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Modem based on sphere packing techniques in high-dimensional Euclidian sub-space for efficient data over voice communication through mobile voice channels

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Reza Kazemi[1] ✉, Mahdi Boloursaz Mashhadi[1,2], Mohsen Heidari Khoozani[1], Fereydoon Behnia[1]

[1]Department of Electronic and Electrical Engineering, Sharif University of Technology, Tehran 11365-9363, Islamic Republic of Iran
[2]Department of Electronic and Electrical Engineering, Advanced Communications Research Institute (ACRI), Sharif University of Technology, Tehran 11365-9363, Islamic Republic of Iran
✉ *E-mail: reza.kazemi.2014@ieee.org*

Abstract: The increased penetration of cellular networks has made voice channels widely available ubiquitously. On the other hand, mobile voice channels possess properties that make them an ideal choice for high priority, low-rate real-time communications. Mobile voice channel with the mentioned properties, could be utilised in emergency applications in vehicular communications area such as the standardised emergency call system planned to be launched in 2015. This study aims to investigate the challenges of data transmission through these channels and proposes an efficient data transfer structure. To this end, a proper statistical model for the channel distortion is proposed and an optimum detector is derived considering the proposed channel model. Optimum symbols are also designed according to the derived rule and analytical bounds on error probability are obtained for the orthogonal signaling and sphere packing techniques. Moreover, analytical evaluation is performed and appropriate simulation results are presented. Finally, it is observed that the proposed structure based on the sphere packing technique achieves superior performance compared with prior works in this field. Although the ideas offered in this study are utilised to cope with voice channel non-idealities, the steps taken in this study could also be applied to channels with similar conditions.

1 Introduction

The rapid penetration of mobile communications in the past decades has led to the development of mobile network infrastructure. This progress has encouraged a good body of research and investments to make the most out of this existing widespread infrastructure by introducing new value added services. As mentioned in [1–3], mobile voice channels benefit from higher priority for acquiring the channel, widespread coverage, availability, reliability and faster connection establishment. These features can meaningfully encourage the use of the mobile voice channel for data communication in low bit-rate high priority services in which it is essential to transmit the required information as soon as possible.

This paper aims at proposing a novel modem for data communication through mobile voice channels. It should be pointed out that the modem expressed in this paper is certainly not a competitor or substitute for data transmission protocols such as general packet radio service, high-speed downlink packet access and so on. However, as a supplementary system to be utilised in specific situations. The broader coverage and higher priority of mobile voice channels are mostly of interest in vehicular applications such as emergency call (eCall) service which is envisaged to operate in both congested and outlying areas. eCall is a system that transmits an automated message over the cellular network voice channel to the public safety answering point following a road crash which includes the minimum set of data informing the precise crash location, airbag deployment and other relevant information. An implementation of this system is the pan-European eCall system scheduled for deployment in 2015 [4, 5]. Another value added service is the WikiWalk system in which the mobile phone is equipped with a small device that modulates the location data and transmits them through the mobile voice channel of the cellular phone to the WikiWalk server in order to receive appropriate voiced guidance instructions [6].

Some other applications are addressed as follows. Katugampala *et al.* [1, 7, 8] designed a data modem for global system of mobile (GSM) voice channel in order to transmit encrypted end-to-end voice or data securely. A new method for transmission of point of sale (POS) transactions from POS terminals to financial hosts over GSM voice channel was introduced in [9]. This channel has also been introduced for network address translator (NAT) traversal allowing mobile users to set up a direct peer-to-peer (P2P) connection with minimal user intervention and without connecting to a middle server by making an end-to-end hole through the NATs [10]. Furthermore, a voice channel can be utilised in vehicular networks for emergency data communications. Other possible applications are online text and multimedia messaging, telemetry, real-time monitoring systems and so on [3].

The applications mentioned above are depicted in Fig. 1. As observed in this figure, all these applications share the common point of utilising the mobile voice channel as a medium for low-rate, high priority digital communications which is addressed by the data over voice (DoV) technology [11]. This technology deals with efficient methods of data communication through commonly used voice channels in wireless communication. The idea of DoV technology was revealed by the early works of Katugampala *et al.* [1] in 2003 and followed up by other researches referenced in the following literature review. Moreover, the international patents registered by Preston *et al.* in 2006 [12], Dorr in 2007 [13], Eatherly *et al.* in 2011 [14], Kondoz *et al.* in 2013 [15] and the standards issued by European Telecommunications Standards Institute (ETSI) in 2009, 2011 [16, 17] increased the practical utilisation of this technology.

Despite the above-mentioned rewarding applications, data communication through mobile voice channels or equivalently from the voice codec of cellular network is complicated because of several limitations to be discussed shortly.

First, the linear predictive coding (LPC) based speech coding process is a lossy compression technique that guarantees a level of

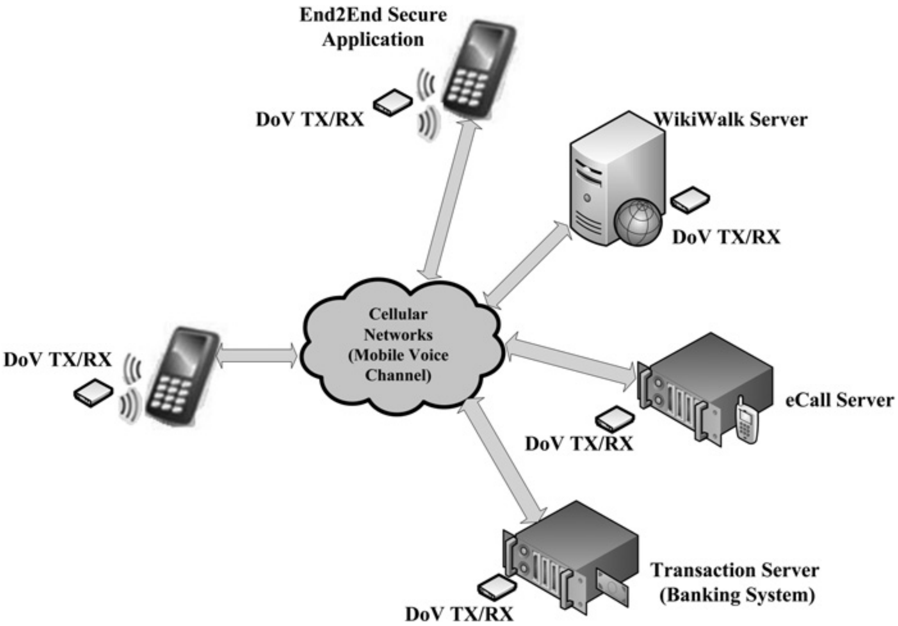

Fig. 1 *General applications of DoV technology*

correlation between the reconstructed speech and the original one. The compression process just guarantees that the two signals will sound similar to a human listener regardless of how different the samples of the two signals may be. This lossy compression introduces notable distortion in the signal. Second, the vocoder speech coding process involves multiple lossy vector and non-vector quantisation steps, which cause additional distortion to the original speech waveform. Third, differential encoding of the speech parameters in the vocoder compression process causes the signal to have memory. In fact, the vocoder compression/decompression process is a non-linear process with a long memory which makes it practically impossible to be represented by an analytic transfer function. Fourth, the linear prediction and differential encoding of speech parameters used in the vocoder compression process both assume strong correlation between input samples which are true for voice, but not for common data signals. Consequently, the more data transmitted, the less correlated the samples are. Moreover, the less the signal fits into the vocoder speech model, the more distortions caused by vocoder and finally the higher the detection error rate. Finally, the vocoder channel is anti-causal because the channel output for each sample depends on preceding and following samples in the same frame.

