Linear Algebra
A gentle introduction

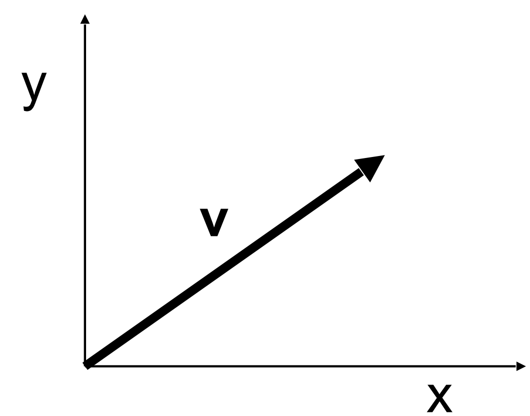
Linear Algebra has become as basic and as applicable as calculus, and fortunately it is easier.
--Gilbert Strang, MIT

Lovingly stolen from
Shivkumar Kalyanaraman

What is a Vector ?

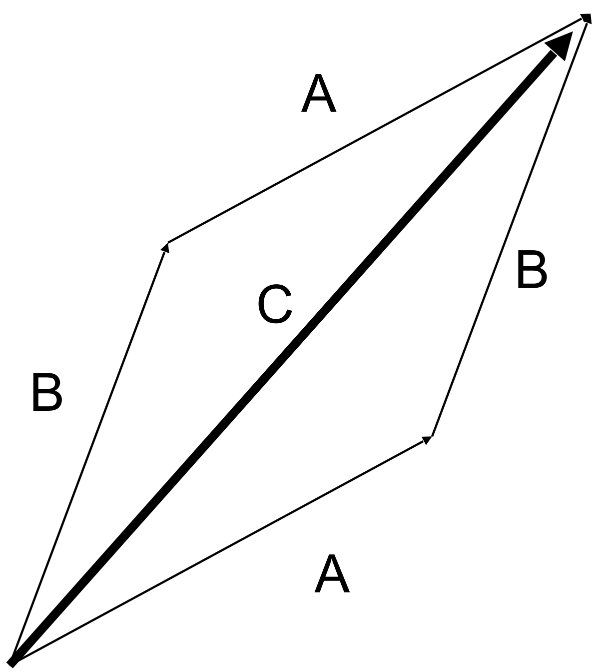
- Think of a vector as a *directed line segment in N-dimensions*! (has “length” and “direction”)
- Basic idea: convert geometry in higher dimensions into algebra!
  - Once you define a “nice” *basis* along each dimension: x-, y-, z-axis …
  - Vector becomes a 1 x N matrix!
  - **v** = [a b c]^T
  - Geometry starts to become linear algebra on vectors like **v**!

$$v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



Vector Addition: A+B

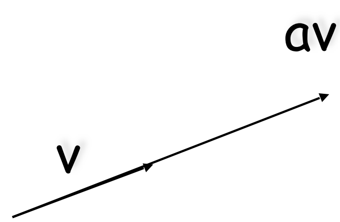
$$\mathbf{A+B} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$



A+B = C
(use the head-to-tail method to combine vectors)

Scalar Product: $a\mathbf{v}$

$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$



Change only the length ("scaling"), but keep *direction fixed*.

Sneak peek: matrix operation (**Av**) can change *length, direction and also dimensionality*!

Vectors: Dot Product

$$A \cdot B = A^T B = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$$

Think of the dot product as a matrix multiplication

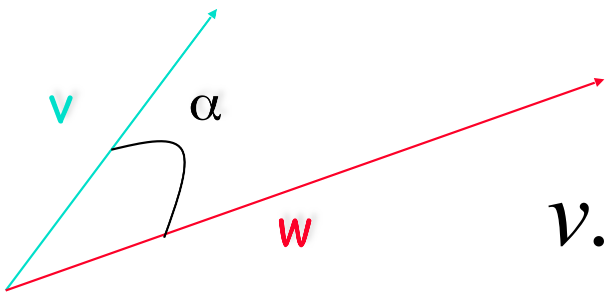
$$\|A\|^2 = A^T A = aa + bb + cc$$

The magnitude is the dot product of a vector with itself

$$A \cdot B = \|A\| \, \|B\| \cos(\theta)$$

The dot product is also related to the angle between the two vectors

Inner (dot) Product: v.w or $w^T v$



$$v.w = (x_1, x_2).(y_1, y_2) = x_1 y_1 + x_2 . y_2$$

The inner product is a **SCALAR!**

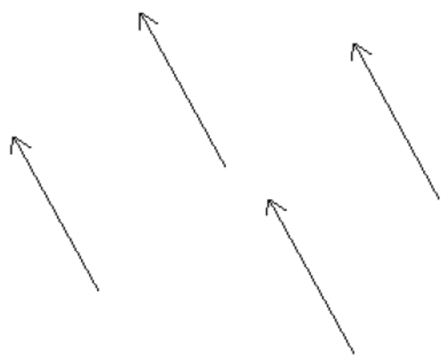
$$v.w = (x_1, x_2).(y_1, y_2) = \|v\| \cdot \|w\| \cos\alpha$$

