Assignment 1

Q1

2.6 The following table gives the joint probability distribution between employment status and college graduation among those either employed or looking for work (unemployed) in the working-age U.S. population for 2012.

Joint Distribution of Employment Status and College Graduation in the U.S. Population Aged 25 and Older, 2012

	Unemployed ($Y = 0$)	Employed ($Y = 1$)	Total
Non–college grads ($X = 0$)	0.053	0.586	0.639
College grads ($X = 1$)	0.015	0.346	0.361
Total	0.068	0.932	1.000

a. Compute $E(Y)$.

b. The unemployment rate is the fraction of the labor force that is unemployed. Show that the unemployment rate is given by $1 - E(Y)$.

c. Calculate $E(Y|X = 1)$ and $E(Y|X = 0)$.

d. Calculate the unemployment rate for (i) college graduates and (ii) non–college graduates.

e. A randomly selected member of this population reports being unemployed. What is the probability that this worker is a college graduate? A non–college graduate?

f. Are educational achievement and employment status independent? Explain.

a) $E(Y) = (0 \times 0.068) + (1 \times 0.932) = 0.932$

b) $1 - E(Y) =$ Unemployment %

$\therefore$ Unemploymet % $= 1 - 0.932 = 0.068$ or 6.8%

c) Conditional probabilities

1) $Pr(Y=0 | X=0) = Pr(X=0 \,\&\, Y=0) / Pr(X=0) = 0.053/0.639 = 0.083$

2) $Pr(Y=1 | X=0) = Pr(X=0 \,\&\, Y=1) / Pr(X=0) = 0.586/0.639 = 0.917$

3) $Pr(Y=0 | X=1) = Pr(X=1 \,\&\, Y=0) / Pr(X=1) = 0.015/0.361 = 0.042$

4) $Pr(Y=1 | X=1) = Pr(X=1 \,\&\, Y=1) / Pr(X=1) = 0.346/0.361 = 0.958$

Conditional exp.:

$$E(Y|x=1) = 0 \times Pr(Y=0|X=1) + 1 \times Pr(Y=1|x=1)$$
$$= 0 + 1 \times 0.958 = 0.958$$

$$E(Y/x=0) = 0 \times Pr(Y=0|X=0) + 1 \times Pr(Y=1|x=0)$$
$$= 0 + 1 \times 0.917 = 0.917$$

d) i) $1 - E(Y|X=1) = 1 - 0.958 = 0.042$

ii) $1 - E(Y|X=0) = 1 - 0.917 = 0.083$

Correction

e) i) College grad $= \dfrac{0.053}{0.068} = 0.78$ × $\dfrac{0.015}{0.068} = 0.22$

ii) Non-college grad $= \dfrac{0.065}{0.068} = 0.22$ × $\dfrac{0.053}{0.068} = 0.78$

f) Emp. status and educational achievement are not independent because the following assumption does not hold true.

if, i.e., $Pr(Y=y|X=x) = Pr(Y=y) Pr(X=x)$

however $Pr(Y=1|X=1) \neq Pr(Y=1) Pr(X=1)$

as $0.958 \neq 0.336$. Hence proved.

27)

2.8 The random variable Y has a mean of 1 and a variance of 4. Let $Z = \frac{1}{2}(Y - 1)$. Show that $\mu_Z = 0$ and $\sigma_Z^2 = 1$.

We know $\mu_y = 1$

finding $\mu_z = E(z)$

$$= E\left(\frac{1}{2}(Y-1)\right)$$

$$= \frac{1}{2} E(Y) - \frac{1}{2}$$

$$= \frac{1}{2}\mu_y - \frac{1}{2}$$

$$= \frac{1}{2} \times 1 - \frac{1}{2}$$

$= 0$ hence proved

$\mu_z = 0$

$\sigma_y^2 = 4$

$$z = \frac{1}{2}y - \frac{1}{2}$$

$$\mathrm{Var}(z) = \sigma_z^2 = \left(\frac{1}{2}\right)^2 (\mathrm{Var}(y))$$

$$\sigma_z^2 = \frac{1}{4}\sigma_y^2$$

$$= \frac{1}{4} \times 4$$

$$\sigma_z^2 = 1$$

3) **2.10** Compute the following probabilities:

a. If Y is distributed $N(1, 4)$, find $\Pr(Y \leq 3)$.

