{P}}_x)$ where $\mathbb{P}_x$ is a probability measure over a set $Y=\mathbb{R}_{t_{i+1}}$ parameterized by $x\in X=\mathbb{R}_{t_i}$ and $\overline{\mathbb{P}}_x$ a probability measure over

Y with normal law $\Gamma_{x\to y}(0,\overline{a})$. This inequality is a consequence of the well-known fact that the probability that minimizes the entropy under constrained second moments is a Gaussian measure. In the following, we prove it using the dual formulation of entropy, recalled hereafter, see [35, lemma 9.4.4.]

$$\begin{cases} \mathcal{H}(\mathbb{P}_x|\overline{\mathbb{P}}_x) = \sup_{f\,\in\, C_b^0(Y)} \beta(f) \\ \beta(f) := \int f(y)\, d\mathbb{P}_x(y) - \int \exp\left(f(y)\right) - 1\, d\overline{\mathbb{P}}_x(y)\,. \end{cases} \tag{42}$$

We plan on restricting the test functions f to quadratic functions

$$y \to p_{x,C,A}(y) = C + A\,(y-x)^2$$

which are not bounded functions. To do so, we remark that the truncation

$$\begin{cases} p_{x,C,A,R}(y) & = p_{x,C,A}(y) \text{ if } |y-x| \le R \\ & = C + A\,R^2 \text{ if } |y-x| \ge R\,, \end{cases} \tag{43}$$

is a pointwise increasing (in the variable R) sequence of functions if $A \ge 0$ and it is also the case for $\exp\left(p_{x,C,A,R}\right)$. It is a pointwise decreasing sequence of functions if $A \le 0$ and it is also the case for $\exp\left(p_{x,C,A,R}\right)$. As a consequence, in both cases, one can apply the monotone convergence theorem to obtain

$$\beta(p_{x,C,A}) = \lim_{R\to\infty} \beta(p_{x,C,A,R}) \le \mathcal{H}(\mathbb{P}_x|\overline{\mathbb{P}}_x)\,. \tag{44}$$

We can now directly optimize on the family of quadratic functions. plugging $f := C + A\,(y-x)^2$ in the integrals we get

$$\beta(C + A\,\frac{(y-x)^2}{2}) = C + h\,\frac{D(x)}{2}\,A - \int \exp\left(C + A\,\frac{(y-x)^2}{2}\right) - 1\, d\overline{\mathbb{P}}_x(y)$$

$(C,A) \to \beta(C + A\,(y-x)^2)$ is strictly concave. The optimality in C gives

$$\begin{aligned} 1 &= \int \exp\left(C + A\,(y-x)^2\right)\, d\overline{\mathbb{P}}_x(y) \\ &= (2\pi h \overline{a})^{-\frac{1}{2}} \exp\left(C\right) \int \exp\left((A - \frac{1}{h\,\overline{a}})\,\frac{(y-x)^2}{2}\right)\, dy\,. \end{aligned}$$

We identify above a Gaussian probability measure over y with mean x and standard deviation α defined by

$$\alpha^2(A) = (\frac{1}{h\,\overline{a}} - A)^{-1}$$

(for small h and $\overline{a} > 0$ this is always well defined). The normalizing constant gives:

$$(2\pi h\overline{a})^{-\frac{1}{2}} \exp\left(C\right) = (2\pi\alpha^2(A))^{-\frac{1}{2}}$$

We can eliminate C and the function to maximize in A now is (the second integral vanishes)

$$A \to \frac{1}{2}\log\left(1 - h\,\overline{a}\,A\right) + h\,\frac{D(x)}{2}\,A$$

the optimal A satisfies

$$1 - h\,\overline{a}\,A = \frac{\overline{a}}{D} \qquad h\,D\,A = \frac{D(x)}{\overline{a}} - 1\,.$$

Therefore we have

$$\sup_{\begin{cases} (C, A) \in (\mathbb{R} \times \mathbb{R}) \\ f := C + A\,(y - x)^2 \end{cases}} \beta(f) = -\log\left(\frac{D(x)}{\overline{a}}\right) + \frac{D(x)}{\overline{a}} - 1\,,$$

which gives the result using the inequality (44).

6.2 Proof of theorem 3 [Convergence of Markov Chain to diffusion]

Some remarks about space-time transformations in the inhomogeneous case.

Suppose we have a inhomogeneous (family in h) of transition probabilities $\mathbb{P}^h_{i\to i+1}(x, dy)$ on $\mathbb{R}^d$ for $i = 0, \ldots, N-1$. We associate to it a discrete-time Markov chain $X_0, \ldots, X_{N-1}$ that is turned itself into a continuous time process $x(t)$ with $x(ih) = X_i$ and linear interpolation for $t \in [ih, (i+1)h)$ for $i = 0, \ldots, N-1$. The discrete sampling $x(0), x(h), \ldots, x((N-1)h))$ is a Markov sequence with inhomogeneous transition

$$\mathbb{P}(x((i+1)h) \in dy \,|\, x(ih) = x) = \mathbb{P}^h_{i\to i+1}(x, dy).$$

We next consider an equivalent homogeneous Markov process/chain via a simple time/space transformation. Define the deterministic process over the integers $N(i) = i$ extended into a continuous time (deterministic) process

$$n(ih) = N(i)h = ih$$

over the times of the form ih and with (trivial) linear interpolation so that simply $n(t) = t$. Define the new process

$$y(t) = (n(t), x(t)).$$

We claim that y_t is a homogeneous Markov sequence on the times of the form ih with state space $[0,1] \times \mathbb{R}^d$ (actually $\{kh, k = 0, \ldots, N-1\} \times \mathbb{R}^d$ but we embed everything into $[0,1] \times \mathbb{R}^d$). Indeed

$$\mathbb{P}(y((i+1)h) \in dvdy \,|\, y(ih) = (u, x))$$

$$
\begin{aligned}
&= \mathbb{P}(n((i+1)h) \in dv, x((i+1)h) \in dy \,|\, n(ih) = u, x(ih) = x)) \\
&= \delta_{u+h}(dv)\mathbb{P}^h_{\lfloor uh^{-1}\rfloor \to \lfloor uh^{-1}\rfloor+1}(x, dy) \\
&=: \Pi^h((u,x), dvdy),
\end{aligned}
$$

and the last expression does not depend on i hence the homogeneity. With the same notation as Stroock-Varadhan, this defines the family of (homogeneous) transitions over which we are going to apply the standard homogeneous Stroock-Varadhan result. It suffices to show that our condition (14) entails the three classical conditions of Stroock-Varadhan, i.e. conditions (2.4)-(2.5)-(2.6) p. 268.

