Discrete Structures

Induction

Example

- $1 = (1 \times 2)/2$
- $1 + 2 = (2 \times 3)/2$
- $1 + 2 + 3 \ = (3 \times 4)/2$
- $1 + 2 + 3 + 4 = (4 \times 5)/2$
- $1 + 2 + 3 + 4 + 5 = 4 \times 5/2 + 5 = 5 \times 6/2$
- $P(n)$: $1 + 2 + \cdots + n = n(n+1)/2$
- Prove $\mathrm{P(n+1)}$ under the assumption $\mathrm{P(n)}$ is true
- Indeed we prove $\forall \mathrm{n} \geq 1\ (\mathrm{P(n)} \Rightarrow \mathrm{P(n+1)})$
- Or $\forall \mathrm{n}\ (\mathrm{P(n)} \rightarrow \mathrm{P(n+1)})$ is tautology
- $1 + 2 + \cdots + n + (n+1) = n \times (n+1)/2 + (n+1) = (n+1)(n+2)/2$

Induction Principle

- Basis Step: $P(1)$ is ture
- Inductive Step: $\forall n \geq 1\colon P(n) \Rightarrow P(n+1)$

Assumption "P(n) is true" is called inductive hypothesis

$$P(1), P(1) \to P(2), P(2) \to P(3), \ldots . \Rightarrow \forall n \geq 1\ P(n)$$

Note: Basis Step can be any integer number m (even negative), just notice that you have to show that

$$\forall n \geq m\colon P(n) \Rightarrow P(n+1)$$

Well-Ordering Principle

- Any non-empty subset of N has a minimum element

Example

- the minimum of prime numbers is 2
- The minimum of odd numbers is 1

Induction Principle can be proved by Well-Ordering Principle and vice versa

- Consider $A=\{n: P(n) \text{ is not true}\}$
- Let m be the smallest element of A
- Since 1 is not in A, the $m > 1$ and $P(m-1)$ is true
- Since $P(m-1) \Rightarrow P(m)$, and $P(m-1)$, then $P(m)$ is true
- Therefore, $m \notin A$ contradicting $m \in A$

More Examples

Def: $H_n = 1 + \frac{1}{2} + \ \dots + \frac{1}{n}$

Problem: $H_{2^n} \geq 1 + \frac{n}{2}$

Solution:

