A New Property of the Discrete Cosine Transform-IV (DCT-IV)

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***Abstract*—This paper presents the derivation of the Duality Property for the DCT-IV, which is a finite-length discrete transform having applications in Digital Signal, Image and Video Processing. This property can be applied for finding the temporal domain signal from its frequency domain counterpart or mutatis mutandis if an existing DCT-IV pair is known to exist, thus receding considerable computational gruntwork and disbursement substantially.**

***Index Terms*—Cost, temporal, inverse, symmetric, transform**

I. Introduction

DISCRETE transforms are mathematical transforms of functions or signals between discrete-temporal and discrete-frequency domains. They are in fact the counterparts of their integral transforms and are derived from them. They play a pivotal role in the dissemination of information from the signals. The DCT-IV is one such mathematical transform that is used for the analysis of unidimensional signals [1]. The section II gives the definition of DCT-IV and its inverse and their expression as a linear transformation. The section III gives the relationship between DCT-IV and DFT. The section IV gives the derivation of the duality property of the DCT-IV. Finally, the veracity of this property is given with an illustrative example in section V, followed by the conclusions drawn in section VI.

II. DCT-IV and Its Inverse as Linear Transformation

The DCT-IV of a unidimensional signal $l(n)$, with

$0 \leq n \leq D-1$, is given as,

$$L(h) = DCT_{IV}\left[l(n)\right]$$

$$L(h) \triangleq \sqrt{\frac{2}{D}} \sum_{n=0}^{D-1} l(n) \cos\left[\frac{(2h+1)(2n+1)\pi}{4D}\right] \tag{1}$$

where, $0 \leq n, h \leq D-1$. The term $\cos\left[\frac{(2h+1)(2n+1)\pi}{4D}\right]$ is called the kernel of DCT-IV. Equation (1) is the Forward Transform of DCT-IV [2]. DCT_{IV} designates the DCT-IV operator. The DCT-IV matrix is given by

$$\mathbb{C} = \sqrt{\frac{2}{D}} \cos\left[\frac{(2h+1)(2n+1)\pi}{4D}\right] \tag{2}$$

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where, $0 \leq n, h \leq D-1$. DCT-IV is indeed a real discrete transform [3]. Expanding (2) as a $D \times D$ matrix, we get,

$$\mathbb{C} = \sqrt{\frac{2}{D}} \begin{bmatrix} \cos(\frac{\pi}{4D}) & \cdots & \cos(\frac{(2D-1)\pi}{4D}) \\ \cos(\frac{3\pi}{4D}) & \cdots & \cos(\frac{(2D-1)3\pi}{4D}) \\ \cos(\frac{5\pi}{4D}) & \cdots & \cos(\frac{(2D-1)5\pi}{4D}) \\ \vdots & \vdots & \vdots \\ \cos(\frac{(2D-1)\pi}{4D}) & \cdots & \cos(\frac{(2D-1)^2\pi}{4D}) \end{bmatrix} \tag{3}$$

For $D = 4$, the DCT-IV Matrix $\mathbb{C}$ becomes,

$$\mathbb{C} = \begin{bmatrix} .6935 & .5879 & .3928 & .1379 \\ .5879 & -.1379 & -.6935 & -.3928 \\ .3928 & -.6935 & .1379 & .5879 \\ .1379 & -.3928 & .5879 & -.6935 \end{bmatrix} \tag{4}$$

Thus, we can represent (2) as a linear transformation as

$$\begin{bmatrix} L(0) \\ L(1) \\ L(2) \\ \vdots \\ L(D-1) \end{bmatrix} = \sqrt{\frac{2}{D}} \begin{bmatrix} \cos(\frac{\pi}{4D}) & \cdots & \cos(\frac{(2D-1)\pi}{4D}) \\ \cos(\frac{3\pi}{4D}) & \cdots & \cos(\frac{(2D-1)3\pi}{4D}) \\ \cos(\frac{5\pi}{4D}) & \cdots & \cos(\frac{(2D-1)5\pi}{4D}) \\ \vdots & \vdots & \vdots \\ \cos(\frac{(2D-1)\pi}{4D}) & \cdots & \cos(\frac{(2D-1)^2\pi}{4D}) \end{bmatrix} \begin{bmatrix} l(0) \\ l(1) \\ l(2) \\ \vdots \\ l(D-1) \end{bmatrix} \tag{5}$$

Some remarkable demeanors of the DCT-IV matrix $\mathbb{C}$ are that it is real, symmetric and orthogonal (unitary) [4]. Thus, we can formulate that

$$\mathbb{C} = \mathbb{C}^{*} = \mathbb{C}^{T} = \mathbb{C}^{-1} = \mathbb{C}^{*T} \tag{6}$$