$$v.w = 0 \Leftrightarrow v \perp w$$

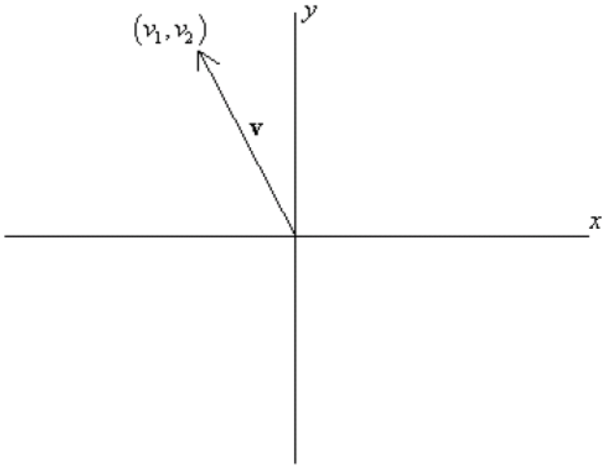
If vectors **v, w** are "columns", then dot product is $\mathbf{w}^T\mathbf{v}$

Bases & Orthonormal Bases

- Basis (or axes): frame of reference



vs



Basis: a space is totally defined by a set of vectors – any point is a *linear combination* of the basis

Ortho-Normal: orthogonal + normal

[**<u>Sneak peek</u>**:

<u>Orthogonal</u>: dot product is zero

<u>Normal</u>: magnitude is one]

$$x = [1 \quad 0 \quad 0]^T \qquad x \cdot y = 0$$
$$y = [0 \quad 1 \quad 0]^T \qquad x \cdot z = 0$$
$$z = [0 \quad 0 \quad 1]^T \qquad y \cdot z = 0$$

What is a Matrix?

- A matrix is a set of elements, organized into rows and columns

$$\begin{array}{cc} & \text{rows} \\ \text{columns} & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$$

Basic Matrix Operations

- Addition, Subtraction, Multiplication: creating new matrices (or functions)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Just add elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Just subtract elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Multiply each row by each column

Matrix Times Matrix

$$\mathbf{L} = \mathbf{M} \cdot \mathbf{N}$$

$$\begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \cdot \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}$$

$$l_{12} = m_{11}n_{12} + m_{12}n_{22} + m_{13}n_{32}$$

Multiplication

- Is AB = BA? Maybe, but maybe not!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & \ldots \\ \ldots & \ldots \end{bmatrix} \qquad \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea + fc & \ldots \\ \ldots & \ldots \end{bmatrix}$$

- Matrix multiplication AB: apply transformation B first, and then again transform using A!
- Heads up: multiplication is NOT commutative!

- **<u>Note</u>**: If A and B both represent either pure "*<u>rotation</u>*" or "*<u>scaling</u>*" they can be interchanged (i.e. AB = BA)

Matrix operating on vectors

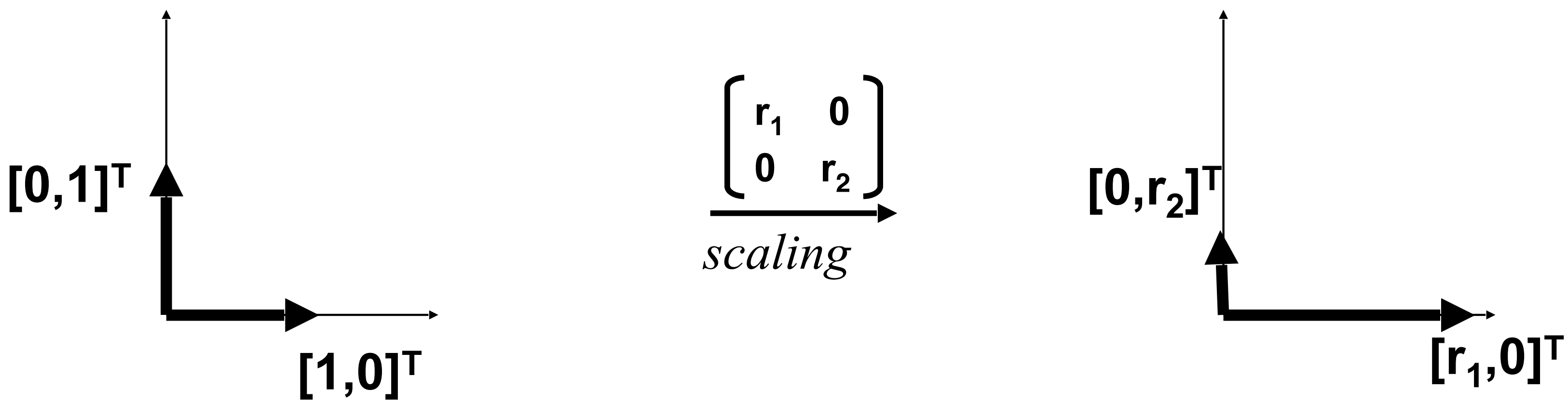
- Matrix is like a <u>function</u> that <u>transforms the vectors on a plane</u>
- Matrix operating on a general point => transforms x- and y-components
- *System of linear equations*: matrix is just the bunch of coeffs !

