# Chapter 2

# Probability Theory



## Discrete Probability Distributions

### Outline

1- Probability Distributions
2- Mean, Variance, Standard Deviation, and Expectation
3- The Binomial Distribution
4- Other Types of Distributions (Optional)

### Learning Objectives

I. Construct a probability distribution for a random variable.

II. Find the mean, variance, standard deviation, and expected value for a discrete random variable.

III. Find the exact probability for X successes in n trials of a binomial experiment.

IV. Find the mean, variance, and standard deviation for the variable of a binomial distribution.

V. Find probabilities for outcomes of variables, using the Poisson, hypergeometric, and multinomial distributions.

### Probability Distributions

- A **random variable** is a variable whose values are determined by chance.
- A **discrete probability** distribution consists of the values a random variable can assume and the corresponding probabilities of the values.
- The sum of the probabilities of all events in a sample space add up to 1. Each probability is between 0 and 1, inclusively.

Recall that when three coins are tossed, the sample space is represented as TTT, TTH, THT, HTT, HHT, HTH, THH, HHH; and if $X$ is the random variable for the number of heads, then $X$ assumes the value 0, 1, 2, or 3.

Probabilities for the values of $X$ can be determined as follows:

| No heads | One head | | | Two heads | | | Three heads |
|---|---|---|---|---|---|---|---|
| TTT | TTH | THT | HTT | HHT | HTH | THH | HHH |
| $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ |
| $\frac{1}{8}$ | $\frac{3}{8}$ | | | $\frac{3}{8}$ | | | $\frac{1}{8}$ |

Hence, the probability of getting no heads is $\frac{1}{8}$, one head is $\frac{3}{8}$, two heads is $\frac{3}{8}$, and three heads is $\frac{1}{8}$. From these values, a probability distribution can be constructed by listing the outcomes and assigning the probability of each outcome, as shown here.

| **Number of heads $X$** | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| **Probability $P(X)$** | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

**Example 1-** Rolling a Die

Construct a probability distribution for rolling a single die.

**Solution**

Since the sample space is 1, 2, 3, 4, 5, 6 and each outcome has a probability of, the distribution is as shown.

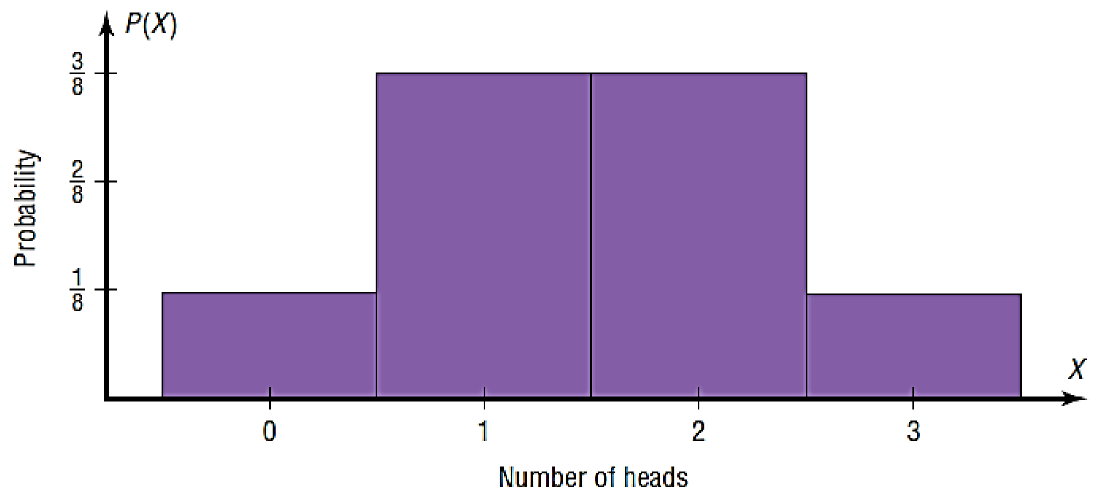
| **Outcome $X$** | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| **Probability $P(X)$** | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

Probability distributions can be shown graphically by representing the values of $X$ on the $x$ axis and the probabilities $P(X)$ on the $y$ axis.

**Example 2- Tossing Coins**
Represent graphically the probability distribution for the sample space for tossing three coins.

| Number of heads $X$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Probability $P(X)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

**Solution**
The values that $X$ assumes are located on the $x$ axis, and the values for $P(X)$ are located on the $y$ axis.




**Two Requirements for a Probability Distribution**

1. The sum of the probabilities of all the events in the sample space must equal 1; that is, $\Sigma P(X) = 1$.
2. The probability of each event in the sample space must be between or equal to 0 and 1. That is, $0 \le P(X) \le 1$.