Previous studies on data transmission over voice channels have taken three different approaches to resolve the mentioned channels' non-idealities. Katugampala *et al.* [1, 7, 8], Ozkan *et al.* [18] and Rashidi *et al.* [19] considered pitch or formant frequency, line spectral frequencies and energy of voice frames as the parameters to be adapted according to the modulated message bits. An analogous approach was proposed by [15, 20] which maps the input data stream on algebraic code excited linear prediction pulse positions. The highest achievable data rate among these methods was 3 kbps with a bit error rate (BER) of 1.2% achieved by [20] in the GSM enhanced full rate (EFR) voice channel. The mentioned researches involve mapping of the input bits onto speech parameters of a certain speech production model and are usually addressed by parameter adaptation approaches. The second approach includes the researches trying to find appropriate independent speech like codewords. LaDue *et al.* [2] and Sapozhnykov and Fienberg [3] optimised a set of codewords by utilising genetic and pattern search algorithms. Shahbazi *et al.* [21] used exhaustive search algorithm over observed human speech waveforms to find optimum symbols. Boloursaz *et al.* [22–24] and Kazemi *et al.* [25] proposed a heuristic algorithm optimising the codebook for a maximised capacity. The highest achieved data rate of this group is 4 kbps with the symbol error rate (SER) of 2.5%. In another approach, the parameters of well-known digital modulation techniques are customised to cope with channel non-idealities. Chmayssani and Baudoin [26] utilised common frequency shift keying (FSK) and quadrature amplitude modulation techniques. Chmayssani and Baudoin [26] and Dhananjay *et al.* [27] introduced the phase continuous context dependent orthogonal frequency division multiplexing (OFDM) amplitude shift keying and Hermes (modified FSK) methods, respectively. Finally, Chen and Guo [28] proposed the OFDM-based method of modulating the message stream on orthogonal multi-frequency sinusoidal carriers. Ali *et al.* [6] introduced a new method based on M-ary frequency shift keying (M-FSK) modulation for data transmission over mobile voice channel. The highest achieved data rate of this group is 3 kbps with a BER of 3×10^{-3} on GSM EFR voice channel. The three methods mentioned above share a common imperfection. They all presume the decision rule beforehand and try to optimise the symbols or other parameters of the presumed model to achieve a locally optimum performance. For instance, the 'parameter adaptation' method priority presumes a certain Euclidian distance as the symbol detection criteria. Similarly, the 'codebook optimisation' method introduced by Ladue *et al.* [2] presumes the correlation as a decision rule which means the assumption of independent identically distributed (i.i.d.) additive Gaussian noise. Finally, the 'modulation optimisation' method is limited to conventional digital modulations. These limitations and prior assumptions may prevent one from achieving the globally optimum detector. The scope of this paper is to analytically investigate the challenges of data transmission over mobile voice channels and propose a framework for efficient data transferring through the mentioned channel .For this purpose, the statistical behaviour of the channel is analysed to model the channel properly. Accordingly, a novel optimum receiver is proposed based on the derived model of the channel. As the next step, the design of optimal symbols for data transmission is discussed and two main approaches are suggested. The first approach is applying orthogonal codes such as Hadamard, Gold and Optical Orthogonal Codes (OOC) [29]. In the second approach, as a more general case, symbols are designed based on sphere packing techniques [30]. After completing the structure of the proposed modem, performance and its error probability in different conditions and data transmission rates is assessed and it is revealed that the proposed modem based on the sphere packing technique outperforms the previously reported researches. The rest of this paper is organised as follows: in Section 2, the statistical model of the channel is derived and a new optimum receiver is designed accordingly. Moreover, the probability of error for this receiver is calculated. In Section 3, the two methods for symbol design are described and several relevant techniques are introduced, thereafter. Section 4, analyses performance of the proposed modem for different approaches and provides bounds on the probability of error for each case. Furthermore, simulation results of the proposed modem are discussed in this section. Finally, Section 5 concludes this paper and highlights the future works.

2 Optimum modem design

In this section, we propose a general approach to cope with non-idealities of voice channel. The proposed method can be applied to other non-linear channels with similar restrictions, but the discussed results are given for adaptive multi rate (AMR) voice codec, which is the one used in later generations of cellular networks. The first step towards this goal is to model the channel in terms of its statistical behaviour, after which the optimum receiver can be designed accordingly with regard to a posterior probability rule. As mentioned earlier, voice dedicated channels are channels with long memory which means the channel output for each specific sample s_j depends not only on the current transmitted symbol S^l, but also on previously transmitted symbols S^{l-1}, S^{l-2}, …. This dependency adds some kind of randomness to the channel output and can be modelled by a probability density function (pdf). This pdf is estimated by simulation experiments on channel simulator [31]. In these simulations, a specific symbol S^l is transmitted through the channel for a statistically sufficient number of times, each time with a different random set of previous symbols and the output histogram is sketched for a specific sample $s_j(f(s_j|S^l))$. This histogram is usually curve-fitted by the best possible Gaussian distribution as demonstrated in Fig. 1.

