On the Necessity of Structured Codes for Communications over MAC with Feedback

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***Abstract*—The problem of three-user multiple-access channel (MAC) with noiseless feedback is investigated. A new coding strategy is presented. The coding scheme builds upon the natural extension of the Cover-Leung (CL) scheme [1]; and uses quasi-linear codes. A new single-letter achievable rate region is derived. The new achievable region strictly contains the CL region. This is shown through an example. In this example, the coding scheme achieves optimality in terms of transmission rates. It is shown that any optimality achieving scheme for this example must have a specific algebraic structure. Particularly, the codebooks must be closed under binary addition.**

I. Introduction

THE problem of three user MAC with noiseless feedback is depicted in Figure 1. This communication channel consists of one receiver and multiple transmitters. After each channel use, the output of the channel is received at each transmitter noiselessly. Gaarder and Wolf [2] showed that the capacity region of the MAC can be expanded through the use of the feedback. This was shown in a binary erasure MAC. Cover and Leung [1] studied the two-user MAC with feedback, and developed a coding strategy using unstructured random codes.

The main idea behind the CL scheme is to use superposition block-Markov encoding. The scheme operates in two stages. In stage one, the transmitters send the messages with a rate outside of the no-feedback capacity region (i.e. higher rates than what is achievable without feedback). The transmission rate is taken such that each user can decode the other user's message using feedback. In this stage, the receiver is unable to decode the messages reliably, since the transmission rates are outside the no-feedback capacity region. Hence, the decoder is only able to form a list of "highly likely" pairs of messages. In the second stage, the encoders fully cooperate to send the messages (as if they are sent by a centralized transmitter). The receiver decodes the message pair from its initial list. After the initiation block, superposition coding is used to transmit the sequences corresponding to the two stages.

The single-letter achievable rate region for the CL scheme was characterized in [1]. Later, it was shown that the CL scheme achieves the feedback capacity for a class of MAC with feedback [3]. However, this is not the case for the general MAC with feedback [4]. Several improvements to the CL achievable region were derived [5], [6]. In [5] and [6], additional stages

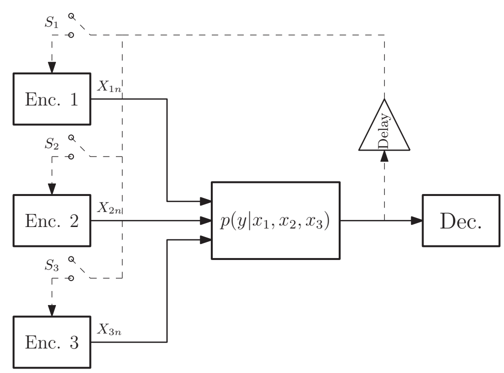

Fig. 1. The three-user MAC with noiseless feedback. If the switch S_i is closed, the feedback is available at the ith encoder, where $i = 1, 2, 3$.

are appended to the CL scheme. In these schemes, the encoders decode each others' messages in several stages. Kramer [7], used the notion of *directed information* to derive the capacity region of the two-user MAC with feedback. However, the characterization is not computable, since it is an infinite letter characterization. Finding a computable characterization of the capacity region remains an open problem.

In this work, we study the problem of three-user MAC with feedback. We propose a new coding scheme which builds upon the CL scheme. We derive a computable single-letter achievable rate region for this scheme, and show that the new region improves upon the previous known achievable regions for this problem. Recently, we showed that the application of structured codes results in improved performance for the problem of transmission of sources over the MAC [8]. Here, we use the ideas proposed in [8] to prove the necessity of structured codes in the problem of MAC with feedback. Specifically, we use *quasi-linear* codes that are proposed in [9].

The coding scheme operates in three stages. In stage one, the encoders send independent messages with rates outside of the CL region. Therefore, encoders are unable to decode each others' messages. However, each encoder can decode the binary sum of the messages of the other two encoders. In stage two, the messages are superimposed on the summation which is decoded in the previous stage. At the end of this stage, the encoders decode each others' messages. Stage three is similar

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to the second stage in CL scheme. We provide an example where the new coding scheme achieves optimal performance, whereas the previous schemes are suboptimal. Finally, we prove that any optimality achieving coding scheme must use encoders whose set of output sequences is linearly closed.

