define $f^{\text{out}}(S) = \sum_{e \text{ out of } S} f(e)$ and $f^{\text{in}}(S) = \sum_{e \text{ into } S} f(e)$. In this terminology, the conservation condition for nodes $v \neq s, t$ becomes $f^{\text{in}}(v) = f^{\text{out}}(v)$; and we can write $v(f) = f^{\text{out}}(s)$.

The Maximum-Flow Problem Given a flow network, a natural goal is to arrange the traffic so as to make as efficient use as possible of the available capacity. Thus the basic algorithmic problem we will consider is the following: Given a flow network, find a flow of maximum possible value.

As we think about designing algorithms for this problem, it's useful to consider how the structure of the flow network places upper bounds on the maximum value of an s-t flow. Here is a basic "obstacle" to the existence of large flows: Suppose we divide the nodes of the graph into two sets, A and B, so that $s \in A$ and $t \in B$. Then, intuitively, any flow that goes from s to t must cross from A into B at some point, and thereby use up some of the edge capacity from A to B. This suggests that each such "cut" of the graph puts a bound on the maximum possible flow value. The maximum-flow algorithm that we develop here will be intertwined with a proof that the maximum-flow value equals the minimum capacity of any such division, called the *minimum cut*. As a bonus, our algorithm will also compute the minimum cut. We will see that the problem of finding cuts of minimum capacity in a flow network turns out to be as valuable, from the point of view of applications, as that of finding a maximum flow.

Designing the Algorithm

Suppose we wanted to find a maximum flow in a network. How should we go about doing this? It takes some testing out to decide that an approach such as dynamic programming doesn't seem to work—at least, there is no algorithm known for the Maximum-Flow Problem that could really be viewed as naturally belonging to the dynamic programming paradigm. In the absence of other ideas, we could go back and think about simple greedy approaches, to see where they break down.

Suppose we start with zero flow: $f(e) = 0$ for all e. Clearly this respects the capacity and conservation conditions; the problem is that its value is 0. We now try to increase the value of f by "pushing" flow along a path from s to t, up to the limits imposed by the edge capacities. Thus, in Figure 7.3, we might choose the path consisting of the edges $\{(s, u), (u, v), (v, t)\}$ and increase the flow on each of these edges to 20, and leave $f(e) = 0$ for the other two. In this way, we still respect the capacity conditions—since we only set the flow as high as the edge capacities would allow—and the conservation conditions—since when we increase flow on an edge entering an internal node, we also increase it on an edge leaving the node. Now, the value of our flow is 20, and we can ask: Is this the maximum possible for the graph in the figure? If we

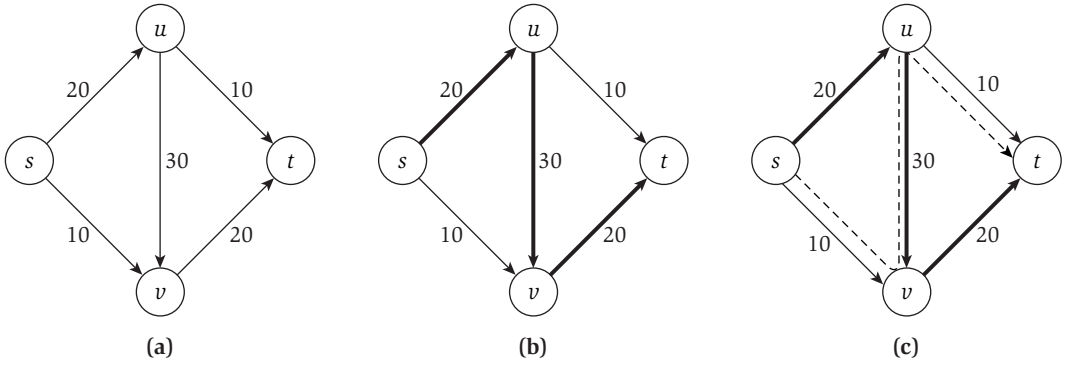

Figure 7.3 (a) The network of Figure 7.2. (b) Pushing 20 units of flow along the path s, u, v, t. (c) The new kind of augmenting path using the edge (u, v) backward.

think about it, we see that the answer is no, since it is possible to construct a flow of value 30. The problem is that we're now stuck—there is no s-t path on which we can directly push flow without exceeding some capacity—and yet we do not have a maximum flow. What we need is a more general way of pushing flow from s to t, so that in a situation such as this, we have a way to increase the value of the current flow.

