Discrete Structures

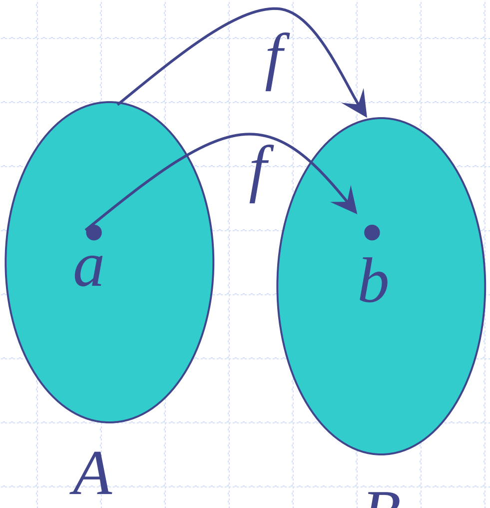
Functions

Definition of Functions

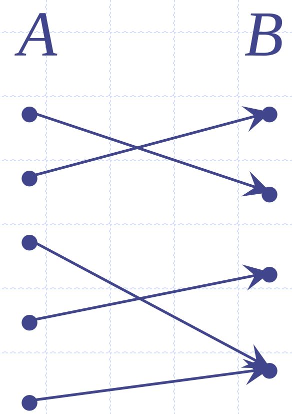
- Given any sets A, B, a function f from *(or "mapping")* A *to* B (f: $A \rightarrow B$) is an assignment of **exactly one** element $\boldsymbol{f(x)} \in \boldsymbol{B}$ to each element $\boldsymbol{x} \in \boldsymbol{A}$.

Graphical Representations

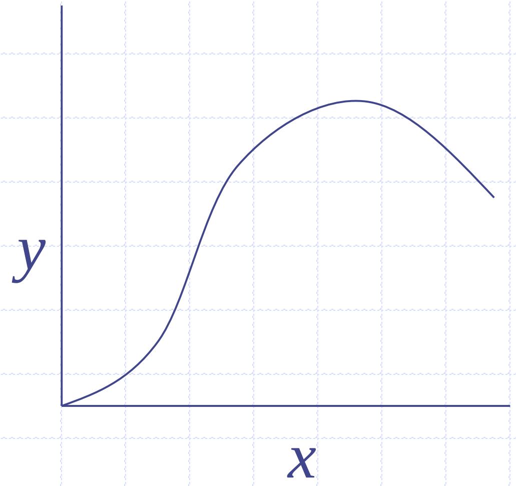
- Functions can be represented graphically in several ways:



Like Venn diagrams



Graph



Plot

Some Function Terminology

- If $f: A \rightarrow B$, and $f(a) = b$ (where $a \in A$ & $b \in B$), then:
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
    - In general, b may have more than one pre-image.
  - The *range* $R \subseteq B$ of f is $\{b \mid \exists a\, f(a)=b\}$.

Range vs. Codomain - Example

- Suppose that: "*f is a function mapping students in this class to the set of grades {A,B,C,D,E}.*"
- At this point, you know *f*'s codomain is: {A,B,C,D,E}, and its range is unknown!
- Suppose the grades turn out all As and Bs.
- Then the range of *f* is {A,B}, but its codomain is still {A,B,C,D,E}!.

Function Addition/Multiplication

- We can add and multiply *functions*

$$f, g : \mathbf{R} \to \mathbf{R}:$$

- $(f + g) : \mathbf{R} \to \mathbf{R}$, where $(f + g)(x) = f(x) + g(x)$
- $(f \times g) : \mathbf{R} \to \mathbf{R}$, where $(f \times g)(x) = f(x) \times g(x)$

Function Composition

- For functions $g{:}A\rightarrow B$ and $f{:}B\rightarrow C$, there is a special operator called *compose* ("$\bigcirc$").
  - It composes (i.e., creates) a new function out of f,g by applying f to the result of g.

$$(f\bigcirc g){:}A\rightarrow C, \text{ where } (f\bigcirc g)(a) = f(g(a)).$$

  - Note $g(a)\in B$, so $f(g(a))$ is defined and $\in C$.
  - **The range of g must be a subset of f's domain!!**
  - Note that $\bigcirc$ (like Cartesian $\times$, but unlike $+,\wedge,\cup$) is non-commuting. (In general, $f\bigcirc g \neq g\bigcirc f$.)

