Proof 10

nibir@nibirsan.org

Given an example of a family of intervals $A_n, n = 1, 2, 3 \ldots$ such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

Proof

The family is $\left[0, \frac{1}{n}\right]$

We have two conditions to prove:

1. $A_{n+1} \subset A_n$
2. $\bigcap_{n=1}^{\infty} A_n = k$ where $k \in \mathbb{R}$

For condition 1

We need to show that for any n, the $n + 1^{\text{th}}$ interval is smaller than the n^{th} interval but also a subset, to prove the first condition.

We do so by taking an arbitrary n.

Then, the interval $\left[0, \frac{1}{n+1}\right]$ is smaller than $\left[0, \frac{1}{n}\right]$, but a subset ($\because \frac{1}{n+1} < \frac{1}{n}$).

Similarly, the interval $\left[0, \frac{1}{n+2}\right]$ is smaller than $\left[0, \frac{1}{n+1}\right]$, but a subset.

Therefore, $\forall n[A_{n+1} \subset A_n]$. **This proves the first condition.**

For condition 2

We know that as $n \to \infty$, $\frac{1}{n} \to 0$. So $\left[0, \frac{1}{n}\right] \to [0, 0]$, which is the last possible interval in the family.

And since $[0, 0] = \{0\}$, the intersection of the family is also $\{0\}$, because $[0, 0]$ is the last interval in the family.

$$\bigcap_{n=1}^{\infty} A_n = \{0\}$$

Which is a single real number. **And this proves the second condition.**

Hence, the family $[0, \frac{1}{n}]$ suffices both the conditions, which is what we wanted to prove.