**Example 3-** **Probability Distributions**

Determine whether each distribution is a probability distribution.

*a.*

| $X$ | 0 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| $P(X)$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ |

*b.*

| $X$ | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $P(X)$ | $-1.0$ | 1.5 | 0.3 | 0.2 |

*c.*

| $X$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X)$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{9}{16}$ |

*d.*

| $X$ | 2 | 3 | 7 |
|---|---|---|---|
| $P(X)$ | 0.5 | 0.3 | 0.4 |

**Solution**

*a*. Yes, it is a probability distribution.
*b*. No, it is not a probability distribution, since $P(X)$ cannot be 1.5 or _1.0.
*c*. Yes, it is a probability distribution.
*d*. No, it is not, since $\sum P(X) = 1.2$.

## Mean, Variance, Standard Deviation, and Expectation

The mean, variance, and standard deviation for a probability distribution are computed differently from the mean, variance, and standard deviation for samples.

### Mean

the mean for a sample or population was computed by adding the values and dividing by the total number of values, as shown in these formulas:

$$\overline{X} = \frac{\Sigma X}{n} \qquad \mu = \frac{\Sigma X}{N}$$

**Formula for the Mean of a Probability Distribution**

The mean of a random variable with a discrete probability distribution is

$$\begin{aligned}\mu &= X_1 \cdot P(X_1) + X_2 \cdot P(X_2) + X_3 \cdot P(X_3) + \cdots + X_n \cdot P(X_n)\\ &= \Sigma X \cdot P(X)\end{aligned}$$

where $X_1, X_2, X_3, \ldots, X_n$ are the outcomes and $P(X_1), P(X_2), P(X_3), \ldots, P(X_n)$ are the corresponding probabilities.

*Note:* $\Sigma X \cdot P(X)$ means to sum the products.

**Rounding Rule for the Mean, Variance, and Standard Deviation for a Probability Distribution is this**: The mean, variance, and standard deviation should be rounded to one more decimal place than the outcome $X$. When fractions are used, they should be reduced to lowest terms.

**Example 4–Rolling a Die**
Find the mean of the number of spots that appear when a die is tossed.

**Solution**
In the toss of a die, the mean can be computed thus.

| Outcome $X$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Probability $P(X)$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

$$\mu = \sum X \cdot P(X)$$

$$= 1 \cdot \tfrac{1}{6} + 2 \cdot \tfrac{1}{6} + 3 \cdot \tfrac{1}{6} + 4 \cdot \tfrac{1}{6} + 5 \cdot \tfrac{1}{6} + 6 \cdot \tfrac{1}{6}$$

$$= \tfrac{21}{6} = 3\tfrac{1}{2} \text{ or } 3.5$$

**Example 5– Children in a Family**
In a family with two children, find the mean of the number of children who will be girls.

**Solution**
The probability distribution is as follows:

| Number of girls $X$ | 0 | 1 | 2 |
|---|---|---|---|
| Probability $P(X)$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |

Hence, the mean is

$$\mu = \Sigma X \cdot P(X) = 0 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{4} = 1$$

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**Example 6– Tossing Coins**

If three coins are tossed, find the mean of the number of heads that occur.

**Solution**

The probability distribution is

| Number of heads $X$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Probability $P(X)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

The mean is

$$\mu = \Sigma X \cdot P(X) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{12}{8} = 1\frac{1}{2} \text{ or } 1.5$$

The value 1.5 cannot occur as an outcome. Nevertheless, it is the long-run or theoretical average.

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**Example 7– Number of Trips of Five Nights or More**

The probability distribution shown represents the number of trips of five nights or more that American adults take per year. (That is, 6% do not take any trips lasting five nights or more, 70% take one trip lasting five nights or more per year, etc.) Find the mean.

| Number of trips $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Probability $P(X)$ | 0.06 | 0.70 | 0.20 | 0.03 | 0.01 |

**Solution**

$$\begin{aligned} \mu &= \Sigma X \cdot P(X) \\ &= (0)(0.06) + (1)(0.70) + (2)(0.20) + (3)(0.03) + (4)(0.01) \\ &= 0 + 0.70 + 0.40 + 0.09 + 0.04 \\ &= 1.23 \approx 1.2 \end{aligned}$$

Hence, the mean of the number of trips lasting five nights or more per year taken by American adults is 1.2.