Following the definition given at the end of p. 267 in Stroock-Varadhan, we have semi-explicit expressions for the drift and diffusion coefficients, (with $x \in \mathbb{R}^d$ that becomes $(u,x) \in [0,1] \times \mathbb{R}^d$) namely

$$
\begin{aligned}
b_h((u,x)) &= \tfrac{1}{h} \int_{|(v,y)-(u,x)| \le 1} ((v,y)-(u,x))\Pi^h((u,x), dvdy) \\
&= \tfrac{1}{h} \begin{cases} \displaystyle\int_{|y-x|\le 1} \int_{[0,1]} (v-u)\delta_{u+h}(dv)\mathbb{P}^h_{\lfloor uh^{-1}\rfloor \to \lfloor uh^{-1}\rfloor+1}(x, dy) \\ \displaystyle\int_{|y-x|\le 1} (y-x) \int_{[0,1]} \delta_{u+h}(dv)\mathbb{P}^h_{\lfloor uh^{-1}\rfloor \to \lfloor uh^{-1}\rfloor+1}(x, dy) \end{cases} \\
&= \begin{cases} \mathbb{P}^h_{\lfloor uh^{-1}\rfloor \to \lfloor uh^{-1}\rfloor+1}(x, \mathcal{B}_{\mathbb{R}^d}(x,1)) \\ \tfrac{1}{h} \displaystyle\int_{|y-x|\le 1} (y-x)\mathbb{P}^h_{\lfloor uh^{-1}\rfloor \to \lfloor uh^{-1}\rfloor+1}(x, dy) \end{cases}
\end{aligned}
$$

and conditions $i)$ and $iv)$ of (14) entail the convergence, for every $R > 0$:

$$
\lim_{h\to 0} \sup_{u\in[0,1], |x|\le R} \|b_h(u,x) - (p_u(x), \tilde{b}^0_u(x))\| = 0
$$

which is (2.4) of Stroock-Varadhan with the state variable $(u,x) \in [0, 1 \times \mathbb{R}^d$ instead of $x \in \mathbb{R}^d$. Likewise, with a slight abuse of notation for the three components of the 2×2 symmetric matrix that defines the diffusion matrix, we have

$$
a_h((u,x)) = \tfrac{1}{h} \begin{cases} \displaystyle\int_{|y-x|\le 1} \int_{[0,1]} (v-u)^2\delta_{u+h}(dv)\mathbb{P}^h_{\lfloor uh^{-1}\rfloor \to \lfloor uh^{-1}\rfloor+1}(x, dy) \\ \displaystyle\int_{|y-x|\le 1} (y-x)^2 \int_{[0,1]} \delta_{u+h}(dv)\mathbb{P}^h_{\lfloor uh^{-1}\rfloor \to \lfloor uh^{-1}\rfloor+1}(x, dy) \\ \displaystyle\int_{|y-x|\le 1} (y-x) \int_{[0,1]} (v-u)\delta_{u+h}(dv)\mathbb{P}^h_{\lfloor uh^{-1}\rfloor \to \lfloor uh^{-1}\rfloor+1}(x, dy) \end{cases}
$$

$$= \begin{cases} h\mathbb{P}^h_{\lfloor uh^{-1}\rfloor \to \lfloor uh^{-1}\rfloor+1}(x, \mathcal{B}_{\mathbb{R}^d}(x,1)) \\ \frac{1}{h}\int_{|y-x|\le 1} (y-x)^2 \mathbb{P}^h_{\lfloor uh^{-1}\rfloor \to \lfloor uh^{-1}\rfloor+1}(x,dy) \\ \int_{|y-x|\le 1} (y-x) \mathbb{P}^h_{\lfloor uh^{-1}\rfloor \to \lfloor uh^{-1}\rfloor+1}(x,dy) \end{cases}$$

$$\to_{h\to 0} \begin{cases} 0 \\ \frac{1}{h}\int_{|y-x|\le 1} (y-x)^2 \mathbb{P}^h_{\lfloor uh^{-1}\rfloor \to \lfloor uh^{-1}\rfloor+1}(x,dy) \\ 0 \end{cases}$$

and conditions ii) of (14) entail the convergence

$$\lim_{h\to 0} \sup_{u\in[0,1],|x|\le R)} \|a_h(u,x) - \begin{pmatrix} \tilde{a}^0(x) & 0 \\ 0 & 0 \end{pmatrix}\| = 0$$

which is (2.5) of Stroock-Varadhan. Finally, we check (2.6) of Stroock-Varadhan, namely for $\varepsilon > 0$ given

$$\lim_{h\to 0} \sup_{|x|\le R, u\in[0,1]} \Delta^\varepsilon_h((u,x),[0,1]\times\mathbb{R}^d \setminus \mathcal{B}_{[0,T]\times\mathbb{R}^d}((u,x),\varepsilon)) = 0. \tag{45}$$

Setting $\mathcal{B}^c(u,x,\varepsilon) = [0,1]\times\mathbb{R}^d \setminus \mathcal{B}_{[0,T]\times\mathbb{R}^d}((u,x),\varepsilon)$, we have

$$\begin{aligned} \Delta^\varepsilon_h((u,x),\mathcal{B}^c(u,x,\varepsilon)) &= \tfrac{1}{h}\Pi^h((u,x),\mathcal{B}^c(u,x,\varepsilon)) \\ &= \tfrac{1}{h}\delta_{u+h}([\varepsilon,1])\mathbb{P}_{\lfloor uh^{-1}\rfloor \to \lfloor uh^{-1}\rfloor+1}(x,\mathbb{R}^d \setminus \mathcal{B}_{\mathbb{R}^d}(x,\varepsilon)) \\ &\le \tfrac{1}{h}\sup_{x\in\mathbb{R}^d}\max_{0\le i\le N-1} \mathbb{P}_{i\to i+1}(x,\mathbb{R}^d \setminus \mathcal{B}_{\mathbb{R}^d}(x,\varepsilon)) \\ &\le \tfrac{1}{h}\sup_{x\in\mathbb{R}^d}\max_{0\le i\le N-1} \mathbb{E}_{\mathbb{P}_{i,i+1}}[|X_{i+1}-X_i|^{4+2\,\alpha}|X_i = x] \\ &\le h^\alpha \varepsilon^{-4-2\,\alpha} \sup_{x\in\mathbb{R}^d}\max_{0\le i\le N-1} hc^h_i(x)\,. \end{aligned}$$

Giving (45) using the definition of c^h_i (13) and the Kurtosis bound constraint in ($\mathcal{V}^h$) which enforces the uniform boundedness of $hc^h_i(x)$.

6.3 Proof of Theorem 4 [Well posedness for the discrete problem]

We give hereafter the proof of the existence of a solution to ($\mathcal{V}^h$). This solution is unique due to the entropy, in contrast to the continuous problem.