The problem is, as Fig. 2 shows, the received samples do not have a well-defined pdf function. Moreover, the joint pdf of samples, that is, $f(S^l)$ is unknown. Therefore it is complicated to derive an analytical model for the channel behaviour and design an optimum modem for it. To resolve this challenge, one approach is to approximate the distribution of the received samples by a conventional pdf or alternatively use an appropriate feature of the received samples which not only has a mathematically well behaving distribution, but also contains almost all of the necessary information from the received samples. Ladue *et al.* [2] selected the first approach and assumed that the received signal has i.i.d. Gaussian distribution; however, as seen in Fig. 1, this is not a

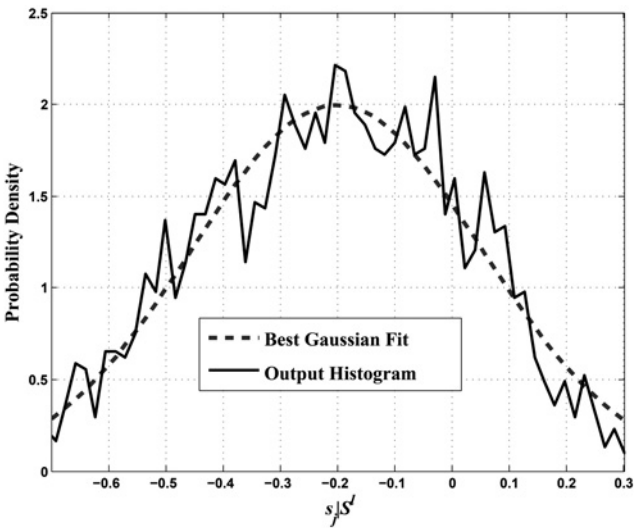

Fig. 2 *Approximate pdf of channel output samples*

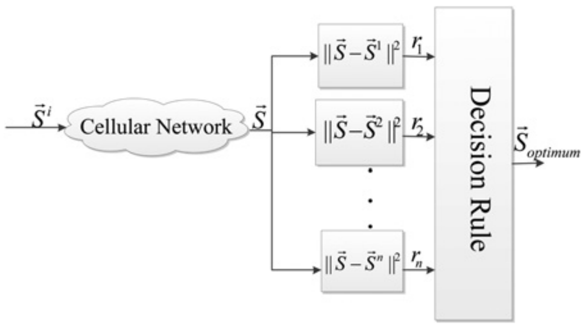

Fig. 3 *Structure of the receiver*

good model for the channel behaviour at least considering single sample distribution. Considering problems of the first approach, we have selected the second approach and propose a new feature to achieve a more accurate approximation of the channel model. We use the Euclidean distance $\|S - S^i\|$ instead of S on the receiver side for decision making. In what follows, we show that this feature leads to a well-defined probability distribution for the received signal while maintaining almost all of its information. Fig. 3 demonstrates the structure of the proposed approach. Defining $r_0, r_1, \ldots, r_M$ as the distance between S and $S^1, \ldots, S^N$, we have

$$r_i = \|S - S^i\| = \sum_{j=1}^{k} \left(s_j - s_j^i\right)^2 \tag{1}$$

where s_j and s_j^i are the jth element of the transmitted and received signal vectors (i.e. S and S^i), respectively, and k is the length of the vector S.

Since the transmitted symbols contain k samples, the signalling space will have at most k dimensions. Now, in the case of orthogonal equal energy signalling, if M is equal to k, the transmitted signals are base-vectors of the sub-space and consequently r_i is the projection of the received signal on S^i and no information will get lost during this space transformation. In addition, as shown in Appendix, distribution of s_j is a distorted Gaussian pdf (Fig. 2), and r_i has a chi-square distribution with K degree of freedom. Therefore, as mentioned before, the selected feature not only leads to a well-defined probability distribution, but also keeps almost all of the information. As outlined in the previous paragraphs, assuming the jth symbol is transmitted, $f(r_i|S^j)$ is a central or non-central chi-squared distribution as below

$$f(r_i|\boldsymbol{s}^j) = \begin{cases} \dfrac{1}{\sigma_1^K 2^{K/2}\Gamma(K/2)} r_i^{(K/2)-1} \mathrm{e}^{-\left(r_i/2\sigma_1^2\right)}, & i = j \\ \dfrac{1}{2\sigma_2^2} \mathrm{e}^{(r_i+b)/\left(2a_2^2\right)} \left(\dfrac{r_i}{\lambda}\right)^{(K/4)-(1/2)} I_{(K/2)-1}\sqrt{\dfrac{\lambda r_i}{\sigma_2^4}}, & i \neq j \end{cases} \tag{2}$$

where $\lambda = \|S^i - S^j\|$. Moreover, σ_1 and σ_2 are constants depending on the characteristics of the applied vocoder. These two parameters show the standard deviation of distortion when the transmitted and received symbols are matched and not matched, respectively, so obviously $\sigma_1 < \sigma_2$. Fig. 4 demonstrates the squared output histogram and the best $\chi^2(1)$ distribution that can be fitted on it.

In this stage, after determining the pdf of the decision variables by a mathematical statement, the optimum receiver can be derived using maximum a posteriori (MAP) rule for r_i's

$$\text{MAP: } \arg\max_i f\left(r_1, \ldots, r_M|\boldsymbol{s}^i\right) \tag{3}$$

Depending on the signalling method used, two cases are considered: orthogonal and general signalling. In the following sections, we

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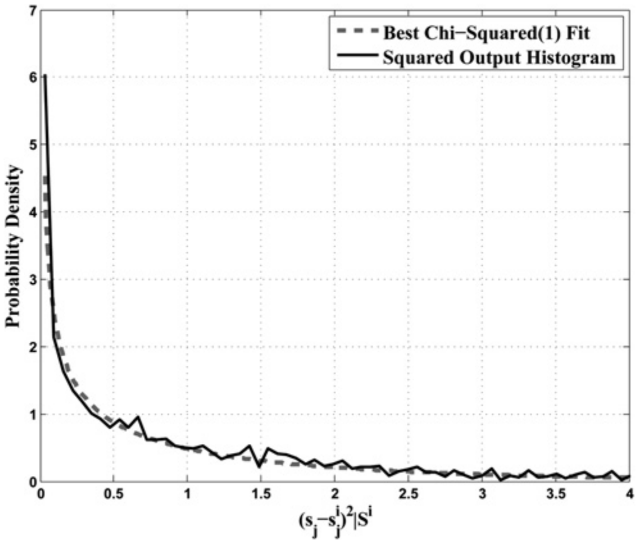

Fig. 4 *Approximate pdf of squared channel output samples*

discuss both cases and derive the optimum receiver structure and the corresponding probability of error for each case.