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## Variance and Standard Deviation

To find the variance for the random variable of a probability distribution, subtract the theoretical mean of the random variable from each outcome and square the difference. Then multiply each difference by its corresponding probability and add the products. The formula is

$$\sigma^2 = \Sigma[(X - \mu)^2 \cdot P(X)]$$

**Formula for the Variance of a Probability Distribution**

Find the variance of a probability distribution by multiplying the square of each outcome by its corresponding probability, summing those products, and subtracting the square of the mean. The formula for the variance of a probability distribution is

$$\sigma^2 = \Sigma[X^2 \cdot P(X)] - \mu^2$$

The standard deviation of a probability distribution is

$$\sigma = \sqrt{\sigma^2} \qquad \text{or} \qquad \sqrt{\Sigma[X^2 \cdot P(X)] - \mu^2}$$

**Example 8–Rolling a Die**

Compute the variance and standard deviation for the probability distribution in Example 4.

**Solution**

Recall that the mean is $\mu = 3.5$, as computed in Example 4. Square each outcome and multiply by the corresponding probability, sum those products, and then subtract the square of the mean.

$$\sigma^2 = \left(1^2 \cdot \tfrac{1}{6} + 2^2 \cdot \tfrac{1}{6} + 3^2 \cdot \tfrac{1}{6} + 4^2 \cdot \tfrac{1}{6} + 5^2 \cdot \tfrac{1}{6} + 6^2 \cdot \tfrac{1}{6}\right) - (3.5)^2 = 2.9$$

To get the standard deviation, find the square root of the variance.

$$\sigma = \sqrt{2.9} = 1.7$$

**Example 9–Selecting Numbered Balls**

A box contains 5 balls. Two are numbered 3, one is numbered 4, and two are numbered 5. The balls are mixed and one is selected at random. After a ball is selected, its number is recorded. Then it is replaced. If the experiment is repeated many times, find the variance and standard deviation of the numbers on the balls.

**Solution**

Let $X$ be the number on each ball. The probability distribution is

| Number on ball $X$ | 3 | 4 | 5 |
|---|---|---|---|
| Probability $P(X)$ | $\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ |

The mean is

$$\mu = \Sigma X \cdot P(X) = 3 \cdot \tfrac{2}{5} + 4 \cdot \tfrac{1}{5} + 5 \cdot \tfrac{2}{5} = 4$$

The variance is

$$\begin{aligned}\sigma &= \Sigma[X^2 \cdot P(X)] - \mu^2 \\ &= 3^2 \cdot \tfrac{2}{5} + 4^2 \cdot \tfrac{1}{5} + 5^2 \cdot \tfrac{2}{5} = 4 \\ &= 16\tfrac{4}{5} - 16 \\ &= \tfrac{4}{5}\end{aligned}$$

The standard deviation is

$$\sigma = \sqrt{\frac{4}{5}} = \sqrt{0.8} = 0.894$$

The mean, variance, and standard deviation can also be found by using vertical columns, as shown.

| $X$ | $P(X)$ | $X \cdot P(X)$ | $X^2 \cdot P(X)$ |
|---|---|---|---|
| 3 | 0.4 | 1.2 | 3.6 |
| 4 | 0.2 | 0.8 | 3.2 |
| 5 | 0.4 | 2.0 | 10 |
| | | $\Sigma X \cdot P(X) = 4.0$ | 16.8 |

$$\sigma^2 = 16.8 - 4^2 = 16.8 - 16 = 0.8$$

and

$$\sigma = \sqrt{0.8} = 0.894$$

**Example 10- On Hold for Talk Radio**

A talk radio station has four telephone lines. If the host is unable to talk (i.e., during a commercial) or is talking to a person, the other callers are placed on hold. When all lines are in use, others who are trying to call in get a busy signal. The probability that 0, 1, 2, 3, or 4 people will get through is shown in the distribution. Find the variance and standard deviation for the distribution.

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(X)$ | 0.18 | 0.34 | 0.23 | 0.21 | 0.04 |

Should the station have considered getting more phone lines installed?

**Solution**

The mean is

$$\begin{aligned}\mu &= \Sigma X \cdot P(X)\\ &= 0 \cdot (0.18) + 1 \cdot (0.34) + 2 \cdot (0.23) + 3 \cdot (0.21) + 4 \cdot (0.04)\\ &= 1.6\end{aligned}$$

The variance is

$$\begin{aligned}\sigma^2 &= \Sigma[X^2 \cdot P(X)] - \mu^2\\ &= [0^2 \cdot (0.18) + 1^2 \cdot (0.34) + 2^2 \cdot (0.23) + 3^2 \cdot (0.21) + 4^2 \cdot (0.04)] - 1.6^2\\ &= [0 + 0.34 + 0.92 + 1.89 + 0.64] - 2.56\\ &= 3.79 - 2.56 = 1.23\\ &= 1.2 \text{ (rounded)}\end{aligned}$$

The standard deviation is $\sigma = \sqrt{\sigma^2}$, or $\sigma = \sqrt{1.2} = 1.1$.