Proof. The cost functional in ($\mathcal{V}^h$) is (16):

$$\mathcal{I}^h(\mathbb{P}^h) \coloneqq h\sum_{i=0}^{N-1}\mathbb{E}_{\mathbb{P}_i^h}\left(F(b_i^h(X_i^h),a_i^h(X_i^h))\right) + h\,\mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h) + \mathcal{D}(\mathbb{P}_0^h,\rho_0) + \mathcal{D}(\mathbb{P}_1^h,\rho_1)$$

with (b_i^h, a_i^h), the discrete drift and quadratic variation increments defined in (10). The strict convexity and lower semi-continuity of $\mathcal{I}^h$ are not immediately seen but obtained, as often in optimal transport, using a linear change of variable in the conditional moments $(b_i^h(X_i^h), a_i^h(X_i^h))$ uncovering the composition of the convex perspective function associated to F and a linear operator. Using (10) and (1), implifying and abusing notations, for $\beta = 1, 2, ...$:

$$\begin{aligned}
\mathbb{P}^h &\to \mathbb{E}_{\mathbb{P}_i^h}\left(F(\frac{1}{h}\mathbb{E}_{\mathbb{P}^h_{i\to i+1}}\left((X_{i+1}^h - X_i^h)^\beta\right))\right) \text{ is decomposed as} \\
\mathbb{P}^h &\to (\mathbb{P}_i^h(x_i), \frac{1}{h}\mathbb{E}_{\mathbb{P}^h_{i,i+1}(x_i,.)}\left((X_{i+1}^h - x_i)^\beta\right))\,,\ \forall x_i \\
&\to \mathbb{E}_{\mathbb{P}_i^h}\left(F(\frac{1}{h}\frac{\mathbb{E}_{\mathbb{P}^h_{i,i+1}(X_i^h,.)}\left((X_{i+1}^h - X_i^h)^\beta\right)}{\mathbb{P}_i^h(X_i^h)})\right).
\end{aligned} \tag{46}$$

where $\mathbb{P}^h_{i,i+1}(x_i,.)$ is to be understood as the measure on $\mathbb{R}_{t_{i+1}}$ obtained by freezing the first variable x_i in the joint $\mathbb{R}_{t_i}\times\mathbb{R}_{t_{i+1}}$ probability $\mathbb{P}^h_{i,i+1}$.

The optimization problem

$$\inf_{\mathbb{P}^h \in \mathcal{P}(\bigotimes_{i=0}^N \mathbb{R}_{t_i})} \mathcal{I}^h(\mathbb{P}^h)$$

therefore has a unique minimizer (the relative entropy is strictly convex). The Markovianity of the minimiser a consequence of the structure of $\mathcal{I}^h$ and the additivity properties of the relative entropy giving

$$\mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h) \geq \textstyle\sum_{i=0}^{N-1}\mathbb{E}_{\mathbb{P}_i^h}\left(\mathcal{H}(\mathbb{P}^h_{i\to i+1}|\overline{\mathbb{P}}^h_{i\to i+1})\right) \text{ with equality if } \mathbb{P}^h \text{ is Markov} \tag{47}$$

(see [11] lemma 3.4). □

6.4 Proof of Lemma 6 [Diffusion Coefficients rescaling]

Scaling. We start with an approximation step by scaling the volatility. Consider a solution $X(t)$ of the continuous problem with a finite cost, then introduce

$$X^{\alpha,\varepsilon}(t) = \sqrt{\alpha}X(t) + \sqrt{\varepsilon}B(t)\,,$$

where $B(t)$ is a d-dimensional Brownian motion, independent of X. As a consequence, $a_t^{\alpha,\varepsilon} = \alpha a_t + \varepsilon \,\mathrm{Id}$. Since $a \in [\lambda \,\mathrm{Id}, \Lambda \,\mathrm{Id}]$, it implies that

$$(\alpha\lambda + \varepsilon)\,\mathrm{Id} \leq a_{\alpha,\varepsilon} \leq (\alpha\Lambda + \varepsilon)\,\mathrm{Id}\ .$$

Now, because $\lambda < \Lambda$, and for δ small enough, a simple computation shows that taking

$$\begin{aligned} \alpha &= 1 - \frac{2\,\varepsilon}{\lambda + \Lambda}, \\ \varepsilon &= \delta\,\frac{\lambda + \Lambda}{\Lambda - \lambda} \end{aligned}$$

with $c_1, c_2 > 0$ there holds

$$\begin{aligned} &\lambda + \delta \leq a_{\alpha,\varepsilon} \leq \Lambda - \delta, \\ &\sup |b_{\alpha,\varepsilon}| = \sqrt{\alpha}\sup |b| \text{ note that } \alpha < 1\,. \end{aligned}$$

Moreover, when $\delta \to 0$, $\alpha \to 1$, $\varepsilon \to 0$. Now, remark that the boundary conditions at final time 1 is lost. The distribution at time $i = 0, 1$ of Y is

$$g_{t,\varepsilon} \star [T_{\sqrt{\alpha}}]_\sharp(\rho(i))\,,$$

where $g_{t,\varepsilon}$ is the gaussian kernel of variance $\varepsilon t\,\mathrm{Id}$ and $T_{\sqrt{\alpha}}(x) = \sqrt{\alpha}x$ is a rescaling.

The same computations apply to a discrete process X^h, and its characteristics a_h, b_h.

Bounds on the characteristics of the discretized process. When we consider a time continuous discretisation a continuous process satisfying the bounds on a, b we might lose the bounds. However, performing first the above rescaling gives enough margin to have the discretized process satisfy the bounds. There holds

$$\mathbb{E}[X^{\alpha,\varepsilon}(t_{i+1}) - X^{\alpha,\varepsilon}(t_i)\,|\,X(t_i)] = \mathbb{E}\left[\int_{t_i}^{t_{i+1}} b_t^{\alpha,\varepsilon}\,dt\right]\,, \tag{48}$$

and

$$\mathbb{E}[(X^{\alpha,\varepsilon}(t_{i+1}) - X^{\alpha,\varepsilon}(t_i))^2\,|\,X(t_i)] = \mathbb{E}\left[\int_{t_i}^{t_{i+1}} 2(X^{\alpha,\varepsilon}(t) - X^{\alpha,\varepsilon}(t_i)b_t^{\alpha,\varepsilon}) + \mathrm{tr}(a_t^{\alpha,\varepsilon})dt\right] \tag{49}$$

$$= \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \mathrm{tr}(a_t^{\alpha,\varepsilon})\,dt\right] + C(\Lambda, B)(h^{3/2})\,. \tag{50}$$

Therefore, for h small one can choose δ_h to rescale the process such that the bound holds for the characteristics of the rescaled discrete process $a^{\alpha,\varepsilon,h}, b^{\alpha,\varepsilon,h}$.

Convergence of F. This follows simply by observing that

$$\begin{aligned}\mathbb{E}_Y \int_t F(a_Y, b_Y) &= \mathbb{E}_Y \int_t G(a_Y, b_Y) \\ &= \mathbb{E}_{X,B} \int_t G(a_Y, b_Y) = \mathbb{E}_{X,B} \int_t G(\alpha a_X + \varepsilon, \sqrt{\alpha} b_X) \quad (51)\end{aligned}$$

and since G is uniformly Lipschitz in its domain, the estimate on the functional $\mathcal{I}$ follows.

Kurtosis. The *Kurtosis* bound (13) is preserved whenever $\alpha < 1$ and ε is small enough depending on α.

Entropy, continuous case. In the time continuous case, the entropy term can be incorporated into F, as a Lipschitz function of a, therefore the convergence is shown as in (51).

