

# STAT 111

## Recitation 4

Mo Huang

Email: [mohuang@wharton.upenn.edu](mailto:mohuang@wharton.upenn.edu)

Office Hours: Wednesdays 3:00 - 4:00 pm, JMHH F96

Slides (adapted from Gemma Moran): [github.com/mohuangx/STAT111-Fall2018](https://github.com/mohuangx/STAT111-Fall2018)

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# Many Random Variables

- ▶ Let  $X$  and  $Y$  be two random variables. Then,

$$\text{mean}(X + Y) = \text{mean}(X) + \text{mean}(Y).$$

- ▶ If  $X, Y$  are also independent,

$$\text{variance}(X + Y) = \text{variance}(X) + \text{variance}(Y).$$

- ▶ For constants  $a, b$ , we have

$$\text{mean}(aX + bY) = a \times \text{mean}(X) + b \times \text{mean}(Y)$$

$$\text{variance}(aX + bY) = a^2 \times \text{variance}(X) + b^2 \times \text{variance}(Y).$$

- ▶ Let  $D = X - Y$ . What is the variance of  $D$ ?

$$\text{variance}(D) = \text{variance}(X) + \text{variance}(Y).$$

# Many Random Variables

- ▶ Let  $X_1, \dots, X_n$  be i.i.d. random variables, each with mean  $\mu$  and variance  $\sigma^2$ .

- ▶ Then for the sum,  $T_n = X_1 + \dots + X_n$ :

$$\text{mean of } T_n = n\mu, \quad \text{variance of } T_n = n\sigma^2$$

- ▶ For the average,  $\bar{X} = \frac{X_1 + \dots + X_n}{n}$ :

$$\text{mean of } \bar{X} = \mu, \quad \text{variance of } \bar{X} = \frac{\sigma^2}{n}.$$

## Questions

Q1: Suppose the company producing a medicine has different means and variances the amount produced on each day of the week:

Day	Mean	Variance
Monday ( $X_1$ )	450	1200
Tuesday ( $X_2$ )	550	800
Wednesday ( $X_3$ )	600	500
Thursday ( $X_4$ )	550	800
Friday ( $X_5$ )	350	1200

Find mean and variance of both the sum  $T_n$  and the average  $\bar{X}$ .

A1:  $X_1, X_2, X_3, X_4$ , and  $X_5$  are no longer *i.i.d.*!

$$\text{Mean}(T_n) = 450 + 550 + 600 + 550 + 350 = 2500$$

$$\text{Var}(T_n) = 1200 + 800 + 500 + 800 + 1200 = 4500$$

$$\text{Mean}(\bar{X}) = 1/n \times \text{Mean}(T_n) = 500$$

$$\text{Var}(\bar{X}) = 1/n^2 \times \text{Var}(T_n) = 4500/25 = 180$$

# Questions

Q2: Let  $P_1$  be the proportion of heads in 50 coin tosses, where  $P(H) = 0.6$ . Find  $Mean(P_1)$  and  $Var(P_1)$ .

A2:  $Mean(P_1) = 0.6$  and  $Var(P_1) = 0.6 \times 0.4/50 = 0.0048$ .

Q3: Let  $P_2$  be the proportion of heads in 20 coin tosses, where  $P(H) = 0.7$ . From earlier,  $Mean(P_2) = 0.7$  and  $Var(P_2) = 0.0105$ . Let  $D = P_1 - P_2$ . Find the mean and variance of  $D$ .

A3:  $Mean(D) = 0.6 - 0.7 = -0.1$   
 $Var(D) = 0.0048 + 0.0105 = 0.0153$

# Continuous Random Variables

- ▶ So far, we have just considered discrete random variables; those whose possible values are countable.
- ▶ A **continuous random variable** can take continuous values in a future experiment.
- ▶ Every continuous random variable  $X$  has an associated **density function**  $f(x)$ .

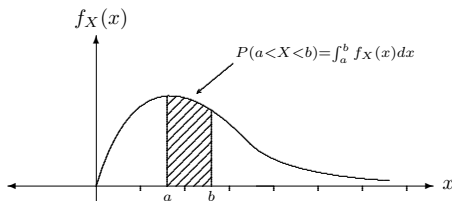
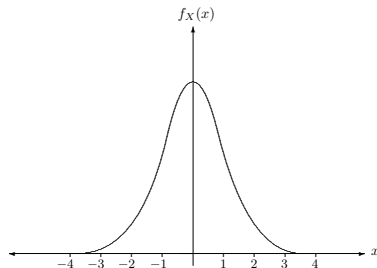


Figure 5:  $P(a < X < b) = \int_a^b f_X(x) dx$ .

# The Normal Distribution

- ▶ A **normal** random variable is a continuous random variable.



The density function for the standard normal distribution with  $\mu = 0$ ,  $\sigma = 1$ .

- ▶ We call a normal random variable with  $\mu = 0$  and  $\sigma^2 = 1$  a **standard normal** random variable.
- ▶ For standard normal random variables, we can use charts (or a computer) to find the area under the density function (i.e. the probabilities).