## Expectation

Another concept related to the mean for a probability distribution is that of expected value or expectation. Expected value is used in various types of games of chance, in insurance, and in other areas, such as decision theory.

> The **expected value** of a discrete random variable of a probability distribution is the theoretical average of the variable. The formula is
>
> $$\mu = E(X) = \Sigma X \cdot P(X)$$
>
> The symbol $E(X)$ is used for the expected value.

The formula for the expected value is the same as the formula for the theoretical mean. The expected value, then, is the theoretical mean of the probability distribution. That is, $E(X) = \mu$.

**Example 11-** Selecting Balls

You have six balls numbered 1-8 and 13 are placed in a box. A ball is selected at random, and its number is recorded and it is replaced Find the expected value of the number that will occur

| Number ($X$) | 1 | 2 | 3 | 5 | 8 | 13 |
|---|---|---|---|---|---|---|
| Probability $P(X)$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

$$E(X) = 1 \cdot \tfrac{1}{6} + 2 \cdot \tfrac{1}{6} + 3 \cdot \tfrac{1}{6} + 5 \cdot \tfrac{1}{6} + 8 \cdot \tfrac{1}{6} + 13 \cdot \tfrac{1}{6} = 5\tfrac{1}{3}$$

**Example 12-** **Winning Tickets**

One thousand tickets are sold at $1 each for four prizes of $100, $50, $25, and $10. After each prize drawing, the winning ticket is then returned to the pool of tickets. What is the expected value if you purchase two tickets?

| Gain $X$ | \$98 | \$48 | \$23 | \$8 | $-$\$2 |
|---|---|---|---|---|---|
| Probability $P(X)$ | $\frac{2}{1000}$ | $\frac{2}{1000}$ | $\frac{2}{1000}$ | $\frac{2}{1000}$ | $\frac{992}{1000}$ |

**Solution**

$$E(X) = \$98 \cdot \frac{2}{1000} + \$48 \cdot \frac{2}{1000} + \$23 \cdot \frac{2}{1000} + \$8 \cdot \frac{2}{1000} + (-\$2) \cdot \frac{992}{1000}$$
$$= -\$1.63$$

An alternate solution is

$$E(X) = \$100 \cdot \frac{2}{1000} + \$50 \cdot \frac{2}{1000} + \$25 \cdot \frac{2}{1000} + \$10 \cdot \frac{2}{1000} - \$2$$
$$= -\$1.63$$

## The Binomial Distribution

Many types of probability problems have only two outcomes or can be reduced to two outcomes. For example, when a coin is tossed, it can land heads or tails. When a baby is born, it will be either male or female. In a basketball game, a team either wins or loses. A true/false item can be answered in only two ways, true or false.

A **binomial experiment** is a probability experiment that satisfies the following four requirements:

1. There must be a fixed number of trials.
2. Each trial can have only two outcomes or outcomes that can be reduced to two outcomes. These outcomes can be considered as either success or failure.
3. The outcomes of each trial must be independent of one another.
4. The probability of a success must remain the same for each trial.

A binomial experiment and its results give rise to a special probability distribution called the *binomial distribution.*

The outcomes of a binomial experiment and the corresponding probabilities of these outcomes are called a **binomial distribution.**

**Notation for the Binomial Distribution**

| | |
|---|---|
| $P(S)$ | The symbol for the probability of success |
| $P(F)$ | The symbol for the probability of failure |
| $p$ | The numerical probability of a success |
| $q$ | The numerical probability of a failure |

$$P(S) = p \qquad \text{and} \qquad P(F) = 1 - p = q$$

| | |
|---|---|
| $n$ | The number of trials |
| $X$ | The number of successes in $n$ trials |

Note that $0 \leq X \leq n$ and $X = 0, 1, 2, 3, \ldots, n$.

The probability of a success in a binomial experiment can be computed with this formula.

**Binomial Probability Formula**

In a binomial experiment, the probability of exactly $X$ successes in $n$ trials is

$$P(X) = \frac{n!}{(n-X)!X!} \cdot p^X \cdot q^{n-X}$$

**Example 13- Survey on Doctor Visits**

A survey found that one out of five Americans say he or she has visited a doctor in any given month. If 10 people are selected at random, find the probability that exactly 3 will have visited a doctor last month.