Entropy in the discrete case. Let us first denote by $\mathbb{P}^h_\varepsilon$ the law of $X + \sqrt{\varepsilon} B$, B a standard Brownian motion. We first use the change of variable

$$X = (x_1, \ldots, x_N) \mapsto Z = (x_1, x_2 - x_1, \ldots, x_N - x_{N-1}).$$

Note that, under this change of coordinates, $\mathbb{P}^h_\varepsilon$ is the convolution of $\mathbb{P}^h$ with a (diagonal) Gaussian kernel G defined on $\mathbb{R}^{dN}$ such that $\int_X \|X\|^2 G(X) dX = \varepsilon$.[1] We first have that

$$\mathcal{H}(\mathbb{P}^h_\varepsilon | \overline{\mathbb{P}}^h) \leq \int_{\mathbb{R}^{dN}} \mathcal{H}(\mathbb{P}^h | [T_X]_\sharp \overline{\mathbb{P}}^h) G(X) dX \,,$$

where T_X denotes the translation by $X \in \mathbb{R}^{dN}$. This inequality is obtained thanks to the convexity of the entropy in the first variable. Let us denote by ρ the image measure of $\mathbb{P}^h$ and ρ_0 the image measure of $\overline{\mathbb{P}}_h$ on $\mathbb{R}^{dN}$,

$$\mathcal{H}(\mathbb{P}^h | [T_X]_\sharp \overline{\mathbb{P}}^h) = \int_z \rho(z) \log\left(\rho(z)/\rho_0(z + X)\right) dz \,.$$

Using the fact that ρ_0 is Gaussian, we have $\rho_0(z + X) = \exp\left(-2N\langle X, z\rangle + N\|X\|^2\right)\rho_0(z)$. We get

$$\mathcal{H}(\mathbb{P}^h | [T_X]_\sharp \overline{\mathbb{P}}^h) = \mathcal{H}(\mathbb{P}^h | \overline{\mathbb{P}}^h) + \int_z \rho(z)(-2N\langle X, z\rangle + N\|X\|^2) dz \,.$$

Integrating over X with respect to a the centered variable gives $\int_X \int_z \rho(z)(-2N\langle X, z\rangle dz G(X) dX = 0$. This implies

$$\int_X \mathcal{H}(\mathbb{P}^h | [T_X]_\sharp \overline{\mathbb{P}}^h) G(X) dX = \mathcal{H}(\mathbb{P}^h | \overline{\mathbb{P}}^h) + \int_X N\|X\|^2 G(X) dX \,.$$

[1] Note that this choice of regularization corresponds to a fixed regularization with a Gaussian kernel on the time interval $[0, 1]$ of variance ε.

Thus, we get

$$h\mathcal{H}(\mathbb{P}^h_\varepsilon|\overline{\mathbb{P}}^h) \leq h\mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h) + \varepsilon\,.$$

We now treat similarly the scaling in α,

$$\begin{aligned}\mathcal{H}([S_{\sqrt{\alpha}}]_\sharp\mathbb{P}^h|\overline{\mathbb{P}}^h) &= \int_z \rho(z)\log\left(\alpha^{-dN/2}\rho(z)/\rho_0(\sqrt{\alpha}z)\right)dz \\ &= -dN/2\log(\alpha) - \int_z \rho(z)\log\left(\rho_0(\sqrt{\alpha}z)\right)dz + \int_z \rho(z)\log\left(\rho(z)\right)dz \\ &= -dN\log(\alpha)/2 + \mathcal{H}(\mathbb{P}^h|[\overline{\mathbb{P}}^h) - \int_z \rho(z)2N(\alpha-1)\|z\|^2 dz\,. \end{aligned} \tag{52}$$

We obtain

$$h\mathcal{H}(\mathbb{P}^h|[S_{\sqrt{\alpha}}]_\sharp\overline{\mathbb{P}}^h) = -d\log(\alpha)/2 + h\mathcal{H}(\mathbb{P}^h|[\overline{\mathbb{P}}^h) - \int_z \rho(z)2(\alpha-1)\|z\|^2 dz\,.$$

The quantity $\int_z \|z\|^2\rho(z)dz$ is equal to the sum of the second moment of $\rho(t=0)$ and the quadratic variation of $\mathbb{P}^h$ which is finite uniformly in h. This completes the proof the Lemma.

Remark 6. *Another possibility for this proof is to apply the rescaling and regularisation also to the reference measure. Then $\mathcal{H}(\mathbb{P}^h_\varepsilon|\overline{\mathbb{P}}^h_\varepsilon) \leq \mathcal{H}(\mathbb{P}^h|\overline{\mathbb{P}}^h)$ is straighforward. We then have to deal a specific relative entropy with a reference diffusion which depends on ε via $\overline{a}_\varepsilon = \alpha\overline{a} + \sigma$ (α and σ depend on ε). Then (25) becomes*

$$\mathcal{I}^0_\varepsilon(\widetilde{\mathbb{P}}^0_\varepsilon) \leq \liminf_{h\searrow 0} \mathcal{I}^h(\mathbb{P}^h) + O(\varepsilon)$$

where $\mathcal{I}^0_\varepsilon = \mathcal{I}^0 + R_\varepsilon$ with $R_\varepsilon = \mathbb{E}_{\widetilde{\mathbb{P}}^0_\varepsilon}(\mathcal{S}^{\mathcal{I}}(\widetilde{a}^0_\varepsilon|\overline{a}_\varepsilon) - \mathcal{S}^{\mathcal{I}}(\widetilde{a}^0_\varepsilon|\overline{a})) = O(\overline{a}_\varepsilon - \overline{a}) = O(\varepsilon)$.

6.5 Proofs of Lemma 7 [Regularisation time continuous case]

The first step concerns space regularization and consists of adding a Gaussian variable of variance σ to the process $X(t)$. The corresponding density at time 0 is given by $g_\sigma \star \rho(0)$ where $\rho(0)$ is the marginal at time 0 of $X(t)$.

The second step involves extrapolating the process X in time with a process that has a finite cost. Choose $\sigma_0 \in [\lambda, \Lambda]$ and define on $[-2\varepsilon, 0]$, the solution of the heat equation starting at time $t = -2\varepsilon$ with the initial condition $g_{\sigma-(2\varepsilon\sigma_0)} \star \rho(0)$. Here we need to assume $2\varepsilon\sigma_0 < \sigma$ which is always satisfied for ε small enough. At time $t = 0$, the density equals $g_\sigma \star \rho(0)$. For the times $t > 1$, we also use a diffusion with coefficient σ_0 so that $X(t)$ can be extended to the time interval $[-2\varepsilon, 1+2\varepsilon]$ with a finite cost outside the interval $[0,1]$ of order $O(\varepsilon)$.

Let $k : \mathbb{R} \to \mathbb{R}_+$ be a smooth nonnegative function with support in the unit ball, such that $\int k(y)dy = 1$. We denote the kernel $\eta_\varepsilon(t,x) = k(t/\varepsilon)/\varepsilon$ with support in the time variable is thus contained in the ball of radius ε.