Function Composition

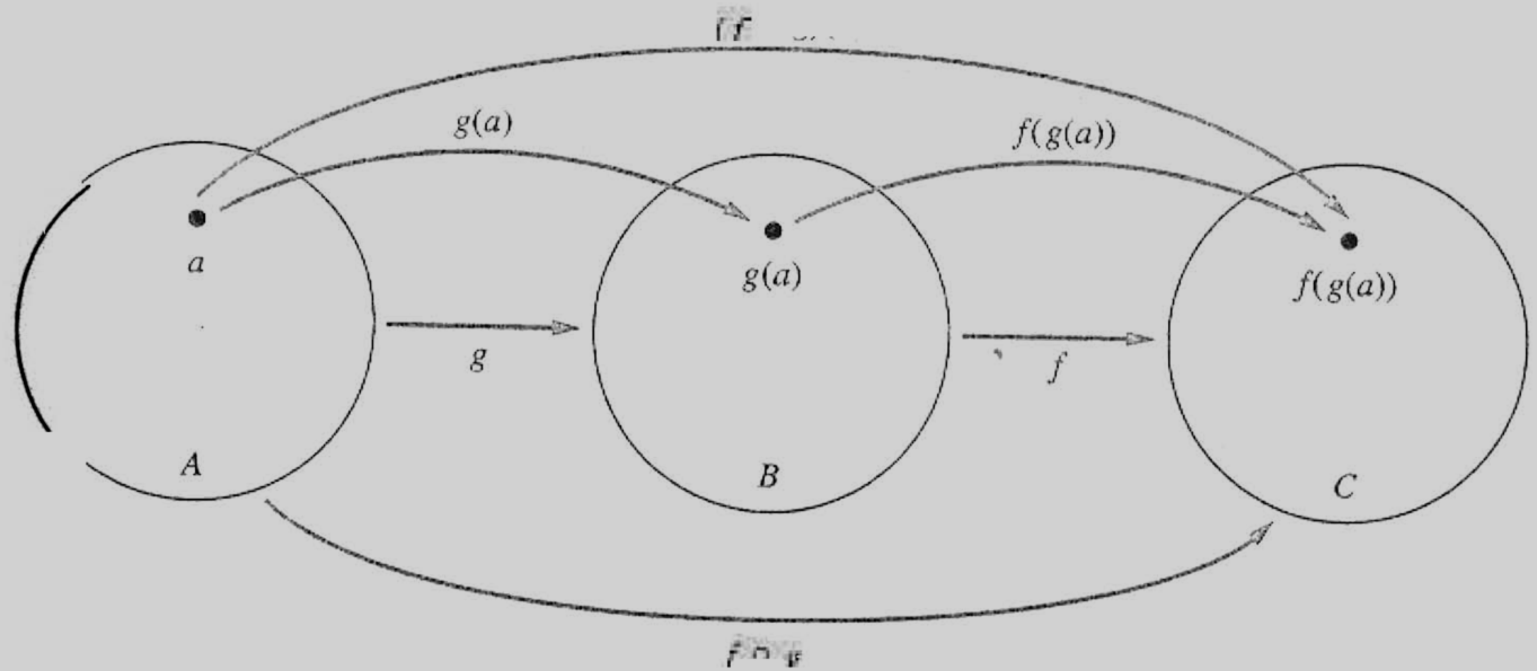


FIGURE The Composition of the Functions f and g.

Images of Sets under Functions

- Given $f: A \to B$, and $S \subseteq A$,
- The *image* of S under f is simply the set of all images (under f) of the elements of S.
  $f(S) :\equiv \{f(s) \mid s \in S\}$
  $:\equiv \{b \mid \exists\, s \in S: f(s)=b\}$.
- Note <u>the range of f can be defined as simply the image (under f) of f's domain</u>!