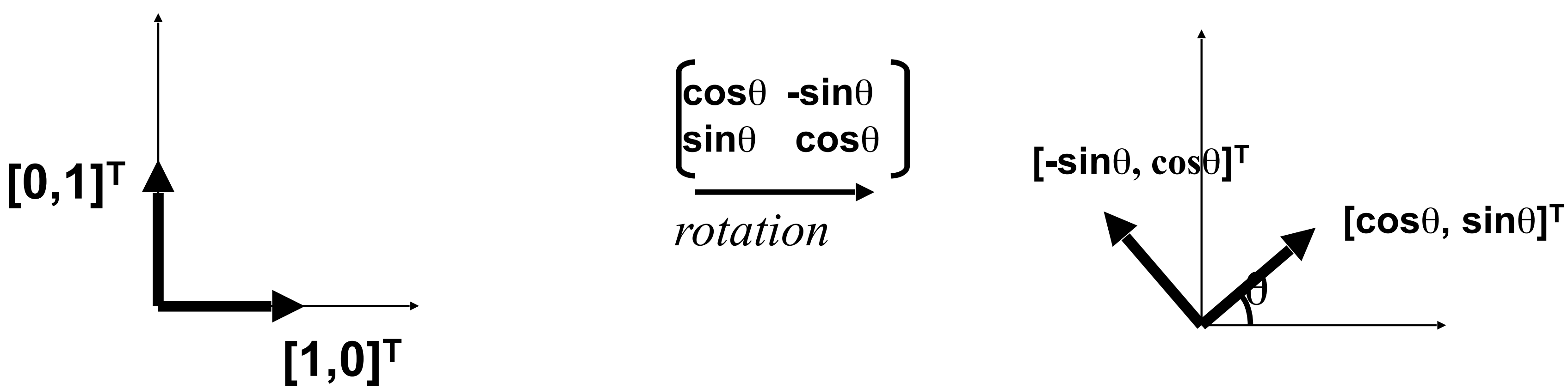
- x' = ax + by
- y' = cx + dy

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

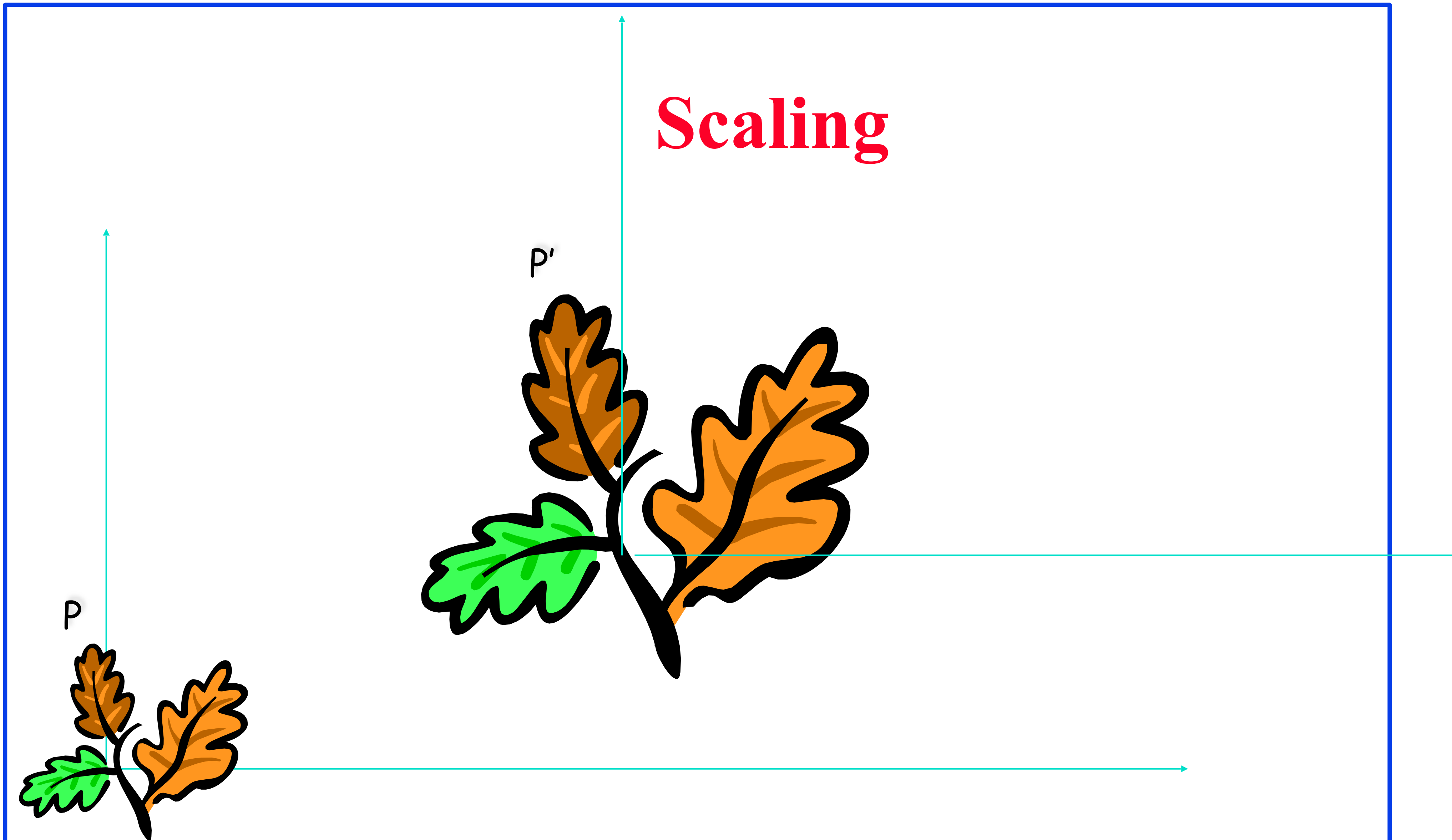
Matrices: Scaling, Rotation, Identity

- Pure scaling, no rotation => "**diagonal**" matrix" (note: x-, y-axes could be scaled differently!)
- Pure rotation, no stretching => "**orthogonal** matrix" **O**
- **Identity** ("do nothing") matrix = unit scaling, no rotation!