We consider

$$\mathbb{P}^0_{t,\varepsilon} := (\mathbb{P}^0_t * \eta_\varepsilon) \qquad b^0_{t,\varepsilon} = \frac{(b^0_t \, d\mathbb{P}^0_t) * \eta_\varepsilon}{d\mathbb{P}^0_{t,\varepsilon}} \qquad a^0_{t,\varepsilon} = \frac{(a^0_t \, d\mathbb{P}^0_t) * \eta_\varepsilon}{d\mathbb{P}^0_{t,\varepsilon}} \,. \tag{53}$$

By convexity, the drift and diffusion coefficients satisfy the hard constraints of the lemma's hypotheses. Moreover, these coefficients (drift and diffusion) are smooth in time (due to the convolution with η_ε) and space (due to the initial regularization in the first step) so that by standard arguments, there exists a unique Markov process solving the corresponding diffusion equation which is well defined on the time interval $[-\varepsilon, 1+\varepsilon]$. We denote such a process by $X^\varepsilon(t)$. By the control on the time extrapolation and the convexity of the functional $\mathcal{I}^0$, we have

$$\int_{-\varepsilon}^{1+\varepsilon} \int_x G(b_{t,\varepsilon}(x), a_{t,\varepsilon}(x)) \, d\rho_{t,\varepsilon}(x) \, dt \leq \mathcal{I}^0(\rho^0_t, b^0_t, a^0_t) + O(\varepsilon) \,.$$

We rescale the time interval from $[-\varepsilon, 1+\varepsilon]$ to $[0,1]$ to control the soft constraint on the boundary terms. Due to the hypotheses (hard constraints on the drift and diffusion coefficients), $Y := X^\varepsilon((1+2\varepsilon)(t+\varepsilon))$ satisfies the hard constraints on the drift $|b_{t,\varepsilon}| \leq B$ and volatility $\lambda \leq a_{t,\varepsilon} \,\mathrm{Id} \leq \Lambda$. Importantly, the distributions of $Y(0)$ and $Y(1)$ are (infinite) mixture of Gaussian convolutions of $\rho(0)$ and $\rho(1)$ the marginals of the process $X(t)$, since the kernel has compact support in time in $[-\varepsilon, \varepsilon]$ and the extension in time is by heat diffusion. As a consequence and from the hypotheses on $\mathcal{D}$, one has for $i = 0, 1$, $|\mathcal{D}(\mathbb{P}^Y_{t=i}, \rho_i) - \mathcal{D}(\mathbb{P}^X_{t=i}, \rho_i)| \leq O(\varepsilon) + O(\sigma)$ (where we recall that ρ_i are the (soft) boundary values). Thus, we obtain the result $\mathcal{I}(\mathbb{P}^Y) \leq \mathcal{I}(\mathbb{P}^X) + O(\varepsilon) + O(\sigma)$.

6.6 Proof of Lemma 8

For the specific entropy, Proposition 2 *i)* gives for the entropic part of $\mathcal{I}(\mathbb{P}^h_\varepsilon)$:

$$\lim_{h \searrow 0} h \, \mathcal{H}(\mathbb{P}^h_\varepsilon | \overline{\mathbb{P}}^h) = \mathcal{S}^{\mathcal{H}}(\mathbb{P}^0_\varepsilon | \overline{\mathbb{P}}). \tag{54}$$

For the integrand part, we first start with the formulae (already in the proof of lemma 6 *iv)*.)

$$\mathbb{E}[X(t_{i+1}) - X(t_i) \,|\, X(t_i)] = \mathbb{E}_{\mathbb{P}^h_{i \to i+1,\varepsilon}} \left[\int_{t_i}^{t_{i+1}} b_{t,\varepsilon}(X_t) \, dt + \int_{t_i}^{t_{i+1}} \sqrt{a_{t,\varepsilon}(X_t)} \, dB_t \right] \tag{55}$$

$$= \mathbb{E}_{\mathbb{P}^h_{i \to i+1,\varepsilon}} \left[\int_{t_i}^{t_{i+1}} b_{t,\varepsilon}(X_t) \, dt \right] , \tag{56}$$

and

$$\mathbb{E}[(X(t_{i+1})-X(t_i))^2 \,|\, X(t_i)] = \mathbb{E}_{\mathbb{P}^h_{i\to i+1,\varepsilon}}\left[\int_{t_i}^{t_{i+1}} 2(X(t)-X_i^h)b_{t,\varepsilon}(X(t)) + \mathrm{tr}(a_{t,\varepsilon}(X_t)dt\right] \tag{57}$$

$$= \mathbb{E}_{\mathbb{P}^h_{i\to i+1,\varepsilon}}\left[\int_{t_i}^{t_{i+1}} \mathrm{tr}(a_{t,\varepsilon}(X_t)\,dt\right] + O(h^{1+\alpha})\,, \tag{58}$$

where $\alpha > 0$ depends on the integrability of b. This formula implies that the discrete characteristic coefficients b_i^h and a_i^h for $\mathbb{P}^h_\varepsilon$ satisfy for h small enough $\lambda Id < a < \Lambda Id$ and $|b| < B$. We also have:

$$h\,\mathbb{E}_{\mathbb{P}^h_{i,\varepsilon}}[F\left(b^h_{i,\varepsilon}(X(t_i)), a^h_{i,\varepsilon}(X(t_i))\right)] =$$
$$h\,\mathbb{E}_{\mathbb{P}^h_{i,\varepsilon}}\left[G\left(\frac{1}{h}\mathbb{E}_{\mathbb{P}^h_{i\to i+1,\varepsilon}}\left[\int_{t_i}^{t_{i+1}} b_{t,\varepsilon}(X(t))\,dt\right], \frac{1}{h}\mathbb{E}_{\mathbb{P}^h_{i\to i+1,\varepsilon}}\left[\int_{t_i}^{t_{i+1}} \mathrm{tr}(a_{t,\varepsilon}(X(t))) + O(h^\alpha)\,dt\right]\right)\right]. \tag{59}$$

Using Jensen's inequality and the Lipschitz assumption on G, we get

$$h\,\mathbb{E}_{\mathbb{P}^h_{i,\varepsilon}}[G(b^h_{i,\varepsilon}(X(t_i)), a^h_{i,\varepsilon}(X(t_i)))] \leq \int_{t_i}^{t_{i+1}} \mathbb{E}_{\mathbb{P}^h_{i,\varepsilon}} G(b_{t,\varepsilon}(X(t), \mathrm{tr}(a_{t,\varepsilon}(X(t))))]dt + O(h^{\alpha+1})\,.$$

The control on the time boundary marginal terms follows directly from lemma 6 and 7.

6.7 Proof of Lemma 9 [Regularisation time discrete case]

Tightness: We need tightness of the quantities $\mathbb{P}^h$, m^h and n^h defined by

$$m_i^h = \frac{1}{h}(\int_{x_{i+1}} (x_{i+1}-x_i)\mathbb{P}^h_{i\to i+1}(x_{i+1}|x_i))\mathbb{P}^h_i(x_i) \tag{60}$$

and

$$n_i^h = \frac{1}{h}(\int_{x_{i+1}} (x_{i+1}-x_i)^2\mathbb{P}^h_{i\to i+1}(x_{i+1}|x_i))\mathbb{P}^h_i(x_i)\,. \tag{61}$$

These quantities are discrete in time; we will define this quantity continuously to obtain tightness in the space of time-space measures. By considering either the Markov chain or the time-linearly interpolated process, we can consider $\mathbb{P}^h$ either as discrete probabilities or as probabilities on the set of continuous paths. Let us define more explicitly the time-dependent path of marginals:

$$\rho^h(t) = \sum_{i=0}^{N} \mathbf{1}_{[t_i,t_{i+1})}\mathbb{P}^h_i\,. \tag{62}$$

To obtain tightness of the ρ^h, we use linear interpolation of $\mathbb{P}^h$ denoted $\tilde{\mathbb{P}}^h$. We start with the tightness of $\mathbb{P}^h$. Recall that we have a control on the α-moments ($\alpha > 4$) of $\mathbb{P}^h$. In particular, the linearly interpolated process $X^h(t)$ satisfies, for $s, t \in [0, T]$, $|s-t| < 1/N$, for some $c > 0$,

$$\mathbb{E}[\|X_s^h - X_t^h\|^\alpha] \leq c|s-t|^{\alpha/2-1}. \tag{63}$$

The Kolmogorov lemma implies that the sequence $\tilde{\mathbb{P}}^h$ only charges Hölder trajectories with Hölder coefficient $1/2 - 2/\alpha - \varepsilon' > 0$ (positive since $\alpha > 4$) for all positive ε' sufficiently small. Hence the sequence $\tilde{\mathbb{P}}^h$ is tight and consequently, so is $\rho^h(t)$.

