STAT 111

Recitation 3

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Slides (adapted from Gemma Moran): github.com/mohuangx/STAT111-Fall2018

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Random Variables: Mean

- ► The mean of a random variable (expected value) is the long-run average of realizations of the random variable over repeated experiments.
- This mean of random variable X is different from the sample average, which is the average of a *finite* number of observations $(\bar{x} = \frac{x_1 + x + 2 + \dots + x_n}{n})$, where x_1, \dots, x_n are observed data).
- Let X be a discrete random variable that can take values $\{v_1, v_2, \dots, v_k\}$. Then the mean of X is given by:

$$\mu = \sum_{i=1}^{k} v_i P(X = v_i)$$

$$= v_1 P(X = v_1) + v_2 P(X = v_2) + \dots + v_k P(X = v_k).$$

▶ If X is binomial with parameters n and θ , then the mean of X is

$$\mu = n\theta$$

Random Variables: Variance

- ► The variance of a random variable is a measure of the *spread* of a distribution that is, how far away values are from the mean.
- Let X be a discrete random variable that can take values $\{v_1, v_2, \dots, v_k\}$. Then the variance of X is given by:

$$\sigma^{2} = (v_{1} - \mu)^{2} P(X = v_{1}) + \dots + (v_{k} - \mu)^{2} P(X = v_{k})$$
(1)

$$\sigma^{2} = v_{1}^{2} P(X = v_{1}) + \dots + v_{k}^{2} P(X = v_{k}) - \mu^{2}$$
(2)

- ▶ The standard deviation of a random variable is the square root of the variance and is denoted by σ .
- ▶ For a binomial random variable $X \sim \mathcal{B}(n, \theta)$:

$$\sigma^2 = n\theta(1-\theta)$$

Random Variables: Questions

Q1: Let X be a random variable with the below distribution. Find the mean and the variance of X using formula (1) and then formula (2).

X	-3	-1	4	5
P(X = x)	0.1	0.3	0.4	0.2

Table: Probability distribution of X.

A1:
$$\mu = -3 \times 0.1 - 1 \times 0.3 + 4 \times 0.4 + 5 \times 0.2 = 2$$
.

$$\sigma^2 = (-3 - 2)^2(0.1) + (-1 - 2)^2(0.3) + (4 - 2)^2(0.4) + (5 - 2)^2(0.2)$$
= 8.6.

$$\sigma^2 = (-3)^2(0.1) + (-1)^2(0.3) + 4^2(0.4) + 5^2(0.2) - 2^2$$

= 8.6.

Random Variables: Questions

Q2: Let X be a random variable with the below distribution. Write the probability distribution of Y = 2X in tableau form.

X	-3	-1	4	5
P(X=x)	0.1	0.3	0.4	0.2

Table: Probability distribution of X.

A2: Probability distribution of *Y*:

У	-6	-2	8	10
P(Y = y)	0.1	0.3	0.4	0.2

Table: Probability distribution of Y.

Properties of mean and variance

- Suppose X is a random variable with mean μ and variance σ^2 . Consider fixed number c.
- Mean properties

 - ▶ Mean of $cX \Longrightarrow c\mu$.
- Variance properties
 - ▶ Variance of $X + c \Longrightarrow \sigma^2$.
 - ▶ Variance of $cX \Longrightarrow c^2 \sigma^2$.
- Q3: Suppose X has mean $\mu_X=3$ and variance $\sigma_X^2=4$. What is the mean and variance of Y=2+5X?

A3:
$$\mu_Y = 2 + 5\mu_X = 2 + 5(3) = 17$$

 $\sigma_Y^2 = 5^2 \sigma_X^2 = 5^2(4) = 100$

- \triangleright Suppose we plan an experiment with a sample size of n.
- We have *n* future outcomes, or random variables, denoted by: X_1, X_2, \dots, X_n .
- We say $\{X_1, \ldots, X_n\}$ are independently and identically distributed (or i.i.d) if:
 - ► All the X_is are independent of each other.
 - **Each** X_i has the same probability distribution.
- For example:
 - ▶ I plan to roll a dice n times. X_i represents the future outcome of the ith roll. X_i is independent of the other rolls of the dice, and it has the same probability of getting a 1, 2, 3, 4, 5, 6 as the other rolls.

▶ Recall the sample average is given by:

$$\bar{x}=\frac{x_1+\cdots+x_n}{n}.$$

- ▶ What if we haven't observed the data $x_1, ..., x_n$ yet?
- ▶ Before an experiment, the average is *also* a random variable:

$$\bar{X} = \frac{X_1 + \dots + X_n}{n}$$

▶ The *sum* is also a random variable:

$$T_n = X_1 + \cdots + X_n$$

Let X and Y be two random variables. Then,

$$mean(X + Y) = mean(X) + mean(Y)$$
.