b. If Y is distributed $N(3, 9)$, find $\Pr(Y > 0)$.

c. If Y is distributed $N(50, 25)$, find $\Pr(40 \leq Y \leq 52)$.

d. If Y is distributed $N(5, 2)$, find $\Pr(6 \leq Y \leq 8)$.

a) $\Pr(Y \leq 3) = 0.8413$

b) $\Pr(Y > 0) = 0.8413$

c) $\Pr(40 \leq Y \leq 52) = 0.6327$

d) $\Pr(6 \leq Y \leq 8) = 0.2228$

4) **2.11** Compute the following probabilities:

a. If Y is distributed χ^2_4, find $\Pr(Y \leq 7.78)$.

b. If Y is distributed χ^2_{10}, find $\Pr(Y > 18.31)$.

c. If Y is distributed $F_{10,\infty}$, find $\Pr(Y > 1.83)$.

d. Why are the answers to (b) and (c) the same?

e. If Y is distributed χ^2_1, find $\Pr(Y \leq 1.0)$. (*Hint:* Use the definition of the χ^2_1 distribution.)

a) $\Pr(Y \leq 7.78) = 0.9$

b) $Pr(Y > 18.31) = 0.04995 \simeq 0.05$

c) $F_{10,\infty}$ = (Data from table 3 provided) = 0.05

d) $F_{10,\infty} = \dfrac{X/m}{Y/n}$ i.e. $m = 10$, $n = \infty$

where X is χ^2_{10} & Y is χ^2_{∞}

$$= \frac{\chi^2_{10}}{10} \div \frac{\chi^2_{\infty}}{\infty}$$

e) $Pr(Y \leq 1) = 0.6827$

5) **2.14** In a population $\mu_Y = 100$ and $\sigma_Y^2 = 43$. Use the central limit theorem to answer the following questions:

a. In a random sample of size $n = 100$, find $\Pr(\overline{Y} \leq 101)$.

b. In a random sample of size $n = 165$, find $\Pr(\overline{Y} > 98)$.

c. In a random sample of size $n = 64$, find $\Pr(101 \leq \overline{Y} \leq 103)$.

a) $Pr(\bar{Y} \leq 101) = 1 - Pr(\bar{Y} > 101)$

$= 1 - 0.93637$

$= 0.06473$

b) $Pr(\bar{Y} > 98) = 0.0000447$

c) Finding critical values for $Pr(\bar{Y}>103)$ & $Pr(\bar{Y}<101)$

$-3.6599657, -1.2199886$

$Pr(101 \leq \bar{Y} \geq 103) = 0.1110847$

6)

2.22 Suppose you have some money to invest—for simplicity, \$1—and you are planning to put a fraction w into a stock market mutual fund and the rest, $1-w$, into a bond mutual fund. Suppose that \$1 invested in a stock fund yields R_s after 1 year and that \$1 invested in a bond fund yields R_b, suppose that R_s is random with mean 0.08 (8%) and standard deviation 0.07, and suppose that R_b is random with mean 0.05 (5%) and standard deviation 0.04. The correlation between R_s and R_b is 0.25. If you place a fraction w of your money in the stock fund and the rest, $1-w$, in the bond fund, then the return on your investment is $R = wR_s + (1-w)R_b$.

a. Suppose that $w = 0.5$. Compute the mean and standard deviation of R.

b. Suppose that $w = 0.75$. Compute the mean and standard deviation of R.

c. What value of w makes the mean of R as large as possible? What is the standard deviation of R for this value of w?

d. (Harder) What is the value of w that minimizes the standard deviation of R? (Show using a graph, algebra, or calculus.)

