ECN 506

Trade Without Money

1.1 (a) *The following solution is very detailed to make sure everyone understands each step as the same process is used frequently in this course.*

The planner's objective is to maximize welfare of the future generations (sum of expected utility of all individuals in future generations). However, the planner is restricted in each period by a resource constraint. To find the planner's resource constraint, we need to know (i) how many of each type of individual (young and old) we have in a given period, and (ii) the endowment of each type.

Consider period $t > 0$. There are N_t young born in period t and each young is endowed with y_1. The number of old is the number of young that were born in the previous period N_{t-1} and each old is endowed with y_2 when old in period t. The total endowment in period t is therefore:

$$N_t y_1 + N_{t-1} y_2. \quad (1)$$

The planner's resource constraint says that the total consumption allocated to the young and old cannot exceed the total endowment. For each young the planner chooses consumption $c_{1,t}$ and for each old the planner chooses consumption $c_{2,t}$. The total consumption in period t is therefore:

$$N_t c_{1,t} + N_{t-1} c_{2,t}. \quad (2)$$

The planner's resource constraint therefore becomes:

$$N_t c_{1,t} + N_{t-1} c_{2,t} \leq N_t y_1 + N_{t-1} y_2. \quad (3)$$

Since there is a constant population then $N_{t-1} = N_t = N$ and the resource constraint becomes:

$$N\, c_{1,t} + N\, c_{2,t} \leq N y_1 + N y_2$$

$$c_{1,t} + c_{2,t} \leq y_1 + y_2 \qquad (4)$$

Finally, assuming a stationary equilibrium where the planner treats all generations the same, we have that $c_{1,t} = c_1$ for all $t \geq 1$ and $c_{2,t} = c_2$ for all $t \geq 1$. The resource constraint becomes:

$$c_1 + c_2 \geq y_1 + y_2 \quad (5)$$

This constraint says that if the planner considers one young and one old individual in any period, the total consumption the planner allocates between these two individuals cannot exceed the total endowment of these two individuals. This is a very intuitive constraint and is likely the one we would have come up with by just using economic intuition and common sense.

(a) ***This part is not in the question but was done in class. I have numbered the steps as these are generally the steps we will use when finding the lifetime budget constraint of an individual in a decentralized economy. There are a lot of symbols and the notation can be tricky, but the math is just basic algebra.***

Consider a decentralized economy where individuals make their own utility maximizing decisions and there is perfect and costless record-keeping. That is, suppose there is a technology that allows all trades between young and old to be tracked and verified. We can show that an individual's lifetime budget constraint is identical to the planner's resource constraint in equation (5) by following the steps below.

(i) Write down the young budget constraint.

Consider a young individual in period t. We denote the transfer the young individual makes by Φ_t. We start by finding the young individual's budget constraint. The individual chooses their consumption $c_{1,t}$ and the transfer they make Φ_t. The individual's endowment when young is y_1. Therefore the young budget constraint is:

$$c_{1,t} + \Phi_t \leq y_1 \quad (6)$$

(ii) Write down the old budget constraint.

Now consider the individual when they are old. Since there is perfect record keeping, an individual who transferred Φ_t to an old person while young receives the transfer Φ^R_{t+1} when old. If the individual did not make a transfer in the last period when young then they would receive nothing when old. We define the relationship between the transfer made while young Φ_t and transfer received while old Φ^R_{t+1} by $\Phi^R_{t+1} = x_{t+1}\Phi_t$. The variable x_{t+1} is a rate of return that tells us how a transfer made while young is converted into a transfer received when old. The old budget constraint is therefore:

$$c_{2,t+1} \leq y_2 + \Phi^R{}_{t+1}$$

$$\Rightarrow c_{2,t+1} \leq y_2 + x_{t+1}\Phi_t \quad (7)$$

We can rearrange this constraint to get Φ_t on its own on the right-hand side:

$$\frac{c_{2,t+1}}{x_{t+1}} - \frac{y_2}{x_{t+1}} \leq \Phi_t \quad (8)$$

(iii) Combine the young and old budget constraints to get a lifetime budget constraint.

We can then combine our expression for Φ_t in equation (8) with our young budget constraint in equation (6) to get the lifetime budget constraint:

$$c_{1,t} + \frac{c_{2,t+1}}{x_{t+1}} - \frac{y_2}{x_{t+1}} \leq y_1$$

$$c_{1,t} + \frac{c_{2,t+1}}{x_{t+1}} \leq y_1 + \frac{y_2}{x_{t+1}} \quad (9)$$

(iv) Use supply and demand to find the price/rate of return.

We are not done yet as our budget constraint in equation (9) still has a price/rate of return in it, x_{t+1}, that we need to find. To find this we will use expressions for the supply and demand of transfers in period t. In period t there are N_t young and each will make transfer Φ_t so the total supply of transfers is $N_t\Phi_t$. Referring back to the young budget constraint in equation (6) the transfer a young individual makes is the difference between their endowment, y_1, and their consumption, $c_{1,t}$. We have that for every period t:

$$\Phi_t = y_1 - c_{1,t} \quad (10)$$

Since there are N_t young individuals in period t, this means the total supply of transfers is $N_t(y_1 - c_{1,t})$.

Now we need to know the demand for transfers. Each old individual demands the transfer Φ^R_t in period t and there are N_{t-1} of them. The total demand for transfers is therefore $N_{t-1}\Phi^R_t = N_{t-1}\Phi_{t-1}x_t$ using the definition of Φ^R_{t+1} above, but in the previous period. Using the fact that $\Phi_{t-1} = y_1 - c_{1,t-1}$ for the same reason as above, we write the total demand for transfers as $N_{t-1}x_t(y_1 - c_{1,t-1})$.

Supply equals demand means:

$$\text{Supply of transfers} = \text{Demand for transfers}$$

$$N_t(y_1 - c_{1,t}) = N_{t-1}x_t(y_1 - c_{1,t-1}). \quad (11)$$

If we have a constant population then $N_t = N_{t-1} = N$ and this becomes:

$$y_1 - c_{1,t} = x_t(y_1 - c_{1,t-1}). \quad (12)$$

In a stationary equilibrium $c_{1,t} = c_1$ for all t so this becomes:

$$y_1 - c_1 = x_t(y_1 - c_1). \quad (13)$$

This means that $x_t = 1$. This is an intuitive result and is what we thought the answer should be even before we started solving in class. If there is a constant population, the endowment is not changing and we're looking for a stationary equilibrium then I would expect that if I transfer 10 units of the consumption good to an old person when I am young, then when I am old I expect to get 10 units from a young person.

(v) Substitute the price/rate of return into our lifetime budget constraint.

If we substitute $x_t = 1$ into our lifetime budget constraint in equation (9) and impose a stationary solution then we get:

$$c_1 + c_2 \leq y_1 + y_2. \quad (14)$$

The individual's lifetime budget constraint is identical to the planner's resource constraint. Again, this is an intuitive result. The individual's lifetime budget constraint says that they cannot consume more in their lifetime then they are endowed with in their lifetime. If there is no trade then we have $c_1 = y_1$ and $c_2 = y_2$. However with trade, the individual can move some consumption to when they are old so c_1 gets smaller and c_2 gets larger. They can do this as long as they don't violate the lifetime budget constraint above. This reallocation allows individuals to get onto a higher indifference curve and increase their lifetime utility. This is an important result because it says that trade calls allow us to achieve the optimal allocation chosen by the planner.