**Solution**

In this case, $n = 10$, $X = 3$, $p = \frac{1}{5}$, and $q = \frac{4}{5}$. Hence,

$$P(3) = \frac{10!}{(10-3)!3!}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^7 = 0.201$$

**Example 14- Survey on Employment**

A survey from Teenage Research Unlimited (Northbrook, Illinois) found that 30% of teenage consumers receive their spending money from part-time jobs. If 5 teenagers are selected at random, find the probability that at least 3 of them will have part-time jobs.

**Solution**

To find the probability that at least 3 have part-time jobs, it is necessary to find the individual probabilities for 3, or 4, or 5, and then add them to get the total probability.

$$P(3) = \frac{5!}{(5-3)!3!}(0.3)^3(0.7)^2 = 0.132$$

$$P(4) = \frac{5!}{(5-4)!4!}(0.3)^4(0.7)^1 = 0.028$$

$$P(5) = \frac{5!}{(5-5)!5!}(0.3)^5(0.7)^0 = 0.002$$

Hence,

$$P(\text{at least three teenagers have part-time jobs}) = 0.132 + 0.028 + 0.002 = 0.162$$

**Example 14-** **Tossing Coins**

A coin is tossed 3 times. Find the probability of getting exactly two heads. Using Table B.

**Solution**

Since $n = 3$, $X = 2$, and $p = 0.5$, the value 0.375 is found

| | | | | | | | $p$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $X$ | 0.05 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.95 |
| 2 | 0 | | | | | | | | | | | |
| | 1 | | | | | | | | | | | |
| | 2 | | | | | | | | | | | |
| 3 | 0 | | | | | | 0.125 | | | | | |
| | 1 | | | | | | 0.375 | | | | | |
| | 2 | | | | | | 0.375 | | | | | |
| | 3 | | | | | | 0.125 | | | | | |

$p = 0.5$ $n = 3$ $X = 2$

**Example 15-** **Survey on Fear of Being Home Alone at Night**

*Public Opinion* reported that 5% of Americans are afraid of being alone in a house at night. If a random sample of 20 Americans is selected, find these probabilities by using the binomial table.

*a*. There are exactly 5 people in the sample who are afraid of being alone at night.
*b*. There are at most 3 people in the sample who are afraid of being alone at night.
*c*. There are at least 3 people in the sample who are afraid of being alone at night.

**Solution**

*a*. $n = 20$, $p = 0.05$, and $X = 5$. From the table, we get 0.002.

*b*. $n = 20$ and $p = 0.05$. "At most 3 people" means 0, or 1, or 2, or 3. Hence, the solution is

$$P(0) + P(1) + P(2) + P(3) = 0.358 + 0.377 + 0.189 + 0.060 = 0.984$$

*c*. $n = 20$ and $p = 0.05$. "At least 3 people" means 3, 4, 5, . . . , 20. This problem can best be solved by finding $P(0) + P(1) + P(2)$ and subtracting from 1.

$$P(0) + P(1) + P(2) = 0.358 + 0.377 + 0.189 = 0.924$$
$$1 - 0.924 = 0.076$$

**Mean, Variance, and Standard Deviation for the Binomial Distribution**

The mean, variance, and standard deviation of a variable that has the *binomial distribution* can be found by using the following formulas.

Mean: $\mu = n \cdot p$ Variance: $\sigma^2 = n \cdot p \cdot q$ Standard deviation: $\sigma = \sqrt{n \cdot p \cdot q}$

**Example 16- Tossing a Coin**

A coin is tossed 4 times. Find the mean, variance, and standard deviation of the number of heads that will be obtained.

**Solution**

With the formulas for the binomial distribution and $n = 4$, $p = \frac{1}{2}$, and $q = \frac{1}{2}$, the results are

$$\mu = n \cdot p = 4 \cdot \tfrac{1}{2} = 2$$
$$\sigma^2 = n \cdot p \cdot q = 4 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = 1$$
$$\sigma = \sqrt{1} = 1$$

From Example 16, when four coins are tossed many, many times, the average of the number of heads that appear is 2, and the standard deviation of the number of heads is 1. Note that these are theoretical values. As stated previously, this problem can be solved by using the formulas for expected value. The distribution is shown.

| No. of heads $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Probability $P(X)$ | $\frac{1}{16}$ | $\frac{4}{16}$ | $\frac{6}{16}$ | $\frac{4}{16}$ | $\frac{1}{16}$ |

$$\mu = E(X) = \Sigma X \cdot P(X) = 0 \cdot \tfrac{1}{16} + 1 \cdot \tfrac{4}{16} + 2 \cdot \tfrac{6}{16} + 3 \cdot \tfrac{4}{16} + 4 \cdot \tfrac{1}{16} = \tfrac{32}{16} = 2$$
$$\sigma^2 = \Sigma X^2 \cdot P(X) - \mu^2$$
$$= 0^2 \cdot \tfrac{1}{16} + 1^2 \cdot \tfrac{4}{16} + 2^2 \cdot \tfrac{6}{16} + 3^2 \cdot \tfrac{4}{16} + 4^2 \cdot \tfrac{1}{16} - 2^2 = \tfrac{80}{16} - 4 = 1$$
$$\sigma = \sqrt{1} = 1$$

Hence, the simplified binomial formulas give the same results.

