On The Reliability Function of Discrete Memoryless Multiple-Access Channel with Feedback

Mohsen Heidari, Achilleas Anastasopoulos, and S. Sandeep Pradhan

***Abstract*—We derive a lower and upper bounds on the reliability function of discrete memoryless multiple-access channel (MAC) with noiseless feedback and variable-length codes (VLCs). For the upper-bound, we use proof techniques of Burnashev for the point-to-point case. Also, we adopt the techniques used to prove the converse for the feedback-capacity of MAC. For the lower-bound on the error exponent, we present a coding scheme consisting of a data and a confirmation stage. In the data stage, any arbitrary feedback capacity-achieving code is used. In the confirmation stage, each transmitter sends one bit of information to the receiver using a pair of codebooks of size two, one for each transmitter. The codewords at this stage are selected randomly according to an appropriately optimized joint probability distribution. The bounds increase linearly with respect to a specific Euclidean distance measure defined between the transmission rate pair and the capacity boundary. The lower and upper bounds match for a class of MACs.**

I. Introduction

Noiseless feedback does not increase the capacity for communications over discrete memoryless channels (DMC) [1]. Furthermore, Dobrushin [4] and later Haroutunian [5] showed that feedback does not improve the error exponent of symmetric channels when fixed-length codes are used. Nevertheless, feedback can be very useful in the context of variable-length codes.

In a remarkable work, Burnashev [2] demonstrated that the error exponent improves for DMCs with feedback and variable-length codes. The error exponent has a simple form

$$E(R) = (1 - \frac{R}{C})C_1, \tag{1}$$

where R is the (average) rate of transmission, C is the capacity of the channel, and C_1 is the maximal relative entropy between conditional output distributions. Berlin et al [6] have provided a simpler derivation of the Burnashev bound that emphasizes the link between the constant C_1 and the binary hypothesis testing problem. Yamamoto and Itoh [7] introduced a coding scheme that its error exponent achieves $E(R)$ in (1). Their scheme consists of two distinct transmission phases that we called the data and the confirmation phase, respectively. In the data stage the message is encoded using a capacity achieving fixed blocklength code. During the confirmation phase, the transmitter sends one bit of information to the receiver. The decoder performs a binary hypothesis test to decide if 0 or 1 is transmitted.

In the context of communications over multi-user channels, the benefits of feedback are more prominent. For instance, Gaarder and Wolf [8] showed that feedback can expand the capacity region of discrete memoryless multiple-access channels (MAC). Willems [9] derived the feedback-capacity region for a class of MACs. Characterizing the capacity region and the error exponent for general MACs remains an open problem. Using *directed information* measures, Kramer [10] was able to characterize the feedback-capacity region of two-user MAC with feedback. However, the characterization is in the form of infinite letter directed information measures which is not computable in general. The error exponent for discrete memoryless MAC without feedback is studied in [13], [14].

In this paper, we study the error exponent of discrete memoryless MAC with noiseless feedback. In particular, we derive an upper-bound and a lower-bound. For that, let $(||\underline{R}||, \theta_R)$ denote the polar coordinate of (R_1, R_2) in $\mathbb{R}^2$. In this setting, the upper-bound is

$$E_u(R_1, R_2) = (1 - \frac{||\underline{R}||}{C(\theta_R)})D_u \tag{2}$$

where $C(\theta_R)$ is the point of the capacity frontier at the angle determined by $\underline{R}$. The lower-bound is the same as E_u but with different constant D_l. The constants D_l and D_u are determined by the relative entropy between the conditional output distributions. We show that for a class of MACs the two bounds coincide.

The paper is organized as follows: In Section II, basic definitions and the problem formulation are provided. In Section III, we derive a lower-bound for the reliability function. In Section IV, we characterize an upper-bound for the reliability function. In Section V, we compare the lower and upper-bound and explore examples for the tightness of the bounds. Finally, Section VI concludes the paper.