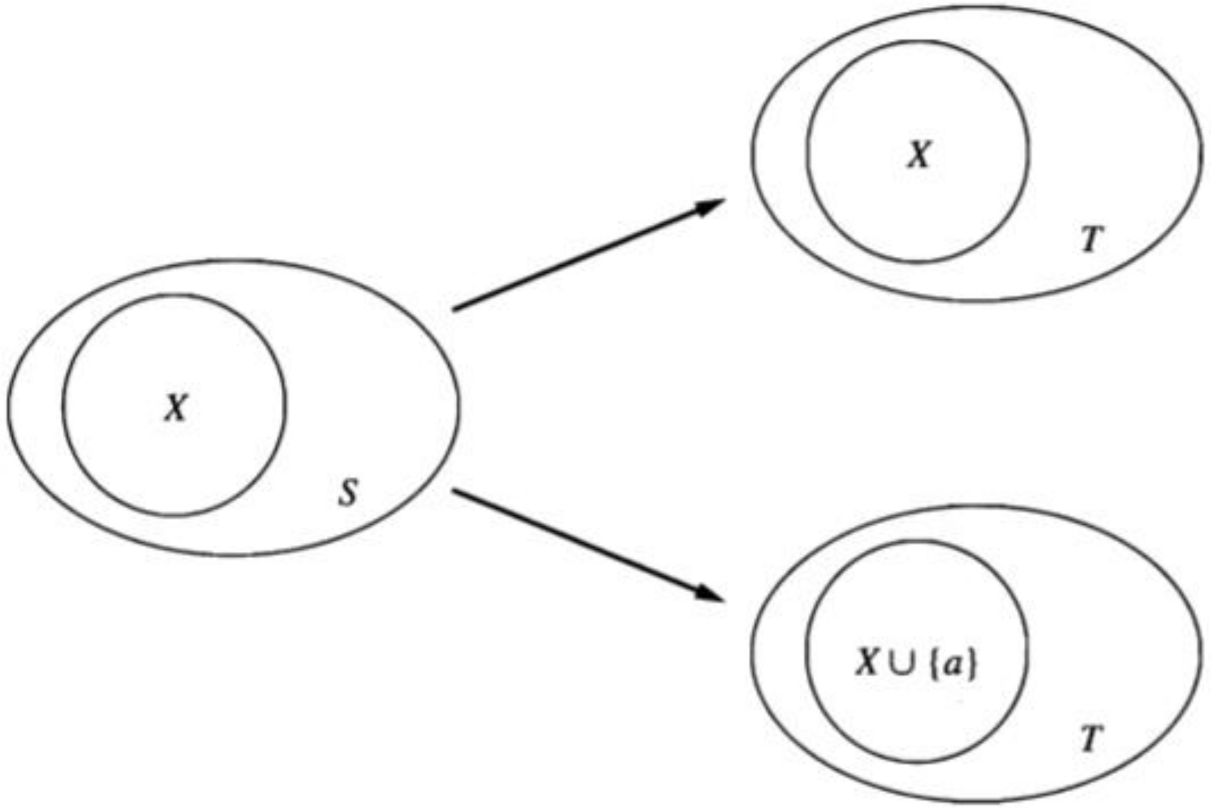
- Basis Step: $1 + \frac{1}{2} \geq 1 + \frac{1}{2}$
- Inductive Step: $H_{2^n} \geq 1 + \frac{n}{2} \Rightarrow H_{2^{n+1}} \geq 1 + \frac{n+1}{2}$

$$H_{2^{n+1}} = H_{2^n} + \left(\frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}}\right) \geq 1 + \frac{n}{2} + \frac{2^n}{2^{n+1}}$$

$$= 1 + \frac{n+1}{2}$$

More Examples

- Problem: #subsets of $T = 2^{|T|}$
- Solution: $T = S \cup \{a\}$



More Examples

- Problem: De Morgan's laws $\overline{\cap A_i} = \cup \overline{A_i}$

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}}$$ by the definition of intersection

$$= \overline{\left(\bigcap_{j=1}^{k} A_j\right)} \cup \overline{A_{k+1}}$$ by De Morgan's law (where the two sets are $\bigcap_{j=1}^{k} A_j$ and A_{k+1})