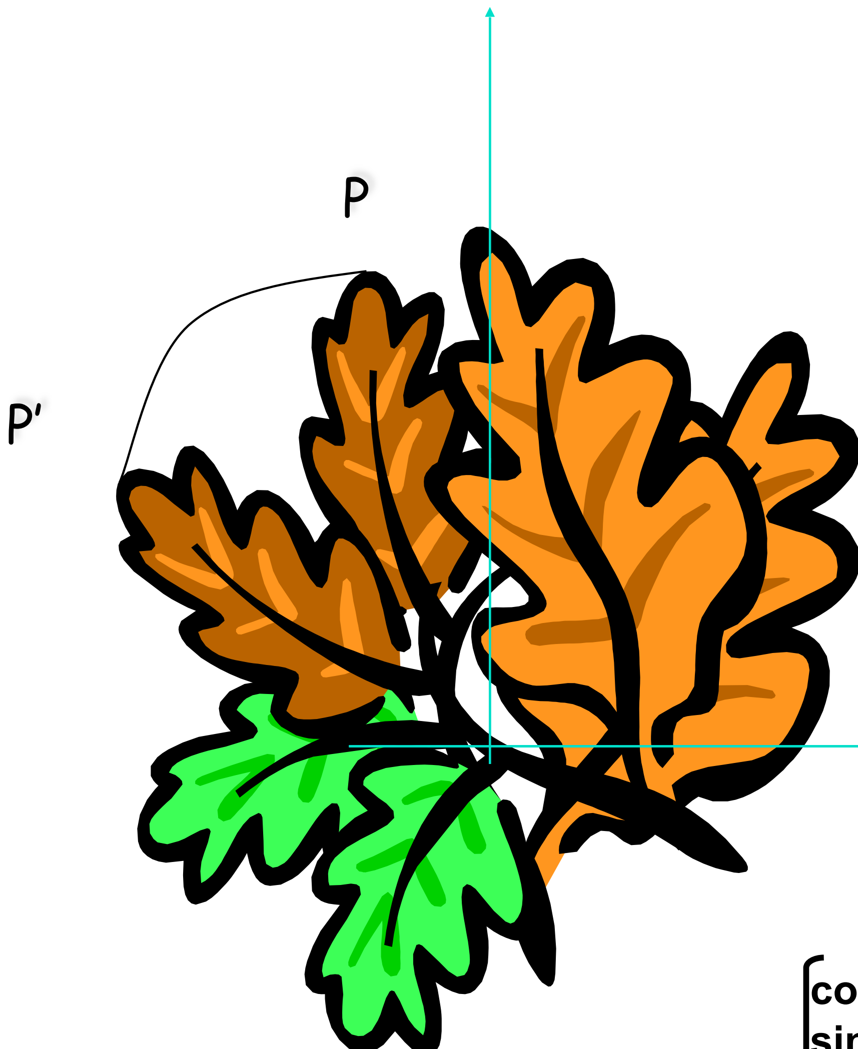
Scaling



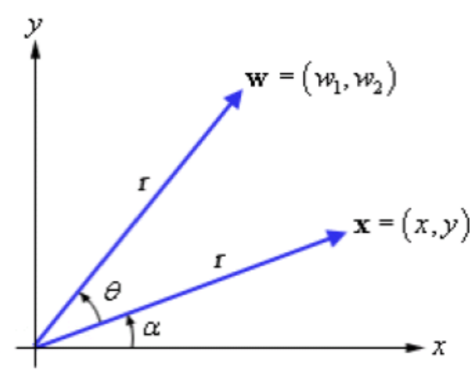
$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

a.k.a: dilation ($r > 1$), contraction ($r < 1$)

