

STAT 111

Recitation 3

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Random Variables: Mean

- ▶ The **mean of a random variable** (expected value) is the long-run average of realizations of the random variable over repeated experiments.
- ▶ This mean of random variable X is different from the sample average, which is the average of a *finite* number of observations ($\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}$, where x_1, \dots, x_n are observed data).
- ▶ Let X be a discrete random variable that can take values $\{v_1, v_2, \dots, v_k\}$. Then the mean of X is given by:

$$\begin{aligned}\mu &= \sum_{i=1}^k v_i P(X = v_i) \\ &= v_1 P(X = v_1) + v_2 P(X = v_2) + \dots + v_k P(X = v_k).\end{aligned}$$

- ▶ If X is binomial with parameters n and θ , then the mean of X is

$$\mu = n\theta$$

Random Variables: Variance

- ▶ The **variance of a random variable** is a measure of the *spread* of a distribution - that is, how far away values are from the mean.
- ▶ Let X be a discrete random variable that can take values $\{v_1, v_2, \dots, v_k\}$. Then the variance of X is given by:

$$\sigma^2 = (v_1 - \mu)^2 P(X = v_1) + \dots + (v_k - \mu)^2 P(X = v_k) \quad (1)$$

or

$$\sigma^2 = v_1^2 P(X = v_1) + \dots + v_k^2 P(X = v_k) - \mu^2 \quad (2)$$

- ▶ The **standard deviation of a random variable** is the square root of the variance and is denoted by σ .
- ▶ For a **binomial** random variable $X \sim \mathcal{B}(n, \theta)$:

$$\sigma^2 = n\theta(1 - \theta)$$

Random Variables: Questions

Q1: Let X be a random variable with the below distribution. Find the mean and the variance of X using formula (1) and then formula (2).

x	-3	-1	4	5
$P(X = x)$	0.1	0.3	0.4	0.2

Table: Probability distribution of X .

A1: $\mu = -3 \times 0.1 - 1 \times 0.3 + 4 \times 0.4 + 5 \times 0.2 = 2.$

$$\begin{aligned}\sigma^2 &= (-3 - 2)^2(0.1) + (-1 - 2)^2(0.3) + (4 - 2)^2(0.4) + (5 - 2)^2(0.2) \\ &= 8.6.\end{aligned}$$

$$\begin{aligned}\sigma^2 &= (-3)^2(0.1) + (-1)^2(0.3) + 4^2(0.4) + 5^2(0.2) - 2^2 \\ &= 8.6.\end{aligned}$$

Random Variables: Questions

Q2: Let X be a random variable with the below distribution. Write the probability distribution of $Y = 2X$ in tableau form.

x	-3	-1	4	5
$P(X = x)$	0.1	0.3	0.4	0.2

Table: Probability distribution of X .

A2: Probability distribution of Y :

y	-6	-2	8	10
$P(Y = y)$	0.1	0.3	0.4	0.2

Table: Probability distribution of Y .

Properties of mean and variance

- ▶ Suppose X is a random variable with mean μ and variance σ^2 . Consider fixed number c .
- ▶ Mean properties
 - ▶ Mean of $X + c \implies \mu + c$.
 - ▶ Mean of $cX \implies c\mu$.
- ▶ Variance properties
 - ▶ Variance of $X + c \implies \sigma^2$.
 - ▶ Variance of $cX \implies c^2\sigma^2$.

Q3: Suppose X has mean $\mu_X = 3$ and variance $\sigma_X^2 = 4$. What is the mean and variance of $Y = 2 + 5X$?

A3: $\mu_Y = 2 + 5\mu_X = 2 + 5(3) = 17$

$$\sigma_Y^2 = 5^2\sigma_X^2 = 5^2(4) = 100$$

Many Random Variables

- ▶ Suppose we plan an experiment with a **sample size** of n .
- ▶ We have n future outcomes, or random variables, denoted by:
 X_1, X_2, \dots, X_n .
- ▶ We say $\{X_1, \dots, X_n\}$ are **independently and identically distributed (or i.i.d)** if:
 - ▶ All the X_i s are independent of each other.
 - ▶ Each X_i has the same probability distribution.
- ▶ For example:
 - ▶ I plan to roll a dice n times. X_i represents the future outcome of the i th roll. X_i is *independent* of the other rolls of the dice, and it has the same probability of getting a 1, 2, 3, 4, 5, 6 as the other rolls.