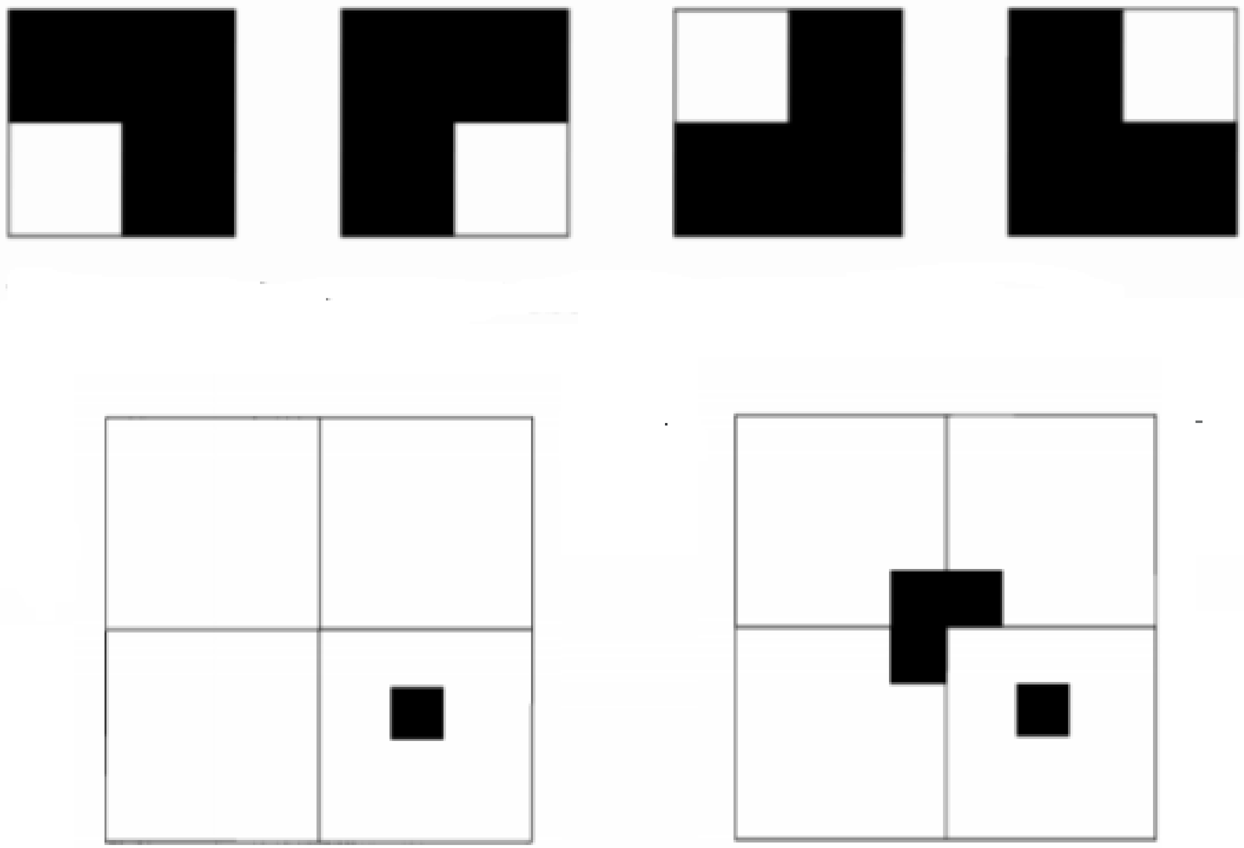
$$= \left(\bigcup_{j=1}^{k} \overline{A_j}\right) \cup \overline{A_{k+1}}$$ by the inductive hypothesis

$$= \bigcup_{j=1}^{k+1} \overline{A_j}$$ by the definition of union.

More Examples

Problem: Tiling a $2^n \times 2^n$ board with one arbitrary cell removed with right triominoes

Solution:



Strong Induction

- Basis Step: $P(1)$ is true
- Inductive Step: $P(1) \wedge P(2) \wedge \ldots \wedge P(k) \Rightarrow P(k+1)$