If we want to apply the criterion in [26], we need to check if the "jump condition" is satisfied on $\mathbb{P}^h$. It is guaranteed by the kurtosis bound. We now deal with quantities m_i^h and n_i^h. We have

$$m^h = \sum_{i=0}^{N} \mathbf{1}_{[t_i, t_{i+1})} m_i^h \,, \quad n^h = \sum_{i=0}^{N} \mathbf{1}_{[t_i, t_{i+1})} n_i^h \,. \tag{64}$$

We want to prove that the measures (defined on $[0,1] \times \mathbb{R}^d$) m^h and n^h are tight. For that, we apply the De la Vallée Poussin lemma. We propose to bound the $1 < 1+\alpha < 2$ moment, using the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$,

$$\int_0^1 \int_{\mathbb{R}^d} |x|^{1+\alpha} n^h(t,x) dt \leq h \left(\sum_{i=1}^{N} \frac{1}{2} \int |x_i|^{2+2\alpha} \mathbb{P}_i^h + \frac{1}{2h^2} \int |x_{i+1} - x_i|^4 \mathbb{P}^h \right) . \tag{65}$$

The second term on the right-hand side is bounded due to the Kurtosis bound. It suffices to prove a uniform bound in time of $(2+2\alpha)$-moment to bound the first term, which is bounded by the 4-moment when $\alpha < 1$ which is true. Using a telescopic sum, there exists a positive constant $M > 0$ such that

$$\int_0^1 \int_{\mathbb{R}^d} |x_i|^4 \mathbb{P}_{i,i+1}^h(x_i, x_{i+1}) dt \leq M \sum_{i=1}^{N} (\int |x_0|^4 + \sum_{j=1}^{i} |x_{i+1} - x_i|^4 \mathbb{P}^h) \,. \tag{66}$$

Our hypothesis bounds the first term on the right-hand side on ρ_0. The second term is bounded by $O(h)$ due to the kurtosis bound. The proof is similar for the quantity m^h.

Scaling the coefficients and space regularization: To prepare for the time extrapolation below, we transform $\mathbb{P}^h$ by using Lemma 6. Denoting $X^h(t)$ any interpolation of $\mathbb{P}^h$, we consider the transformation of the type (for well-chosen parameters α, δ, see Lemma 6)

$$Y(t) = \sqrt{\alpha} X^h(t) + \sqrt{\delta} B(t) + B_0 \,, \tag{67}$$

with $B(t)$ a standard Brownian motion and B_0 a centered Gaussian variable of variance $\varepsilon\,\mathrm{Id}$. Lemma 6 gives that Y satisfies the estimates on the functional and the hard constraints on the drift and the volatility. The kurtosis bound is also trivially satisfied.

Time extrapolation: We want to again define the convolution in time and space of the probability $\mathbb{P}^h$ extended on the path space by linear interpolation of the curves.

However, it is not defined outside the time interval $[0,1]$, which is needed to regularize the process in time and control the density values at times $0,1$. We use the two first steps of the proof of Lemma 7 to construct an extrapolation on the time interval $[-2\varepsilon,0]$ and $[1,1+2\varepsilon]$ with finite cost $\mathcal{I}^0(\mathbb{P})+O(\varepsilon)+O(\delta)$. We now discretize it in time as in the Gamma-Limsup proof so that the cost $\mathcal{I}^h$ on $[-2\varepsilon,0]$ and $[1,1+2\varepsilon]$ is of the order $O(\varepsilon)+O(h)$. This discretization gives us a time probability measure in discrete time still denoted by $\mathbb{P}^h$.

Time regularization: We use $\eta_\varepsilon(s,x) \coloneqq k(s/\varepsilon)/\varepsilon$ as in Lemma 7. We now introduce a time-dependent curve with values in the convex set of plans, i.e. probability measures on the product space,

$$\Pi^h(t,x,y) = \sum_i \mathbf{1}_{[t_i,t_{i+1})}\mathbb{P}^h_{i,i+1}(x,y)\,, \tag{68}$$

which is a collection of time-dependent plans indexed by $t \in [-2\varepsilon, 1+2\varepsilon]$. To a given plan $\pi(x,y) \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$, one can define its conditional moments (exactly as in Formula (60)), in probabilistic notations

$$m^h(\pi) \coloneqq \frac{1}{h}\int_y (y-x)\pi(x,dy)\,,$$

and

$$n^h(\pi) \coloneqq \frac{1}{h}\int_y (y-x)^2\pi(x,dy)\,.$$

Now, we observe that

$$m^h(t) = m^h(\Pi(t))\,, \quad n^h(t) = n^h(\Pi(t))\,.$$

In addition, the marginalization of Π on the first variable gives $\rho^h(t)$. Note also that the following property holds by definition

$$\forall t \quad [p_2]_\sharp\Pi(t) = [p_1]_\sharp\Pi(t+h)\,,$$

where the $[p_i]_\sharp, i=1,2$ is the pushforward operator on the first and second variables. We now regularize in time this object

$$\Pi^h_\varepsilon = \eta_\varepsilon \star \Pi^h\,.$$

Importantly, the regularization on Π^h induces a regularization on the quantities ρ, m, n: By linearity, we have that

$$[\pi_1]_\sharp\Pi^h_\varepsilon = \eta_\varepsilon \star \rho^h \quad m^h(\Pi^h_\varepsilon) = \eta_\varepsilon \star m^h \quad n^h(\Pi^h_\varepsilon) = \eta_\varepsilon \star n^h\,.$$

In addition, it defines a discrete Markov chain: By linearity, we have the property: $\forall t \quad [p_2]_\sharp\Pi_\varepsilon(t) = [p_1]_\sharp\Pi_\varepsilon(t+h)$. From Π_ε we construct a probability on the path space:

we evaluate $\Pi_\varepsilon(ih)$ which is a plan and we note that

$$[p_2]_\sharp \Pi_\varepsilon(ih) = [p_1]_\sharp \Pi_\varepsilon((i+1)h)\,,$$

by the property mentioned above. By the standard gluing lemma [35], we construct a corresponding Markov chain using the joint probabilities $\Pi_\varepsilon(ih)$ that we denote $\mathbb{P}^h_{\delta,\varepsilon}$. By convexity of the integral part of the functional, we estimate the cost

$$\mathcal{I}^h(\mathbb{P}^h_{\delta,\varepsilon}) \leq \mathcal{I}^h(\mathbb{P}^h) + O(\varepsilon) + O(h) + O(\delta)\,, \tag{69}$$

where the term $O(\varepsilon) + O(h)$ is due to the extension in time of $\mathbb{P}^h$ and its time discretization. On $\rho^h_{\delta,\varepsilon}, m^h_{\delta,\varepsilon}, n^h_{\delta,\varepsilon}$ which are smoothing of a tight sequence of measures, we can extract a subsequence that is bounded in $L^\infty_{t,x}$ and smooth in both time and space, uniformly converging on compact subsets.

