STAT 111

Recitation 11

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Hypothesis testing outline

- Test of binomial/proportion
- Test of means/regression
  - One-sample t test
  - Two-sample t test (unpaired)
  - Regression t test
  - Paired two-sample t test
- Test of equality of two binomial parameters (two-by-two table)

Regression t test

- Linear Regression Model: for the ith observation,
  - Mean of $Y_i = \alpha + \beta x_i$, Variance of $Y_i = \sigma^2$.
  - β is estimated by $b = s_{xy}/s_{xx}$
  - α is estimated by $a = \bar{y} - b\bar{x}$
  - σ^2 is estimated by $s_r^2 = \frac{s_{yy} - b^2 s_{xx}}{n-2}$
- We want to test $H_0 : \beta = 0$ (x has no effect on Y).
- The test statistic is

$$t = \frac{b}{s_r/\sqrt{s_{xx}}} \quad \text{with } n-2 \text{ degrees of freedom.}$$

Example

- We observe
  $\overline{x} = 21, \overline{y} = 79.7, s_{xx} = 80, s_{yy} = 83.54, s_{xy} = 25, n = 20.$

Step 1 $H_0 : \beta = 0$ vs. $H_1 : \beta \neq 0$. Two-sided test.

Example

- We observe
  $\overline{x} = 21, \overline{y} = 79.7, s_{xx} = 80, s_{yy} = 83.54, s_{xy} = 25, n = 20.$

Step 1 $H_0 : \beta = 0$ vs. $H_1 : \beta \neq 0$. Two-sided test.

Step 2 Choose $\alpha = 0.05$.

Example

- We observe
  $\overline{x} = 21, \overline{y} = 79.7, s_{xx} = 80, s_{yy} = 83.54, s_{xy} = 25, n = 20.$

Step 1 $H_0 : \beta = 0$ vs. $H_1 : \beta \neq 0$. Two-sided test.

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Step 3 Test-statistic is

$$t = \frac{b}{s_r / \sqrt{s_{xx}}}$$

Example

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Step 3 Test-statistic is

$$t = \frac{b}{s_r / \sqrt{s_{xx}}}$$

$$b = \frac{s_{xy}}{s_{xx}} = \frac{25}{80} = 0.3125$$

Example

- We observe
  $\overline{x} = 21, \overline{y} = 79.7, s_{xx} = 80, s_{yy} = 83.54, s_{xy} = 25, n = 20.$

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Step 3 Test-statistic is

$$t = \frac{b}{s_r/\sqrt{s_{xx}}}$$

$$b = \frac{s_{xy}}{s_{xx}} = \frac{25}{80} = 0.3125$$

$$s_r = \sqrt{\frac{s_{yy} - b^2 s_{xx}}{n-2}} = \sqrt{\frac{83.54 - 0.3125^2(80)}{20-2}} = 2.051$$

Example

- We observe
  $\overline{x} = 21, \overline{y} = 79.7, s_{xx} = 80, s_{yy} = 83.54, s_{xy} = 25, n = 20.$

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$$t = \frac{b}{s_r/\sqrt{s_{xx}}}$$

$$b = \frac{s_{xy}}{s_{xx}} = \frac{25}{80} = 0.3125$$

$$s_r = \sqrt{\frac{s_{yy} - b^2 s_{xx}}{n-2}} = \sqrt{\frac{83.54 - 0.3125^2(80)}{20-2}} = 2.051$$

$$t = \frac{b}{s_r/\sqrt{s_{xx}}} = \frac{0.3125}{2.051/\sqrt{80}} = 1.363$$

Example

- We observe
  $\overline{x} = 21, \overline{y} = 79.7, s_{xx} = 80, s_{yy} = 83.54, s_{xy} = 25, n = 20.$

Step 1 $H_0 : \beta = 0$ vs. $H_1 : \beta \neq 0$. Two-sided test.

Step 2 Choose $\alpha = 0.05$.

Step 3 Test-statistic is $t = 1.363$.

Step 4 Find the critical region.

Example

- We observe
  $\overline{x} = 21, \overline{y} = 79.7, s_{xx} = 80, s_{yy} = 83.54, s_{xy} = 25, n = 20.$

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Step 4 Find the critical region.

How many degrees of freedom do we have?

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How many degrees of freedom do we have? $n - 2 = 18$

Example

- We observe
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Step 2 Choose $\alpha = 0.05$.

Step 3 Test-statistic is $t = 1.363$.

Step 4 Find the critical region.

How many degrees of freedom do we have? $n - 2 = 18$

So we need to look at t_{18}. What is the critical region?