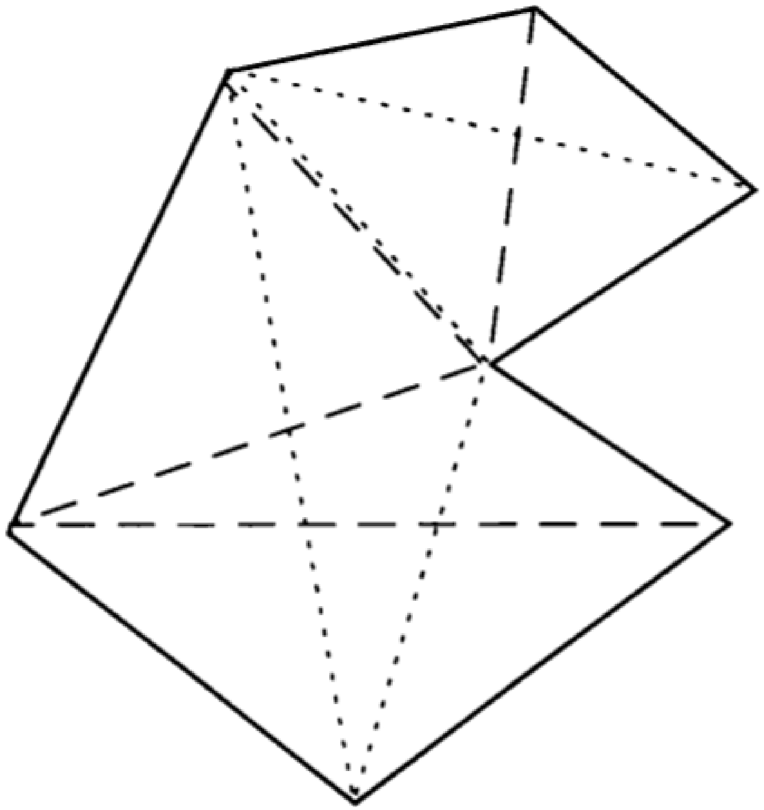
Example

Problem: A simple polygon with n vertices can be partitioned into triangles (so-called triangulated) whose vertices are vertices of the simple polygon.

Two Triangulations

Denoted by

Dots and Dashes



Example (Cont)

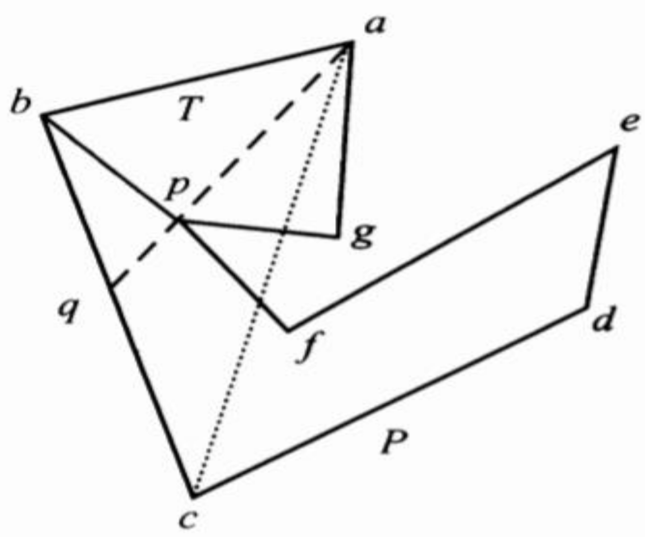
Solution:

- Basis step: $n=3$ is obvious
- Inductive step: any polygon with at most n vertices can be triangulated $\Rightarrow$ any polygon with $n+1$ vertices can be triangulated.

Example (Cont)

How to prove the inductive step:

- there are vertices v and v' such that the segment vv' is inside the polygon.
- Decompose the polygon with n+1 vertices into two polygon with at most n vertices by drawing vv'
- Triangulate each subpolygon



T is the triangle abc

p is the vertex of P inside T such that the $\angle bap$ is smallest

bp must be an interior diagonal of P

Faulty Proof

Problem: P(n): Any set of n non-parallel lines meet in a common point

Solution:

- Basis Step: P(2) is true
- Inductive Step: P(k) ⇒ P(k+1)

$\{l_1, l_2, \dots, l_{k-1,} \, l_k\}$ meet at intersection of l_1, l_2

$\{l_1, l_2, \dots, l_{k-1,} \, l_{k+1}\}$ meet at intersection of l_1, l_2

Then $\{l_1, l_2, \dots, l_{k-1,} \, l_k, l_{k+1}\}$ meet at intersection of l_1, l_2

