STAT 111

Recitation 2

Mo Huang

Email: mohuang@wharton.upenn.edu

Office Hours: Wednesdays 3:00 - 4:00 pm, JMHH F96

Slides: `github.com/mohuangx/STAT111-Spring2019`

February 1, 2019

Parameters (θ), Random Variables (X), and Data (x)

- A parameter represents some underlying numerical constant of a phenomenon. Represented by Greek letters.
- A random variable is a numerical outcome of interest in a future experiment. Can be modeled by a probability distribution which depends on the parameter. Represented by capital letters.
- Data is the realization or observed value of a random variable after performing the experiment. Represented by lower-case letters.

Fair coin-flipping example:

- A parameter θ is the underlying probability of obtaining a head. $\theta = 0.5$.
- A random variable X is the number of heads obtained in 10 tosses. Can be modeled by a binomial distribution dependent on θ.
- Data $x = 6$ is observing 6 heads in 10 tosses, a realization of X.

Random Variables

- Two types of random variables: discrete and continuous.
- A discrete random variable is a random variable that can only take on a countable set of numbers.
- The probability distribution of a discrete random variable is the range of values it can take *and* the probabilities of these values.
- For example:
  - Let X be the number of heads I toss if I toss a fair coin 3 times.

x	0	1	2	3
$P(X = x)$	0.125	0.375	0.375	0.125

Table: Probability distribution of X using the tableau method.

The Binomial Distribution

- The binomial distribution arises if:
  1. We plan to conduct a fixed number of experiments. We denote the number of experiments as n.
  2. In each experiment, there are two outcomes: “success” or “failure”.
  3. The experiments are independent.
  4. The probability of a success is the same for each experiment.
- For example:
  1. I plan to toss a coin n times.
  2. I can toss either a head or a tail.
  3. Each coin toss is independent of the next - it doesn’t matter whether I get a head or a tail on the previous toss.
  4. The probability of getting a head is the same for each toss.

The Binomial Distribution

- Let X be a binomial random variable where $\theta = P(\text{success})$ and there are n experiments. Then the probability distribution of X is given by:

$$P(X = x) = \binom{n}{x} \theta^x (1-\theta)^{n-x}, \quad \text{for } x = 0, 1, 2, \ldots, n.$$

  - θ is called a parameter: a constant whose value may be known or unknown.
  - $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ is said as "n choose x": it is the number of ways x successes can occur in n experiments.

Note: $X \sim \mathcal{B}(n, \theta)$ means "X is a binomial random variable with n experiments and probability of success θ."

The Binomial Distribution: Questions

Q1: Let X be the number of heads if I toss an unbiased coin 3 times ($n = 3, \theta = 0.5$). Find the probability distribution of X in "tableau" form.

A1:

$$P(X = 0) = \binom{3}{0}(0.5)^0(0.5)^3 = 0.125$$

$$P(X = 1) = \binom{3}{1}(0.5)^1(0.5)^2 = 0.375$$

$$P(X = 2) = \binom{3}{2}(0.5)^2(0.5)^1 = 0.375$$

$$P(X = 3) = \binom{3}{3}(0.5)^3(0.5)^0 = 0.125$$

x	0	1	2	3
$P(X = x)$	0.125	0.375	0.375	0.125

Table: Probability distribution of X using the tableau method.

The Binomial Distribution: Tables

Q2: $X \sim \mathcal{B}(18, 0.15)$. Find $P(X = 4)$.

A2: $P(X = 4) = 0.1592$

						θ
n	i	0.05	0.10	0.15	0.20	0.25
18	0	0.3972	0.1501	0.0536	0.0180	0.0056
	1	0.3763	0.3002	0.1704	0.0811	0.0338
	2	0.1683	0.2835	0.2556	0.1723	0.0958
	3	0.0473	0.1680	0.2406	0.2297	0.1704
	4	0.0093	0.0700	$\boxed{0.1592}$	0.2153	0.2130

Q3: Find $P(X \leq 4)$.