Time-rescaling: Similarly to Lemma 7, we rescale time so that the quantities are all defined on the $[0,1]$. The cost of this rescaling is $O(\varepsilon)$. Note that by rescaling, the discretization timestep is also rescaled, by abuse of notation, we still keep the same letter h. More importantly, the rescaling increases the volatility linearly in ε and we need to check that the hard constraints on the volatility are preserved. This is indeed the case since Lemma 6 has been applied beforehand.

Riemann integral: Recall the functional $\mathcal{J}$ as having three arguments

$$\mathcal{J}(\rho, m, n) = \int_{t,x} F(m/\rho, n/\rho) + \mathcal{S}^{\mathcal{I}}_{\overline{a}}(n/\rho)\, d\rho(t,x)dt\,,$$

where $\mathcal{S}^{\mathcal{I}}_{\overline{a}}$ denotes the specific entropy with reference diffusion $\overline{a}$ using n, ρ as arguments. Note that we want to study $\mathcal{J}(\rho(\Pi(t)), m^h(\Pi(t)), n^h(\Pi(t)))$. Then, the discrete formulation $\mathcal{I}^h$ (as well as the specific entropy) can be viewed as a Riemann integral in time:

$$\begin{aligned}\mathcal{I}^h(\mathbb{P}^h) \geq \sum_{i=0}^N h \int_x & F(m^h(\Pi(ih))/\rho(\Pi(ih)), n^h(\Pi(ih))/\rho(\Pi(ih))) \\ & + \mathcal{S}^{\mathcal{I}}_{\overline{a}}(n^h(\Pi(ih))/\rho(\Pi(ih)))d\rho(\Pi(ih))\,,\end{aligned} \tag{70}$$

where $\rho(\Pi(ih))$ is the marginal of $\Pi(ih)$ on the first variable. In addition, we will use below that this Riemann sum coincides with the functional $\mathcal{J}$ evaluated on $(\widetilde{\rho}, \widetilde{m}, \widetilde{n})$. By regularization, $(m^h_{\delta,\varepsilon}(t), n^h_{\delta,\varepsilon}(t), \rho^h_{\delta,\varepsilon}(t))$ is smooth in time. It is also the case for the function integrated in space. To prove it, we use that the drift and volatility coefficients are bounded in L^∞. We have

$$\begin{aligned}&|\int_x F(m^h_{\delta,\varepsilon}(t)/\rho_{\delta,\varepsilon}(t), n^h_{\delta,\varepsilon}(t)/\rho_{\delta,\varepsilon}(t)) + \mathcal{S}^{\mathcal{I}}_{\overline{a}}(m^h_{\delta,\varepsilon}(t)/\rho_{\delta,\varepsilon}(t), n^h_{\delta,\varepsilon}(t)/\rho_{\delta,\varepsilon}(t))d\rho_{\delta,\varepsilon}(t) \\ & - F(m^h_{\delta,\varepsilon}(t')/\rho_{\delta,\varepsilon}(t'), n^h_{\delta,\varepsilon}(t')/\rho_{\delta,\varepsilon}(t')) + \mathcal{S}^{\mathcal{I}}_{\overline{a}}(m^h_{\delta,\varepsilon}(t')/\rho_{\delta,\varepsilon}(t'), n^h_{\delta,\varepsilon}(t')/\rho_{\delta,\varepsilon}(t'))d\rho_{\delta,\varepsilon}(t')|\end{aligned}$$

$$\leq M \int_x |\rho_{\delta,\varepsilon}(t') - \rho_{\delta,\varepsilon}(t))| \leq MC(\varepsilon)|t-t'|\,, \quad (71)$$

where M bounds the integrand (which is possible since all the terms are bounded independently of h, ε, δ) and $C(\varepsilon)$ is a constant coming from the convolution kernel and depending on ε. Therefore, we get

$$\mathcal{I}^h(\mathbb{P}^h_{\delta,\varepsilon}) \geq \mathcal{J}(\widetilde{\rho}^h_{\delta,\varepsilon}, \widetilde{m}^h_{\delta,\varepsilon}, \widetilde{n}^h_{\delta,\varepsilon}) \geq \mathcal{J}(\rho^h_{\delta,\varepsilon}, m^h_{\delta,\varepsilon}, n^h_{\delta,\varepsilon}) + O(h)\,,$$

which gives the inequality (24). Note that the $O(h)$ is due to the difference between the Riemann sum and the integral. By lower-semicontinuity of $\mathcal{J}$, one has

$$\mathcal{J}(\rho_{\delta,\varepsilon}, m_{\delta,\varepsilon}, n_\varepsilon) \leq \lim_{h\to 0} \mathcal{J}(\rho^h_{\delta,\varepsilon}, m^h_{\delta,\varepsilon}, n^h_{\delta,\varepsilon})\,.$$

In particular, the two previous inequalities imply *iii*) of the lemma and

$$\mathcal{J}(\rho_{\delta,\varepsilon}, m_{\delta,\varepsilon}, n_{\delta,\varepsilon}) \leq \lim_{h\to 0} \mathcal{I}^h(\mathbb{P}^h_{\delta,\varepsilon})\,.$$

Conclusion: The coefficients $m^h_{\delta,\varepsilon}, n^h_{\delta,\varepsilon}$, and $\rho^h_{\delta,\varepsilon}$ converge uniformly in space and time: It implies that $b_h(ih) = \frac{m^h_{\delta,\varepsilon}(ih)}{\rho^h_{\delta,\varepsilon}(ih)}$ and $a_h(ih) = \frac{n^h_{\delta,\varepsilon}(ih)}{\rho^h_{\delta,\varepsilon}(ih)}$ converge uniformly on every compact in time and space since ρ_ε is bounded below by a positive constant on every compact set. Thus, we can apply Theorem 3 to obtain *ii*) of the lemma: the sequence $\mathbb{P}^h_\varepsilon$ uniformly converges to $\mathbb{P}_\varepsilon$ the probability on the path space of a diffusion process. By uniform convergence, the drift and diffusion coefficients can be identified by $b_{\delta,\varepsilon}(t,x) = \frac{m_{\delta,\varepsilon}(t,x)}{\rho^h_{\delta,\varepsilon}(t,x)}$ and $a_{\delta,\varepsilon}(t,x) = \frac{n_{\delta,\varepsilon}(t,x)}{\rho^h_{\delta,\varepsilon}(t,x)}$. Finally, we get

$$\mathcal{J}(\rho_{\delta,\varepsilon}, m_{\delta,\varepsilon}, n_{\delta,\varepsilon}) \leq \lim \inf_{h\to 0} \mathcal{I}^h(\mathbb{P}^h) + O(\varepsilon) + O(\delta)\,.$$

6.8 Additional Lemmas

Lemma 11. *b_t and a_t are the characteristic coefficient of the semi-martingale X with law $\mathbb{P} \in \mathcal{P}^1$. For every $p \geq 1$ and $q, q' > 1$, we have, in full generality:*

$$\mathbb{E}\big[|X_t - X_s|^p\big] \leq C_p\Big((t-s)^{p(1-\frac{1}{q})}\big(\mathbb{E}[\int_0^1 |b_u(X_u)|^{\max(p,q)}du]\big)^{\min(p/q,1)} + (t-s)^{\frac{p}{2}(1-\frac{1}{q'})}\big(\mathbb{E}[\int_0^1 |a_u(X_u)|^{\max(p/2,q')}du]\big)^{\min(p/2q'),1)}\Big), \quad (72)$$

where $C_p > 0$ only depends on p.