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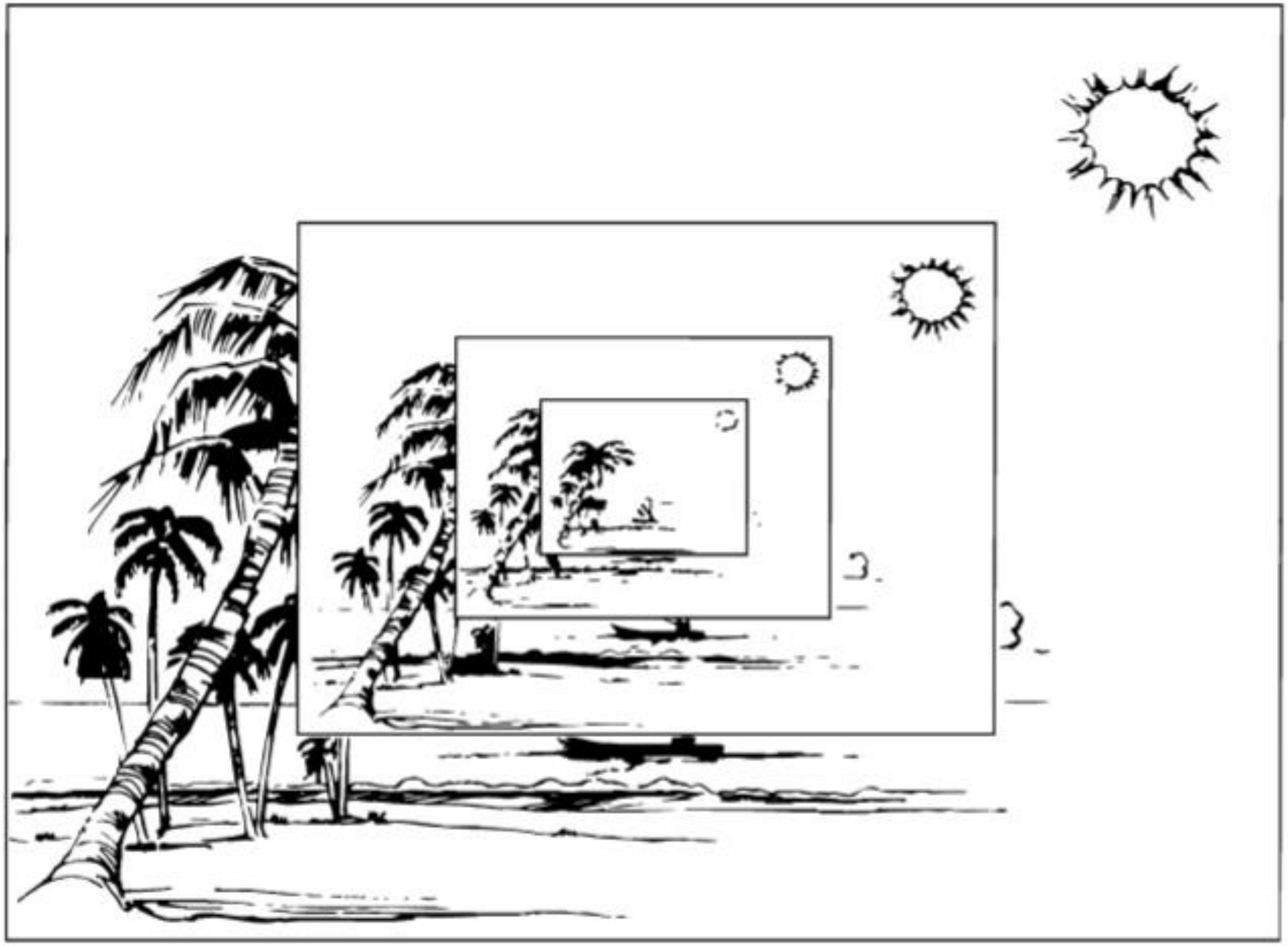
Above argument is true when k is at least 3. Then P(3) is basis step which of course is false.

Recursive Def. and Structural Indu.