The rest of the paper is organized as follows: Section II presents basic definitions. Section III presents an example of a MAC with feedback, and discusses the necessity of structured codes for that setup. Section IV contains the main result of the paper, and characterizes a new achievable rate region. Finally, Section V concludes the paper.

II. Preliminaries and Model

A three-user discrete memoryless MAC is defined by: 1) three input alphabets $\mathcal{X}_1, \mathcal{X}_2$, and $\mathcal{X}_3$, 2) an output alphabet $\mathcal{Y}$, and 3) a conditional probability distribution $p(y|x_1, x_2, x_3)$ for all $(y, x_1, x_2, x_3) \in \mathcal{Y} \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$. Such setup is denoted by $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, P_{Y|X_1X_2X_3})$. Let y^n be the output sequence corresponding to n uses of the channel, and x_i^n be the ith input sequence of the channel, $i = 1, 2, 3$. Then, the following condition is satisfied:

$$p(y_n|y^{n-1}, x_1^{n-1}, x_2^{n-1}, x_3^{n-1}) = p(y_n|x_{1n}, x_{2n}, x_{3n}). \quad (1)$$

Figure 1 illustrates this setup. Note that in this setup noiseless feedback is available at a subset of the encoders. The switches $S_i, i = 1, 2, 3$ determine which encoder receives the feedback.

Definition 1. *A (N, M_1, M_2, M_3) transmission system for a given three-user MAC with feedback is defined as a sequence of encoding functions and a decoding function. If the feedback is available at the ith user, the corresponding encoding functions are defined as*

$$f_{i,n} : \{1, 2, \dots, M_i\} \times \mathcal{Y}^{n-1} \to \mathcal{X}_i,$$

where $i = 1, 2, 3$, and $n = 1, 2, \dots, N$. If the feedback is not available at ith encoder, the corresponding encoding functions are defined as

$$f'_{i,n} : \{1, 2, \dots, M_i\} \to \mathcal{X}_i.$$

The decoding function is defined as

$$g : \mathcal{Y}^N \to \{1, 2, \dots, M_1\} \times \{1, 2, \dots, M_2\} \times \{1, 2, \dots, M_3\}.$$

Let Θ_i denote the message for ith transmitter, $i = 1, 2, 3$. We assume Θ_i is drawn randomly and uniformly from $\{1, 2, ..., M_i\}$. Furthermore, we assume Θ_1, Θ_2, and Θ_3 are mutually independent. The average probability of error for this setup is defined as

$$\bar{P} \triangleq \frac{1}{M_1M_2M_3} \sum_{\theta_1,\theta_2,\theta_3} p(g(\mathbf{Y}^N) \neq (\theta_1, \theta_2, \theta_3)|\theta_1, \theta_2, \theta_3).$$

Definition 2. *A rate triple (R_1, R_2, R_3) is said to be achievable for a given MAC with feedback, if for any $\epsilon > 0$ there exists an (N, M_1, M_2, M_3) transmission system such that*

$$\bar{P} < \epsilon, \quad \frac{1}{n} \log_2 M_i \geq R_i - \epsilon, \quad i = 1, 2, 3.$$

The capacity region of the MAC with feedback is the closure of the set of all achievable rate pairs (R_1, R_2, R_3).

III. An Example of a MAC with Feedback

In this section, we show that coding strategies based on structured codes are necessary for the problem of MAC with feedback. We first provide an example of a MAC with feedback. Then, we propose a coding scheme using linear codes, and show that such coding scheme achieves optimality in terms of achievable rates.

