Discrete Structure

Predicate Logic

Predicate Logic

- *Predicate logic* is an extension of propositional logic that permits concisely reasoning about whole *classes* of entities.

  E.g., "$x>1$", "$x+y=10$"
- Such statements are neither true or false when the values of the variables are not specified.

Applications of Predicate Logic

- It is *the* formal notation for writing perfectly clear, concise, and unambiguous mathematical *definitions*, *axioms*, and *theorems* for *any* branch of mathematics.
- Supported by some of the more sophisticated *database query engines.*
- Basis for *automatic theorem provers* and many other Artificial Intelligence systems.

Subjects and Predicates

- The proposition

  "The dog is sleeping"

  has two parts:
  - "the dog" denotes the *subject* - the *object* or *entity* that the sentence is about.
  - "is sleeping" denotes the *predicate*- a property that the subject can have.

Propositional Functions

- A *predicate* is modeled as a *function* $P(\cdot)$ from objects to propositions.
  - $P(x)$ = "x is sleeping" (where x is any object).

- The *result of applying* a predicate P to an object $x=a$ is the *proposition* $P(a)$.
  - *e.g.* if $P(x)$ = "$x > 1$", then $P(3)$ is the *proposition* "3 is greater than 1."

- Note: The predicate P **itself** (*e.g.* P="is sleeping") is **not** a proposition (not a complete sentence).

Propositional Functions

- Predicate logic includes propositional functions of **any** number of arguments.

  e.g. let $P(x,y,z)$ = "x gave y the grade z",

  x="Mike", y="Mary", z="A",

  $P(x,y,z)$ = "Mike gave Mary the grade A."

Universe of Discourse

- The collection of values that a variable x can take is called x's *universe of discourse*.

  e.g., let $P(x)$ = "$x+1>x$". we could define the course of universe as the set of integers.

Quantifier Expressions

- *Quantifiers* allow us to *quantify* (count) *how many* objects in the universe of discourse satisfy a given predicate:
  - "$\forall$" is the FOR$\forall$LL or *universal* quantifier. $\forall x\, P(x)$ means <u>*for all* x in the u.d.</u>, *P* holds.
  - "$\exists$" is the $\exists$XISTS or *existential* quantifier. $\exists x\, P(x)$ means <u>there *exists* an x in the u.d.</u> (that is, one or more) such that $P(x)$ is true.

Universal Quantifier ∀: Example

- Let $P(x)$ be the *predicate* "x is full."
- Let the u.d. of x be parking spaces at the city.
- The *universal quantification of* $P(x)$,
  $\forall x\, P(x)$, is the *proposition:*
  - "All parking spaces at the city are full." or
  - "Every parking space at the city is full." or
  - "For each parking space at the city, that space is full."

The Universal Quantifier $\forall$

- To prove that a statement of the form $\forall x\, P(x)$ is false, it suffices to find a **counterexample** (i.e., one value of x in the universe of discourse such that $P(x)$ is false)
  - e.g., $P(x)$ is the predicate "x>0"

Existential Quantifier ∃ Example

- Let *P*(*x*) be the *predicate* "*x* is full."
- Let the u.d. of x be parking spaces at the city.
- The *universal quantification of P*(*x*),
  ∃*x P*(*x*), is the *proposition*:
  - "Some parking space at the city is full." or
  - "There is a parking space at the city that is full." or
  - "At least one parking space at the city is full."

Quantifier Equivalence Laws

- Definitions of quantifiers: If u.d.=a,b,c,…
  $\forall x\, P(x) \Leftrightarrow P(\mathrm{a}) \wedge P(\mathrm{b}) \wedge P(\mathrm{c}) \wedge \ldots$
  $\exists x\, P(x) \Leftrightarrow P(\mathrm{a}) \vee P(\mathrm{b}) \vee P(\mathrm{c}) \vee \ldots$
- We can prove the following laws:
  $\forall x\, P(x) \Leftrightarrow \neg\exists x\, \neg P(x)$
  $\exists x\, P(x) \Leftrightarrow \neg\forall x\, \neg P(x)$
- Which *propositional* equivalence laws can be used to prove this?

More Equivalence Laws

- $\neg\exists x\, P(x) \Leftrightarrow \forall x \neg P(x)$
  $\neg\forall x\, P(x) \Leftrightarrow \exists x \neg P(x)$

- $\forall x \forall y\, P(x,y) \Leftrightarrow \forall y \forall x\, P(x,y)$
  $\exists x \exists y\, P(x,y) \Leftrightarrow \exists y \exists x\, P(x,y)$

- $\forall x\, (P(x) \wedge Q(x)) \Leftrightarrow (\forall x\, P(x)) \wedge (\forall x\, Q(x))$
  $\exists x\, (P(x) \vee Q(x)) \Leftrightarrow (\exists x\, P(x)) \vee (\exists x\, Q(x))$

- $\forall x\, (P(x) \vee Q(x)) \Leftrightarrow (\forall x\, P(x)) \vee (\forall x\, Q(x))$???
  $\exists x\, (P(x) \wedge Q(x)) \Leftrightarrow (\exists x\, P(x)) \wedge (\exists x\, Q(x))$???