The DCT-IV is an invertible transform and hence, its inverse also exists. Thus, its inversion is given by the Inverse Discrete Cosine Transform-IV (IDCT-IV) operation The IDCT-IV of $L(h)$ is formally defined as [5],

$$l(n) = IDCT_{IV}\left[L(h)\right]$$

$$l(n) \triangleq \frac{2}{D} \sum_{h=0}^{D-1} L(h) \cos\left[\frac{(2h+1)(2n+1)\pi}{4D}\right] \tag{7}$$

Equation (7) can also be written as linear transformation as [6]

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$$\begin{bmatrix} l(0) \\ l(1) \\ l(2) \\ \vdots \\ l(D-1) \end{bmatrix} =$$

$$\sqrt{\frac{2}{D}} \begin{bmatrix} \cos(\frac{\pi}{4D}) & \dots & \cos(\frac{(2D-1)\pi}{4D}) \\ \cos(\frac{3\pi}{4D}) & \dots & \cos(\frac{(2D-1)3\pi}{4D}) \\ \cos(\frac{5\pi}{4D}) & \dots & \cos(\frac{(2D-1)5\pi}{4D}) \\ \vdots & \vdots & \vdots \\ \cos(\frac{(2D-1)\pi}{4D}) & \dots & \cos(\frac{(2D-1)^2\pi}{4D}) \end{bmatrix} \begin{bmatrix} L(0) \\ L(1) \\ L(2) \\ \vdots \\ L(D-1) \end{bmatrix} \quad (8)$$

For $D = 4$, the IDCT-IV Matrix $\mathbb{C}^{-1}$ becomes,

$$\mathbb{C}^{-1} = \begin{bmatrix} .6935 & .5879 & .3928 & .1379 \\ .5879 & -.1379 & -.6935 & -.3928 \\ .3928 & -.6935 & .1379 & .5879 \\ .1379 & -.3928 & .5879 & -.6935 \end{bmatrix} \quad (9)$$

$l(n)$ and $L(h)$ form a DCT-IV pair and are written as

$$l(n) \xleftrightarrow{DCT_{IV}} L(h) \quad (10)$$

Thus, the DCT-IV is an involutionary discrete transform. This simply follows from the fact that

$$DCT_{IV}[DCT_{IV}[\{l(n)\}]] = l(n) \quad (11)$$

There are many such involutionary transforms that exist in the mathematical literature [7].

III. Relationship Between DCT-IV And DFT

The renowned Discrete Fourier Transform (DFT) of an unidimensional signal $l(n)$, with $0 \leq n \leq D-1$, is defined as [8]

$$L(h) = DFT[l(n)]$$

$$L(h) = \sum_{n=0}^{D-1} l(n) W_D^{hn} = \sum_{n=0}^{D-1} l(n) e^{-j2\pi hn/D} \quad (12)$$

where, $0 \leq n, h \leq D-1$.

The Inverse Discrete Fourier Transform (IDFT) is given by the relation,

$$l(n) = \frac{1}{D}\sum_{h=0}^{D-1} L(h) W_D^{-hn} = \frac{1}{D}\sum_{h=0}^{D-1} L(h) e^{j2\pi hn/D} \quad (13)$$

where, $0 \leq n, h \leq D-1$.

Equation (1) can also be written as

$$L(h) \triangleq \sqrt{\frac{2}{D}} \sum_{n=0}^{D-1} l(n) \cos\left[\frac{\pi}{D}(n+\frac{1}{2})(h+\frac{1}{2})\right] \quad (14)$$

Making usage of the trigonometric identity [9],

$$\cos\left(\frac{\pi\mathcal{E}}{D}\right) = \frac{1}{4}[W_{8D}^{4\mathcal{E}} - W_{8D}^{4D-4\mathcal{E}} - W_{8D}^{4D+4\mathcal{E}} + W_{8D}^{8D-4\mathcal{E}}] \quad (15)$$

$$L(h) = \sqrt{\frac{2}{D}} \cdot \frac{1}{4}[\sum_{n=0}^{D-1}\{W_{8D}^{(2n+1)(2h+1)} - W_{8D}^{(4D-2n-1)(2h+1)} - W_{8D}^{(4D+2n+1)(2h+1)} + W_{8D}^{(8D-2n-1)(2h+1)}\}]$$

This can be simplified and written as

$$L(h) = K\sum_{n=0}^{\tilde{D}-1} \hat{l}(n) W_{8D}^{n(2h+1)} \quad (16)$$

where, K is a scaling factor (constant) and $\hat{l}(n)$ is the real-even signal of length $\tilde{D} = 8D$ defined in the interval $0 \leq n \leq D-1$ as [10]

$$\hat{l}(2n+1) = \hat{l}(8D-2n-1) = \frac{1}{4} l(n) \quad (17)$$

and

$$\hat{l}(4D-2n-1) = \hat{l}(4D+2n+1) = -\frac{1}{4} l(n) \quad (18)$$

It is important to note that $\hat{l}(2n) = 0, \forall\, 0 \leq n \leq 4D-1$.