Example

- We observe
  $\overline{x} = 21, \overline{y} = 79.7, s_{xx} = 80, s_{yy} = 83.54, s_{xy} = 25, n = 20.$

Step 1 $H_0 : \beta = 0$ vs. $H_1 : \beta \neq 0$. Two-sided test.

Step 2 Choose $\alpha = 0.05$.

Step 3 Test-statistic is $t = 1.363$.

Step 4 Find the critical region.

How many degrees of freedom do we have? $n - 2 = 18$

So we need to look at t_{18}. What is the critical region?

$t \geq t_{18,0.025} = 2.101$ and $t \leq -t_{18,0.025} = -2.101$.

Example

- We observe
  $\overline{x} = 21, \overline{y} = 79.7, s_{xx} = 80, s_{yy} = 83.54, s_{xy} = 25, n = 20.$

Step 1 $H_0 : \beta = 0$ vs. $H_1 : \beta \neq 0$. Two-sided test.

Step 2 Choose $\alpha = 0.05$.

Step 3 Test-statistic is $t = 1.363$.

Step 4 Find the critical region: $t \geq 2.101$ and $t \leq -2.101$

Step 5 Do we reject H_0?

Example

- We observe
  $\overline{x} = 21, \overline{y} = 79.7, s_{xx} = 80, s_{yy} = 83.54, s_{xy} = 25, n = 20.$

Step 1 $H_0 : \beta = 0$ vs. $H_1 : \beta \neq 0$. Two-sided test.

Step 2 Choose $\alpha = 0.05$.

Step 3 Test-statistic is $t = 1.363$.

Step 4 Find the critical region: $t \geq 2.101$ and $t \leq -2.101$

Step 5 Do we reject H_0? No, $t = 1.363$ is not in the critical region.

Paired two sample t test

- Suppose we have two samples where there is a natural pairing of data between the two samples. Let μ_d be the mean difference between the two samples.
- For example, we have n patients and we are interested in determining if a drug decreases cholesterol levels. We collect cholesterol levels before $(x_{11}, \ldots, x_{1n})$ and after $(x_{21}, \ldots, x_{2n})$ administering the drug.
- We want to test $H_0 : \mu_d = 0$.
- Consider $d_i = x_{2i} - x_{1i}$, the difference in measurement between sample 2 and sample 1 for subject i.
  - Estimate of μ_d: $\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i$
  - Estimate of σ^2: $s_d^2 = \frac{d_1^2+d_2^2+\cdots+d_n^2-n(\bar{d})^2}{n-1}$
- The test statistic is

$$t = \frac{\bar{d}}{s_d/\sqrt{n}}$$

Example

- Suppose we have 10 patients and we are interested in determining if a drug decreases cholesterol levels. We collect the following cholesterol levels before and after administering the drug:

Patient	1	2	3	4	5	6	7	8	9	10
Before	204	243	253	212	239	241	256	267	231	251
After	200	235	256	200	232	210	249	270	233	243
Difference	-4	-8	3	-12	-7	-31	-7	3	2	-8

Step 1 $H_0 : \mu_d = 0$ vs. $H_1 : \mu_d < 0$. One-sided test.

Example

- Suppose we have 10 patients and we are interested in determining if a drug decreases cholesterol levels. We collect the following cholesterol levels before and after administering the drug:

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Step 1 $H_0 : \mu_d = 0$ vs. $H_1 : \mu_d < 0$. One-sided test.

Step 2 Choose $\alpha = 0.01$.

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$$t = \frac{\bar{d}}{s_d/\sqrt{n}}$$

$$\bar{d} = -6.9 \qquad s_d = 9.96$$

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Step 1 $H_0 : \mu_d = 0$ vs. $H_1 : \mu_d < 0$. One-sided test.

Step 2 Choose $\alpha = 0.01$.

Step 3 Test-statistic is

$$t = \frac{\bar{d}}{s_d/\sqrt{n}}$$

$$\bar{d} = -6.9 \qquad s_d = 9.96$$

$$t = \frac{\bar{d}}{s_d/\sqrt{n}} = \frac{-6.9}{9.96/\sqrt{10}} = -2.191$$

Example

- Suppose we have 10 patients and we are interested in determining if a drug decreases cholesterol levels. We collect the following cholesterol levels before and after administering the drug:

Patient	1	2	3	4	5	6	7	8	9	10
Before	204	243	253	212	239	241	256	267	231	251
After	200	235	256	200	232	210	249	270	233	243
Difference	-4	-8	3	-12	-7	-31	-7	3	2	-8

Step 1 $H_0 : \mu_d = 0$ vs. $H_1 : \mu_d < 0$. One-sided test.