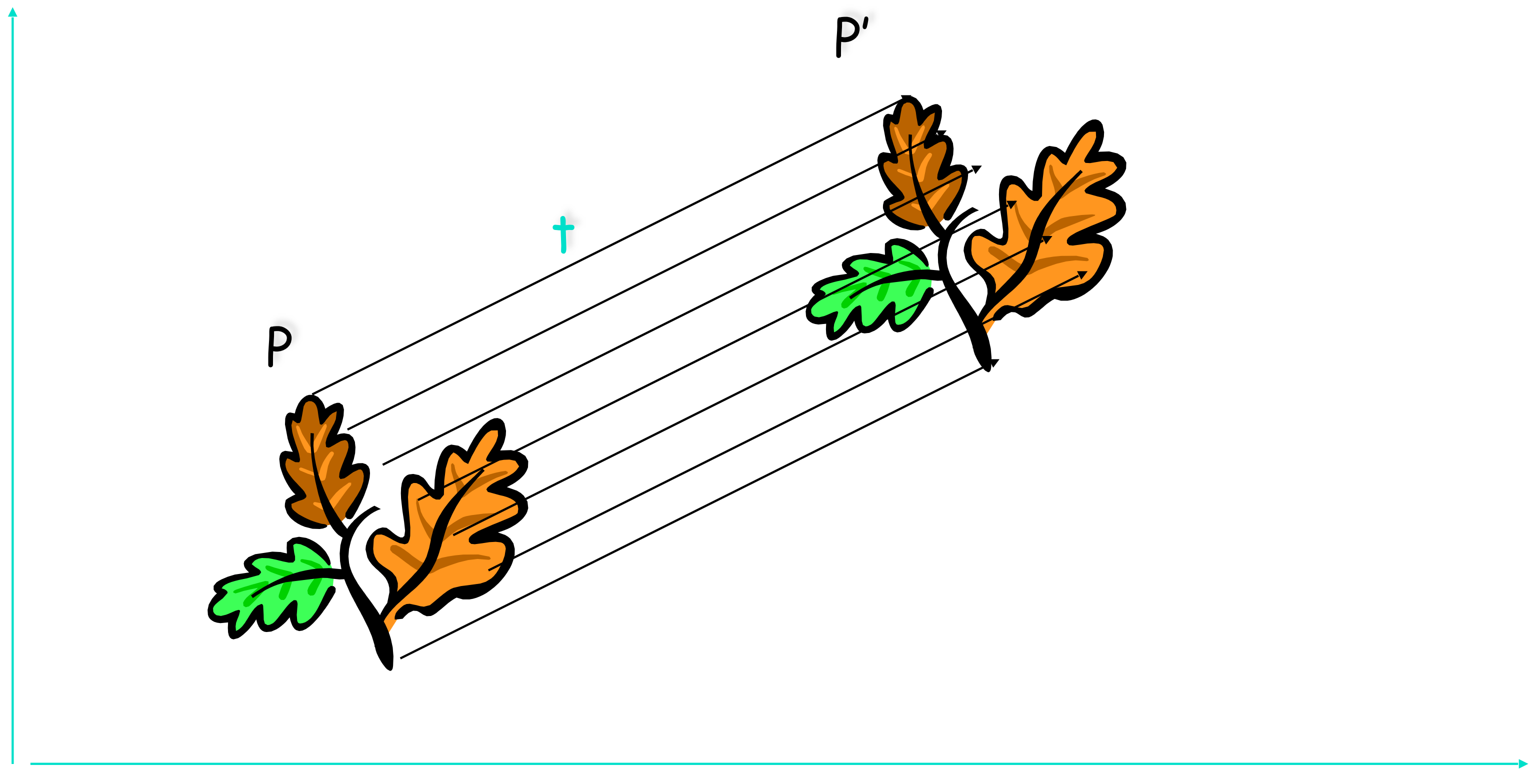
Rotation



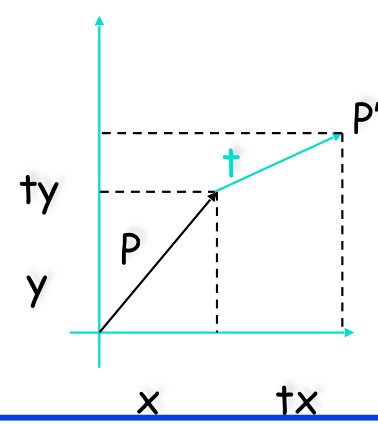
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



2D Translation



$$\mathbf{P}' = (x + t_x, y + t_y) = \mathbf{P}+\mathbf{t}$$



Inverse of a Matrix

- Identity matrix:
  $\mathbf{AI = A}$
- Inverse exists only for <u>square matrices</u> that are <u>non-singular</u>
  - Maps N-d space to another N-d space bijectively
- Some matrices have an inverse, such that:
  $\mathbf{AA^{-1} = I}$
- Inversion is tricky:
  $\mathbf{(ABC)^{-1} = C^{-1}B^{-1}A^{-1}}$

  Derived from non-commutativity property

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinant of a Matrix