2.1 Orthogonal signalling

For orthogonal signalling case assumes that S^i is chosen such that $S^i \perp S^j$ for $i \neq j$, $i, j \in \{1, 2, \ldots, N\}$; hence considering $r_i = -2\langle S^i, S\rangle + \text{const}$, r_i's are orthogonal and then $f(r_1, \ldots, r_M|S^j)$ can be written as

$$f\left(r_1, \ldots, r_M|\boldsymbol{s}^j\right) = \prod_{i=1,\ldots,M} f\left(r_i|\boldsymbol{s}^j\right) = f\left(r_j|\boldsymbol{s}^j\right) \prod_{i=1,\ldots,M,\, i\neq j} f\left(r_i|\boldsymbol{s}^j\right) \tag{4}$$

Using (2), the following statement is derived (see (5))

As a simple case where $M = 2$, for $j = 1$ and $j = 2$ the MAP detector is reduced to

$$\frac{\frac{1}{\sigma_1^K 2^{\frac{K}{2}}\Gamma\left(\frac{K}{2}\right)} r_1^{\frac{K}{2}-1} e^{-\frac{r_1}{2\sigma_1^2}} \frac{1}{2\sigma_2^2} e^{\frac{r_2+\lambda}{2\sigma_2^2}} \left(\frac{r_2}{\lambda}\right)^{\frac{K}{4}-\frac{1}{2}} I_{\frac{K}{2}-1}\left(\sqrt{\frac{\lambda r_2}{\sigma_2^4}}\right)}{\frac{1}{\sigma_1^K 2^{\frac{K}{2}}\Gamma\left(\frac{K}{2}\right)} r_2^{\frac{K}{2}-1} e^{-\frac{r_2}{2\sigma_1^2}} \frac{1}{2\sigma_2^2} e^{\frac{r_1+\lambda}{2a_2^2}} \left(\frac{r_1}{\lambda}\right)^{\frac{K}{4}-\frac{1}{2}} I_{\frac{K}{2}-1}\left(\sqrt{\frac{\lambda r_1}{\sigma_2^4}}\right)} \gtrless 1 \tag{6}$$

By some manipulations the above equation simplifies to (see (7))

where $I_{k/2-1}$ is the modified Bessel function. Generally, $I_\alpha(x)$ is computed [32] as

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{8}$$

In which $J_\alpha(x)$ is the Bessel function of the first kind. Replacing $I_\alpha(x)$ by (8) and doing some algebraic simplifications, (7) yields to (see (9))

Since $\sigma_1 \leq \sigma_2$, the denominators are strictly increasing functions of r_1 and r_2. Thus, the result is the simple minimum distance detector as

$$r_1 \gtrless r_2 \tag{10}$$

Therefore the MAP detector is

$$\text{MAP: } \arg\min_j \left\| S - S^j \right\| \tag{11}$$

In the following section, an upper bound on the probability of error for this receiver will be proposed. Having the simplified MAP detector in (10) and using the union bound, the probability of error can be bounded. Consider, $P_e = \sum_i P(S^i)P(e|S^i)$, without loss of generality, assume $P(e|S^1) = \max_i P(e|s^i)$. Then applying the union bound yields

$$P_e \leq (M-1)P(e|S^1) = (M-1)\int_0^\infty \int_{r_2}^\infty f(r_1|S^1) f(r_2|S^1)\, dr_1\, dr_2 \tag{12}$$

Changing the order of the two integrals and the bounds accordingly, we have

$$P_e \leq (M-1)\int_0^\infty f(r_1|S^1) \int_0^{r_1} f(r_2|S^1)\, dr_2\, dr_1 \tag{13}$$

Considering (2) and doing some algebraic simplifications lead to

$$P_e \leq (M-1)\int_0^\infty \left[1 - Q_{k/2}(\sqrt{\lambda}, \sqrt{r_1})\right] f(r_1)\, dr_1 \tag{14}$$

where $Q_{k/2}(a, b)$ is the Marcum Q-function which is defined as [33]

$$Q_M(a, b) = \exp\left(-\frac{a^2+b^2}{2}\right) \sum_{k=1-M}^{\infty} \left(\frac{a}{b}\right)^k I_k(ab) \tag{15}$$

Now, since $\int_0^\infty f(r_1)\, dr_1 = 1$, we have (see (16) at the bottom of the next page)

Fig. 5 shows $\ln P_e$ against different values of K, where $\tilde{\lambda} = \frac{3.5\lambda}{\frac{2}{2}}$ when $_1 = _2$. To simplify the above equation, p_e is curve-fitted using the non-linear least-square method. Therefore applying the

$$f\left(r_1, \ldots, r_M|\boldsymbol{s}^j\right) = \frac{1}{\sigma_1^K 2^{K/2}\Gamma(K/2)} r_i^{(K/2)-1} e^{-\left(r_i/2\sigma_1^2\right)} \prod_{i=1,\ldots,M,\, i\neq j} \frac{1}{2\sigma_2^2} e^{\left(r_i+\lambda\right)/\left(2\sigma_2^2\right)} \left(\frac{r_i}{\lambda}\right)^{(K/4)-(1/2)} I_{(K/2)-1}\left(\sqrt{\frac{\lambda r_i}{a_2^4}}\right) \tag{5}$$

$$\frac{r_1^{(K/4)-(1/2)}}{e^{(r_1/2)\left(\left(1/\sigma_1^2\right)-\left(1/\sigma_2^2\right)\right)} I_{(K/2)-1}\left(\sqrt{\lambda r_1/\sigma_2^4}\right)} \gtrless \frac{r_2^{(K/4)-(1/2)}}{e^{(r_1/2)\left(\left(1/\sigma_1^2\right)-\left(1/\sigma_2^2\right)\right)} I_{(K/2)-1}\left(\sqrt{\lambda r_2/\sigma_2^4}\right)} \tag{7}$$

$$\frac{e^{(r_2/2)\left(\left(1/\sigma_1^2\right)-\left(1/\sigma_2^2\right)\right)} \sum_{m=0}^{\infty} \left(1/2\right)^{2m+(k/2)-1} \left((\lambda)^{m+(K/4)-1}/\left(m!\Gamma\left(m+(k/2)\right)\right)\right) {r_2}^m}{e^{(r_1/2)\left(\left(1/\sigma_1^2\right)-\left(1/\sigma_2^2\right)\right)} \sum_{m=0}^{\infty} \left(1/2\right)^{2m+(k/2)-1} \left(\left(\lambda/\sigma_2^4\right)^{m+(K/4)-1}/m!\Gamma\left(m+(k/2)\right)\right) {r_1}^m} \gtrless 1 \tag{9}$$

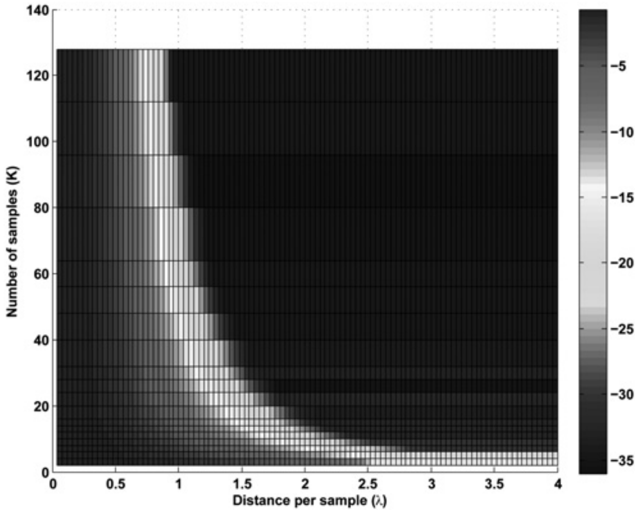

Fig. 5 *Fitted plot of $\ln P_e$ against distance per sample and number of samples*