II. Problem Formulation and Definitions

Consider a discrete memoryless MAC with input alphabets $\mathcal{X}_1, \mathcal{X}_2$, and output alphabet $\mathcal{Y}$. The channel conditional probability distribution is denoted by $Q(y|x_1, x_2)$ for all $(y, x_1, x_2) \in \mathcal{Y} \times \mathcal{X}_1 \times \mathcal{X}_2$. Such a setup is denoted by $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, Q)$. Let y^t and x_i^t, $i = 1, 2$, be the channel

The authors are with the Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI, 48105 USA e-mail: mohsenhd, anastas, pradhanv@umich.edu

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output and the inputs sequences after t uses of the channel, respectively. Then, the following condition is satisfied:

$$P(y_t|y^{t-1}, x_1^{t-1}, x_2^{t-1}) = Q(y_t|x_{1t}, x_{2t}). \tag{3}$$

We assume that the output of the channel as a feedback is available at the encoders with one unit of delay.

Definition 1. *An (M_1, M_2, N)- variable-length code (VLC) for a MAC $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, Q)$ with feedback is defined by*

- *A pair of messages W_1, W_2 selected randomly with uniform distribution from $\{1, 2, \ldots, M_i\}, i = 1, 2$.*
- *Two sequences of encoding functions*

$$e_{i,t} : \{1, 2, \ldots, M_i\} \times \mathcal{Y}^{t-1} \to \mathcal{X}_i, \quad t \in \mathbb{N},\ i = 1, 2,$$

  one for each transmitter.
- *A sequence of decoding functions*

$$d_t : \mathcal{Y}^t \to \{1, 2, ..., M_1\} \times \{1, 2, ..., M_2\}, \quad t \in \mathbb{N}.$$

- *A stopping time T with respect to (w.r.t) the filtration $\mathcal{F}_t$ defined as the σ-algebra of Y^t for $t \in \mathbb{N}$. Furthermore, it is assumed that T satisfies $\mathbb{E}[T] \leq N$.*

For each $i = 1, 2$, given a message W_i, the tth output of Transmitter i is denoted by $X_{i,t} = e_{i,t}(W_i, Y^{t-1})$.

Let $(\hat{W}_{1,t}, \hat{W}_{2,t}) = d_t(Y^t)$. Then, the decoded messages at the decoder are denoted by $\hat{W}_1 = \hat{W}_{1,T}$, and $\hat{W}_2 = \hat{W}_{2,T}$. In what follows, for any (M_1, M_2, N) VLC, we define average rate-pair, error probability, and error exponent. Average rates for an (M_1, M_2, N) VLC are defined as

$$R_i \triangleq \frac{\log_2 M_i}{\mathbb{E}[T]}, \quad i = 1, 2.$$

The probability of error is defined as

$$P_e = P\left((\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)\right).$$

The error exponent of a VLC with probability of error P_e and stopping time T is defined as $E \triangleq -\frac{\log_2 P_e}{\mathbb{E}[T]}$.

Definition 2. *A reliability function $E(R_1, R_2)$ is said to be achievable for a given MAC, if for any $R_1, R_2 > 0$ and $\epsilon > 0$ there exists an (M_1, M_2, N)-VLC such that*

$$-\frac{\log_2 P_e}{N} \geq E(R_1, R_2) - \epsilon, \text{ and } \frac{\log_2 M_i}{N} \geq R_i - \epsilon,$$

where $i = 1, 2$, and P_e is the error probability of the VLC.

Definition 3. *The reliability function of a MAC with feedback is defined as the supremum of all achievable reliability functions $E(R_1, R_2)$.*

A. The Feedback-Capacity Region of MAC

We summarize Kramer's results presented in [10] for the feedback capacity of MAC. We use *directed information* and *conditional directed information* as defined in [10]. The normalized directed information from a sequence $\mathbf{X}^n$ to a sequence $\mathbf{Y}^n$ when causally conditioned on $\mathbf{Z}^n$ is denoted by

$$I_n(X \to Y||Z) \triangleq \frac{1}{n} I(\mathbf{X}^n \to \mathbf{Y}^n||\mathbf{Z}^n). \tag{4}$$

The feedback-capacity region of a discrete memoryless MAC with feedback $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, Q)$ is denoted by $\mathcal{C}$, and is the closure of the set of all rate-pairs (R_1, R_2) such that

$$\begin{aligned} R_1 &\leq I_L(X_1 \to Y||X_2) \\ R_2 &\leq I_L(X_2 \to Y||X_1) \\ R_1 + R_2 &\leq I_L(X_1 X_2 \to Y), \end{aligned}$$

where L is a positive integer, and $P_{X_1^L X_2^L Y^L}$ factors as

$$\prod_{l=1}^{L} P_{1,l}(x_{1l}|x_1^{l-1} y^{l-1}) P_{2,l}(x_{2l}|x_2^{l-1} y^{l-1}) Q(y_l|x_{1,l} x_{2,l}). \tag{5}$$