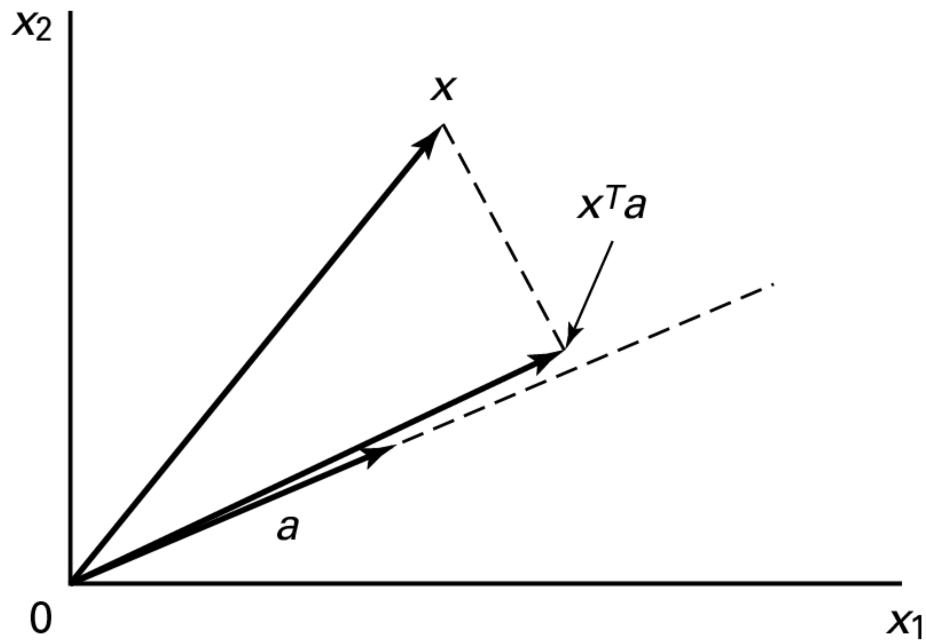
- Used for inversion
- If det(A) = 0, then A has no inverse

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ad - bc$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

http://www.euclideanspace.com/maths/algebra/matrix/functions/inverse/threeD/index.htm

Projection: Using Inner Products (I)



Projection of x along the direction **a** ($\|\mathbf{a}\| = 1$).

$\mathbf{p} = \mathbf{a}\,(a^T x)$
$\|a\| = a^T a = 1$

Homogeneous Coordinates

- The transformation matrices become 3x3 matrices, and we have a translation matrix!

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

New point　　Transformation　　Original point

Exercise: Try composite translation.