Example 1. Consider the three-user MAC with feedback problem depicted in Figure 2. In this setup, there is a MAC with three inputs. The ith input is denoted by the pair (X_{i1}, X_{i2}), where $i = 1, 2, 3$. The output of the channel is denoted by the vector (Y_1, Y_{21}, Y_{22}). Noiseless feedback is available only at the third transmitter. The MAC in this setup consists of two

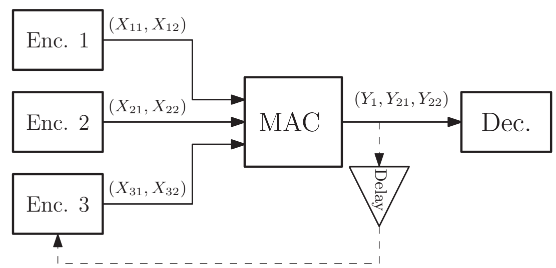

Fig. 2. The MAC with feedback setup for Example 1.

parallel sub-channels. The first channel is a three-user binary additive MAC with inputs (X_{11}, X_{21}, X_{31}), and output Y_1. The output is related to the inputs via the relation

$$Y_1 = X_{11} \oplus X_{21} \oplus X_{31} \oplus \tilde{N}_\delta,$$

where $\tilde{N}_\delta$ is a Bernoulli random variable with bias δ, and is independent of the inputs.

The second channel is a MAC with (X_{12}, X_{22}, X_{32}) as the inputs, and (Y_{21}, Y_{22}) as the output. The relation between the output and the input of the channel is depicted in Figure 3. The channel operates in two states. If the condition $X_{32} = X_{12} \oplus X_{22}$ holds, the channel would be in the first state (the left channel in Figure 3); otherwise it would be in the second state (the right channel in Figure 3). In each state the channel consists of two parallel sub-channels. The sub-channels are binary additive channels with independent noises. Note that the random variables N_δ and N'_δ are Bernoulli random variables with identical bias δ. Whereas, $N_{1/2}$ and $N'_{1/2}$ are Bernoulli random variables with bias $\frac{1}{2}$. We assume that $\tilde{N}_\delta, N_\delta, N'_\delta, N_{1/2}$, and $N'_{1/2}$ are mutually independent, and are independent of all the inputs.

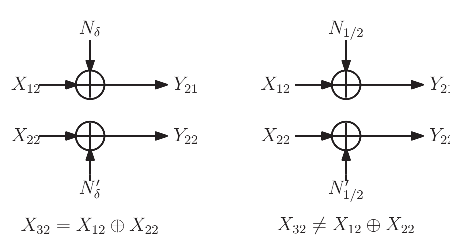

Fig. 3. The second channel for Example 1. If the condition $X_{32} = X_{12} \oplus X_{22}$ holds, the channel would be the one on the left; otherwise it would be the right channel.

The setup in this example matches with the model depicted in Figure 1 (where only the switch S_3 is connected). The argument follows by noting that the MAC channel in this example is a discrete memoryless channel with three inputs. In addition, the feedback is available at one of the encoders without any noise.

We use linear codes to propose a new coding strategy for the setup given in Example 1. The scheme uses a large number L of blocks. The length of each block is n. Each encoder has two outputs, one for each channel. We use identical linear codes with length n and rate $\frac{k}{n}$ for each transmitter. The coding scheme at each block is performed in two stages. In the first stage, each transmitter encodes the fresh message at the beginning of the block l, where $1 \leq l \leq L$. The encoding process is performed using the identical linear codes. At the end of the block l, the feedback is received by the third user. In stage 2, the third user uses the feedback from the first channel (that is Y_1) to decode the binary sum of the messages of the other encoders. Then, it encodes the summation, and sends it through its second output. If the decoding process is successful at the third user, then the relation $X_{32} = X_{12} \oplus X_{22}$ holds with probability one. This is because identical linear codes are used to encode the messages. As a result of this equality, the channel in Figure 3 is in the first state with probability one. In what follows, we show that the rate

$$(1-h(\delta), 1-h(\delta), 1-h(\delta))$$

is achievable using this strategy.

Preposition 1. *For the channel given in Example 1, the rate triple* $(1-h(\delta), 1-h(\delta), 1-h(\delta))$ *is achievable.*

Proof: The proof is given in Appendix A. ■

Remark 1. We can show that the triple $(1-h(\delta), 1-h(\delta), 1-h(\delta))$ is a corner-point in the capacity region of the channel in Example 1. This implies that the proposed coding scheme for this example achieves optimality in terms of achievable rates.