# Questions

►  $P(Z < -1.75)$

$$P(Z < -1.75) = 0.0401$$

►  $P(Z > 0.85)$

$$P(Z > 0.85) = 1 - 0.8023 = 0.1977$$

►  $P(-1.43 < Z < 0.92)$

$$P(-1.43 < Z < 0.92) = 0.8212 - .0764 = .7448$$



# Standardization

- ▶ What if a normal random variable has a different mean and variance?
- ▶ We need to *standardize* it.
- ▶ Let  $X \sim N(\mu, \sigma^2)$ : that is,  $X$  is a normal random variable with mean  $\mu$ , variance  $\sigma^2$ . Let

$$Z = \frac{X - \mu}{\sigma}$$

- ▶ Then  $Z$  is a *standard* normal random variable.

## Example

- ▶  $X$  is a normal random variable with  $\mu = 5$  and  $\sigma^2 = 9$ . Find  $P(X > 8)$ .

$$\begin{aligned}P(X > 8) &= P\left(\frac{X - 5}{3} > \frac{8 - 5}{3}\right) \\&= P(Z > 1) \\&= 0.16\end{aligned}$$

# Questions

- $X \sim N(2, 16)$ . Find  $P(-1 < X < 6)$ .

$$\begin{aligned} P(-1 < X < 6) &= P\left(\frac{-1-2}{4} < \frac{X-2}{4} < \frac{6-2}{4}\right) \\ &= P(-0.75 < Z < 1) \\ &= 0.8413 - 0.2266 \\ &= 0.6147 \end{aligned}$$

# Questions

- $Y \sim N(7, 25)$ . Find  $P(8 < Y < 13)$ .

$$\begin{aligned}P(8 < Y < 13) &= P\left(\frac{8-7}{5} < \frac{Y-7}{5} < \frac{13-7}{5}\right) \\&= P(0.2 < Z < 1.2) \\&= 0.8849 - 0.5793 \\&= 0.3056\end{aligned}$$

# Two-Standard-Deviation Rule

- ▶ From the chart:

$$P(Z < -1.96) = 0.025, \quad P(Z > 1.96) = 0.025.$$

- ▶ Then:

$$P(-1.96 < Z < 1.96) = 0.95.$$

- ▶ Approximate  $1.96 \approx 2$  and “unstandardize”:

$$P\left(-2 < \frac{X - \mu}{\sigma} < 2\right) = 0.95$$

⇒

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.95.$$

- ▶ The probability that a normal random variable is within 2 standard deviations of the mean is 95%.

# Normal Distribution: Sums and Averages

- ▶ Let  $X_1, \dots, X_n$  be independent and *normally distributed*. Let

$$T_n = X_1 + \dots + X_n, \quad \bar{X} = \frac{X_1 + \dots + X_n}{n}, \quad D = X_2 - X_1.$$

- ▶ Then  $T_n$ ,  $\bar{X}$  and  $D$  are **also normal random variables**.
- ▶ Let  $X_1, \dots, X_n \stackrel{i.i.d}{\sim} N(\mu, \sigma^2)$ . Then:

$$T_n \sim N(n\mu, n\sigma^2)$$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$D \sim N(0, 2\sigma^2)$$

## Example

- Suppose we know the weight  $X$  of an adult man chosen at random is normally distributed with mean 160 pounds and variance 64 pounds<sup>2</sup>.

- a) Find  $P(156 < X < 164)$ .

$$\begin{aligned}P(156 < X < 164) &= P\left(\frac{156 - 160}{8} < \frac{X - 160}{8} < \frac{164 - 160}{8}\right) \\&= P(-0.5 < Z < 0.5) \\&= 0.3830\end{aligned}$$

- b) Find the probability that the average weight of 16 men chosen at random is between 156 and 164 pounds.

$$\begin{aligned}\bar{X} &\sim N(160, 64/16 = 4) \\P(156 < \bar{X} < 164) &= P(-2 < Z < 2) \\&= 0.95\end{aligned}$$

## Example

- ▶ Suppose we know the weight  $X$  of an adult man chosen at random is normally distributed with mean 160 pounds and variance 64 pounds<sup>2</sup>.
- c) Calculate the numbers  $A$  and  $B$  such that  $P(A < X < B) \approx 0.95$ .

$$0.95 \approx P\left(-2 < \frac{X - \mu}{\sigma} < 2\right) = P(-2\sigma + \mu < X < 2\sigma + \mu)$$

$$A = -2(8) + 160 = 144, \quad B = 2(8) + 160 = 176$$

- d) Calculate the numbers  $C$  and  $D$  such that the average of 256 randomly chosen adults is between  $C$  and  $D$  with probability approximately 0.95.

$$\bar{X}_{256} \sim N(160, 64/256 = 1/4)$$

$$C = -2\sigma + \mu = -2(1/2) + 160 = 159$$

$$D = 2\sigma + \mu = 2(1/2) + 160 = 161$$