Proof of the lemma. In the following, the notation C_p denotes a positive number that only depends on p and that may vary at each occurence. We start with the following observation: for any measurable random process f_u, we have, for any $q > 1$, by Hölder's inequality

$$\mathbb{E}\big[|\int_s^t f_u du|^p\big] \leq (t-s)^{p(1-\frac{1}{q})} \begin{cases} \mathbb{E}[\int_0^1 |f_u|^p du] & \text{if } p \geq q \\ \mathbb{E}[\int_0^1 |f_u|^q du]^{p/q} & \text{if } p \leq q \end{cases}$$

$$= (t-s)^{p(1-\frac{1}{q})}(\mathbb{E}[\int_0^1 |f_u|^{\max(p,q)} du])^{\min(1,p/q)} \tag{73}$$

Next, from

$$X_t - X_s = \int_s^t b_u(X_u)du + \int_s^t a_u(X_u)^{1/2} dB_u$$

and thus

$$\mathbb{E}[|X_t - X_s|^p] \leq C_p(\mathbb{E}[|\int_s^t b_u(X_u)du|^p] + \mathbb{E}[|\int_s^t a_u(X_u)^{1/2} dB_u|^p])$$

since $p \geq 1$, we successively have

$$\mathbb{E}[|\int_s^t b_u(X_u)du|^p] \leq (t-s)^{p(1-\frac{1}{q})}(\mathbb{E}[\int_0^1 |b_u(X_u)|^{\max(p,q)} du])^{\min(1,p/q)}$$

and

$$\mathbb{E}[|\int_s^t a_u(X_u)^{1/2} du|^p] \leq C_p \mathbb{E}[|\int_s^t a_u(X_u) du|^{p/2}]$$
$$\leq (t-s)^{\frac{p}{2}(1-\frac{1}{q'})}(\mathbb{E}[\int_0^1 |a_u(X_u)|^{\max(p/2,q')} du])^{\min(1,p/2q')}$$

by (73) and the Burkholder-Davis-Gundy inequality. The conclusion follows.

□

A brief excerpt from chapter 9 of [35] and [36, 37]:

Definition 1 (perspectives functions over measures a.k.a. entropies)**.** *Assuming that $F(.) = \theta(|.|) : \mathbb{R}^d \to [0, \infty]$ is a convex lower semi-continuous function with superlinear growth at infinity. The perspective function (aka the entropy) of F evaluated at the measures (M, ν) is defined as*

$$(\nu, M) \to \mathcal{F}(M|\nu) = \begin{cases} \mathbb{E}_\nu(F(\frac{dM}{d\nu})) \text{ if } M \ll \nu. \\ +\infty \text{ else.} \end{cases}$$

The classic example is the Shanon relative entropy $F(.) = -log\big(.|\bar{M}\big)$. The "nice" properties of $\mathcal{F}$ are inherited from its dual formulation (42),

Definition 2 (Weak Fokker-Planck Solutions). *We will refer to this property as: "$(\nu_t, b_t, a_t) \in FP(\rho_0, \rho_1$").*

A curve $(\nu_t)_{t\in[0,1]}$ of probability measures is a weak solution of the Fokker-Planck with drift coefficients (b_t) and diffusion coefficients (a_t) if $(b_t, a_t) \in L^1_{t,x}(\nu_t)$ and for all $f \in C_b^{1,2}((0,1)\times\mathbb{R}^d)$

$$\int_0^1 \int [\partial_t f + b_t\, \partial_x f + a_t\, \partial_{xx} f]\, d\nu_t(x)\, dt = 0$$

In our case we add the initial-final marginal conditions :

$$\nu_{0,1} = \rho_{0,1}$$

It is not restrictive to assume that (ν_t) is narrowly continuous. For more details and $d > 1$ see [37] definition 2.2 and remark 2.3.

Proposition 12. *i) $\mathcal{F}$ in definition 1 is jointly convex and lower semi continuous in (M, ν).*

ii) (M^h, ν^h) two sequences of Borel positive measure in $\mathbb{R}^d$, such that, ν^h weakly converges to ν^0, M^h is a.c. w.r.t. ν_h for all h and

$$\sup_h \mathbb{E}_{\nu^h}(F(\frac{dM^h}{d\nu^h})) < \infty.$$

Then M^0 is a.c. w.r.t. ν^0 and

$$\liminf_{h\searrow 0} \mathbb{E}_{\nu^h}(F(\frac{dM^h}{d\nu^h})) \geq \mathbb{E}_{\nu^0}(F(\frac{dM^0}{d\nu^0}))$$

iii) Let η_ε be a smooth regularisation kernel with k bounded derivatives. Denote $M_\varepsilon = M * \eta_\varepsilon$, $\nu_\varepsilon = \nu * \eta_\varepsilon$. then

$$\mathcal{F}(M_\varepsilon|\nu_\varepsilon) \leq \mathcal{F}(M|\nu)$$

iv) If (ν_t) is a weak solution of $FP_{\rho_{0,1}}(b_t, a_t)$, $(b_t, a_t) \in \mathcal{L}^p(\nu_t)$ $(\nu_{t,\varepsilon})$ is a weak solution of $FP_{\rho_{0,1,\varepsilon}}(b_{t,\varepsilon}, a_{t,\varepsilon})$ where for all t $(\nu_{t,\varepsilon} := (\nu_t * \eta_\varepsilon)$ and we mollify the "moments":

$$b_{t,\varepsilon} = \frac{(b_t\, d\nu_t) * \eta_\varepsilon}{d\nu_{t,\varepsilon}} \qquad a_{t,\varepsilon} = \frac{(a_t\, d\nu_t) * \eta_\varepsilon}{d\nu_{t,\varepsilon}}.$$

$(b_{t,\varepsilon}, a_{t,\varepsilon})$ are in $\mathcal{L}^p(\nu_{t,\varepsilon})$ and well defined uniformly bounded C^k densities:

$$\sup_{t\in[0,1]} \|(b_{t\varepsilon}, a_{t,\varepsilon})\|_{C_b^k(B)} < +\infty$$

for all bounded set $B \in \mathbb{R}^d$.

v) Applying iii) to the setting in iv) we get

$$\mathbb{E}_{\nu_{t,\varepsilon}}(F(b_{t,\varepsilon})) \leq \mathbb{E}_{\nu_t}(F(b_t))$$

Proof. These results are a direct application of the dual form of the entropy given in lemma 9.4.3 and 9.4.4 [35], see also lemma A.1 [37]. $\square$