One-to-One Functions

- A function is *one-to-one (1-1)*, or *injective*, or *an injection*, iff every element of its range has **only one** pre-image.
- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
  - Domain & range have same cardinality. What about codomain?

One-to-One Functions (cont'd)

- Formally: given $f: A \to B$

"f is injective" $:\equiv (\neg\exists x,y{:}\ x \neq y \land f(x)=f(y))$ or

"f is injective" $:\equiv (\forall\ x,y{:}\ \neg(x \neq y) \lor \neg(f(x)=f(y)))$ or

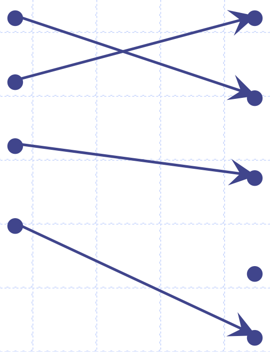
"f is injective" $:\equiv (\forall\ x,y{:}\ \neg(x \neq y) \lor (f(x) \neq f(y)))$ or

"f is injective" $:\equiv (\forall\ x,y{:}\ (x \neq y) \to (f(x) \neq f(y)))$ or

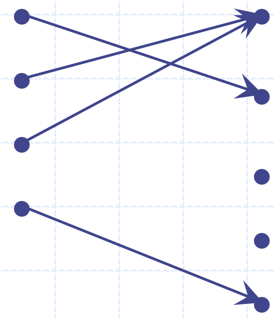
"f is injective" $:\equiv (\forall\ x,y{:}\ (f(x)=f(y)) \to (x = y))$

One-to-One Illustration

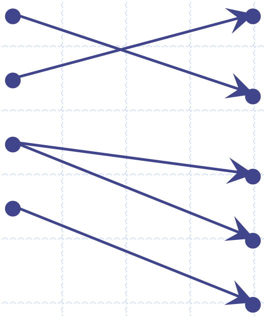
- Graph representations of functions that are (or not) one-to-one:



One-to-one



Not one-to-one



Not even a function!

Sufficient Conditions for 1-1ness

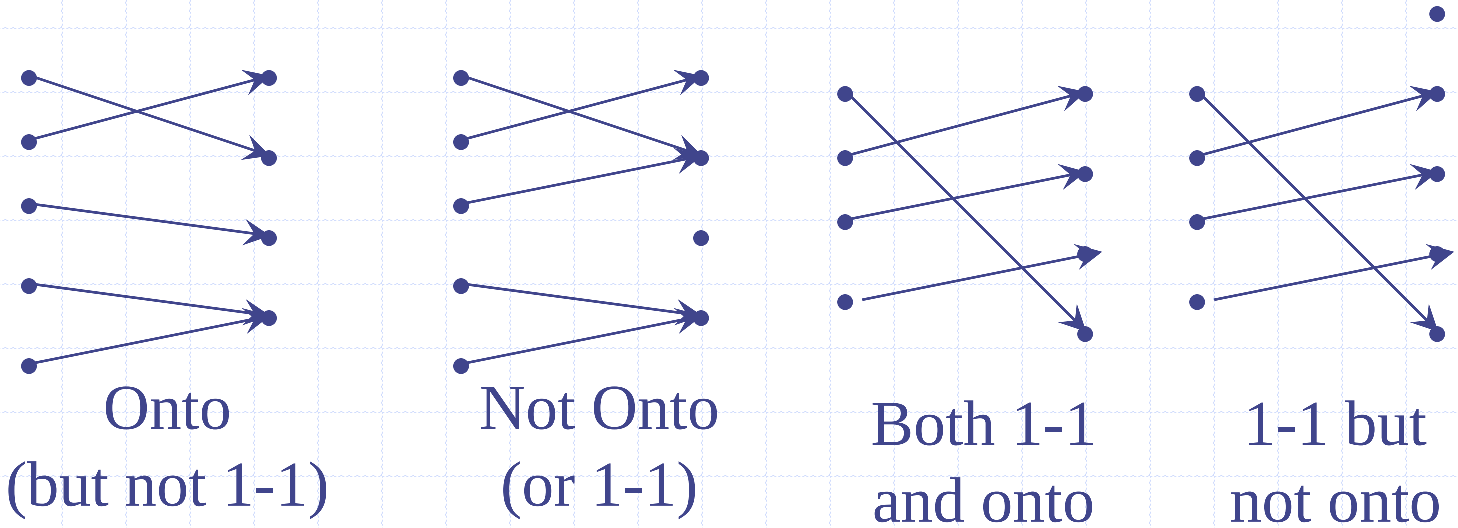
- Definitions (for functions *f* over numbers):
  - *f* is *strictly* (or *monotonically*) *increasing* iff $x>y \rightarrow f(x)>f(y)$ for all x,y in domain;
  - *f* is *strictly* (or *monotonically*) *decreasing* iff $x>y \rightarrow f(x)<f(y)$ for all x,y in domain;
- If *f* is either strictly increasing or strictly decreasing, then *f* is one-to-one.
  - *e.g.* $f(x)=x^3$

