STAT 111

Recitation 3

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Slides (adapted from Gemma Moran): `github.com/mohuangx/STAT111-Fall2018`

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Random Variables: Mean

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- Let X be a discrete random variable that can take values $\{v_1, v_2, \ldots, v_k\}$. Then the mean of X is given by:

$$\begin{aligned}\mu &= \sum_{i=1}^{k} v_i P(X = v_i) \\ &= v_1 P(X = v_1) + v_2 P(X = v_2) + \cdots + v_k P(X = v_k).\end{aligned}$$

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- If X is binomial with parameters n and θ, then the mean of X is

$$\mu = n\theta$$

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$$\sigma^2 = (v_1 - \mu)^2 P(X = v_1) + \cdots + (v_k - \mu)^2 P(X = v_k) \qquad (1)$$

or

$$\sigma^2 = v_1^2 P(X = v_1) + \cdots + v_k^2 P(X = v_k) - \mu^2 \qquad (2)$$

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- For a binomial random variable $X \sim \mathcal{B}(n, \theta)$:

$$\sigma^2 = n\theta(1 - \theta)$$

Random Variables: Questions

Q1: Let X be a random variable with the below distribution. Find the mean and the variance of X using formula (1) and then formula (2).

x	-3	-1	4	5
$P(X = x)$	0.1	0.3	0.4	0.2

Table: Probability distribution of X.

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A1: $\mu = -3 \times 0.1 - 1 \times 0.3 + 4 \times 0.4 + 5 \times 0.2 = 2.$

$$\sigma^2 = (-3-2)^2(0.1) + (-1-2)^2(0.3) + (4-2)^2(0.4) + (5-2)^2(0.2)$$
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Q2: Let X be a random variable with the below distribution. Write the probability distribution of $Y = 2X$ in tableau form.

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Table: Probability distribution of X.

A2: Probability distribution of Y:

y	-6	-2	8	10
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Q3: Suppose X has mean $\mu_X = 3$ and variance $\sigma_X^2 = 4$. What is the mean and variance of $Y = 2 + 5X$?

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A3: $\mu_Y = 2 + 5\mu_X = 2 + 5(3) = 17$

$\sigma_Y^2 = 5^2\sigma_X^2 = 5^2(4) = 100$

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- For example:
  - I plan to roll a dice n times. X_i represents the future outcome of the ith roll. X_i is *independent* of the other rolls of the dice, and it has the same probability of getting a $1, 2, 3, 4, 5, 6$ as the other rolls.

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  - For constants a, b, we have

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Many Random Variables

- Let $X_1, \ldots, X_n$ be i.i.d. random variables, each with mean μ and variance σ^2.
- Then for the sum, T_n:

$$\text{mean of } T_n = n\mu, \quad \text{variance of } T_n = n\sigma^2$$

- For the average, $\bar{X}$:

$$\text{mean of } \bar{X} = \mu, \quad \text{variance of } \bar{X} = \frac{\sigma^2}{n}.$$

Questions

Q4: Suppose we have a company producing a medicine. Each day the *mean* amount of medicine produced is 500 mg and the *variance* is 900 mg^2. Assume the amount produced each day is independent.

(i) Let T_n be the total amount of medicine produced in a week. Find the mean and variance of T_n.

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$Mean(\bar{X}) = 500$
$Var(\bar{X}) = 900/5 = 180$

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A5:

$$Mean(P) = 0.7, \quad Var(P) = 0.7 \times 0.3/20 = 0.0105.$$

Questions

Q6: Suppose the company producing a medicine has different means and variances the amount produced on each day of the week:

Day	Mean	Variance
Monday (X_1)	450	1200
Tuesday (X_2)	550	800
Wednesday (X_3)	600	500
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A6: X_1, X_2, X_3, X_4, and X_5 are no longer *i.i.d.*!

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$Mean(\bar{X}) = 1/n \times Mean(T_n) = 500$
$Var(\bar{X}) = 1/n^2 \times Var(T_n) = 4500/25 = 180$

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Q8: Let P_2 be the proportion of heads in 20 coin tosses, where $P(H) = 0.7$. From earlier, $Mean(P_2) = 0.7$ and $Var(P_2) = 0.0105$. Let $D = P_1 - P_2$. Find the mean and variance of D.

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A8: $Mean(D) = 0.6 - 0.7 = -0.1$
$Var(D) = 0.0048 + 0.0105 = 0.0153$

Continuous Random Variables

- So far, we have just considered discrete random variables; those whose possible values are countable.
- A continuous random variable can take continuous values in a future experiment.
- Every continuous random variable X has an associated density function $f(x)$.

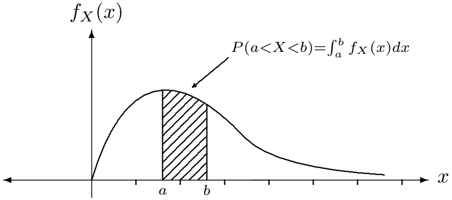
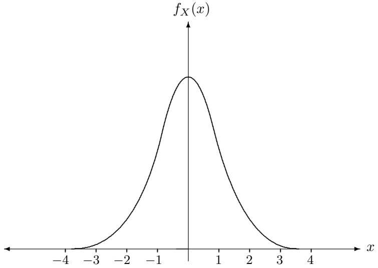


Figure 5: $P(a < X < b) = \int_a^b f_X(x)dx$.

The Normal Distribution

- A normal random variable is a continuous random variable.



The density function for the standard normal distribution with $\mu = 0$, $\sigma = 1$.

- We call a normal random variable with $\mu = 0$ and $\sigma^2 = 1$ a standard normal random variable.
- For standard normal random variables, we can use charts (or a computer) to find the area under the density function (i.e. the probabilities).

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  $P(Z > 0.85) = 1 - 0.8023 = 0.1977$

- $P(-1.43 < Z < 0.92)$

  $P(-1.43 < Z < 0.92) = 0.8212 - .0764 = .7448$