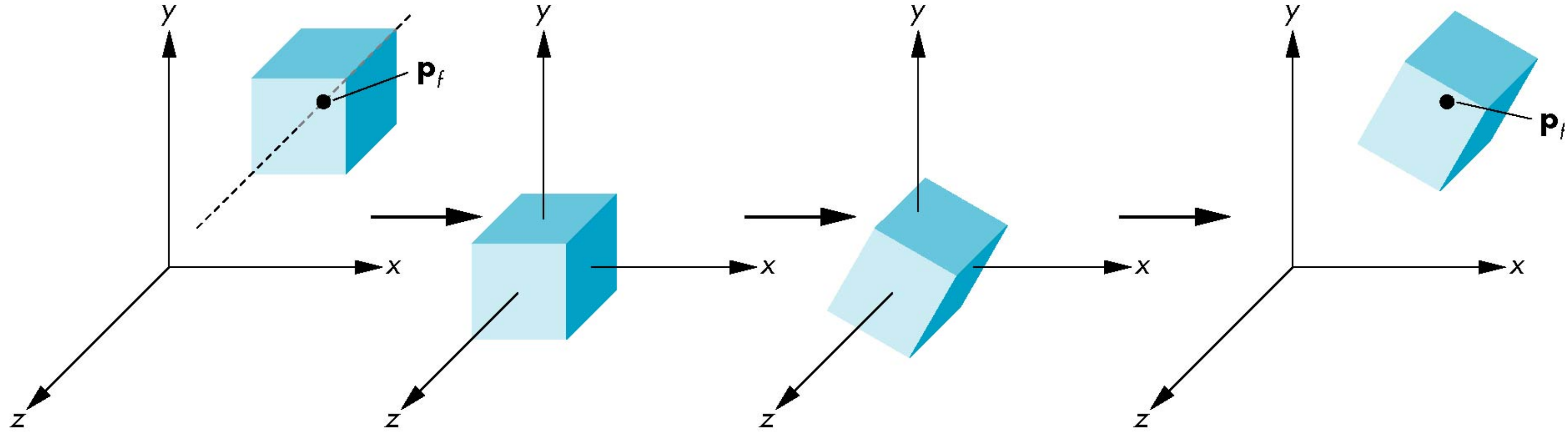
Rotation About a Fixed Point other than the Origin

Move fixed point to origin

Rotate

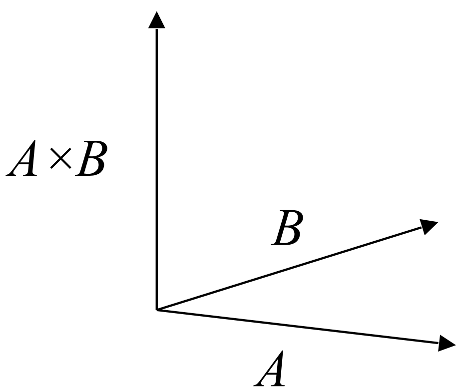
Move fixed point back

$\mathbf{M} = \mathbf{T}(p_f)\ \mathbf{R}(\theta)\ \mathbf{T}(-p_f)$



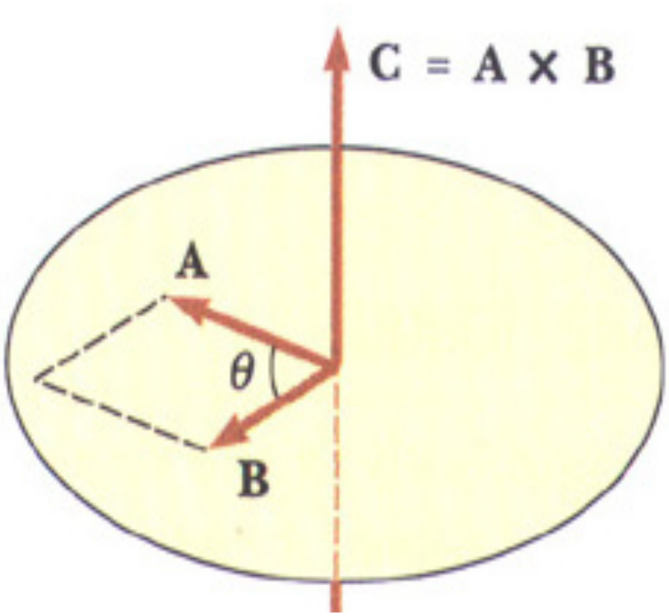
Vectors: Cross Product

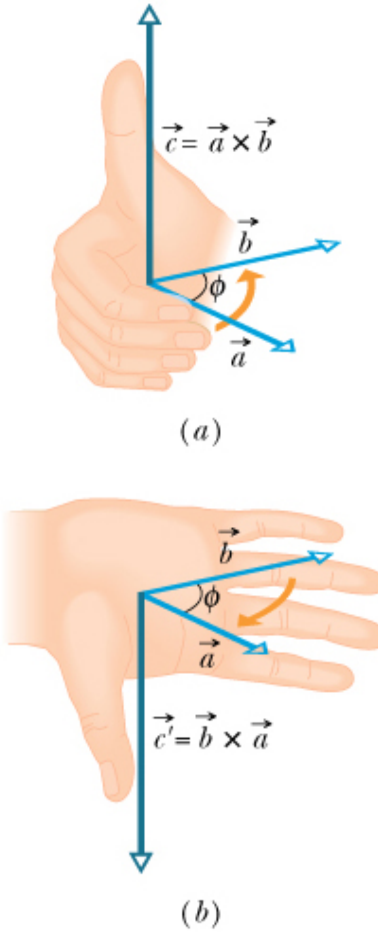
- The cross product of vectors A and B is a vector C which is perpendicular to A and B
- The magnitude of C is proportional to the sin of the angle between A and B
- The direction of C follows the **right hand rule** if we are working in a right-handed coordinate system