Definition 4. *Let $\lambda_1, \lambda_2, \lambda_3 \geq 0$, and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Define*

$$\begin{aligned} C_{\underline{\lambda}} = \sup_{L \in \mathbb{N}} \sup_{P_{X_1^L X_2^L Y^L}} & \lambda_1 I_L(X_1 \to Y|X_2) + \lambda_2 I_L(X_2 \to Y|X_1) \\ &+ \lambda_3 I_L(X_1 X_2 \to Y), \end{aligned}$$

where $P_{X_1^L X_2^L Y^L}$ factors as in (5).

Fact 1. The feedback-capacity of a discrete memoryless MAC with feedback is the same as the closure of the set of rate-pairs (R_1, R_2) such that the inequality

$$\lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 (R_1 + R_2) \leq C_{\underline{\lambda}}$$

holds for all $\lambda_1, \lambda_2, \lambda_3 \geq 0$, with $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

B. Notational Conventions

For more convenience, we denote a rate-pair (R_1, R_2) by (R_1, R_2, R_3), where $R_3 = R_1 + R_2$. For a $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, Q)$ MAC we use the following notational convenience

$$I_L^1 \triangleq I_L(X_1 \to Y||X_2), \tag{6}$$
$$I_L^2 \triangleq I_L(X_2 \to Y||X_1), \tag{7}$$
$$I_L^3 \triangleq I_L(X_1 X_2 \to Y). \tag{8}$$

The Kullback–Leibler divergence for the MAC with transition probability matrix Q is defined as

$$D_Q(x_1, x_2||z_1, z_2) = \sum_{y \in \mathcal{Y}} Q(y|x_1, x_2) \log_2 \frac{Q(y|x_1, x_2)}{Q(y|z_1, z_2)},$$

where $(x_1, x_2), (z_1, z_2) \in \mathcal{X}_1 \times \mathcal{X}_2$. For notational convenience we denote

$$\begin{aligned} D_1(x_1, x_2||z_1, z_2) &= D_Q(x_1, x_2||z_1, x_2) \\ D_2(x_1, x_2||z_1, z_2) &= D_Q(x_1, x_2||x_1, z_2) \\ D_3(x_1, x_2||z_1, z_2) &= D_Q(x_1, x_2||z_1, z_2). \end{aligned}$$

III. A Lower-Bound for the Reliability Function

We build upon Yamamoto-Itoh transmission scheme for point-to-point (ptp) channel coding with feedback [7]. The scheme sends the messages W_1, W_2 through blocks of length n. The transmission process is performed in two stages: 1) The "data transmission" stage taking up to $n(1-\gamma)$ channel uses, 2) The "confirmation" stage taking up to $n\gamma$ channel uses, where γ is a design parameter taking values from $[0,1]$.

Stage 1: For the first stage, we use any coding scheme that achieves the feedback-capacity of the MAC. The length of this coding scheme is at most $n(1-\gamma)$. Let $\hat{W}_1, \hat{W}_2$ denote the decoder's estimation of the messages at the end of the first stage. Define the following random variables:

$$H_i = 1\{\hat{W}_i \neq W_i\}, \quad i = 1, 2.$$

Because of the feedback, $\hat{W}_1$ and $\hat{W}_2$ are known at each transmitter. Therefore, at the end of the first stage, transmitter i has access to $W_i, \hat{W}_1, \hat{W}_2$, and H_i, where $i = 1, 2$.

Stage 2: The objective of the second stage is to inform the receiver whether the hypothesis $\Theta_0 : (\hat{W}_1, \hat{W}_2) = (W_1, W_2)$ or $\Theta_1 : (\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)$ is correct. For that, each transmitter employs a code of size two and length γn. The codewords of such codebooks are denoted by two pairs of sequences $(\underline{x_1}(0), \underline{x_2}(0))$ and $(\underline{x_1}(1), \underline{x_2}(1))$ each with elements belonging to $\mathcal{X}_1 \times \mathcal{X}_2$. Fix a joint-type P_n defined over the set $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_1 \times \mathcal{X}_2$ and for sequences of length γn. The sequences $(\underline{x_1}(0), \underline{x_2}(0), \underline{x_1}(1), \underline{x_2}(1))$ are selected randomly among all the sequences with joint-type P_n. During this stage and given H_1, Transmitter 1 sends $\underline{x_1}(H_1)$. Similarly, Transmitter 2 sends $\underline{x_2}(H_2)$.