A3:

$$
\begin{aligned}
P(X \leq 4) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) \\
&= 0.0536 + 0.1704 + 0.2556 + 0.2406 + 0.1592 \\
&= 0.8794
\end{aligned}
$$

The Binomial Distribution: Tables

Q2: $X \sim \mathcal{B}(12, 0.8)$. Find $P(X = 3)$.

A2: Finding 3 successes with $\theta = 0.8$ is the same as finding 9 failures with θ of failure 0.2. Hence, $P(X = 3) = 0.0001$.

		θ						
n	i	0.05	0.10	0.15	0.20	0.25	0.30	0.35
12	0	0.5404	0.2824	0.1422	0.0687	0.0317	0.0138	0.0057
	1	0.3413	0.3766	0.3012	0.2062	0.1267	0.0712	0.0368
	2	0.0988	0.2301	0.2924	0.2835	0.2323	0.1678	0.1088
	3	0.0173	0.0852	0.1720	0.2362	0.2581	0.2397	0.1954
	4	0.0021	0.0213	0.0683	0.1329	0.1936	0.2311	0.2367
	5	0.0002	0.0038	0.0193	0.0532	0.1032	0.1585	0.2039
	6	0.0000	0.0005	0.0040	0.0155	0.0401	0.0792	0.1281
	7	0.0000	0.0000	0.0006	0.0033	0.0115	0.0291	0.0591
	8	0.0000	0.0000	0.0001	0.0005	0.0024	0.0078	0.0199
	9	0.0000	0.0000	0.0000	$\boxed{0.0001}$	0.0004	0.0015	0.0048
	10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0008
	11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
	12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Random Variables: Mean

- The mean of a random variable (expected value) is the long-run average of realizations of the random variable over repeated experiments.

- This mean of random variable X is different from the sample mean, which is the average of a *finite* number of observations (x).

- Let X be a discrete random variable that can take values $\{v_1, v_2, \ldots, v_k\}$. Then the mean of X is given by:

$$\begin{aligned} \mu &= \sum_{i=1}^{k} v_i P(X = v_i) \\ &= v_1 P(X = v_1) + v_2 P(X = v_2) + \cdots + v_k P(X = v_k). \end{aligned}$$

Note: We use μ to denote the mean of a random variable.
Note: Can think of it as a weighted average, weighted by probability.

Random Variables: Questions

Q2: Let X be the outcome of one roll of a biased dice where the probability of the number j turning up is $j/21$. Find the mean of X.

A2:

$$\begin{aligned}\mu &= 1 \times 1/21 + 2 \times 2/21 + 3 \times 3/21 + 4 \times 4/21 + 5 \times 5/21 + 6 \times 6/21 \\ &= 91/21\end{aligned}$$

Q3: Let X be a binomial random variable with $n = 2$ and $\theta = 0.8$. Find the mean of X using the binomial table.

A3:

$$\begin{aligned}\mu &= 0 \times 0.04 + 1 \times 0.32 + 2 \times 0.64 \\ &= 1.6\end{aligned}$$

The Binomial Distribution: Mean

- For the binomial distribution, we have a simpler formula for the mean: if $X \sim \mathcal{B}(n, \theta)$, then the mean of X is

$$\mu = n\theta.$$

- For example:
    - On the previous slide, we calculated the mean of a binomial random variable with $n = 2$ and $\theta = 0.8$ to be 1.6. This is exactly $n\theta$.

Note: This formula is *only* for the binomial distribution.

Random Variables: Variance

- The variance of a random variable is a measure of the *spread* of a distribution - that is, how far away values are from the mean.

- Let X be a discrete random variable that can take values $\{v_1, v_2, \ldots, v_k\}$. Then the variance of X is given by:

$$\sigma^2 = (v_1 - \mu)^2 P(X = v_1) + \cdots + (v_k - \mu)^2 P(X = v_k) \quad (1)$$

or

$$\sigma^2 = v_1^2 P(X = v_1) + \cdots + v_k^2 P(X = v_k) - \mu^2 \quad (2)$$

- The standard deviation of a random variable is the square root of the variance and is denoted by σ.

- For a binomial random variable $X \sim \mathcal{B}(n, \theta)$:

$$\sigma^2 = n\theta(1 - \theta)$$

Note: The variance is always positive - you can't have a negative spread of a distribution.