► If X, Y are also independent,

$$variance(X + Y) = variance(X) + variance(Y).$$

For constants a, b, we have

$$mean(aX + bY) = a \times mean(X) + b \times mean(Y)$$

 $variance(aX + bY) = a^2 \times variance(X) + b^2 \times variance(Y).$

▶ Let D = X - Y. What is the variance of D?

$$variance(D) = variance(X) + variance(Y).$$

- ▶ Let $X_1, ..., X_n$ be i.i.d. random variables, each with mean μ and variance σ^2 .
- ▶ Then for the sum, T_n :

mean of
$$T_n = n\mu$$
, variance of $T_n = n\sigma^2$

▶ For the average, \bar{X} :

mean of
$$\bar{X} = \mu$$
, variance of $\bar{X} = \frac{\sigma^2}{n}$.

- Q4: Suppose we have a company producing a medicine. Each day the *mean* amount of medicine produced is 500 mg and the *variance* is 900 mg². Assume the amount produced each day is independent.
 - (i) Let T_n be the total amount of medicine produced in a week. Find the mean and variance of T_n .

$$Mean(T_n) = 5 \times 500 = 2500$$

 $Var(T_n) = 5 \times 900 = 4500$

(ii) Let \bar{X} be the average amount of medicine produced in a 5-day week. Find the mean and variance of \bar{X} .

$$Mean(\bar{X}) = 500$$

 $Var(\bar{X}) = 900/5 = 180$

Proportions

- ➤ Sometimes it is necessary to consider the *proportion* of "successes" in a Binomial trial (instead of the total number of successes)
- ▶ Proportions are a type of average!
- ► Let

$$Y_i = \begin{cases} 1 & \text{if the } i \text{th trial is a "success"} \\ 0 & \text{if the } i \text{th trial is a "failure"} \end{cases}$$

- ▶ Then we have $Y_1, ..., Y_n$ where $Y_i \sim Binomial(1, \theta)$.
- ▶ The proportion of successes is the average:

$$P=\frac{Y_1+\cdots+Y_n}{n}$$

Proportions

 $ightharpoonup Y_i \sim Binomial(1, \theta)$. Recall:

$$Mean(Y_i) = \theta$$
, $Var(Y_i) = \theta(1 - \theta)$.

▶ The proportion of successes is the average:

$$P = \frac{Y_1 + \dots + Y_n}{n}$$

► Then,

$$Mean(P) = \theta, \quad Var(P) = \frac{\theta(1-\theta)}{n}$$

.

Q5: Suppose we plan to toss a coin 20 times and Prob(H) = 0.7. Let P be the *proportion* of heads that we toss. Find the mean and variance of P.

A5:

$$Mean(P) = 0.7$$
, $Var(P) = 0.7 \times 0.3/20 = 0.0105$.

Q6: Suppose the company producing a medicine has different means and variances the amount produced on each day of the week:

Day	Mean	Variance
Monday (X_1)	450	1200
Tuesday (X_2)	550	800
Wednesday (X_3)	600	500
Thursday (X_4)	550	800
Friday (X_5)	350	1200

Find mean and variance of both the sum T_n and the average \bar{X} .

A6:
$$X_1, X_2, X_3, X_4$$
, and X_5 are no longer i.i.d.!

$$Mean(T_n) = 450 + 550 + 600 + 550 + 350 = 2500$$

 $Var(T_n) = 1200 + 800 + 500 + 800 + 1200 = 4500$

$$Mean(\bar{X}) = 1/n \times Mean(T_n) = 500$$

 $Var(\bar{X}) = 1/n^2 \times Var(T_n) = 4500/25 = 180$

- Q7: Let P_1 be the proportion of heads in 50 coin tosses, where P(H) = 0.6. Find $Mean(P_1)$ and $Var(P_1)$.
- A7: $Mean(P_1) = 0.6$ and $Var(P_1) = 0.6 \times 0.4/50 = 0.0048$.
- Q8: Let P_2 be the proportion of heads in 20 coin tosses, where P(H)=0.7. From earlier, $Mean(P_2)=0.7$ and $Var(P_2)=0.0105$. Let $D=P_1-P_2$. Find the mean and variance of D.
- A8: Mean(D) = 0.6 0.7 = -0.1Var(D) = 0.0048 + 0.0105 = 0.0153

Continuous Random Variables

- ► So far, we have just considered discrete random variables; those whose possible values are countable.
- ► A continuous random variable can take continuous values in a future experiment.
- Every continuous random variable X has an associated density function f(x).

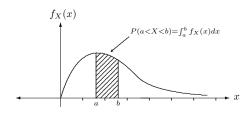
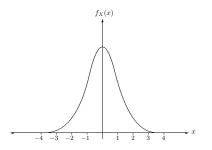


Figure 5: $P(a < X < b) = \int_a^b f_X(x) dx$.

The Normal Distribution

▶ A normal random variable is a continuous random variable.



The density function for the standard normal distribution with $\mu = 0$, $\sigma = 1$.

- We call a normal random variable with $\mu = 0$ and $\sigma^2 = 1$ a standard normal random variable.
- For standard normal random variables, we can use charts (or a computer) to find the area under the density function (i.e. the probabilities).

▶
$$P(Z < -1.75)$$

$$P(Z < -1.75) = 0.0401$$

P(Z > 0.85)

$$P(Z > 0.85) = 1 - 0.8023 = 0.1977$$

P(-1.43 < Z < 0.92)

$$P(-1.43 < Z < 0.92) = 0.8212 - .0764 = .7448$$