$$\|A \times B\| = \|A\| \, \|B\| \sin(\theta)$$

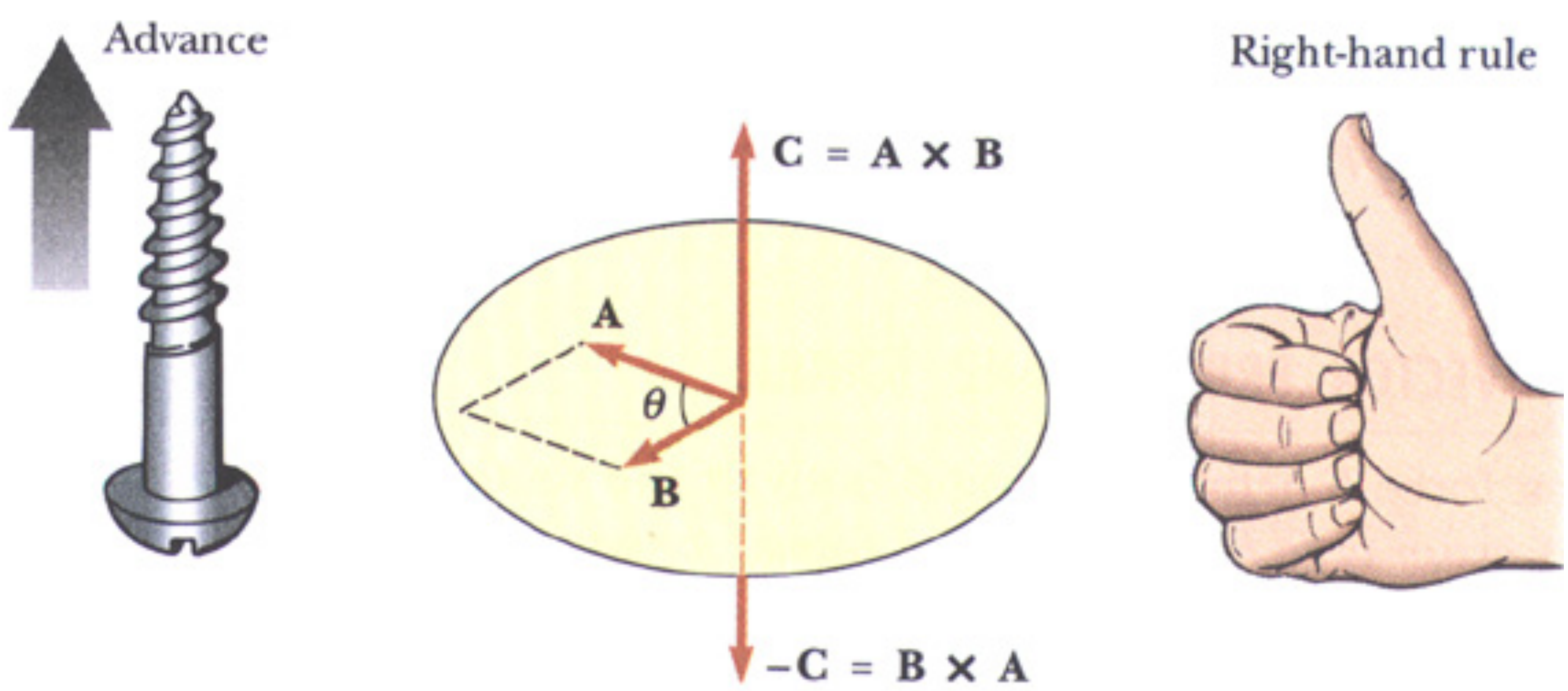
MAGNITUDE OF THE CROSS PRODUCT





DIRECTION OF THE CROSS PRODUCT

- The right hand rule determines the direction of the cross product



HW& References:

- Watch these lectures:
  - https://www.youtube.com/watch?v=fNk_zzaMoSs&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=2
  - https://www.youtube.com/watch?v=k7RM-ot2NWY&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=3
  - https://www.youtube.com/watch?v=kYB8IZa5AuE&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=4
  - https://www.youtube.com/watch?v=XkY2DOUCWMU&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=5
- (optional, if you want to master the material) Prof. Gilbert Strang's course videos:
- http://ocw.mit.edu/OcwWeb/Mathematics/18-06Spring-2005/VideoLectures/index.htm

  - Esp. the lectures on eigenvalues/eigenvectors, singular value decomposition & applications of both. (second half of course)