STAT 111

Recitation 1

Mo Huang

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Office Hours: Wednesdays 3:00 - 4:00 pm, JMHH F96

Slides (adapted from Gemma Moran): `github.com/mohuangx/STAT111-Fall2018`

September 7, 2018

Logistics

- Section 201: 11:00 am - 11:50 am
- Section 202: 12:00 pm - 12:50 pm

- Every Friday, I will:
  1. Collect the homework for that week;
  2. Give you your graded homework; and
  3. Give you the homework for the week after (also posted on Canvas).

- If you do not collect your homework, it will be available in the STAT 111 box in the Statistics Department (4th Floor, JMHH).

- If you know in advance that you will not be able to attend a recitation, give your homework to Professor Ewens on Thursdays or put it in my mailbox (at entrance to the Statistics Department)

- Questions about course materials should be asked during office hours or in an email to Professor Ewens.

Statistics and Probability

- Probability theory is *deductive* (or "top-down logic"). We start with a theory and then consider the implications resulting from that theory.

- Statistics is *inductive* (or "bottom-up logic"). We start with finite observations and then attempt to make objective statements about the world from a sample.

- Statistical methods use probability theory to draw conclusions from observed data.

Statistics and Probability: Example

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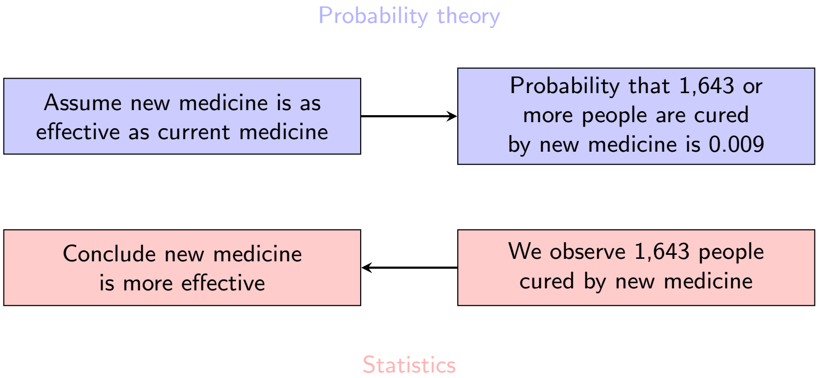

Probability: Axioms

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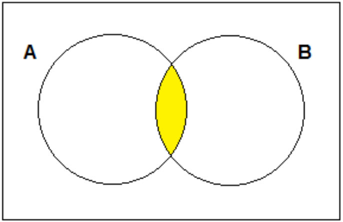
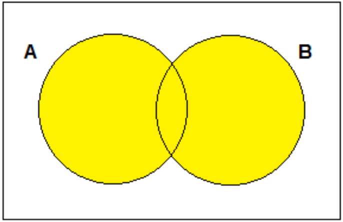
- $0 \leq P(A) \leq 1$, where A is any event.
- $P(A^C) = 1 - P(A)$ where A^C is the complement of A.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Intersection $P(A \cap B)$



Union $P(A \cup B)$



Probability: Independence

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- A and B are independent if and only if $P(A \cap B) = P(A)P(B)$
- Conditional probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Suppose A and B are independent. Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

So we have two definitions of independence:

1. $P(A \cap B) = P(A)P(B)$
2. $P(A|B) = P(A)$

Note: This year, conditional probability is not on the syllabus but it's still an interesting and useful concept!

Probability: Questions

Q1: A smoke alarm consists of two parts, A and B. If there is smoke, part A will detect it with probability 0.96 and part B will detect it with probability 0.98. The probability that they both detect it is 0.95. What is the probability that smoke will not be detected?

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$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.96 + 0.98 - 0.95 \\ &= 0.99 \\ P([A \cup B]^C) &= 1 - P(A \cup B) \\ &= 0.01 \end{aligned}$$

Probability: Questions

Q2: A six-sided die is biased such that an odd number is twice as likely to occur as an even number. That is, the probability of an odd number is $2/9$ and an even number is $1/9$. Let A be the event that an even number occurs. Let B be the event that a number greater than or equal to 4 occurs. Find $P(A)$, $P(B)$, $P(A \cup B)$ and $P(A \cap B)$.

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$$\begin{aligned} P(A) &= P(2, 4 \text{ or } 6) \\ &= 1/9 + 1/9 + 1/9 \\ &= 3/9 \end{aligned}$$

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$$\begin{aligned} P(B) &= P(4, 5 \text{ or } 6) \\ &= 1/9 + 2/9 + 1/9 \\ &= 4/9 \\ P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 3/9 + 4/9 - 2/9 \\ &= 5/9 \end{aligned}$$

Probability: Questions

A2: $P(A) = 3/9, \quad P(B) = 4/9, \quad P(A \cap B) = 2/9, \quad P(A \cup B) = 5/9$

Q3: Following Q2, find $P(A|B)$.

Probability: Questions

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Q3: Following Q2, find $P(A|B)$.

A3:

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{2/9}{4/9} \\ &= 1/2 \end{aligned}$$

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Q4: Are A and B independent?

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Q4: Are A and B independent?

A4: No, because $P(A|B) \neq P(A)$. (Recall $P(A) = 1/3$).

Probability: Questions

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A5: We want to find $P(HH|\text{at least one } H)$.

$$\begin{aligned} P(\text{at least one } H) &= P(HH) + P(HT) + P(TH) \\ &= (0.4)^2 + (0.4)(0.6) + (0.6)(0.4) \\ &= 0.64. \end{aligned}$$

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$$\begin{aligned} P(\text{at least one } H) &= P(HH) + P(HT) + P(TH) \\ &= (0.4)^2 + (0.4)(0.6) + (0.6)(0.4) \\ &= 0.64. \\ P(HH|\text{at least one } H) &= \frac{P(HH \cap \text{at least one } H)}{P(\text{at least one } H)} \\ &= \frac{P(HH)}{P(\text{at least one } H)} \\ &= \frac{(0.4)^2}{0.64} \\ &= 0.25 \end{aligned}$$

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- Data $x = 6$ is observing 6 heads in 10 tosses, a realization of X.

Random Variables

- Two types of random variables: discrete and continuous.
- A discrete random variable is a random variable that can only take on a countable set of numbers.
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x	0	1	2	3
$P(X = x)$	0.125	0.375	0.375	0.125

Table: Probability distribution of X.

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- Let X be a discrete random variable that can take values $\{v_1, v_2, \ldots, v_k\}$. Then the mean of X is given by:

$$\begin{aligned}\mu &= \sum_{i=1}^{k} v_i P(X = v_i) \\ &= v_1 P(X = v_1) + v_2 P(X = v_2) + \cdots + v_k P(X = v_k).\end{aligned}$$

Note: We use μ to denote the mean of a random variable.
Note: Can think of it as a weighted average, weighted by probability.

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- Let X be a binomial random variable where $\theta = P(\text{success})$ and there are n experiments. Then the probability distribution of X is given by:

$$P(X = x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}, \quad \text{for } x = 0, 1, 2, \ldots, n.$$

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Note: $X \sim \mathcal{B}(n, \theta)$ means "X is a binomial random variable with n experiments and probability of success θ."

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Note: This formula is *only* for the binomial distribution.