- Recursive Def: Defining an object in terms of itself



Functions

Fibonacci Numbers

- $f(n+1) = f(n) + f(n-1), f(1) = f(2) = 1$

Problem: $f(n) > \alpha^{n-2}$ where $\alpha = (\sqrt{5}+1)/2$

Solution:

$$f(n+1) = f(n) + f(n-1) > \alpha^{n-2} + \alpha^{n-3}$$
$$= \alpha^{n-3}(\alpha+1) = \alpha^{n-3}\alpha^2 = \alpha^{n-1}$$

Sets

- Basis Step: $3 \in S$
- Recursive Step: $x \in S, y \in S \Rightarrow x + y \in S$

Problem: Prove $S = \{3k | k \in N\}$

- $A = \{3k | k \in N\}$
- $3 \in S, 3k \in S \Rightarrow 3(k+1) \in S$
- Then $A \subseteq S$
- Prove $S \subseteq A$

String

Defining Σ^*

- Basis Step: $\lambda \in \Sigma^*$
- Recursive Step: $x \in \Sigma^*, a \in \Sigma \Rightarrow xa \in \Sigma^*$

Defining the length

- $L(\lambda) = 0$
- $L(xa) = L(x) + 1$

Problem: $L(xy) = L(x) + L(y)$

- Basis: $L(x\lambda) = L(x) + L(\lambda) = L(x)$
- Inductive Step: $y = wa, w \in \Sigma^*, \mathrm{a} \in \Sigma$

  $$L(xy) = L(xwa) = L(xw) + 1 = L(x) + L(w) + 1$$
  $$= L(x) + L(y)$$

Propositional Expression

- Basis Step: T, F, and s (a propositional variable)
- Recursive Step:

$$E, F \Rightarrow (\sim E), (E \rightarrow F), (E \wedge F), (E \vee F). (E \leftrightarrow F)$$

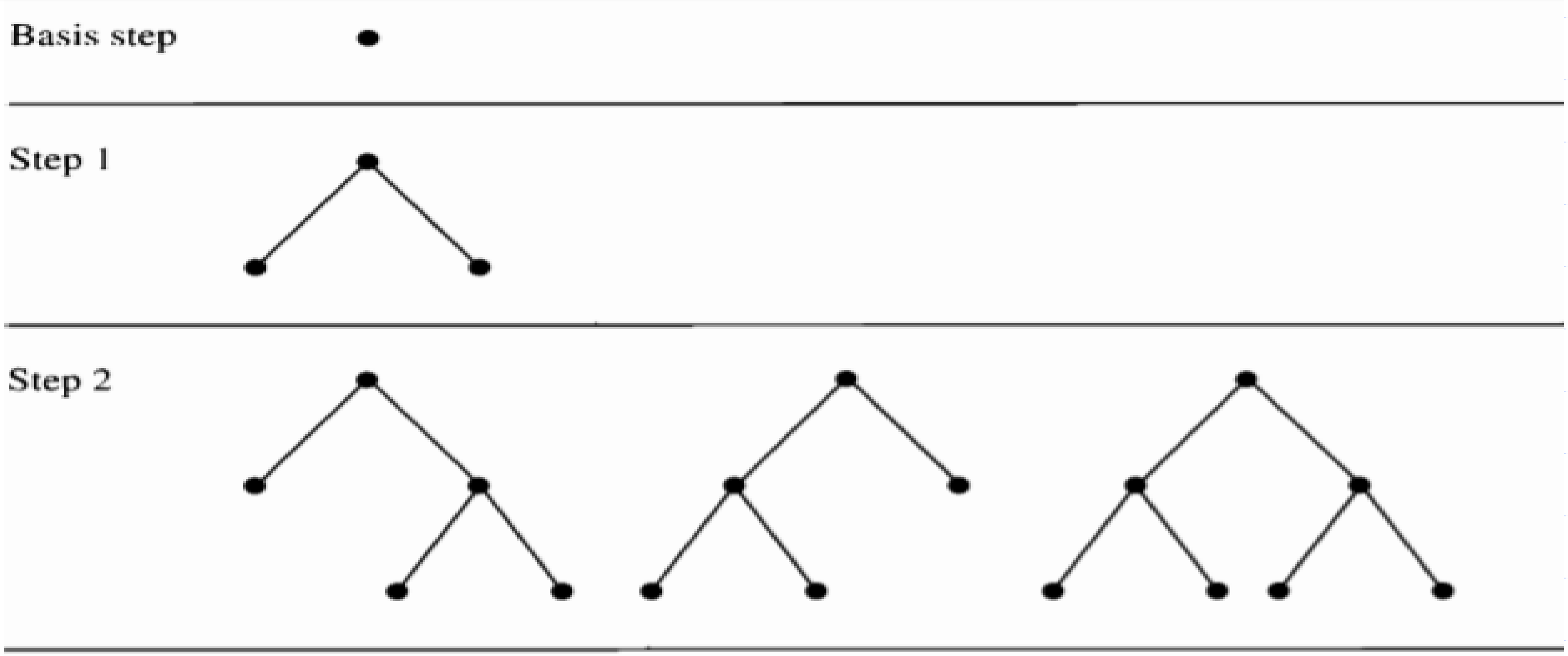
Problem: Show any expression contains an equal number of left and right parentheses