Decoding: Upon receiving the channel output, the receiver estimates H_1, H_2. Denote this estimation by $\hat{H}_1, \hat{H}_2$. If $(\hat{H}_1, \hat{H}_2) = (0, 0)$, then the hypothesis $\hat{\Theta} = \Theta_0$ is declared. Otherwise, $\hat{\Theta} = \Theta_1$ is declared. Because of the feedback, $\hat{\Theta}$ is also available at each encoders. If $\hat{\Theta} = \Theta_0$, then transmission stops and a new data packet is transmitted at the next block. Otherwise, the message is transmitted again at the next block. The process continues until $\hat{\Theta} = \Theta_0$ occurs.

The confirmation stage in the proposed scheme can be viewed as a decentralized binary hypothesis problem in which a binary hypothesis $\{\Theta_0, \Theta_1\}$ is observed partially by two distributed agents and the objective is to convey the true hypothesis to a central receiver. This problem is qualitatively different from the sequential binary hypothesis testing problem as identified in [6] for ptp channel. Note also that in the confirmation stage we use a different coding strategy than the one used in Yamamoto-Itoh scheme [7]. Here, all four codewords have a joint-type P_n. It can be shown that repetition codes, and more generally, constant composition codes are strictly suboptimal in this problem.

Theorem 1. *The following is a lower-bound for the reliability function of any discrete memoryless MAC:*

$$E_l(R_1, R_2) = \min_{\substack{\lambda_1, \lambda_2, \lambda_3 \geq 0 \\ \lambda_1 + \lambda_2 + \lambda_3 = 1}} D_l\Big(1 - \frac{\sum_i \lambda_i R_i}{C_{\underline{\lambda}}}\Big), \tag{9}$$

where,

$$D_l \triangleq \sup_{P_{X_1 X_2 Z_1 Z_2}} \min_{i=1,2,3} \mathbb{E}\left[D_i(X_1, X_2 || Z_1, Z_2)\right], \tag{10}$$

and the supremum is taken over all probability distributions $P_{X_1 X_2 Z_1 Z_2}$ defined over $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_1 \times \mathcal{X}_2$.

IV. An Upper-bound for the Reliability Function

In this part of the paper, we establish an upper-bound for the reliability function of any discrete memoryless MAC. Define

$$D_i \triangleq \max_{\substack{x_1, z_1 \in \mathcal{X}_1, \\ x_2, z_2 \in \mathcal{X}_2}} D_i(x_1, x_2 || z_1, z_2), \quad i = 1, 2, 3. \tag{11}$$

Theorem 2 (Upper-bound). *For any (N, M_1, M_2) VLC with probability of error P_e, and any $\epsilon > 0$, there exists a function δ such that the following is an upper-bound for the reliability function of the VLC*

$$\begin{aligned} E(R_1, R_2) \leq & \min_{\substack{\lambda_1, \lambda_2, \lambda_3 \geq 0 \\ \lambda_1 + \lambda_2 + \lambda_3 = 1}} \min_{j \in \{1,2,3\}} D_j \left(1 - \frac{\lambda_j R_j}{C_\lambda}\right) \\ & + \delta(P_e, M_1 M_2, \epsilon), \end{aligned} \tag{12}$$

where (R_1, R_2) is the rate pair of the VLC and δ satisfies

$$\lim_{\epsilon \to 0} \lim_{P_e \to 0} \lim_{M_1 M_2 \to \infty} \delta(P_e, M_1 M_2, \epsilon) = 0.$$

Corollary 1. *From Theorem 2, the following is an upper-bound for the error exponent of a MAC:*

$$E_u(R_1, R_2) = \min_{\substack{\lambda_1, \lambda_2, \lambda_3 \geq 0 \\ \lambda_1 + \lambda_2 + \lambda_3 = 1}} D_u \left(1 - \frac{\sum_{i=1}^3 \lambda_i R_i}{C_\lambda}\right) + \delta,$$

where $D_u = \max\{D_1, D_2, D_3\}$, and δ is as in Theorem 2.