The above coding strategy has two main properties: 1) The codebooks used in the first and the second encoders are closed under the binary addition, 2) The third user uses feedback to decode only the binary sum of others' messages. One implication of Remark 1 is that such coding scheme achieves optimality. In the next subsection, we show a stronger statement. We show that every coding scheme that achieves $(1-h(\delta), 1-h(\delta), 1-h(\delta))$ in the setup given in Example 1, should carry these properties.

A. Converse

Suppose there exists a (N, M_1, M_2, M_3) transmission system with rates close to $R_i = 1-h(\delta)$, and average probability of error close to 0, more precisely the following inequalities are satisfied

$$\bar{P} < \epsilon, \quad \frac{1}{n}\log_2 M_i \geq 1-h(\delta)-\epsilon, \quad i = 1,2,3,$$

where $\epsilon > 0$ is sufficiently small, and $\bar{P}$ is the average probability of error as in Definition 2. Since there is no feedback at the first and second encoders, the transmission system predetermines a codebook for user 1 and 2. Note that there are two outputs for encoder 1 and 2. Suppose $\mathcal{C}_{12}$ and $\mathcal{C}_{22}$ are the codebooks assigned to the second output of encoder 1 and encoder 2, respectively.

Let $\mathbf{X}_{i2}^N$ be the second output of encoder i, where $i = 1,2,3$. Let $X_{i2,l}$ denote the lth component of $\mathbf{X}_{i2}^N$, where $1 \leq l \leq N,\ i = 1,2,3$. The following lemmas hold for this transmission system.

Lemma 1. *For any fixed* $c > 0$*, define*

$$\mathcal{I}_c^N := \{l \in [1:N] : P(X_{32,l} \neq X_{12,l} \oplus X_{22,l}) \geq c\}.$$

Then, the inequality $\frac{|\mathcal{I}_c^N|}{N} \leq \frac{\eta(\epsilon)}{2c(1-h(\delta))}$ *holds, where* $\eta(\epsilon)$ *is a function such that,* $\eta(\epsilon) \to 0$*, as* $\epsilon \to 0$*.*

Proof: The proof is given in Appendix B. ■

The lemma implies that in order to achieve $(1-h(\delta), 1-h(\delta), 1-h(\delta))$, the third user needs to decode $X_{12,l} \oplus X_{22,l}$ for "*almost all*" $l \in [1:N]$. This requirement is necessary to insure that the channel given in Figure 3 is in the first state.

In the next step, we use the results of Lemma 1, and drive two necessary conditions for decoding $X_{12} \oplus X_{22}$.

Lemma 2. *The following inequalities hold:*

$$\frac{1}{N}\Big|\ \log||\mathcal{C}_{12} \oplus \mathcal{C}_{22}|| - \log||\mathcal{C}_{12}||\ \Big| \leq \lambda_1(\epsilon),$$
$$\frac{1}{N}\Big|\ \log||\mathcal{C}_{12} \oplus \mathcal{C}_{22}|| - \log||\mathcal{C}_{22}||\ \Big| \leq \lambda_2(\epsilon),$$

where $\lambda_j(\epsilon) \to 0$*, as* $\epsilon \to 0, j = 1,2$*.*

Proof: Please refer to [10]. ■

As a result of this lemma, $\log||\mathcal{C}_{12} \oplus \mathcal{C}_{22}||$ needs to be close to $\log||\mathcal{C}_{12}||$ and $\log||\mathcal{C}_{22}||$. This implies that $\mathcal{C}_{12}$ and $\mathcal{C}_{22}$ possesses an algebraic structure, and are *almost* close under the binary addition. Not that for the case of unstructured random codes $||\mathcal{C}_{12} \oplus \mathcal{C}_{22}|| \approx ||\mathcal{C}_{12}|| \times ||\mathcal{C}_{22}||$. Hence, unstructured random coding schemes are suboptimal in this example.