a) $E(R) = E(0.5 \cdot R_s + 0.5 R_b)$

$= 0.5\, E(R_s) + 0.5\, E(R_b)$

$= 0.5 \times 0.08 + 0.5 \times 0.05$

$\mu_R = 0.065$

$Var(R) = 0.5^2 \cdot Var(R_s) + 0.5^2 \cdot Var(R_b) + 2 \times 0.5^2 \times Cov(R_s, R_b)$

$\sigma_R^2 = 0.25 \times 0.0049 + 0.25 \times 0.0016 + 0.125 \times 0.07 \times 0.04$

$\sqrt{\sigma_R^2} = \sqrt{0.001975}$ | $\sigma_R = 0.04444$

b) $R = 0.75 R_s + 0.25 R_b$

$E(R) = \mu_R = 0.75 \times 0.08 + 0.25 \times 0.05$

$= 0.0725$

$Var(R) = 0.75^2 \cdot Var(R_s) + 0.25^2 Var(R_b) + 2 \times 0.75 \times 0.25 \times Cov(R_s, R_b)$

$Cov(R_s, R_b) = \sigma_s^R \times \sigma_b^R \times corr(R_s, R_b)$

$= 0.07 \times 0.04 \times 0.25$

$Var(R) = 0.0421375$

$\sigma_R = 0.20527$

c) When $R = R_s$ as $R_s > R_b$

therefore $w = 1$ and $\sigma_R = \sigma_{R_s} = 0.07$

d) $Var(R) = w^2 Var(R_s) + (1-w)^2 Var(R_b) + 2w(1-w) Cov(R_s, R_b)$

$Corr(R_s, R_b) \cdot (\sigma_{R_s} \times \sigma_{R_b}) = Cov(R_s, R_b)$

$Var(R) = w^2 Var(R_s) + (1-w)^2 Var(R_b) + (2w(1-w) \times 0.25 \times 0.07 \times 0.09)$

$\Rightarrow w^2 0.0049 + (1-w)^2 0.0016 + 0.0014 w(1-w)$

$Var(R)' = 0.0102 w + 0.0046$ assume $= 0$, we get $w = -0.45$

$Var(R)'' = 0.0102$ $\therefore$ It is minimum as $Var(R)'' > 0$

when $w = -0.4509$

min. Var (R) is the same as its corresponding min. std. dev value. Hence proved.

Resolve

2.14 In a population $\mu_Y = 100$ and $\sigma_Y^2 = 43$. Use the central limit theorem to answer the following questions:

a. In a random sample of size $n = 100$, find $\Pr(\overline{Y} \le 101)$.

b. In a random sample of size $n = 165$, find $\Pr(\overline{Y} > 98)$.

c. In a random sample of size $n = 64$, find $\Pr(101 \le \overline{Y} \le 103)$.

a) $n = 100$, $\sigma_{\bar{Y}}^2 = \dfrac{\sigma_Y^2}{n} = \dfrac{43}{100} = 0.43$

$$\Pr(\bar{Y} \le 101) = \Pr\left[\frac{\bar{Y} - 100}{\sqrt{0.43}} \le \frac{101 - 100}{\sqrt{0.43}}\right]$$

$$= \Pr\left[\frac{\bar{Y} - 100}{\sqrt{0.43}} \le 1.525\right]$$

Z value / critical value = 1.525

$\therefore$ $\sigma = 1$, $\mu = 0$, low : -10^{10}, up 1.525

$$\Pr(\bar{Y} \le 101) = 0.9364$$

b) $$\Pr(\bar{Y} > 98) = \Pr\left[\frac{\bar{Y} - 100}{\sqrt{0.26}} > \frac{98 - 100}{\sqrt{0.26}}\right]$$

$$\Pr(\bar{Y} > 98) = \Pr\left[\frac{\bar{Y} - 100}{\sqrt{0.26}} > -3.92\right] = 1.0$$

c) $$\Pr\left[\frac{101 - 100}{\sqrt{0.6719}} \le \frac{\bar{Y} - 100}{\sqrt{0.6719}} \le \frac{103 - 100}{\sqrt{0.6719}}\right]$$

$\Rightarrow$ 0.1111274