Proof 9

nibir@nibirsan.org

Given an example of a family of intervals $A_n, n = 1, 2, 3 \ldots$ such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n = \phi$. Prove that your example has the stated property.

Proof

The interval family is $\left(\frac{1}{n}, \frac{1}{n}\right)$

We have two conditions to prove:

1. $A_{n+1} \subset A_n$
2. $\bigcap_{n=1}^{\infty} A_n = \phi$

For condition 1

What $A_{n+1} \subset A_n$ is saying, is that the $n + 1^{\text{th}}$ interval is the subset of the n^{th} interval in the family. Which suggests that the intervals are getting smaller and smaller.

Thus, to prove the first condition, we need to show that for any n, the $n + 1^{\text{th}}$ interval is smaller than the n^{th} interval but also a subset.

This is simple.

If n is taken arbitrarily, the interval $\left(\frac{1}{n+1}, \frac{1}{n+1}\right)$ is smaller than $\left(\frac{1}{n}, \frac{1}{n}\right)$, but a subset because $\frac{1}{n+1} < \frac{1}{n}$.

Similarly, the interval $\left(\frac{1}{n+2}, \frac{1}{n+2}\right)$ is smaller than $\left(\frac{1}{n+1}, \frac{1}{n+1}\right)$, but a subset.

Therefore, $\forall n[A_{n+1} \subset A_n]$. **This proves the first condition.**

For condition 2

We know that, $\frac{1}{n} \to 0$ as $n \to \infty$ (limit). So $\left(\frac{1}{n}, \frac{1}{n}\right) \to (0, 0)$, which is the last possible interval in the family.

Since $(0, 0)$ does not contain any number, $(0, 0) = \phi$

Thus, the intersection is also ϕ, because $(0, 0)$ is the last interval in the family.

$$\bigcap_{n=1}^{\infty} A_n = \phi$$

This proves the second condition.

Therefore, the interval family $\left(\frac{1}{n}, \frac{1}{n}\right)$ suffices both the conditions, which is what we wanted to prove.