QUATERNION CONSTRAINTS

If $Quat = p * q$ where p and q are unit quaternions

then $Quat = Left(P) * q$

and $Quat = Right(q) * p$.

where $Right(P) = \begin{bmatrix} q_0 & -q_v \\ q_v & q_0 I - \hat{q}_v \end{bmatrix}$ where $\hat{q}_v$ is the cross product matrix of q.

If $\vec{v} = \vec{\omega} \times \vec{r}$

then $\vec{v} = \hat{\vec{\omega}} * \vec{r}$

where $\hat{\vec{\omega}}$ is the 3×3 cross product matrix of $\vec{\omega}$.

which is

$$\underbrace{\begin{bmatrix} 0 & -\omega_2 & \omega_1 \\ \omega_2 & 0 & -\omega_0 \\ -\omega_1 & \omega_0 & 0 \end{bmatrix}}_{\hat{\omega} \longrightarrow \text{skew symmetric matrix.}} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix}$$

$q^{(1)}$ and $q^{(2)}$ are world space orientations of body 1 and body 2

Relative orientation in A's frame :=

$$\boxed{q = q^{(1)\dagger} * q^{(2)}}$$

$\dot{q} = \frac{1}{2} \omega * q$ where ω is pure quaternion with ω_v = angular velocity.

$\omega^\dagger = -\omega$

and $(\dot{q})^\dagger = \left(\frac{1}{2} \omega * q\right)^\dagger$

$= \frac{1}{2} q^{(\dagger)} * \omega^\dagger = -\frac{1}{2} q^\dagger * \omega$

$$\dot{q} = \dot{q}^{(1)\dagger} * q^{(2)} + q^{(1)\dagger} * \dot{q}^{(2)}$$

$$= \left(-\tfrac{1}{2} q_1^\dagger * \omega_1\right) * q^{(2)} + q^{(1)\dagger} * \left(-\tfrac{1}{2} \omega_2 * q^{(2)}\right)$$

$$= -\frac{1}{2} q_1^\dagger * \omega_1 * q^{(2)} - \frac{1}{2} q^{(1)\dagger} * \omega_2 * q^{(2)}$$

$$\dot{q} = -\frac{1}{2} q_1^{+} * (\omega_1 - \omega_2) * q^{(2)}$$

Now $\dot{q} = \frac{1}{2} \omega * q$

$$\boxed{\dot{q}\, q^{+} = \frac{1}{2} \omega}$$

multiply both sides by q^{+}

$$\dot{q}\, q^{+} = -\frac{1}{2} q_1^{+} * (\omega_1 - \omega_2) * q^{(2)} * q^{+}$$

$$\frac{1}{2}\omega = -\frac{1}{2} q_1^{+} * (\omega_1 - \omega_2) * q^{(2)} * \left(q^{(1)+} * q^{(2)}\right)^{+}$$

$$\frac{1}{2}\omega = -\frac{1}{2} q_1^{+} * (\omega_1 - \omega_2) * q^{(2)} * q^{(2)+} * \left(q^{(1)+}\right)^{+}$$

$$\frac{1}{2}\omega = -\frac{1}{2} q_1^{+} * (\omega_1 - \omega_2) * q^{(1)}$$

$$\omega = - q_1^{(1)+} * (\omega_1 - \omega_2) * q^{(1)}$$

this is of the form $v' = q * v * q^{+}$ which is basically rotating the vector $\vec{v}$ by the quaternion

this can also be done alternately by multiply the vector $\vec{v}$ by the Rotation matrix R which represents the quaternion

$$q * \vec{v} * q^{+} = R^{(1)} * \vec{v}$$

$$q^{+} * \vec{v} * q = \underbrace{R^{(1)T}}_{\text{transpose}} * \vec{v}$$

$$\omega = - q^{(1)+} * (\omega_1 - \omega_2) * q^{(1)}$$

$$\boxed{\omega = - R^{T} * (\omega_1 - \omega_2)}$$

Now, $\dot{q} = \frac{1}{2} \omega q$

$$= \frac{1}{2} * \text{Right}(q) * \omega$$

$$\text{Right}(q) = \begin{bmatrix} q_0 & -q_{v_x} & -q_{v_y} & -q_{v_z} \\ q_{v_x} & q_0 & q_{v_z} & -q_{v_y} \\ q_{v_y} & -q_{v_z} & q_0 & q_{v_x} \\ q_{v_z} & q_{v_y} & -q_{v_x} & q_0 \end{bmatrix}$$

$$\text{Right}(q) = \begin{bmatrix} q_0 & -q_v^{T} \\ q_v & q_0 I - \hat{q}_v \end{bmatrix}$$