Essentially, we'd like to perform the following operation denoted by a dotted line in Figure 7.3(c). We push 10 units of flow along (s, v); this now results in too much flow coming into v. So we "undo" 10 units of flow on (u, v); this restores the conservation condition at v but results in too little flow leaving u. So, finally, we push 10 units of flow along (u, t), restoring the conservation condition at u. We now have a valid flow, and its value is 30. See Figure 7.3, where the dark edges are carrying flow before the operation, and the dashed edges form the new kind of augmentation.

This is a more general way of pushing flow: We can push *forward* on edges with leftover capacity, and we can push *backward* on edges that are already carrying flow, to divert it in a different direction. We now define the *residual graph*, which provides a systematic way to search for forward-backward operations such as this.

The Residual Graph Given a flow network G, and a flow f on G, we define the *residual graph* G_f of G with respect to f as follows. (See Figure 7.4 for the residual graph of the flow on Figure 7.3 after pushing 20 units of flow along the path s, u, v, t.)

- The node set of G_f is the same as that of G.
- For each edge $e = (u, v)$ of G on which $f(e) < c_e$, there are $c_e - f(e)$ "leftover" units of capacity on which we could try pushing flow forward.

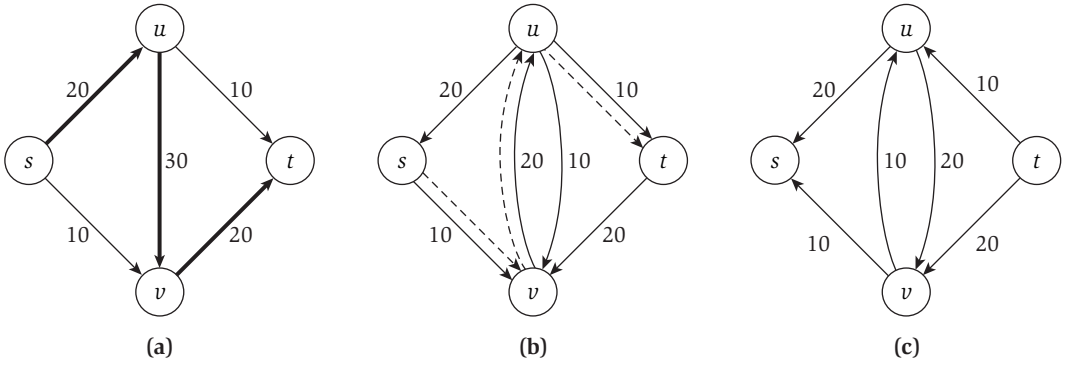

Figure 7.4 (a) The graph G with the path s, u, v, t used to push the first 20 units of flow. (b) The residual graph of the resulting flow f, with the residual capacity next to each edge. The dotted line is the new augmenting path. (c) The residual graph after pushing an additional 10 units of flow along the new augmenting path s, v, u, t.

So we include the edge $e = (u, v)$ in G_f, with a capacity of $c_e - f(e)$. We will call edges included this way *forward edges*.

- For each edge $e = (u, v)$ of G on which $f(e) > 0$, there are $f(e)$ units of flow that we can "undo" if we want to, by pushing flow backward. So we include the edge $e' = (v, u)$ in G_f, with a capacity of $f(e)$. Note that e' has the same ends as e, but its direction is reversed; we will call edges included this way *backward edges*.

This completes the definition of the residual graph G_f. Note that each edge e in G can give rise to one or two edges in G_f: If $0 < f(e) < c_e$ it results in both a forward edge and a backward edge being included in G_f. Thus G_f has at most twice as many edges as G. We will sometimes refer to the capacity of an edge in the residual graph as a *residual capacity*, to help distinguish it from the capacity of the corresponding edge in the original flow network G.