Remark 2. The three-user extension of CL scheme is suboptimal. Because, the conditions in Lemma 2 are not satisfied.

IV. A New Achievable Rate Region

The coding scheme proposed for Example 1 is built upon linear codes. We extend this scheme, and propose a new coding strategy which subsumes the CL scheme. Using this scheme, we derive a new computable single-letter achievable rate region for the three-user MAC with feedback problem.

Definition 3. *For a given set* $\mathcal{U}$ *and a three-user MAC with feedback* $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, P_{Y|X_1X_2X_3})$*, define* $\mathscr{P}$ *as the collection of all distributions* P *of the form*

$$p(u)p(v_1,v_2,v_3)\prod_{i=1}^{3} p(t_i)p(x_i|u,t_i,v_i)p(y|x_1,x_2,x_3),$$

for all $y \in \mathcal{Y}, u \in \mathcal{U}, t_i \in \mathbb{F}_2, v_i \in \mathbb{F}_2, x_i \in \mathcal{X}_i,\ i = 1,2,3,$ *where 1)* T_1, T_2, T_3 *are mutually independent with uniform distribution over* $\mathbb{F}_2$ *, 2)* V_1, V_2, V_3 *are pairwise independent, 3)* $p(v_i) = \frac{1}{2}$*, and 3)* $p(v_1,v_2,v_3) = \frac{1}{4}$*.*

Fix a distribution $P \in \mathscr{P}$. Denote $S_i = (X_i, T_i, V_i)$ for $i = 1,2,3$. Consider two sets of random variables

(U, S_1, S_2, S_3, Y) and $(\tilde{U}, \tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{Y})$. Suppose the distribution of each set of the random variables is P. Then with this notation we have

$$P_{US_1S_2S_3Y} = P_{\tilde{U}\tilde{S}_1\tilde{S}_2\tilde{S}_3\tilde{Y}} = P$$

Theorem 1. *Consider a MAC $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, P_{Y|X_1X_2X_3})$, and a distribution $P \in \mathscr{P}$. For any subset $\mathcal{A} \subseteq \{1,2,3\}$, and for any distinct elements $i, j, k \in \{1,2,3\}$ the following bounds hold*

$$\begin{aligned}
R_{\mathcal{A}} &\leq I(X_{\mathcal{A}}; Y|US_{\mathcal{A}^c}\tilde{V}_1\tilde{V}_2\tilde{V}_3) + I(U; Y|\tilde{U}\tilde{Y}) \\
R_i + R_j &\leq I(T_i \oplus T_j; Y|UT_kX_k\tilde{V}_1\tilde{V}_2\tilde{V}_3) \\
&\quad + I(\tilde{X}_i\tilde{X}_j; \tilde{Y}|\tilde{U}\tilde{S}_k\tilde{V}_1\tilde{V}_2\tilde{V}_3V_k) \\
&\quad + I(\tilde{X}_i\tilde{X}_j; Y|\tilde{U}\tilde{S}_k\tilde{V}_1\tilde{V}_2\tilde{V}_3US_k\tilde{Y}) \\
R_i + R_j &\leq \frac{H(W_i) + H(W_j)}{H(W_i \oplus W_j)} I(T_i \oplus T_j; Y|UT_kX_k),
\end{aligned}$$

where 1) W_i, is a Bernoulli random variable that is independent of all other random variables, 2) the equality $V_i = \tilde{T}_j \oplus \tilde{T}_k$ holds with probability one, and 3) the Markov chain

$$\tilde{U}, \tilde{S}_1, \tilde{S}_2, \tilde{S}_3 \leftrightarrow V_1, V_2, V_3 \leftrightarrow U, T_i, X_i,$$

holds for $i = 1, 2, 3$.

Proof: The proof is available in [10]. ∎

Remark 3. The rate region in Theorem 1 contains the three-user extension of the CL region. For that set V_1, V_2, V_3 to be independent of all other random variables. This gives a distribution in $\mathscr{P}$.