A. Proof of the Upper-Bound

Consider any (N, M_1, M_2) VLC with probability of error P_e, and stopping time T. Suppose the message W_2 at Encoder 2 is made available to all terminals. For the new setup, as W_2 is available at Decoder, the average probability of error is $P_e^1 \triangleq P\{\hat{W}_1 \neq W_1\}$. Note that $P_e \geq P_e^1$. We refer to such setup as W_2-assisted MAC. Given the stochastic process $\{Y^t, W_2\}_{t>0}$, define the following stopping time

$$T_1^\delta \triangleq \inf\big\{n : \max_{1 \leq i \leq M_1} P(W_1 = i | Y^n, W_2) \geq 1 - \delta\big\},$$

where $\delta > 0$ is a fixed real number. Let $\tau_1 \triangleq \min\{T, T_1^\delta\}$, where T is the original stopping time defined for the chosen (N, M_1, M_2) VLC. Note that τ_1 is a stopping time w.r.t the filtration $\{\mathcal{F}_{W_2} \times \mathcal{F}_t\}_{t>0}$. The following lemma provides a lower-bound on the probability of error for such setup.

Lemma 1. *Given a MAC with $D_3 < \infty$, and for any (N, M_1, M_2) VLC with probability of error P_e the following holds*

$$P_e \geq \frac{\zeta \delta}{4} e^{-D_1 \mathbb{E}[T - \tau_1]}, \tag{13}$$

where $\zeta \triangleq \min_{x_1, x_2, y} Q(y | x_1, x_2)$.

Next, we apply the same argument for the case where W_1 is available at all the terminals. For that, define

$$T_2^\delta \triangleq \inf\{n : \max_{1\le j\le M_2} P(W_2=j|Y^n, W_1)\ge 1-\delta\},$$

and let $\tau_2 \triangleq \min\{T, T_2^\delta\}$. By symmetry, Lemma 1 holds for this case and we obtain

$$P_e \ge \frac{\zeta\delta}{4} e^{-D_2\mathbb{E}[T-\tau_2]}. \tag{14}$$

Next, define the following stopping time

$$T_3^\delta \triangleq \inf\{n : \max_{i,j} P(W_1=i, W_2=j|Y^n)\ge 1-\delta\}.$$

and let $\tau_3 = \min\{T, T_3^\delta\}$. Using a similar argument as in the above, we can show that

$$P_e \ge \frac{\zeta\delta}{4} e^{-D_3\mathbb{E}[T-\tau_3]}. \tag{15}$$

Therefore, the maximum of the right-hand sides of (13), (14) and (15) gives a lower-bound on P_e. The lower-bound depends on the expectation of the stopping times $\tau_i, i=1,2,3$. In what follows, we provide a lower-bound on $\mathbb{E}[\tau_i]$. For $t>0$, define the following random processes:

$$\begin{aligned} H_t^1 &\triangleq H(W_1|\ \mathcal{F}_{W_2}\times\mathcal{F}_t),\\ H_t^2 &\triangleq H(W_2|\ \mathcal{F}_{W_1}\times\mathcal{F}_t),\\ H_t^3 &\triangleq H(W_1, W_2|\ \mathcal{F}_t). \end{aligned}$$

Lemma 2. *Given a (M_1, M_2, N)-VLC, for any $\epsilon>0$ there exist L and a probability distribution $P_{X_1^L X_2^L Y^L}$ that factors as in* (5) *such that the following inequalities hold almost surely for $1\le t\le N$*

$$\begin{aligned} \mathbb{E}[H_{t+1}^1 - H_t^1|\mathcal{F}_{W_2}\times\mathcal{F}_t] &\ge -(I_L^1+\epsilon),\\ \mathbb{E}[H_{t+1}^2 - H_t^2|\mathcal{F}_{W_1}\times\mathcal{F}_t] &\ge -(I_L^2+\epsilon),\\ \mathbb{E}[H_{t+1}^3 - H_t^3|\mathcal{F}_t] &\ge -(I_L^3+\epsilon), \end{aligned}$$

where $i=1,2,3$, and I_L^i is defined as in (6)-(8).