Onto (Surjective) Functions

- A function $f: A \rightarrow B$ is *onto* or *surjective* or *a surjection* iff its **range is equal to its codomain** ($\forall b \in B, \exists a \in A: f(a)=b$).
- An *onto* function maps the set A onto (over, covering) the *entirety* of the set B, not just over a piece of it.
  - *e.g.*, for domain & codomain $\mathbf{R}$, x^3 is onto, whereas x^2 isn't. (Why not?)

Illustration of Onto

- Some functions that are or are not *onto* their codomains:



Bijections

- A function *f* is *a one-to-one correspondence*, or *a bijection*, or *reversible*, or *invertible*, iff **it is both one-to-one and onto**.

Inverse of a Function

- For bijections $f{:}A\rightarrow B$, there exists an *inverse* of f, written $f^{-1}{:}B\rightarrow A$, which is the unique function such that:

$$f^{-1} \circ f = I$$

Inverse of a function (cont'd)

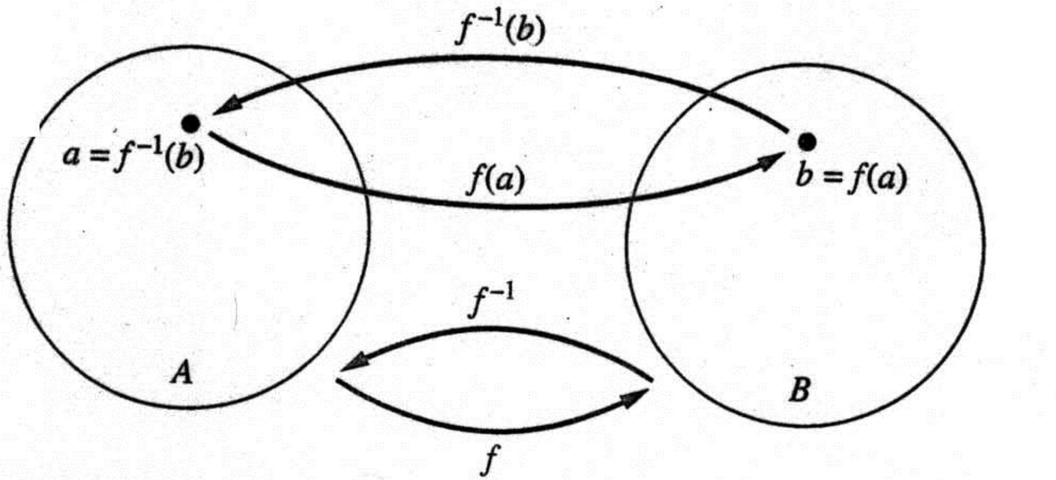


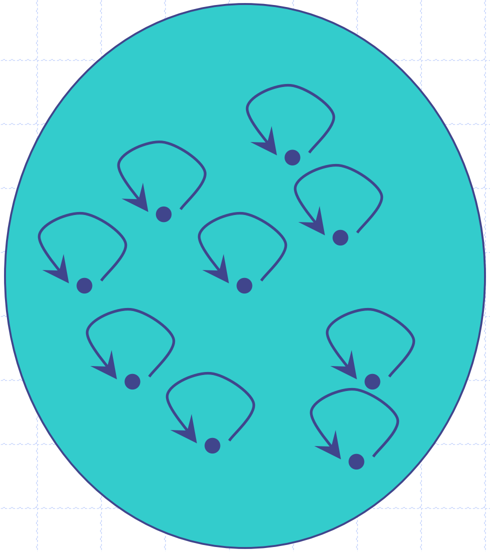
FIGURE 6 **The Function f^{-1} Is the Inverse of Function f.**

