STAT 111

Recitation 2

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Slides (adapted from Gemma Moran): `github.com/mohuangx/STAT111-Fall2018`

September 14, 2018

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- A parameter θ is the underlying probability of obtaining a head. $\theta = 0.5$.
- A random variable X is the number of heads obtained in 10 tosses. Can be modeled by a binomial distribution dependent on θ.
- Data $x = 6$ is observing 6 heads in 10 tosses, a realization of X.

Random Variables

- Two types of random variables: discrete and continuous.
- A discrete random variable is a random variable that can only take on a countable set of numbers.
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x	0	1	2	3
$P(X = x)$	0.125	0.375	0.375	0.125

Table: Probability distribution of X using the tableau method.

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The Binomial Distribution

- Let X be a binomial random variable where $\theta = P(\text{success})$ and there are n experiments. Then the probability distribution of X is given by:

$$P(X = x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}, \quad \text{for } x = 0, 1, 2, \ldots, n.$$

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Note: $X \sim \mathcal{B}(n, \theta)$ means "X is a binomial random variable with n experiments and probability of success θ."

The Binomial Distribution: Questions

Q1: Let X be the number of heads if I toss an unbiased coin 3 times ($n = 3, \theta = 0.5$). Find the probability distribution of X in "tableau" form.

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$$P(X = 0) = \binom{3}{0}(0.5)^0(0.5)^3$$
$$= 0.125$$

$$P(X = 1) = \binom{3}{1}(0.5)^1(0.5)^2$$
$$= 0.375$$

$$P(X = 2) = \binom{3}{2}(0.5)^2(0.5)^1$$
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$$P(X = 3) = \binom{3}{3}(0.5)^3(0.5)^0$$
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A2: $P(X = 4) = 0.1592$

		θ				
n	i	0.05	0.10	0.15	0.20	0.25
18	0	0.3972	0.1501	0.0536	0.0180	0.0056
	1	0.3763	0.3002	0.1704	0.0811	0.0338
	2	0.1683	0.2835	0.2556	0.1723	0.0958
	3	0.0473	0.1680	0.2406	0.2297	0.1704
	4	0.0093	0.0700	$\boxed{0.1592}$	0.2153	0.2130

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Q3: Find $P(X \leq 4)$.

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Q3: Find $P(X \leq 4)$.

A3:

$$
\begin{aligned}
P(X \leq 4) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) \\
&= 0.0536 + 0.1704 + 0.2556 + 0.2406 + 0.1592 \\
&= 0.8794
\end{aligned}
$$

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A2: Finding 3 successes with $\theta = 0.8$ is the same as finding 9 failures with θ of failure 0.2. Hence, $P(X = 3) = 0.0001$.

		θ						
n	i	0.05	0.10	0.15	0.20	0.25	0.30	0.35
12	0	0.5404	0.2824	0.1422	0.0687	0.0317	0.0138	0.0057
	1	0.3413	0.3766	0.3012	0.2062	0.1267	0.0712	0.0368
	2	0.0988	0.2301	0.2924	0.2835	0.2323	0.1678	0.1088
	3	0.0173	0.0852	0.1720	0.2362	0.2581	0.2397	0.1954
	4	0.0021	0.0213	0.0683	0.1329	0.1936	0.2311	0.2367
	5	0.0002	0.0038	0.0193	0.0532	0.1032	0.1585	0.2039
	6	0.0000	0.0005	0.0040	0.0155	0.0401	0.0792	0.1281
	7	0.0000	0.0000	0.0006	0.0033	0.0115	0.0291	0.0591
	8	0.0000	0.0000	0.0001	0.0005	0.0024	0.0078	0.0199
	9	0.0000	0.0000	0.0000	$\boxed{0.0001}$	0.0004	0.0015	0.0048
	10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0008
	11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
	12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

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- The mean of a random variable (expected value) is the long-run average of realizations of the random variable over repeated experiments.

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- Let X be a discrete random variable that can take values $\{v_1, v_2, \ldots, v_k\}$. Then the mean of X is given by:

$$\begin{aligned} \mu &= \sum_{i=1}^{k} v_i P(X = v_i) \\ &= v_1 P(X = v_1) + v_2 P(X = v_2) + \cdots + v_k P(X = v_k). \end{aligned}$$

Note: We use μ to denote the mean of a random variable.
Note: Can think of it as a weighted average, weighted by probability.

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$$= 91/21$$

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Q3: Let X be a binomial random variable with $n = 2$ and $\theta = 0.8$. Find the mean of X using the binomial table.

A3:

$$\begin{aligned}\mu &= 0 \times 0.04 + 1 \times 0.32 + 2 \times 0.64 \\ &= 1.6\end{aligned}$$

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Note: This formula is *only* for the binomial distribution.

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- Let X be a discrete random variable that can take values $\{v_1, v_2, \ldots, v_k\}$. Then the variance of X is given by:

$$\sigma^2 = (v_1 - \mu)^2 P(X = v_1) + \cdots + (v_k - \mu)^2 P(X = v_k) \tag{1}$$

or

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$$\sigma^2 = n\theta(1 - \theta)$$

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Note: The variance is always positive - you can't have a negative spread of a distribution.