Levenberg–Marquardt algorithm [34], (16) is approximated as the following equation and the resulting R-square of the curve-fitting is 0.9998

$$P_\mathrm{e} = \frac{1}{2}\exp\{-\tilde{\lambda}k^{-0.37}\} \tag{17}$$

2.2 General signalling

Note that the previous proposed MAP is valid only for orthogonal signalling. For a more general case where designed signals are not orthogonal, a different approach must be taken into consideration. For this purpose, suppose that each symbol $S^j = [s_1^j, \ldots, s_k^j]$ corresponds to a point in a K-dimensional sub-space. Note that, since $\|s_i^j\| \le 1$, the sub-space is a k-dimensional square, that is, $\mathbb{A} = \{S^j = (s_1^j, \ldots, s_k^j): |s_i^j\| \le 1\}$. Now for each S^j, consider a separate sphere C_j with centre S^j such that they cover the sub-space $\mathbb{A}$ as much as possible and have no common point. This is a well-known problem in mathematics addressed by sphere packing problem [30, 35]. Suppose that r_{θ_i} is the radius of sphere C_i then $r_{\theta_i} \le \min_{i, i \ne j}\|S^i - S^j\|$. Let $r_\theta = \min_i r_{\theta_i}$. Assume that the probability distribution of r_i when S^j is transmitted and $j \ne i$ is

$$f(r_i|s^j) = \begin{cases} \dfrac{1}{2\sigma_2^2}\mathrm{e}^{(r_i+b)/(2\sigma_2^2)}\left(\dfrac{r_i}{\lambda}\right)^{(K/4)-(1/2)} I_{(K/2)-1}\left(\sqrt{\dfrac{\lambda r_i}{\sigma_2^4}}\right), & r_i \le r_\theta \\ \beta, & \text{otherwise} \end{cases} \tag{18}$$

where k is the number of samples and β is a constant such that $f(r_i|S^j)\,\mathrm{d}r_i = 1$. Note that the distribution of r_i for $r_i \le r_\theta$ is a non-central chi-square distribution which is discussed in more detail in Appendix. Moreover, for the case of $i = j$, we have

$$f(r_i|\boldsymbol{s}^j) = \begin{cases} \dfrac{1}{\sigma_1^K 2^{(K/2)}\Gamma(K/2)} r_i^{(K/2)-1}\mathrm{e}^{-\left(r_i/2\sigma_1^2\right)}, & r_i \le r_\theta \\ \tilde{\beta}, & \text{otherwise} \end{cases} \tag{19}$$

In which $\tilde{\beta}$ is selected such that $f(r_i|S^i)\,\mathrm{d}r_i = 1$. The main point of using these spheres is that when r_i is not in the sphere C_i, the probability of r does not depend on r_i. To explain this, we show that these spheres can be used as base-vectors of the sub-space $\mathbb{A}$. The received signal can be shown as $U = [u_i]_{1\times M}$ where $u_i = r_\theta - \|S - S^i\|$, if S is in sphere C_i otherwise $u_i = 0$. In this coordinate system, $U = [0, \ldots, 0, r_\theta - \|S - S^i\|^2, 0, \ldots, 0]$. Since the spheres are separated and the received signal is only in one sphere, this coordinate system completely describes sub-space $\mathbb{A}$. Sphere packing in this way guarantees separated spheres and thus the above assumption holds [35]. First, it is needed to derive the pdf of $f(r_1, r_M/S^j)$. Assume that S^j is transmitted. By conditioning on $r_j \le r_\theta$ we obtain (see (20))

Note that since spheres C_i's have no common point, then the received point is in one sphere only. Assuming that the uncovered spaces are infinitely small, if $r_j \le r_\theta$ then $r_i > r_\theta$ for all $i \ne j$. Moreover, since $f(r_i/S^j, r_i \ge r_\theta) = \beta$, the pdf of r only depends on r_j when $r_j \le r_\theta$. Thus

$$f\left(r_1, \ldots, r_M|S^j, r_j < r_\theta\right) = \frac{f(r_j|S^j, r_j \le r_\theta)}{\int_0^{r_\theta} f(r_j|S^j)\,\mathrm{d}r_j} \tag{21}$$

For the second term of (20), applying the above condition for other r_i's yields

$$f\left(r_1, \ldots, r_M|S^j, r_j > r_\theta\right) = \sum_{i=0, i\ne j}^{M} \frac{f(r_i|S^j, r_i \le r_\theta)}{\int_0^{r_\theta} f(r_i|S^j)\,\mathrm{d}r_i} \tag{22}$$

Note that r_i's have the same statistics, and therefore (22) is reduced to (see (23))

In addition, we denote δ as $\int_0^{r_\theta} f(r_i|S^j)\,\mathrm{d}r_i$ and α as $\int_0^{r_\theta} f(r_j|S^j)\,\mathrm{d}r_j$. Finally, the above simplifications for (20) lead to the following (see (24))

Now the receiver can be determined using the MAP detector

$$\text{MAP: } \arg\max_j f(r_1, \ldots, r_M|S^j) \tag{25}$$

The above MAP can be simplified by conditioning on r_i's. Suppose

$$P_\mathrm{e} \le (M-1)\left[1 - \int_0^\infty Q_{k/2}\left(\sqrt{\frac{\lambda}{\sigma_2^2}}, \sqrt{\frac{r_1}{\sigma_2^2}}\right)\frac{1}{\sigma_1^K 2^{K/2}\Gamma(K/2)} r_1^{(K/2)-1}\mathrm{e}^{-\left(r_1/2\sigma_1^2\right)}\,\mathrm{d}r\right] \tag{16}$$

$$f\left(r_1, \ldots, r_M|S^j\right) = P\left(r_j < r_\theta\right) f\left(r_1, \ldots, r_M|S^j, r_j < r_\theta\right) + \left(1 - P\left(r_j < r_\theta\right)\right) f\left(r_1, \ldots, r_M|S^j, r_j > r_\theta\right) \tag{20}$$

$$f\left(r_1, \ldots, r_M|S^j, r_j > r_\theta\right) = \frac{1}{(M-1)\int_0^{r_\theta} f(r_i|S^j)\,\mathrm{d}r_i} \sum_{i=0, i\ne j}^{M} f(r_i|S^j, r_i \le r_\theta) \tag{23}$$

$$f(r_1, \ldots, r_M|S^j) = f(r_j|S^j, r_j \le r_\theta) + \frac{1-\alpha}{(M-1)\delta}\sum_{i=0, i\ne j}^{M} f(r_i|S^j, r_i \le r_\theta) \tag{24}$$