In what follows, we give the insight behind the the coding scheme used in Theorem 1. This coding scheme is built upon quasi-linear codes, and the CL scheme. The description of the scheme is provided in a more complete version of this work [10]. Here, we explain an special case of the scheme in which linear codes are used. Similar to Example 1, each encoder is equipped with a linear code. We use identical generator matrices for each linear code. The coding scheme is performed in three stages at each block. At the first stage, each transmitter encodes the fresh message at the beginning of the block l, where $1 \leq l \leq L$. The encoding process is performed using the identical linear codes. This stage is similar to the first stage of the coding scheme proposed for Example 1. At the second stage, upon receiving the feedback from the block $l-1$, each encoder decodes the sum of the codewords sent by the other two encoders. This stage is also similar to the scheme proposed for Example 1. At the third stage, each encoder uses the feedback from block $l-1$ and $l-2$ to decode the message of the other two encoders at block $l-2$. This is an additional stage comparing to the scheme in Example 1. As a result of these stages, at each block l, the followings are known at the ith encoder: 1) the sum of the codewords of the other encoders at block $l-1$. 2) The message of the other encoders at block $l-2$. Note that at the block l, the decoder can only create a list of highly likely messages sent at block $l-2$. This list is also known at each encoder, since the encoders have access to the feedback. As all the messages at block $l-2$ are known at all encoders, the transmitters can cooperate in the transmission of the information to resolve the uncertainty of the decoder about the messages.

V. Conclusion

We proposed a new single-letter achievable rate region for the three-user discrete memoryless MAC with noiseless feedback. We used an example to show that this achievable region strictly contains the CL region. In the example, the proposed coding scheme achieves optimality in terms of transmission rates. Moreover, we proved that any optimality achieving scheme for this example must have a specific algebraic structure. Particularly, the codebooks must be closed under binary addition.

Appendix A
Proof of Preposition 1

Outline of the proof: We start by proposing a coding scheme. There are L blocks of transmissions in this scheme, with new messages available at each user at the beginning of each block. The scheme sends the messages with n uses of the channel. Let $\mathbf{W}_{i,[l]}^k$ denotes the message of the ith transmitter at the lth block, where $i = 1,2,3$, and $1 \leq l \leq L$. Let $\mathbf{W}_{i,[l]}^k$ take values randomly and uniformly from $\mathbb{F}_2^k$. In this case, the transmission rate of each user is $R_i = \frac{k}{n}, i = 1,2,3$. The first and the second outputs of the ith encoder in block l is denoted by $\mathbf{X}_{i1,[l]}^n$ and $\mathbf{X}_{i2,[l]}^n$, respectively.

Codebook Construction: Select a $k \times n$ matrix $\mathbf{G}$ randomly and uniformly from $\mathbb{F}_2^{k\times n}$. This matrix is used as the generator matrix of a linear code. Each encoder is given the matrix $\mathbf{G}$. Therefore, the encoders use an identical linear code generated by $\mathbf{G}$.

Encoder 1 and 2: For the first block set $\mathbf{X}_{i2,[1]}^n = 0$, for $i = 1,2,3$. For the block l, encoder 1 sends $\mathbf{X}_{11,[l]}^n = \mathbf{W}_{1,[l]}^k\mathbf{G}$ through its first output. For the second output, encoder 1 sends $\mathbf{X}_{11,[l-1]}^n$ from block $l-1$, that is $\mathbf{X}_{12,[l]}^n = \mathbf{X}_{11,[l-1]}^n$. Similarly, the outputs of the second encoder are $\mathbf{X}_{21,[l]}^n = \mathbf{W}_{2,[l]}^k\mathbf{G}$, and $\mathbf{X}_{22,[l]}^n = \mathbf{X}_{21,[l-1]}^n$.