**Example 17- Rolling a Die**

A die is rolled 360 times. Find the mean, variance, and standard deviation of the number of 4s that will be rolled.

**Solution**

This is a binomial experiment since getting a 4 is a success and not getting a 4 is considered a failure. Hence $n = 360$, $p = \frac{1}{6}$, and $q = \frac{5}{6}$.

$$\mu = n \cdot p = 360 \cdot \tfrac{1}{6} = 60$$

$$\sigma^2 = n \cdot p \cdot q = 360 \cdot \left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right) = 50$$

$$\sigma = \sqrt{n \cdot p \cdot q} = \sqrt{50} = 7.07$$

On average, sixty 4s will be rolled. The standard deviation is 7.07.

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**Example 18- Likelihood of Twins**

The *Statistical Bulletin* published by Metropolitan Life Insurance Co. reported that 2% of all-American births result in twins. If a random sample of 8000 births is taken, find the mean, variance, and standard deviation of the number of births that would result in twins.

**Solution**

This is a binomial situation, since a birth can result in either twins or not twins (i.e., two outcomes).

$$\mu = n \cdot p = (8000)(0.02) = 160$$

$$\sigma^2 = n \cdot p \cdot q = (8000)(0.02)(0.98) = 156.8$$

$$\sigma = \sqrt{n \cdot p \cdot q} = \sqrt{156.8} = 12.5$$

For the sample, the average number of births that would result in twins is 160, the variance is 156.8, or 157, and the standard deviation is 12.5, or 13 if rounded.

## Other Types of Distributions (Optional)

In addition to the binomial distribution, other types of distributions are used in statistics. Three of the most commonly used distributions are the multinomial distribution, the Poisson distribution, and the hypergeometric distribution.

### The Multinomial Distribution

- The **multinomial distribution** is similar to the binomial distribution but has the advantage of allowing one to compute probabilities when there are more than two outcomes. For example, a survey might require the responses of "approve," "disapprove," or "no opinion."
- The binomial distribution is a special case of the multinomial distribution.

**Formula for the Multinomial Distribution**

If $X$ consists of events $E_1, E_2, E_3, \ldots, E_k$, which have corresponding probabilities $p_1, p_2, p_3, \ldots, p_k$ of occurring, and $X_1$ is the number of times $E_1$ will occur, $X_2$ is the number of times $E_2$ will occur, $X_3$ is the number of times $E_3$ will occur, etc., then the probability that $X$ will occur is

$$P(X) = \frac{n!}{X_1! \cdot X_2! \cdot X_3! \cdots X_k!} \cdot p_1^{X_1} \cdot p_2^{X_2} \cdots p_k^{X_k}$$

where $X_1 + X_2 + X_3 + \cdots + X_k = n$ and $p_1 + p_2 + p_3 + \cdots + p_k = 1$.

**Example 19–Leisure Activities**

In a large city, 50% of the people choose a movie, 30% choose dinner and a play, and 20% choose shopping as a leisure activity. If a sample of 5 people is randomly selected, find the probability that 3 are planning to go to a movie, 1 to a play, and 1 to a shopping mall.

**Solution**

We know that $n = 5, X_1 = 3, X_2 = 1, X_3 = 1, p_1 = 0.50, p_2 = 0.30$, and $p_3 = 0.20$. Substituting in the formula gives

$$P(X) = \frac{5!}{3! \cdot 1! \cdot 1!} \cdot (0.50)^3(0.30)^1(0.20)^1 = 0.15$$

**Example 20–CD Purchases**

In a music store, a manager found that the probabilities that a person buys 0, 1, or 2 or more CDs are 0.3, 0.6, and 0.1, respectively. If 6 customers enter the store, find the probability that 1 won't buy any CDs, 3 will buy 1 CD, and 2 will buy 2 or more CDs.