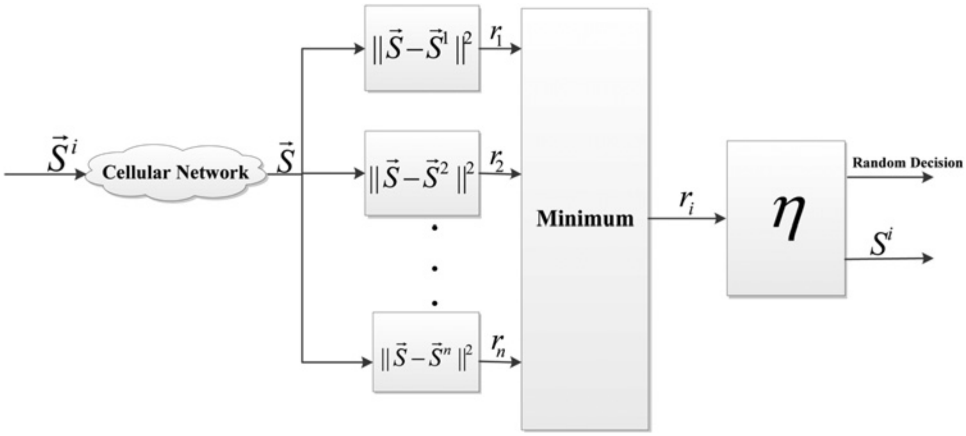

Fig. 6 *Receiver structure for the general signalling case*

the received signal r is in sphere C_k then (20) yields

$$f(r_1,\ \dots,\ r_M|S^j) = \frac{1}{\delta} f(r_k|S^j,\ r_k \le r_\theta) \tag{26}$$

Hence, the MAP is arg max $f(r_k|S^j,\ r_k \le r_\theta)$. The following steps describe the procedure for this receiver

(1) Find the $r_{i^*} = \min_{i=1,\dots,M} r_i$

(2) Decision: $\begin{cases} S^{i^*},\ r_{i^*} \le r_\theta \\ \text{Random selection among} \quad \{S^j\}_{j=1,\dots,M, j\ne i^*} \end{cases}$

The proposed receiver is shown in Fig. 6. As previously discussed, this receiver uses r_i instead of the original received signal (S_i). The minimum r_i is determined and based on the threshold r_θ and the received signal is detected as described in step 2 of the above procedure.

The error probability in this case can be calculated similar to the orthogonal case

$$P_e = \sum_i P(S^i) P(e|S^i) \tag{27}$$

As before assumed $P(e|S^1) = \max_i P(e|S^i)$. Since $f(r_i|\boldsymbol{S}^i)$ is centralised chi-square function as in (19), then having its cumulative distribution function [22], we obtain

$$P(e|S^i) = \int_{r_\theta}^{\infty} f(r_i|S^i)\,\mathrm{d}r_i = 1 - \frac{\gamma((k/2),\ (r_\theta/2))}{\Gamma((k/2),\ (r_\theta/2))} \tag{28}$$

where $\Gamma(k/2)$ and $\gamma((k/2),\ (r_\theta/2))$ are upper and lower incomplete Gamma functions

$$\Gamma(s,\ x) = \int_x^\infty t^{s-1}\ \exp -t\,\mathrm{d}t,\ \ \gamma(s,\ x) = \int_0^x t^{s-1}\ \exp -t\,\mathrm{d}t \tag{29}$$

As a result, in the case of equiprobable symbols, the error probability is bounded by

$$P_e \le \left(1 - \frac{\gamma((k/2),\ (r_\theta/2))}{\Gamma((k/2),\ (r_\theta/2))}\right) \tag{30}$$

In which k is the number of samples.

3 Symbol design

In this section, based on the two signalling methods listed in the previous section, two different methods for symbol design are presented. In the first method, the symbols are designed based on the orthogonal codes. In this method, any kind of orthogonal codes such as binary, bipolar, ternary and multi-level are applicable. Therefore our symbols can be generated using known orthogonal coding techniques such as Hadamard, Gold, M-sequence, Prime and OOC [29]. As a result of (16), P_e depends on the distance between the code-words. The energy of a code is an important factor for determining P_e. The energy of codes is determined by code weights for unipolar codes and the length of code-words for bipolar codes. The other important factor in orthogonal symbol design is the number of symbols and the mutual code-word distances. For example in the case of Hadamard codes with length k, the codes' energy equals k and the mutual codeword distances are $d^2 = 2k$. In this case, the number of symbols is $M = k$. In the second method, we use sphere packing technique. To design optimum symbols in sub-space $\mathbb{A}$, it is necessary to maximise the distance of adjacent symbols. In other words, our problem is equivalent to packing a certain number of spheres in sub-space $\mathbb{A}$ with maximum possible radius. $\mathbb{A}$ can be a set such that the energy of each symbol is limited. Hence, the problem is reduced to a sphere-in-sphere packing (SSP) problem. In the case where the amplitude of each sample is limited, the problem is reduced to sphere-in-cube packing (SCP) problem. Sphere packing problem is discussed in more details in the following section.

3.1 *Sphere packing problem*

A typical sphere packing problem is to find the arrangement of identical size spheres in a certain sub-space such that as large proportion of the space as possible is filled. According to the shape of the sub-space, several problems can be defined for example, SSP or SCP problems. The proportion of the space filled by the spheres is called the density of the packing (σ_n). Several bounds on the density of sphere packing are introduced in the literature. We discuss some of the tightest bounds for our problem [35]. Rogers *et al.* [36] claimed that $\sigma_n = 2^{(-3n/2)} n!^2 (n+1)^{(1/2)} J_n F_n(\alpha)$ is an upper bound when the number of samples (k) is equal to the dimension of the problem (n), where $J_n = \pi^{(1/2)n}/\Gamma(1+(1/2)n)$ and F_n is Schlafli function defined recursively by (see (31))

In the asymptotic case ($n \to \infty$), we have $\sigma_n \simeq 2^{-(1/2)n}(n/e)$. Similarly, another upper bound for σ_n was derived by Blitchfeld

$$F_0(\alpha) = F_1(\alpha) = 1,\ F_{n+1}(\alpha) = \frac{2}{\pi} \int_{(1/2)\sec^{-1} n}^{\alpha} F_{n-1}(\phi)\,\mathrm{d}\theta,\ \ \phi = \frac{1}{2}\sec^{-1}(\sec 2\theta - 2) \tag{31}$$

as [37]

$$\sigma_n \leq 2 \times 2^{-(1/2)n-1}(n+2) \tag{32}$$

Torquato *et al.* [38] and Kabatjanski and Levenshtein [39] and introduced another upper bound as follows

$$\sigma_n \leq 2^{(-0.599+o(1))n} \tag{33}$$

Moreover, the two following lower bounds were introduced in [30, 35]

$$\sigma_n \geq \frac{2(n-1)}{2^n} s(n), \quad s(n) = \sum_{t=1}^{\infty} \frac{1}{t^n} \tag{34}$$

$$\sigma_n \geq 2^{n-(0.77865\ldots)n-0.0848n^{(1/3)}-0.0312\log_2 n+\ldots} \tag{35}$$

where n is the dimension of the sub-spaces and $\sigma_n = (r^n V_s/V_t)$, $r = (d/2)$ being the radius of sphere. Here, d is the distance between the centres of spheres or distance between symbols and $V_s = (\pi^{(n/2)}/\Gamma((n/2)+1))$ which is the volume of an n-dimensional sphere with radius one. Moreover, V_t is the total volume of the sub-space. If our sub-space is a sphere then $V_t = V_s$, otherwise if the sub-space is a hypercube the total volume is equal to one. It is needed to mention that in each range, a specific bound is tighter than others [40–42].