Encoder 3: The third encoder sends $\mathbf{X}_{31,[l]}^n = \mathbf{W}_{3,[l]}^k\mathbf{G}$ though its first output. This encoder receives the feedback from the block $l-1$ of the channel. This encoder wishes to decode $\mathbf{W}_{1,[l-1]}^k \oplus \mathbf{W}_{2,[l-1]}^k$ using $\mathbf{Y}_{1,[l-1]}^n$. For this purpose, it subtracts $\mathbf{X}_{31,[l-1]}^n$ from $\mathbf{Y}_{1,[l-1]}^n$. Denote the resulting vector by $\mathbf{Z}^n$. Then, it finds a unique vector $\tilde{\mathbf{w}}^k \in \mathbb{F}_2^k$ such that $(\tilde{\mathbf{w}}^k\mathbf{G}, \mathbf{Z}^n)$ is ϵ-typical with respect to P_{XZ}, where X is uniform over $\mathbb{F}_2$, and $Z = X \oplus \tilde{N}_\delta$. If the decoding process is successful, the third encoder sends $\mathbf{X}_{32,[l]}^n = \tilde{\mathbf{w}}_{[l-1]}^k\mathbf{G}$. Otherwise, an event $E_{1,[l]}$ is declared.

Decoder: The decoder receives the outputs of the channel from the lth block, that is $\mathbf{Y}_{1,[l]}^n$ and $\mathbf{Y}_{2,[l]}^n$. The decoding is performed in three steps. First, the decoder uses $\mathbf{Y}_{2,[l]}^n$ to decode $\mathbf{W}_{1,[l-1]}^k$, and $\mathbf{W}_{2,[l-1]}^k$. In particular, it finds unique $\tilde{\mathbf{w}}_1^k, \tilde{\mathbf{w}}_2^k \in \mathbb{F}_2^k$ such that $(\tilde{\mathbf{w}}_1^k\mathbf{G}, \tilde{\mathbf{w}}_2^k\mathbf{G}, \mathbf{Y}_{2,[l]}^n)$ are jointly ϵ-typical with respect to $P_{X_{12}X_{22}Y_2}$. Otherwise, an error event $E_{2,[l]}$ will be declared.

Suppose the first part of the decoding process is successful. At the second step, the decoder calculates $\mathbf{X}_{11,[l-1]}^n$, and $\mathbf{X}_{21,[l-1]}^n$. This is possible, because $\mathbf{X}_{11,[l-1]}^n$, and $\mathbf{X}_{21,[l-1]}^n$

are functions of the messages. The decoder, then, subtracts $\mathbf{X}^n_{11,[l-1]} \oplus \mathbf{X}^n_{21,[l-1]}$ from $Y_{1,[l-1]}$. The resulting vector is

$$\tilde{\mathbf{Y}}^n = \mathbf{X}^n_{31,[l-1]} \oplus \tilde{N}^n_\delta.$$

In this situation, the channel from X_{31} to $\tilde{Y}$ is a binary additive channel with δ as the bias of the noise. At the third step, the decoder uses $\tilde{\mathbf{Y}}^n$ to decode the message of the third user, i.e., $\mathbf{W}^k_{3,[l-1]}$. In particular, the decoder finds unique $\tilde{\mathbf{w}}^k_3 \in \mathbb{F}^k_2$ such that $(\tilde{\mathbf{w}}^k_3 \mathbf{G}, \tilde{\mathbf{Y}}^n)$ are jointly ϵ-typical with respect to $P_{X_{31}\tilde{Y}}$. Otherwise, an error event $E_{3,[l]}$ is declared.

Error Analysis: We can show that this problem is equivalent to a point-to-point channel coding problem, where the channel is described by $Z = X \oplus \tilde{N}_\delta$. The average probability of error approaches zero, if $\frac{k}{n} \leq 1 - h(\delta)$.

Suppose there is no error in the decoding process of the third user. That is $E^c_{1,[l]}$ occurs. Therefore, $\mathbf{X}^n_{32,[l]} = \mathbf{X}^n_{22,[l]} \oplus \mathbf{X}^n_{12,[l]}$ with probability one. As a result, the channel in Fig. 3 is in the first state. This implies that the corresponding channel consists of two parallel binary additive channel with independent noises and bias δ. Similar to the argument for E_1, it can be shown that $P(E_{2,[l]}|E^c_{1,[l]}) \to 0$, if $\frac{k}{n} \leq 1 - h(\delta)$. Lastly, we can show that conditioned on $E^c_{1,[l]}$ and $E^c_{2,[l]}$, the probability of $E_{3,[l]}$ approaches zero, if $\frac{k}{n} \leq 1 - h(\delta)$.