Lemma 3. *For any $t\ge 1$ and $i=1,2,3$, the following inequality holds almost surely w.r.t $\mathcal{F}_{W_1}\times\mathcal{F}_{W_2}\times\mathcal{F}_t$*

$$\log H_t^i - \log H_{t+1}^i \le \max_{\substack{j,l\in[1:M_1]\\ k,m\in[1:M_2]}} \max_{y\in\mathcal{Y}} \frac{\hat{Q}_{j,k}(y)}{\hat{Q}_{l,m}(y)},$$

where $\hat{Q}_{j,k}(y) \triangleq P(W_1=j, W_2=k|Y=y)$ for all $i\in[1:M_1], j\in[1:M_2]$ and $y\in\mathcal{Y}$.

From Lemma 2 and the fact that $H_t^i \le \log_2 M_i < \infty$, the processes $\{H_t^i + (I_L^1+\epsilon)t\}_{t>0}$ are submartingales for $i=1,2,3$. In addition, from Lemma 3 and the inequalities $\mathbb{E}[\tau_i]\le\mathbb{E}[T]\le N<\infty$, we can apply Doob's Optional Stopping Theorem for each submartingale $\{H_t^i+(I_L^1+\epsilon)t\}_{t>0}$. Then, we get:

$$\log M_i \le \mathbb{E}[H_{\tau_i}^i] + \mathbb{E}[\tau_i](I_L^i+\epsilon) \tag{16}$$

where $M_3 \triangleq M_1M_2$. In this context, $\log M_i, i=1,2,3$, can be viewed as the entropy at time $t=0$ that is H_0^i.

Lemma 4. *The following inequality holds for each $i=1,2,3$*

$$\mathbb{E}[H_{\tau_i}^i] \le h_b(\delta) + (\delta + \frac{P_e}{\delta})\log_2 M_i.$$

As a result of the above lemma and (16), the inequality $\mathbb{E}[\tau_i] \ge \frac{\log M_i}{I_L^i+\epsilon} - \frac{h_b(\delta)}{I_L^i+\epsilon}$ holds. Finally, combining this inequality with (13)-(15) completes the proof of the theorem.

V. The Shape of the Lower and Upper Bounds

In this Section, we point out a few remarks on $E_u(R_1,R_2)$ and the lower-bound $E_l(R_1,R_2)$ defined in Theorem 1. Furthermore, we provide an alternative representation for the bounds and show that the lower and upper-bounds match for a class of MACs.

We first provide an alternative representation for the lower/upper-bound. For that, suppose (R_1,R_2) is a point inside the capacity region $\mathcal{C}$. By $(||\underline{R}||, \theta_R)$ denote the polar coordinate of (R_1,R_2) in $\mathbb{R}^2$. It is shown in the following Remark that the optimum $\underline{\lambda}$ in E_u and E_l is independent of the Euclidean norm of (R_1,R_2), i.e., $\|\underline{R}\|$.

Remark 1. Given an arbitrary $\alpha>0$ and a rate pair (R_1,R_2) in the capacity region, the optimum $\underline{\lambda}$ for $E_l(R_1,R_2)$ is the same as the one for $E_l(\alpha R_1, \alpha R_2)$.

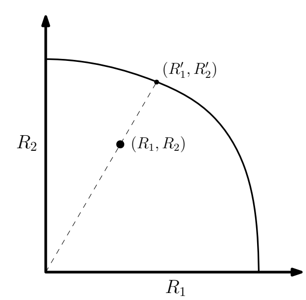

Fig. 1. Given a rate pair (R_1,R_2) which is inside the capacity region, consider the line passing (R_1,R_2) and the origin. Then, (R_1',R_2') is the point of intersection of this line with the boundary of the capacity region.

Now, consider the line passing (R_1,R_2) and the origin. Let (R_1',R_2') denote the point of intersection of this line with the boundary of the capacity region. Fig. 1 shows how (R_1',R_2') is determined. Since, $R_i' = \alpha R_i, i=1,2$ for some $\alpha>0$, then the optimum $\underline{\lambda}$ in $E_l(R_1',R_2')$ is the same as the one in $E_l(R_1,R_2)$. Therefore, from this argument and the fact that $R_i = \frac{R_i'}{\alpha}, i=1,2$, we can rewrite $E_l(R_1,R_2)$ as

$$\begin{aligned} E_l(R_1,R_2) &= \min_{\substack{\lambda_1,\lambda_2,\lambda_3\ge 0\\ \lambda_1+\lambda_2+\lambda_3=1}} D_l\left(1-\frac{1}{\alpha}\frac{\sum_{i=1}^3 \lambda_i R_i'}{C_\lambda}\right),\\ &\overset{(a)}{=} D_l\left(1-\frac{1}{\alpha}\right), \end{aligned}$$

where (a) follows, since (R_1',R_2') is on the capacity boundary. Note that $\alpha = \frac{\|\underline{R}\|}{\|\underline{R'}\|}$. Therefore, $E_l(R_1,R_2) = D_l\left(1-\frac{\|\underline{R}\|}{\|\underline{R'}\|}\right)$. Moreover, note that $\|\underline{R'}\|$ depends on (R_1,R_2) only through