**Solution**

It is given that $n = 6, X_1 = 1, X_2 = 3, X_3 = 2, p_1 = 0.3, p_2 = 0.6$, and $p_3 = 0.1$. Then

$$P(X) = \frac{6!}{1!3!2!} \cdot (0.3)^1(0.6)^3(0.1)^2$$
$$= 60 \cdot (0.3)(0.216)(0.01) = 0.03888$$

**Example 21–Selecting Colored Balls**

A box contains 4 white balls, 3 red balls, and 3 blue balls. A ball is selected at random, and its color is written down. It is replaced each time. Find the probability that if 5 balls are selected, 2 are white, 2 are red, and 1 is blue.

**Solution**

We know that $n = 5, X_1 = 2, X_2 = 2, X_3 = 1; p_1 = \frac{4}{10}, p_2 = \frac{3}{10}$, and $p_3 = \frac{3}{10}$; hence,

$$P(X) = \frac{5!}{2!2!1!} \cdot \left(\frac{4}{10}\right)^2\left(\frac{3}{10}\right)^2\left(\frac{3}{10}\right)^1 = \frac{81}{625}$$

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## The Poisson Distribution

A discrete probability distribution that is useful when $n$ is large and $p$ is small and when the independent variables occur over a period of time is called the **Poisson distribution.**

**Formula for the Poisson Distribution**

The probability of $X$ occurrences in an interval of time, volume, area, etc., for a variable where $\lambda$ (Greek letter lambda) is the mean number of occurrences per unit (time, volume, area, etc.) is

$$P(X; \lambda) = \frac{e^{-\lambda}\lambda^X}{X!} \qquad \text{where } X = 0, 1, 2, \ldots$$

The letter $e$ is a constant approximately equal to 2.7183.

**Example 22–Typographical Errors**

If there are 200 typographical errors randomly distributed in a 500-page manuscript, find the probability that a given page contains exactly 3 errors.

**Solution**

First, find the mean number $\lambda$ of errors. Since there are 200 errors distributed over 500 pages, each page has an average of

$$\lambda = \frac{200}{500} = \frac{2}{5} = 0.4$$

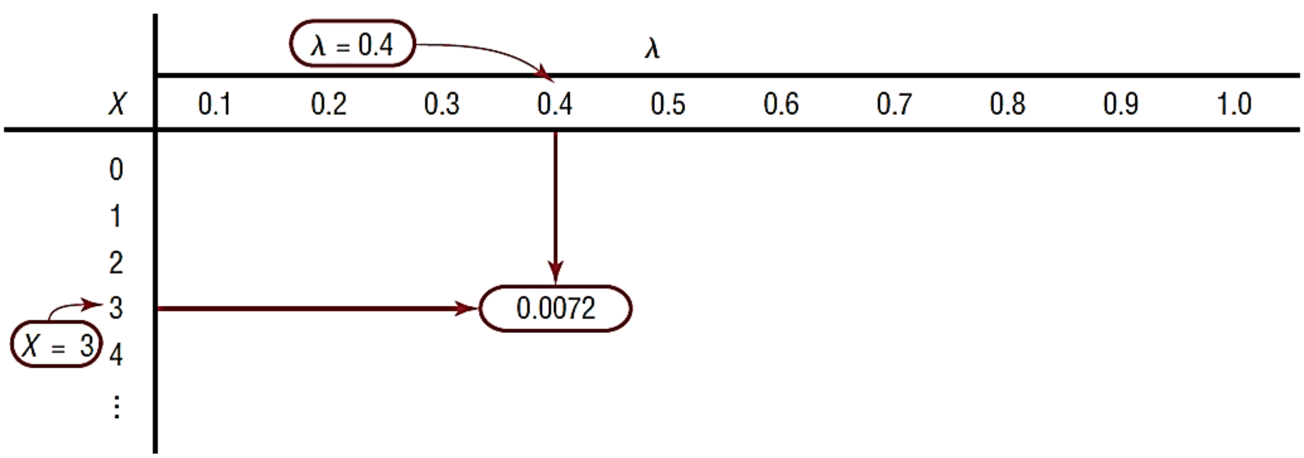
or 0.4 error per page. Since $X = 3$, substituting into the formula yields

$$P(X; \lambda) = \frac{e^{-\lambda}\lambda^X}{X!} = \frac{(2.7183)^{-0.4}(0.4)^3}{3!} = 0.0072$$

Thus, there is less than a 1% chance that any given page will contain exactly 3 errors.

Since the mathematics involved in computing Poisson probabilities is somewhat complicated, tables have been compiled for these probabilities. Table C in Appendix C gives $P$ for various values for $\lambda$ and $X$.

In Example 22, where $X$ is 3 and l is 0.4, the table gives the value 0.0072 for the probability.



**Example 23–Toll-Free Telephone Calls**

A sales firm receives, on average, 3 calls per hour on its toll-free number. For any given hour, find the probability that it will receive the following.