So $\dot{q} = \frac{1}{2} * R(q) * \omega$

$$\boxed{R(q) = \begin{bmatrix} q_0 & -q_v \\ q_v & q_0 I - \hat{q}_v \end{bmatrix}}$$

$$\dot{q} = \frac{1}{2} * \begin{bmatrix} q_0 & -q_v^T \\ q_v & q_0 I - \hat{q}_v \end{bmatrix} \begin{bmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\dot{q} = \frac{1}{2} * \begin{bmatrix} -q_v^T \cdot \omega \\ [q_0 I - \hat{q}_v] \cdot \omega \end{bmatrix}$$ $\underline{\omega \text{ is a 3d vector.}}$

$$\omega = -R^{(1)T} * (\omega_1 - \omega_2)$$

$$\dot{q}_0 = +\frac{1}{2} q_v^T * R^{(1)T} * (\omega_1 - \omega_2)$$

$$\dot{q}_v = -\frac{1}{2} \cdot [q_0 I - \hat{q}_v] * R^{(1)T} (\omega_1 - \omega_2)$$

$\hat{q}_v$ is a skew symmetric matrix.

$$\boxed{\hat{q}_v^T = -\hat{q}_v}$$

So, Jacobian for $\dot{q}_v$ $\underline{G_{1,quat,v}}$.

$$= \begin{bmatrix} 0 & \underbrace{-\frac{1}{2} [q_0 I - \hat{q}_v] * R^{(1)T}} & 0 & \underbrace{\frac{1}{2} [q_0 I - \hat{q}_v] * R^{(1)T}} \end{bmatrix}$$

$$= \begin{Bmatrix} 0 & -\frac{1}{2} \left(R^{(1)} * [q_0 I + \hat{q}_v]\right)^T & 0 & \frac{1}{2} \left[R^{(1)} * (q_0 I + q_v)\right]^T \end{Bmatrix}$$

q_v represents the axis of rotation for the relative quaternion.

For a hinge, this q_v (relative quaternion in the frame of body A) should be parallel to the hinge axis in A's frame.

$\bar{u}_1$ and $\bar{v}_1$ are two vectors orthogonal to h_1 in frame of body A.

So for a hinge the eq^n is that

$$\boxed{q_v \cdot \bar{u}_1 = 0} \quad \text{and} \quad \boxed{q_v \cdot \bar{v}_1 = 0}$$

these 2 constraint equations will make sure that the relative quaternion's axis is parallel to the Hinge Axis.

$$C_1 = q_v \cdot \bar{u}_1$$

$$\dot{C}_1 = \dot{q}_v \cdot \bar{u}_1 + q_v \cdot \dot{\bar{u}}_1$$

$$\boxed{\dot{\bar{u}}_1 = 0}$$

since $\bar{u}_1$ is local to body A, it is constant.
so $\frac{d(\bar{u}_1)}{dt} = 0$

$$\dot{C}_1 = \boxed{\dot{q}_v \cdot \bar{u}_1 = 0} \quad (1)$$

similarly

$$\boxed{\dot{C}_2 = \dot{q}_v \cdot \bar{v}_1 = 0} \quad (2)$$

So, the final Jacobian is :⟶

$$\begin{bmatrix} 0 & -\frac{1}{2}\left(R^{(1)} * [q_0 I + \hat{q}_v]\right)^T \cdot \bar{u}_1 & 0 & \frac{1}{2} * \left(R^{(1)} * [q_0 I + \hat{q}_v]\right)^T \cdot \bar{u}_1 \\ 0 & -\frac{1}{2}\left(R^{(1)} * [q_0 I + \hat{q}_v]\right)^T \cdot \bar{v}_1 & 0 & \frac{1}{2} * \left(R^{(1)} * [q_0 I + \hat{q}_v]\right)^T \cdot \bar{v}_1 \end{bmatrix}$$