4 Simulation results

In this section, performance of the presented symbol design methods is analysed and bounds on the error probability of each technique are derived. First performance of the orthogonal codes is discussed in Section 4.1 and an upper bound on P_e is derived based on Hadamard codes. Then, several bounds on P_e for the general case (sphere packing) are derived in Section 4.2.

4.1 Performance of orthogonal signalling

As discussed in Section 3, P_e depends on the applied method for symbol design. As an example, we derive P_e for Hadamard codes. Let k be the code length (symbol length), consequently, the number of symbols is $M = k$ and the mutual symbol distance is $d^2 = 2k$. Substitution in the previously derived formula for error probability (17) gives

$$P_e = \frac{1}{2}\exp\left\{-\frac{7k}{\sigma^2}k^{-0.37}\right\} \tag{36}$$

Note that in this case the data rate is given by $R = R_s((\log_2 k)/k)$. The upper bound derived in (14) on symbol error probability is sketched in Fig. 7 against the data transmission rate. In the simulations, different modes of AMR vocoder channel are used as typical examples of voice dedicated channels. Note that for each vocoder mode, the parameter is estimated by simulation experiments on the channel output, making it possible to evaluate the upper bound in (36). The derived bound is also compared with empirical SER obtained by simulations with Hadamard symbols.

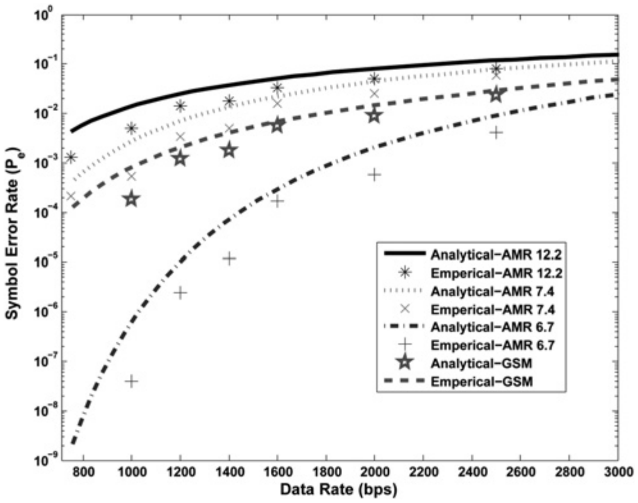

Fig. 7 *Comparison between the empirical SER and the derived upper bound*

4.2 Performance of sphere packing method

Note that as mentioned in Section 3, P_e generally depends on the designed symbols for transmission of data over the channel. There is a reverse relationship between P_e and the distance of designed symbols and therefore the local density (σ_n). Thus, the bounds introduced in Section 3 on σ_n are used to derive approximations and bounds on the error probability as below:

(1) *Roger bound (L1):* Recall from Section 3.1 that $\sigma_n = 2^{-(3n/2)}n!^2 (n+1)^{(1/2)} J_n F_n(\alpha)$. In this case, the number of samples (k) is equal to the dimension of the problem (n). Bear in mind that $R = R_s(\log_2 M/n)$ thus $M = 2^{(nR/R_s)}$. By considering the asymptotic case of Roger's bound and assuming symbol design in case of sphere packing in sphere (SSP) we obtain

$$\sigma_n = M\left(\frac{d}{2}\right)^n \frac{\pi^{(n/2)}}{\Gamma((n/2)+1)} \leq 2^{-(1/2)n}\frac{n}{e} \tag{37}$$

where $d = 2r_\theta$, hence we have

$$d \leq \sqrt[n]{2^{(1/2)n}\frac{n\Gamma((n/2)+1)}{e\pi^{(n/2)}2^{(nR/R_s)}}} = \frac{2^{(1/2)-(R/R_s)}}{\sqrt{\pi}}\sqrt[n]{\frac{n\Gamma((n/2)+1)}{e}} = \tau \tag{38}$$

And if the problem is sphere packing in Euclidean space (SCP), then

$$d \leq \sqrt[n]{2^{(1/2)n}\frac{n}{e2^{(nR/R_s)}}} = 2^{(1/2)-(R/R_s)}\sqrt[n]{\frac{n}{e}} = \tau \tag{39}$$

Inserting (38) in (30) yields

$$P_e \geq 1 - \frac{\gamma\big((n/2),\ (\tau/4\sigma^2)\big)}{\Gamma(n/2)} \tag{40}$$

Note that τ is an upper bound for d and substituting it in (30) yields a lower bound on P_e. The above formula (40) depends on symbol length (n) and the distance between adjacent symbols (d). For a fixed rate (R), the number of symbols (M) varies according to (37), which in turn changes (d) according to (38). Therefore for each fixed rate, the optimum P_e results by optimising (n) as below

$$P_{e_{opt}} = \min_n\left(1 - \frac{\gamma\big(n/2,\ (\tau/4\sigma^2)\big)}{\Gamma(n/2)}\right) \tag{41}$$

(2) *Bitchfeld bound (L2) and Kabatjanski bound (L3):* The following bounds on (d) can be derived by similar calculations using other bounds introduced in Section 3 on σ_n. These bounds on (d) similarly lead to different lower bounds on the error probability by

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substitution of the resulting (τ) in (30)

$$\mathbf{L2}:\ \tau = \begin{cases} \dfrac{2^{(1/2)-(R/R_s)}}{\sqrt{\pi}} \sqrt[n]{(n+2)\Gamma\left(\dfrac{n}{2}+1\right)}, & \text{SSP} \\ 2^{(1/2)-(R/R_s)} \sqrt[n]{n+2}, & \text{SCP} \end{cases} \quad (42)$$

$$\mathbf{L3}:\ \tau = \begin{cases} \dfrac{2^{0.41-(R/R_s)}}{\sqrt{\pi}} \sqrt[n]{\Gamma\left(\dfrac{n}{2}+1\right)}, & \text{SSP} \\ 2^{0.41-(R/R_s)}, & \text{SCP} \end{cases} \quad (43)$$