Step 2 Choose $\alpha = 0.01$.

Step 3 Test-statistic is $t = -2.191$.

Step 4 Find the critical region.

Example

- Suppose we have 10 patients and we are interested in determining if a drug decreases cholesterol levels. We collect the following cholesterol levels before and after administering the drug:

Patient	1	2	3	4	5	6	7	8	9	10
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Difference	-4	-8	3	-12	-7	-31	-7	3	2	-8

Step 1 $H_0 : \mu_d = 0$ vs. $H_1 : \mu_d < 0$. One-sided test.

Step 2 Choose $\alpha = 0.01$.

Step 3 Test-statistic is $t = -2.191$.

Step 4 Find the critical region.

How many degrees of freedom do we have?

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Step 1 $H_0 : \mu_d = 0$ vs. $H_1 : \mu_d < 0$. One-sided test.

Step 2 Choose $\alpha = 0.01$.

Step 3 Test-statistic is $t = -2.191$.

Step 4 Find the critical region.

How many degrees of freedom do we have? $n - 1 = 9$

Example

- Suppose we have 10 patients and we are interested in determining if a drug decreases cholesterol levels. We collect the following cholesterol levels before and after administering the drug:

Patient	1	2	3	4	5	6	7	8	9	10
Before	204	243	253	212	239	241	256	267	231	251
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Step 1 $H_0 : \mu_d = 0$ vs. $H_1 : \mu_d < 0$. One-sided test.

Step 2 Choose $\alpha = 0.01$.

Step 3 Test-statistic is $t = -2.191$.

Step 4 Find the critical region.

How many degrees of freedom do we have? $n - 1 = 9$

So we need to look at t_9. What is the critical region?

Example

- Suppose we have 10 patients and we are interested in determining if a drug decreases cholesterol levels. We collect the following cholesterol levels before and after administering the drug:

Patient	1	2	3	4	5	6	7	8	9	10
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Difference	-4	-8	3	-12	-7	-31	-7	3	2	-8

Step 1 $H_0 : \mu_d = 0$ vs. $H_1 : \mu_d < 0$. One-sided test.

Step 2 Choose $\alpha = 0.01$.

Step 3 Test-statistic is $t = -2.191$.

Step 4 Find the critical region.

How many degrees of freedom do we have? $n - 1 = 9$

So we need to look at t_9. What is the critical region?

$t \leq -t_{9,0.01} = -2.821$.

Example

- Suppose we have 10 patients and we are interested in determining if a drug decreases cholesterol levels. We collect the following cholesterol levels before and after administering the drug:

Patient	1	2	3	4	5	6	7	8	9	10
Before	204	243	253	212	239	241	256	267	231	251
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Difference	-4	-8	3	-12	-7	-31	-7	3	2	-8

Step 1 $H_0 : \mu_d = 0$ vs. $H_1 : \mu_d < 0$. One-sided test.

Step 2 Choose $\alpha = 0.01$.

Step 3 Test-statistic is $t = -2.191$.

Step 4 Find the critical region: $t \leq -2.821$

Step 5 Do we reject H_0?

Example

- Suppose we have 10 patients and we are interested in determining if a drug decreases cholesterol levels. We collect the following cholesterol levels before and after administering the drug:

Patient	1	2	3	4	5	6	7	8	9	10
Before	204	243	253	212	239	241	256	267	231	251
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Step 1 $H_0 : \mu_d = 0$ vs. $H_1 : \mu_d < 0$. One-sided test.

Step 2 Choose $\alpha = 0.01$.

Step 3 Test-statistic is $t = -2.191$.

Step 4 Find the critical region: $t \leq -2.821$

Step 5 Do we reject H_0? No, $t = -2.191$ is not in the critical region.

Testing for equality of two binomial parameters using two-by-two tables

- Suppose we have two binomial parameters θ_1 and θ_2 and we want to test if they are equal.
- For example, we want to see if there is a difference in voter turnout between men and women. Let θ_1 be the voter turnout for men and θ_2 be the voter turnout for women. The two-by-two table would be

	voted	did not vote	total
men	o_{11}	o_{12}	r_1
women	o_{21}	o_{22}	r_2
total	c_1	c_2	n

- We want to test $H_0 : \theta_1 = \theta_2$.
- The test statistic is

$$z = \frac{(o_{11} \times o_{22} - o_{21} \times o_{12})\sqrt{n}}{\sqrt{r_1 \times r_2 \times c_1 \times c_2}}$$

Example

- Suppose we have the following data on voter turnout:

	voted	did not vote	total
men	170	140	310
women	120	110	230
total	290	250	540

Step 1 $H_0 : \theta_1 = \theta_2$ vs. $H_1 : \theta_1 \neq \theta_2$. Two-sided test.