*a.* At most 3 calls *b.* At least 3 calls *c.* 5 or more calls

**Solution**

*a.* "At most 3 calls" means 0, 1, 2, or 3 calls. Hence,

$$\begin{aligned} &P(0; 3) + P(1; 3) + P(2; 3) + P(3; 3) \\ &\quad = 0.0498 + 0.1494 + 0.2240 + 0.2240 \\ &\quad = 0.6472 \end{aligned}$$

*b.* "At least 3 calls" means 3 or more calls. It is easier to find the probability of 0, 1, and 2 calls and then subtract this answer from 1 to get the probability of at least 3 calls.

$$P(0; 3) + P(1; 3) + P(2; 3) = 0.0498 + 0.1494 + 0.2240 = 0.4232$$

and

$$1 - 0.4232 = 0.5768$$

*c.* For the probability of 5 or more calls, it is easier to find the probability of getting 0, 1, 2, 3, or 4 calls and subtract this answer from 1. Hence,

$$\begin{aligned} &P(0; 3) + P(1; 3) + P(2; 3) + P(3; 3) + P(4; 3) \\ &\quad = 0.0498 + 0.1494 + 0.2240 + 0.2240 + 0.1680 \\ &\quad = 0.8152 \end{aligned}$$

and

$$1 - 0.8152 = 0.1848$$

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The Poisson distribution can also be used to approximate the binomial distribution when the expected value $\lambda = n \cdot p$ is less than 5, as shown in Example 5–29. (The same is true when $n \cdot q < 5$.)

**Example 24–Left-Handed People**

If approximately 2% of the people in a room of 200 people are left-handed, find the probability that exactly 5 people there are left-handed.

**Solution**

Since $\lambda = n \cdot p$, then $\lambda = (200)(0.02) = 4$. Hence,

$$P(X; \lambda) = \frac{(2.7183)^{-4}(4)^5}{5!} = 0.1563$$

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## The Hypergeometric Distribution

When sampling is done *without* replacement, the binomial distribution does not give exact probabilities, since the trials are not independent. The smaller the size of the population, the less accurate the binomial probabilities will be.

**Formula for the Hypergeometric Distribution**

Given a population with only two types of objects (females and males, defective and nondefective, successes and failures, etc.), such that there are $a$ items of one kind and $b$ items of another kind and $a + b$ equals the total population, the probability $P(X)$ of selecting without replacement a sample of size $n$ with $X$ items of type $a$ and $n - X$ items of type $b$ is

$$P(X) = \frac{{}_aC_X \cdot {}_bC_{n-X}}{{}_{a+b}C_n}$$

**Example 25- Assistant Manager Applicants**

Ten people apply for a job as assistant manager of a restaurant. Five have completed college and five have not. If the manager selects 3 applicants at random, find the probability that all 3 are college graduates.

**Solution**

Assigning the values to the variables gives

$a = 5$ college graduates $\quad n = 3$

$b = 5$ nongraduates $\quad X = 3$

and $n - X = 0$. Substituting in the formula gives

$$P(X) = \frac{{}_5C_3 \cdot {}_5C_0}{{}_{10}C_3} = \frac{10}{120} = \frac{1}{12}$$

**Example 26- House Insurance**

A recent study found that 2 out of every 10 houses in a neighborhood have no insurance. If 5 houses are selected from 10 houses, find the probability that exactly 1 will be uninsured.

**Solution**

In this example, $a = 2$, $b = 8$, $n = 5$, $X = 1$, and $n - X = 4$.

$$P(X) = \frac{{}_2C_1 \cdot {}_8C_4}{{}_{10}C_5} = \frac{2 \cdot 70}{252} = \frac{140}{252} = \frac{5}{9}$$

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**Example 27- Defective Compressor Tanks**

A lot of 12 compressor tanks is checked to see whether there are any defective tanks. Three tanks are checked for leaks. If 1 or more of the 3 is defective, the lot is rejected. Find the probability that the lot will be rejected if there are actually 3 defective tanks in the lot.

**Solution**

Since the lot is rejected if at least 1 tank is found to be defective, it is necessary to find the probability that none are defective and subtract this probability from 1.

Here, $a = 3$, $b = 9$, $n = 3$, and $X = 0$; so

$$P(X) = \frac{{}_3C_0 \cdot {}_9C_3}{{}_{12}C_3} = \frac{1 \cdot 84}{220} = 0.38$$

Hence,

$$P(\text{at least 1 defective}) = 1 - P(\text{no defectives}) = 1 - 0.38 = 0.62$$

There is a 0.62, or 62%, probability that the lot will be rejected when 3 of the 12 tanks are defective.