θ_R; in particular, it equals to $C(\theta_R)$ which is a function of θ_R. With this notation, we can rewrite E_l as

$$E_l(R_1, R_2) = D_l\left(1 - \frac{\|\underline{R}\|}{C(\theta_R)}\right)$$

Using a similar argument for E_u, we have

$$E_u(R_1, R_2) = D_u\left(1 - \frac{\|\underline{R}\|}{C(\theta_R)}\right) + \delta.$$

As a conclusion of the above argument, the lower (upper) bound increases linearly with respect to a specific Euclidean distance measure defined between the transmission rate pair and the capacity boundary.

A. On the Tightness of the Bounds on the Error Exponent

In what follows, we provide examples of classes of channels for which the lower and upper bound coincide.

Example 1. Consider a MAC in which the output is (Y_1, Y_2) and the transition probability matrix is described by the product $Q_{Y_1|X_1}Q_{Y_2|X_2}$. This MAC consists of two parallel (independent) point-to-point channels. Suppose, C_1 and C_2 are the capacity of the first and the second parallel channel, respectively. For this MAC, one can use two parallel Yamamoto-Itoh schemes, one for each channel. Based on the results for the point-to-point case, it is not difficult to show that the error exponent for such MAC satisfies

$$E(R_1, R_2) \geq \min\{D_1(1 - \frac{R_1}{C_1}), D_2(1 - \frac{R_2}{C_2})\}, \qquad (17)$$

where C_1 and C_2 are the point-to-point capacity of the channel corresponding to $Q_{Y_1|X_1}$ and $Q_{Y_2|X_2}$, respectively. Note that this lower-bound is not covered by the proposed coding strategy given in Section III. For such MAC, the upper-bound given in (12) is simplified to

$$E(R_1, R_2) \leq \min_{\lambda_1, \lambda_2 \geq 0} \min_{j \in \{1,2\}} D_j\left(1 - \frac{\lambda_j R_j}{\lambda_1 C_1 + \lambda_2 C_2}\right) + \delta.$$

The right-hand side of the above inequality is further upper-bounded by substituting $(\lambda_1, \lambda_2) = (0, 1)$ or $(\lambda_1, \lambda_2) = (1, 0)$. Therefore, we obtain

$$E(R_1, R_2) \leq \min_{j \in \{1,2\}} D_j\left(1 - \frac{R_j}{C_j}\right) + \delta$$

By letting $\delta \to 0$ as in Theorem 2, the above bound can be made arbitrary close to the lower-bound given in (17).

Example 2. For an integer $m > 2$, consider an m-ary additive MAC, as is shown in Fig. 2, where the transition probability of the channel is described by

$$Y = X_1 \oplus_m X_2 \oplus_m N_p,$$

where all the random variables take values from $\mathbb{Z}_m$, and N_p is a random variable with $P(N_p = i) = p$ for any $i \in \mathbb{Z}_m,\ i \neq 0$ and $P(N_p = 0) = 1 - (m-1)p$. It can be shown that for this channel

$$D_l = D_u = (1 - mp)\log\frac{1 - (m-1)p}{p}.$$

Hence, the upper-bound in Corollary 1 can be made arbitrary close to the lower-bound in Theorem 1.

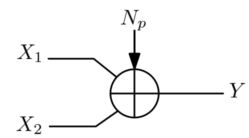

Fig. 2. The diagram of an m-ary additive MAC as described in Example 2.

VI. Conclusion

We derive lower and upper bounds on the reliability function of discrete memoryless MAC with noiseless feedback and variable-length codes. For the lower-bound, we adapt Yamamoto and Itoh's coding scheme consisting of a data and a confirmation phase. For the upper-bound, we adopt the proof techniques of Burnashev for the reliability function of the point-to-point case. The two bounds have the same shape with the difference being the constants at zero rate. We show that the bounds are tight for a class of MACs.

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