Example

- Suppose we have the following data on voter turnout:

	voted	did not vote	total
men	170	140	310
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total	290	250	540

Step 1 $H_0 : \theta_1 = \theta_2$ vs. $H_1 : \theta_1 \neq \theta_2$. Two-sided test.

Step 2 Choose $\alpha = 0.05$.

Example

- Suppose we have the following data on voter turnout:

	voted	did not vote	total
men	170	140	310
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Step 1 $H_0 : \theta_1 = \theta_2$ vs. $H_1 : \theta_1 \neq \theta_2$. Two-sided test.

Step 2 Choose $\alpha = 0.05$.

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$$z = \frac{(o_{11} \times o_{22} - o_{21} \times o_{12})\sqrt{n}}{\sqrt{r_1 \times r_2 \times c_1 \times c_2}}$$

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Step 1 $H_0 : \theta_1 = \theta_2$ vs. $H_1 : \theta_1 \neq \theta_2$. Two-sided test.

Step 2 Choose $\alpha = 0.05$.

Step 3 Test-statistic is

$$z = \frac{(o_{11} \times o_{22} - o_{21} \times o_{12})\sqrt{n}}{\sqrt{r_1 \times r_2 \times c_1 \times c_2}}$$

$$z = \frac{(170 \times 110 - 120 \times 140)\sqrt{540}}{\sqrt{310 \times 230 \times 290 \times 250}} = 0.6141$$

Example

- Suppose we have the following data on voter turnout:

	voted	did not vote	total
men	170	140	310
women	120	110	230
total	290	250	540

Step 1 $H_0 : \theta_1 = \theta_2$ vs. $H_1 : \theta_1 \neq \theta_2$. Two-sided test.

Step 2 Choose $\alpha = 0.05$.

Step 3 Test-statistic is $z = 0.6141$.

Step 4 Find the critical region and p-value.

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- Suppose we have the following data on voter turnout:

	voted	did not vote	total
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Step 1 $H_0 : \theta_1 = \theta_2$ vs. $H_1 : \theta_1 \neq \theta_2$. Two-sided test.

Step 2 Choose $\alpha = 0.05$.

Step 3 Test-statistic is $z = 0.6141$.

Step 4 Find the critical region and p-value.

$z \leq z_{0.025} = -1.96$ and $z \geq z_{0.975} = 1.96$.

Example

- Suppose we have the following data on voter turnout:

	voted	did not vote	total
men	170	140	310
women	120	110	230
total	290	250	540

Step 1 $H_0 : \theta_1 = \theta_2$ vs. $H_1 : \theta_1 \neq \theta_2$. Two-sided test.

Step 2 Choose $\alpha = 0.05$.

Step 3 Test-statistic is $z = 0.6141$.

Step 4 Find the critical region and p-value.

$z \leq z_{0.025} = -1.96$ and $z \geq z_{0.975} = 1.96$.

p-value $= 2P(Z \geq |0.6141|) \approx 2(0.27) = 0.54$

Example

- Suppose we have the following data on voter turnout:

	voted	did not vote	total
men	170	140	310
women	120	110	230
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Step 1 $H_0 : \theta_1 = \theta_2$ vs. $H_1 : \theta_1 \neq \theta_2$. Two-sided test.

Step 2 Choose $\alpha = 0.05$.

Step 3 Test-statistic is $z = 0.6141$.

Step 4 Find the critical region p-value: $z \leq -1.96$ and $z \geq 1.96$.
p-value $= 0.54$.

Step 5 Do we reject H_0?

Example

- Suppose we have the following data on voter turnout:

	voted	did not vote	total
men	170	140	310
women	120	110	230
total	290	250	540

Step 1 $H_0 : \theta_1 = \theta_2$ vs. $H_1 : \theta_1 \neq \theta_2$. Two-sided test.

Step 2 Choose $\alpha = 0.05$.

Step 3 Test-statistic is $z = 0.6141$.

Step 4 Find the critical region p-value: $z \leq -1.96$ and $z \geq 1.96$.
p-value $= 0.54$.

Step 5 Do we reject H_0? No, $z = 0.6141$ is not in the critical region and the p-value is greater than 0.05.