Augmenting Paths in a Residual Graph Now we want to make precise the way in which we push flow from s to t in G_f. Let P be a simple s-t path in G_f—that is, P does not visit any node more than once. We define `bottleneck`(P, f) to be the minimum residual capacity of any edge on P, with respect to the flow f. We now define the following operation `augment`(f, P), which yields a new flow f' in G.

```
augment(f, P)
  Let b = bottleneck(P, f)
  For each edge (u, v) ∈ P
    If e = (u, v) is a forward edge then
      increase f(e) in G by b
```

```
    Else ((u, v) is a backward edge, and let e = (v, u))
      decrease f(e) in G by b
    Endif
  Endfor
  Return(f)
```

It was purely to be able to perform this operation that we defined the residual graph; to reflect the importance of `augment`, one often refers to any *s*-*t* path in the residual graph as an *augmenting path*.

The result of `augment`(f, P) is a new flow f' in G, obtained by increasing and decreasing the flow values on edges of P. Let us first verify that f' is indeed a flow.

(7.1) *f' is a flow in G.*

Proof. We must verify the capacity and conservation conditions.

Since f' differs from f only on edges of P, we need to check the capacity conditions only on these edges. Thus, let (u, v) be an edge of P. Informally, the capacity condition continues to hold because if $e = (u, v)$ is a forward edge, we specifically avoided increasing the flow on e above c_e; and if (u, v) is a backward edge arising from edge $e = (v, u) \in E$, we specifically avoided decreasing the flow on e below 0. More concretely, note that `bottleneck`(P, f) is no larger than the residual capacity of (u, v). If $e = (u, v)$ is a forward edge, then its residual capacity is $c_e - f(e)$; thus we have

$$0 \leq f(e) \leq f'(e) = f(e) + \texttt{bottleneck}(P, f) \leq f(e) + (c_e - f(e)) = c_e,$$

so the capacity condition holds. If (u, v) is a backward edge arising from edge $e = (v, u) \in E$, then its residual capacity is $f(e)$, so we have

$$c_e \geq f(e) \geq f'(e) = f(e) - \texttt{bottleneck}(P, f) \geq f(e) - f(e) = 0,$$

and again the capacity condition holds.

We need to check the conservation condition at each internal node that lies on the path P. Let v be such a node; we can verify that the change in the amount of flow entering v is the same as the change in the amount of flow exiting v; since f satisfied the conservation condition at v, so must f'. Technically, there are four cases to check, depending on whether the edge of P that enters v is a forward or backward edge, and whether the edge of P that exits v is a forward or backward edge. However, each of these cases is easily worked out, and we leave them to the reader. ■

This augmentation operation captures the type of forward and backward pushing of flow that we discussed earlier. Let's now consider the following algorithm to compute an *s-t* flow in *G*.

```
Max-Flow
  Initially f(e) = 0 for all e in G
  While there is an s–t path in the residual graph G_f
    Let P be a simple s–t path in G_f
    f' = augment(f, P)
    Update f to be f'
    Update the residual graph G_f to be G_f'
  Endwhile
  Return f
```

We'll call this the *Ford-Fulkerson Algorithm*, after the two researchers who developed it in 1956. See Figure 7.4 for a run of the algorithm. The Ford-Fulkerson Algorithm is really quite simple. What is not at all clear is whether its central `While` loop terminates, and whether the flow returned is a maximum flow. The answers to both of these questions turn out to be fairly subtle.

Analyzing the Algorithm: Termination and Running Time

First we consider some properties that the algorithm maintains by induction on the number of iterations of the `While` loop, relying on our assumption that all capacities are integers.

(7.2) *At every intermediate stage of the Ford-Fulkerson Algorithm, the flow values* $\{f(e)\}$ *and the residual capacities in* G_f *are integers.*

Proof. The statement is clearly true before any iterations of the `While` loop. Now suppose it is true after j iterations. Then, since all residual capacities in G_f are integers, the value `bottleneck`(P, f) for the augmenting path found in iteration $j+1$ will be an integer. Thus the flow f' will have integer values, and hence so will the capacities of the new residual graph. ■

We can use this property to prove that the Ford-Fulkerson Algorithm terminates. As at previous points in the book we will look for a measure of *progress* that will imply termination.

First we show that the flow value strictly increases when we apply an augmentation.

(7.3) *Let* f *be a flow in* G*, and let* P *be a simple s-t path in* G_f*. Then* $v(f') = v(f) +$ `bottleneck`(P, f)*; and since* `bottleneck`$(P, f) > 0$*, we have* $v(f') > v(f)$.