Many Random Variables

- ▶ Recall the sample average is given by:

$$\bar{x} = \frac{x_1 + \cdots + x_n}{n}.$$

- ▶ What if we haven't observed the data x_1, \dots, x_n yet?
- ▶ Before an experiment, the average is *also* a random variable:

$$\bar{X} = \frac{X_1 + \cdots + X_n}{n}$$

- ▶ The *sum* is also a random variable:

$$T_n = X_1 + \cdots + X_n$$

Many Random Variables

- ▶ Let X and Y be two random variables. Then,

$$\text{mean}(X + Y) = \text{mean}(X) + \text{mean}(Y).$$

- ▶ If X, Y are also independent,

$$\text{variance}(X + Y) = \text{variance}(X) + \text{variance}(Y).$$

- ▶ For constants a, b , we have

$$\text{mean}(aX + bY) = a \times \text{mean}(X) + b \times \text{mean}(Y)$$

$$\text{variance}(aX + bY) = a^2 \times \text{variance}(X) + b^2 \times \text{variance}(Y).$$

- ▶ Let $D = X - Y$. What is the variance of D ?

$$\text{variance}(D) = \text{variance}(X) + \text{variance}(Y).$$

Many Random Variables

- ▶ Let X_1, \dots, X_n be i.i.d. random variables, each with mean μ and variance σ^2 .
- ▶ Then for the sum, T_n :

$$\text{mean of } T_n = n\mu, \quad \text{variance of } T_n = n\sigma^2$$

- ▶ For the average, \bar{X} :

$$\text{mean of } \bar{X} = \mu, \quad \text{variance of } \bar{X} = \frac{\sigma^2}{n}.$$

Questions

Q4: Suppose we have a company producing a medicine. Each day the *mean* amount of medicine produced is 500 mg and the *variance* is 900 mg². Assume the amount produced each day is independent.

- (i) Let T_n be the total amount of medicine produced in a week. Find the mean and variance of T_n .

$$\text{Mean}(T_n) = 5 \times 500 = 2500$$

$$\text{Var}(T_n) = 5 \times 900 = 4500$$

- (ii) Let \bar{X} be the average amount of medicine produced in a 5-day week. Find the mean and variance of \bar{X} .

$$\text{Mean}(\bar{X}) = 500$$

$$\text{Var}(\bar{X}) = 900/5 = 180$$

Proportions

- ▶ Sometimes it is necessary to consider the *proportion* of “successes” in a Binomial trial (instead of the total number of successes)
- ▶ Proportions are a type of average!
- ▶ Let

$$Y_i = \begin{cases} 1 & \text{if the } i\text{th trial is a “success”} \\ 0 & \text{if the } i\text{th trial is a “failure”} \end{cases}$$

- ▶ Then we have Y_1, \dots, Y_n where $Y_i \sim \text{Binomial}(1, \theta)$.
- ▶ The proportion of successes is the average:

$$P = \frac{Y_1 + \dots + Y_n}{n}$$

Proportions

- ▶ $Y_i \sim \text{Binomial}(1, \theta)$. Recall:

$$\text{Mean}(Y_i) = \theta, \quad \text{Var}(Y_i) = \theta(1 - \theta).$$

- ▶ The proportion of successes is the average:

$$P = \frac{Y_1 + \cdots + Y_n}{n}$$

- ▶ Then,

$$\text{Mean}(P) = \theta, \quad \text{Var}(P) = \frac{\theta(1 - \theta)}{n}$$

.

Q5: Suppose we plan to toss a coin 20 times and $\text{Prob}(H) = 0.7$. Let P be the *proportion* of heads that we toss. Find the mean and variance of P .

A5:

$$\text{Mean}(P) = 0.7, \quad \text{Var}(P) = 0.7 \times 0.3/20 = 0.0105.$$

Questions

Q6: Suppose the company producing a medicine has different means and variances the amount produced on each day of the week:

Day	Mean	Variance
Monday (X_1)	450	1200
Tuesday (X_2)	550	800
Wednesday (X_3)	600	500
Thursday (X_4)	550	800
Friday (X_5)	350	1200

Find mean and variance of both the sum T_n and the average \bar{X} .

A6: X_1, X_2, X_3, X_4 , and X_5 are no longer *i.i.d.*!

$$\text{Mean}(T_n) = 450 + 550 + 600 + 550 + 350 = 2500$$

$$\text{Var}(T_n) = 1200 + 800 + 500 + 800 + 1200 = 4500$$

$$\text{Mean}(\bar{X}) = 1/n \times \text{Mean}(T_n) = 500$$

$$\text{Var}(\bar{X}) = 1/n^2 \times \text{Var}(T_n) = 4500/25 = 180$$

Questions

Q7: Let P_1 be the proportion of heads in 50 coin tosses, where $P(H) = 0.6$. Find $Mean(P_1)$ and $Var(P_1)$.

A7: $Mean(P_1) = 0.6$ and $Var(P_1) = 0.6 \times 0.4/50 = 0.0048$.

Q8: Let P_2 be the proportion of heads in 20 coin tosses, where $P(H) = 0.7$. From earlier, $Mean(P_2) = 0.7$ and $Var(P_2) = 0.0105$. Let $D = P_1 - P_2$. Find the mean and variance of D .

A8: $Mean(D) = 0.6 - 0.7 = -0.1$
 $Var(D) = 0.0048 + 0.0105 = 0.0153$