Rooted Trees

- Basis Step: A single node is a rooted tree
- Recursive Step: If $T_1, T_2, \ldots, T_k$ are rooted trees with roots of $r_1, r_2, \ldots, r_k$, then a tree with root r and edges from r to $r_1, r_2, \ldots, r_k$ is a rooted tree

Fully Binary Tree

- Basis Step: A single node is a fully binary tree
- Recursive Step: If T_1, T_2 are fully binary trees with roots of r_1, r_2, then a tree with root r and edges from r to r_1, r_2 is a fully binary tree



Fully Binary Tree

Height (h(T))

- Basis Step: If T is a single node, h(T)=0
- Recursive Step: If T constructed from T_1, T_2, then h(T)=max(h(T_1), h(T_2))+1

Size or the number of nodes (n(T))

Problem: $n(T) \le 2^{h(T)+1} - 1$

$$n(T) = 1 + n(T_1) + n(T_2)$$
$$\le 1 + \left(2^{h(T_1)+1} - 1\right) + (2^{h(T_2)+1} - 1)$$
$$\le 1 + \left(2^{h(T)} - 1\right) + (2^{h(T)} - 1) = (2^{h(T)+1} - 1)$$

Muti-Variables

For Fibonacci Num:

$$F(n+m) = F(n+1)F(m) + F(n)F(m-1)$$

Fix n and induction on m

- Basis Step:

$$F(n+1) = F(n+1)F(1) + F(n)F(0) = F(n+1)$$

- Inductive Step

$$\begin{aligned} F(n+m) &= F((n+1)+(m-1)) \\ &= F(n+2)F(m-1) + F(n+1)F(m-2) \\ &= (F(n+1)+F(n))F(m-1) + F(n+1)F(m-2) \\ &= F(n+1)(F(m-1)+F(m-2)) + F(n)F(m-1) \\ &= F(n+1)F(m) + F(n)F(m-1) \end{aligned}$$

Muti-Variables

For Fibonacci Num:

$$F(n+m) = F(n+1)F(m) + F(n)F(m-1)$$

- induction on $n+m$
- Basis Step:

For $n+m < 2$ check

$$F(n+m) = F(n+1)F(m) + F(n)F(m-1)$$

- Inductive Step:

$$F(n+m) = F(n+m-1) + F(n+m-2)$$
$$= F\big(n+(m-1)\big) + F\big(n+(m-2)\big)$$
$$= F(n+1)F(m-1) + F(n)F(m-2)$$
$$+ F(n+1)F(m-2) + F(n)F(m-3)$$
$$= F(n+1)\big(F(m-1) + F(m-2)\big)$$
$$+ F(n)\big(F(m-2) + F(m-3)\big)$$
$$= F(n+1)F(m) + F(n)F(m-1)$$