The Identity Function

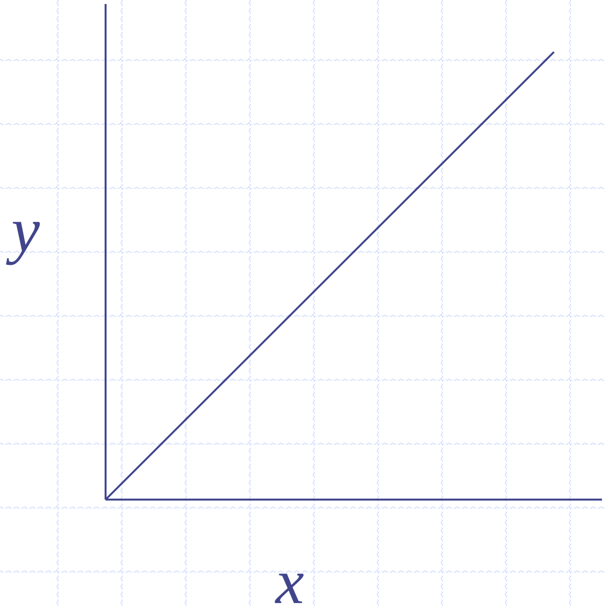
- For any domain A, the *identity function* $I: A \rightarrow A$ (variously written, I_A, $\mathbf{1}$, $\mathbf{1}_A$) is the unique function such that $\forall a \in A$: $I(a)=a$.
- Note that the identity function is both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:



Domain and range



Graphs of Functions

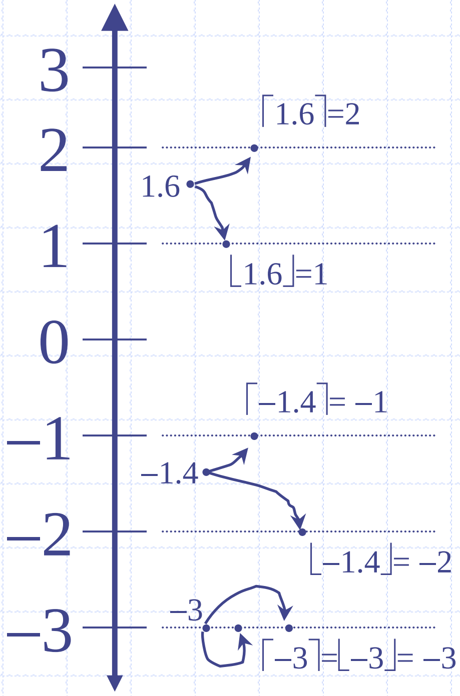
- We can represent a function f:$A \rightarrow B$ as a set of ordered pairs $\{(a,f(a)) \mid a \in A\}$.
- Note that $\forall a$, there is only one pair $(a, f(a))$.
- For functions over numbers, we can represent an ordered pair (x,y) as a point on a plane. A function is then drawn as a curve (set of points) with only one y for each x.

A Couple of Key Functions

- In discrete math, we frequently use the following functions over real numbers:
  - $\lfloor x \rfloor$ ("floor of x") is the largest integer $\leq x$.
  - $\lceil x \rceil$ ("ceiling of x") is the smallest integer $\geq x$.