(3) *Bezdek bound (U1) and Sloane bound (U2):* In addition to the above approximations, the lower bounds of (34) and (35) on σ_n can be similarly used to derive lower bounds on (d) denoted by (λ). This time, substitution of () in (30) leads to upper bounds on the error probability indicated by **U1** and **U2**, respectively. Equation (34) gives

$$\mathbf{U1}:\ \lambda = \begin{cases} \dfrac{2^{-(R/R_s)}}{\sqrt{\pi}} \sqrt[n]{2(n-1)\varsigma(n)\left(\dfrac{n}{2}\right)!}, & \text{SSP} \\ 2^{-(R/R_s)} \sqrt[n]{2(n-1)\varsigma(n)}, & \text{SCP} \end{cases} \quad (44)$$

With a similar approach (35)

$$\mathbf{U2}:\ \lambda = \begin{cases} \dfrac{2^{\frac{n-(0.77865)n-0.0848n^{1/3}-0.0312\log_2 n}{n}-\frac{R}{R_s}}}{\sqrt{\pi}} \sqrt[n]{\Gamma\left(\dfrac{n}{2}+1\right)}, & \text{SSP} \\ 2^{\frac{n-(0.77865)n-0.0848n^{1/3}-0.0312\log n}{n}-\frac{R}{R_s}}, & \text{SCP} \end{cases} \quad (45)$$

(4) *Comparison results:* The three lower bounds (L1, L2, L3) and the two upper bounds (U1, U2) introduced in this section are evaluated and compared with the empirical SERs achieved for AMR 12.2. Fig. 8 sketches these bounds and shows that the derived bounds tighten as the transmission data rate increases. This helps us to approximate the SER accurately for large data rates. It should be noted that the bounds are sketched for the SSP case and the parameter is again estimated by simulation results on AMR 12.2 voice channel .as one notes, the obtained results in this method as illustrated in Fig. 8, are so outstanding. The achieved results based on sphere packing technique are not only better than the orthogonal codes results, but also more efficient than all previously mentioned works in the introduction part. Besides the better performance, the proposed modem based on sphere packing technique has two major advantages. At first the analytical statements for the performance analysis of this method is extracted and illustrated via lower and upper bounds. Second, the proposed method is scalable and one can customise it to his/her own application. At the end of this section, it should be pointed out that, although, we propose a general approach to conquer the voice channel non-idealities, the offered ideas could also be applied to other communication channels with similar limitations.

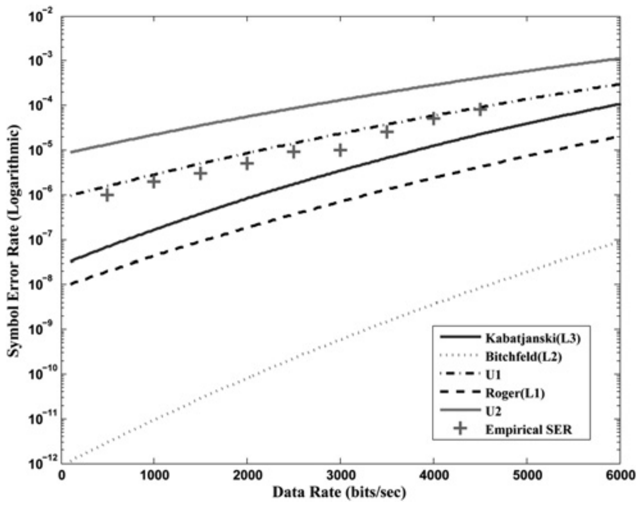

Fig. 8 *Comparison between the empirical SER and the derived bounds for AMR 12.2*

5 Conclusion and future works

In this paper, we introduced the idea of utilising cellular networks voice channel for device-to-device data communication. To this end, proper statistical model for these channels was proposed and an optimum receiver based on this model was presented. In the next step, for proper communication in these channels, two different symbol design methods were proposed using orthogonal coding and sphere packing techniques. Moreover, performances of the proposed techniques were analytically assessed using BER and data rate as the criteria. Finally, it was shown that the obtained results utilising the proposed structure based on the sphere packing technique has better performance compared with previously reported works in this area. It should be noted that the presented modem in this paper is not restricted to voice codec channels and could be generalised to the other communication channels with similar limitations. Further works can be considered to develop the proposed methods for commercial applications and increase their technological readiness. These future works include: offering a new coding technique with respect to the error occurrence pattern, evaluating capacity of the discussed channel and proposing a method for synchronisation considering the limited bandwidth constrains and non-linearities, modelling the statistical property mobile voice channel in a more detailed way and considering the vocoder rate adoption in practical situation.

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7 Appendix

7.1 Appendix 1: non-central chi-squared distributions

It is well known from the classic theory of probability that the sum of squares of k independent standard normal random variables forms a chi-square random variable with k degrees of freedom [43]. That is if $r = \sum_{i=1}^{k} Z_i^2$, $Z_i \sim \mathcal{N}(0,\ 1)$ and Z_i's are i.i.d. then, $r \sim \chi^2(k)$. Now consider the case where for some $1 \le j \le k$, Z_j is not precisely a Gaussian random variable. In other words, Z_j has a distorted Gaussian pdf modelled. Now consider the pdf of Z_j^2. It can be rewritten as the sum of two parts $Z_j^2 = f_j + n_j$, in which n_j is the distortion part and f_j follows $\chi^2(1)$. In what follows, it is shown that if k is large enough, the pdf of r approaches $\chi^2(k)$ distribution, as $r = \sum_{i=1}^{k} Z_i^2$. Using characteristic function concept we have

$$\Phi_r = \prod_{i=1}^{k} \Phi_{Z_i^2} = \prod_{i=1}^{k} \left(\Phi_{f_i} + \Phi_{n_i}\right) = \sum_{i=1}^{k} \binom{k}{i} \Phi_{f_i}^{i} \Phi_{n_i}^{k-i} \tag{46}$$

where Θ_x denotes characteristic function of x. Now, as noted in [43], characteristic function of $\chi^2(1)(\Phi_{f_i})$ is $1/\sqrt{1-2if}$, which acts like a low-pass filter in frequency domain. In the other hand, Φ_{n_i} behaves similar to a high-pass filter because of its noisy distortion nature. Therefore by neglecting the cross-terms of these two opposite filters, (46) can be approximated as

$$\Phi_r = \sum_{i=1}^{k} \binom{k}{i} \Phi_{f_i}^{i} \Phi_{n_i}^{k-i} \simeq \Phi_{f_i}^{k} + \Phi_{n_i}^{k} \tag{47}$$

As the distortion n_j is so smaller than f_i, we can omit the term $\Phi_{n_i}^{k}$ in the presence of $\Phi_{f_i}^{k}$ and the proof is completed.

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