As a result of the above argument, the average probability of error approaches 0, if $\frac{k}{n} \leq 1 - h(\delta)$. This implies that the rates $R_i = 1 - h(\delta), i = 1, 2, 3$ are achievable, and the proof is completed. ■

APPENDIX B
PROOF OF LEMMA 1

Proof: Let R_i be the rate of the ith encoder. We have $R_i \geq 1 - h(\delta) - \epsilon$. We apply the generalized Fano's inequality (Lemma 4.3 in [7]) for decoding of the messages. More precisely, as $\bar{P} \leq \epsilon$, we have

$$\frac{1}{\log_2(M_1 M_2 M_3)} H(\Theta_1, \Theta_2, \Theta_3 | \mathbf{Y}^N) \leq h(\bar{P}) \leq h(\epsilon)$$

Let $\lambda(\epsilon) \triangleq \frac{\log_2(M_1 M_2 M_3)}{N} h(\epsilon)$. By the definition of the rate we have

$$\begin{aligned} R_1 + R_2 + R_3 &= \frac{1}{N} H(\Theta_1, \Theta_2, \Theta_3) \\ &\leq \frac{1}{N} I(\Theta_1, \Theta_2, \Theta_3; \mathbf{Y}^N) + \lambda(\epsilon) \\ &\overset{(a)}{\leq} \frac{1}{N} I(\mathbf{X}^N_1, \mathbf{X}^N_2, \mathbf{X}^N_3; \mathbf{Y}^N) + \lambda(\epsilon) \\ &\overset{(b)}{\leq} 3 - \frac{1}{N} H(\mathbf{Y}^N | \mathbf{X}^N) + \lambda(\epsilon), \end{aligned} \tag{2}$$

where (a) follows by the data processing inequality, and for (b) we use the fact that Y is a vector of three binary random variables, which implies $\frac{1}{N} H(Y^N) \leq 3$. As the channel is memoryless, and since (1) holds, we have

$$\frac{1}{N} H(\mathbf{Y}^N | \mathbf{X}^N) = \frac{1}{N} \sum_{l=1}^{N} H(Y_l | X_{1,l} X_{2,l} X_{3,l}).$$

Let $P(X_{32,l} \neq X_{12,l} \oplus X_{12,l}) = q_l$, for $l \in [1 : N]$. Denote $\bar{q}_l = 1 - q_l$. We can show that,

$$H(Y_l | X_{1,l} X_{2,l} X_{3,l}) = (1 + 2\bar{q}_l) h(\delta) + 2q_l.$$

We use the above argument, and the last inequality in (2) to give the following bound

$$\begin{aligned} R_1 + R_2 + R_3 &\leq 3 - \frac{1}{N} \sum_{l=1}^{N} [(1 + 2\bar{q}_l) h(\delta) + 2q_l] + \lambda(\epsilon) \\ &= 3 - 3h(\delta) - \frac{1}{N} 2(1 - h(\delta)) \sum_{l=1}^{N} q_l + \lambda(\epsilon) \end{aligned}$$

By assumption $R_1 + R_2 + R_3 \geq 3(1 - h(\delta) - \epsilon)$. Therefore, using the above bound we obtain,

$$\frac{3\epsilon + \lambda(\epsilon)}{2(1 - h(\delta))} \geq \frac{1}{N} \sum_{l=1}^{N} q_l \overset{(a)}{\geq} \frac{1}{N} \sum_{l \in \mathcal{I}^N_c} q_l,$$

where (a) holds, because we remove the summation over all $l \notin \mathcal{I}^N_c$. We defined $\mathcal{I}^N_c$ as in the statement of this lemma. Note that if $l \in \mathcal{I}^N_c$, then $q_l \geq c$. Finally, we obtain

$$\frac{|\mathcal{I}^N_c|}{N} \leq \frac{3\epsilon + \lambda(\epsilon)}{2c(1 - h(\delta))}$$

■

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