Scope of Quantifiers

- The part of a logical expression to which a quantifier is applied is called the scope of this quantifier.

  e.g., $(\forall x\, P(x)) \wedge (\exists y\, Q(y))$

  e.g., $(\forall x\, P(x)) \wedge (\exists x\, Q(x))$

Free and Bound Variables

- An expression like $P(x)$ is said to have a *free variable* x (meaning x is undefined).
- A quantifier (either $\forall$ or $\exists$) *operates* on an expression having one or more free variables, and *binds* one or more of those variables, to produce an expression having one or more *bound variables*.

Examples of Binding

- $P(x,y)$ has 2 free variables, x and y.
- $\forall x\, P(x,y)$ has 1 free variable, and one bound variable. [which is which?]
- "$P(x)$, where $x=3$" is another way to bind x.
- An expression with <u>zero</u> free variables is an actual proposition.
- An expression with one or more free variables is still only a predicate: $\forall x\, P(x,y)$

More to Know About Binding

- $\forall x \exists x P(x)$ - x is not a free variable in $\exists x P(x)$, therefore the $\forall x$ binding isn't used.
- $(\forall x P(x)) \wedge Q(x)$ - The variable x is outside of the *scope* of the $\forall x$ quantifier, and is therefore free. Not a proposition.
- $(\forall x P(x)) \wedge (\exists x Q(x))$ - Legal because there are 2 <u>different</u> x's!
- Quantifiers bind as loosely as needed: parenthesize $\forall x \, \big(\, P(x) \wedge Q(x) \, \big)$

Nested Quantifiers

Exist within the scope of other quantifiers

- Let the u.d. of x & y be people.
- Let $P(x,y)$="x likes y" (a predicate with 2 f.v.)
- Then $\exists y\, P(x,y)$ = "There is someone whom x likes." (a predicate with 1 free variable, x)
- Then $\forall x\, (\exists y\, P(x,y))$ = "Everyone has someone whom they like."

Order of Quantifiers Is Important!!

If $P(x,y)$="x relies upon y," express the following in unambiguous English:

$\forall x(\exists y\, P(x,y))$= Everyone has *someone* to rely on.

$\exists y(\forall x\, P(x,y))$= There's a poor overworked soul whom *everyone* relies upon (including himself)!

$\exists x(\forall y\, P(x,y))$= There's some needy person who relies upon *everybody* (including himself).

$\forall y(\exists x\, P(x,y))$= Everyone has *someone* who relies upon them.

$\forall x(\forall y\, P(x,y))$= *Everyone* relies upon *everybody*, (including themselves)!

Natural language is ambiguous!

- "Everybody likes somebody."
  - For everybody, there is somebody they like,
    - $\forall x \exists y\, Likes(x,y)$
  - or, there is somebody (a popular person) whom everyone likes?
    - $\exists y \forall x\, Likes(x,y)$ [Probably more likely.]

Notational Conventions

- Consecutive quantifiers of the same type can be combined: $\forall x \forall y \forall z\, P(x,y,z) \Leftrightarrow \forall x,y,z\, P(x,y,z)$ or even $\forall xyz\, P(x,y,z)$
- Sometimes the universe of discourse is restricted within the quantification, *e.g.*,
  - $\forall x{>}0\ P(x)$ is shorthand for "For all x that are greater than zero, $P(x)$."
  - $\exists x{>}0\ P(x)$ is shorthand for "There is an x greater than zero such that $P(x)$."

Defining New Quantifiers

As per their name, quantifiers can be used to express that a predicate is true of any given *quantity* (number) of objects.

Define $\exists! x\, P(x)$ to mean "$P(x)$ is true of *exactly one* x in the universe of discourse."

$\exists! x\, P(x) \Leftrightarrow \exists x\, (P(x) \wedge \neg \exists y\, (P(y) \wedge y \neq x))$

"There is an x such that $P(x)$, where there is no y such that $P(y)$ and y is other than x."

Some Number Theory Examples

- Let u.d. = the *natural numbers* 0, 1, 2, …
- "A number x is *even*, $E(x)$, if and only if it is equal to 2 times some other number."
  $\forall x\,(E(x) \leftrightarrow (\exists y\; x=2y))$
- "A number is *prime*, $P(x)$, iff it isn't the product of two non-unity numbers."
  $\forall x\,(P(x) \leftrightarrow (\neg\exists y,z\; x=yz \wedge y\neq 1 \wedge z\neq 1))$

Calculus Example

- Precisely defining the concept of a limit using quantifiers:

$$\left(\lim_{x \to a} f(x) = L\right) \Leftrightarrow$$

$$\left(\begin{array}{l} \forall \varepsilon > 0 : \exists \delta > 0 : \forall x : \\ \left(|x - a| < \delta\right) \to \left(|f(x) - L| < \varepsilon\right) \end{array}\right)$$

References

- *Sections 1.3 and 1.4 of the text book "Discrete Mathematics and its Applications" by Rosen, 6th edition.*
- The original slides were prepared by Bebis