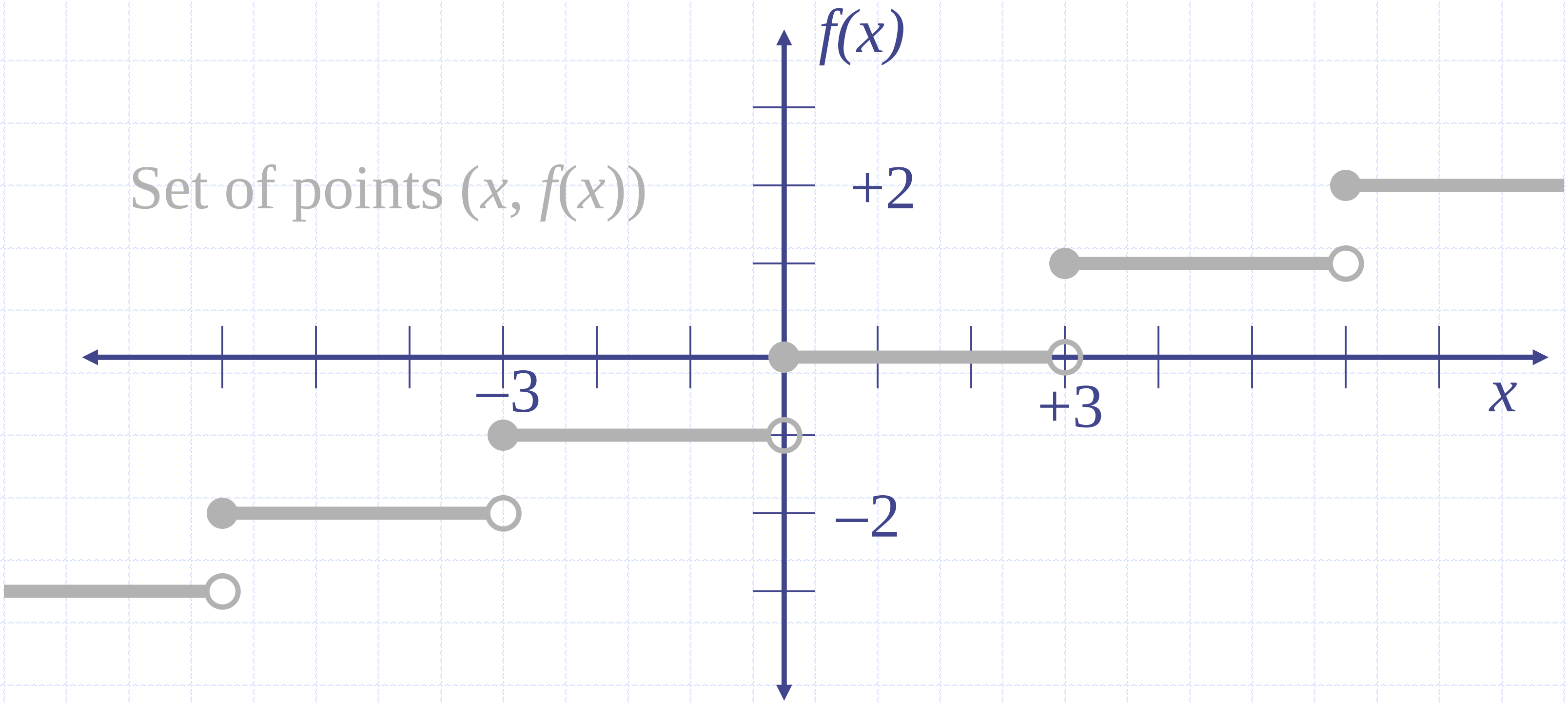
Visualizing Floor & Ceiling

- Real numbers "fall to their floor" or "rise to their ceiling."
- Note that if $x \notin \mathbf{Z}$,
  $\lfloor -x \rfloor \neq -\lfloor x \rfloor$ &
  $\lceil -x \rceil \neq -\lceil x \rceil$
- Note that if $x \in \mathbf{Z}$,
  $\lfloor x \rfloor = \lceil x \rceil = x$.



Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:



References

- *Section 2.3 of the text book "Discrete Mathematics and its Applications" by Rosen, 6th edition.*
- The original slides were prepared by Bebis