From the above discussion, it is self-evident that DCT-IV is an exceptional case of the DFT with real inputs of symmetries. This is so because the DCT-IV can be computed by using DFT and hence, by the usage of the Fast Fourier Transform (FFT) Algorithms at a vigorous tread with ameliorated exactitude. This makes DCT-IV necessitous for any practical applications in Electromagnetics, Speech Processing, Image Processing, etc. It can also be utilized in Pure and Applied Mathematics for solving differential equations and partial differential equations [11].

IV. Duality Property For DCT-IV

A. *Statement*

If the DCT-IV of $l(n)$ is given by $L(h)$, then, the DCT-IV of $L(n)$ is given by $l(h)$. Mathematically, $DCT_{IV}[L(n)] = l(h)$.

B. *Proof*

From (1) and (7), it is given that

$$L(h) = \sqrt{\frac{2}{D}} \sum_{n=0}^{D-1} l(n) \cos\left[\frac{(2h+1)(2n+1)\pi}{4D}\right] \quad (19)$$

$$l(n) = \sqrt{\frac{2}{D}} \sum_{h=0}^{D-1} L(h) \cos\left[\frac{(2h+1)(2n+1)\pi}{4D}\right] \quad (20)$$

Exchanging n with h contrariwise in (20) yields,

$$l(h) = \sqrt{\frac{2}{D}} \sum_{n=0}^{D-1} L(n) \cos\left[\frac{(2h+1)(2n+1)\pi}{4D}\right] \quad (21)$$

A simple observation of (19) with (20) and (21) reveals that its right-hand side is the DCT-IV of $L(n)$. Hence, it allows us to write symbolically as

$$DCT_{IV}[L(n)] = l(h) \quad (22)$$

V. Illustration Of The Duality Property Of DCT-IV

Consider the discrete-time signal of length $D = 3$, which is given by the function,

$$l(n) = \delta(n-1) + 2\,\delta(n-2) + 3\delta(n-3) = \{1,2,3\} \quad (23)$$

Taking DCT-IV on both sides of (23), we get,

$$L(h) = \sqrt{\frac{2}{D}} \sum_{n=0}^{2} l(n) \cos[\frac{(2h+1)(2n+1)\pi}{12}] \quad (24)$$

where, $0 \leq n, h \leq 2$. Simplification of (24) yields the following result,

$$L(h) = \{2.577, -2.309, 1.422\} \quad (25)$$

Consider the IDCT-IV computation of discrete-frequency signal,

$$L(h) = \{1,2,3\} = \delta(h) + 2\delta(h-1) + 3\delta(h-2) \quad (26)$$

The length of the sequence is $D = 3$. Taking IDCT-IV on both sides of (26), we obtain,

$$l(n) = \sqrt{\frac{2}{3}} \sum_{h=0}^{2} L(h) \cos[\frac{(2h+1)(2n+1)\pi}{12}] \quad (27)$$

where, $0 \leq n, h \leq 2$. Simplifying (27), we get,

$$l(n) = \{2.577, \ -2.309, \ 1.422\} \quad (28)$$

Next, we apply the Duality Theorem for DCT-IV for the computation of the IDCT-IV of

$$L(h) = \{1,2,3\} = \delta(h) + 2\delta(h-1) + 3\delta(h-2) \quad (29)$$

It is proved in (25) that

$$\{1,2,3\}_{time} \xleftrightarrow{DCT_{IV}} \{2.577, -2.309, 1.422\}_{frequency} \quad (30)$$

Or, we can write (30) as

$$[\delta(n) + 2\delta(n-1) + 3\delta(n-2)]$$
$$\xleftrightarrow{DCT_{IV}} [2.577\delta(h) - 2.309\delta(h-1) + 1.422\delta(h-2)]$$

Applying Duality Theorem derived in the previous section to (30), we get,

$$\{1,2,3\}_{frequency} \xleftrightarrow{DCT_{IV}} \{2.577, -2.309, 1.422\}_{time} \quad (31)$$

Or, we can write (31) as

$$[\delta(h) + 2\delta(h-1) + 3\delta(h-2)]$$
$$\xleftrightarrow{DCT_{IV}} [2.577\delta(n) - 2.309\delta(n-1) + 1.422\delta(n-2)]$$

That is,

$$l(n) = IDCT_{IV}[L(h)]$$
$$\Rightarrow l(n) = IDCT_{IV}[\delta(h) + 2\delta(h-1) + 3\delta(h-2)]$$
$$\Rightarrow l(n) = 2.577\delta(n) - 2.309\delta(n-1) + 1.422\delta(n-2)$$
$$\Rightarrow l(n) = \{2.577, \ -2.309, \ 1.422\} \quad (32)$$

It is easily seen that (32) is exactly same as (28) and the Duality Theorem for DCT-IV is verified. Figs. 1, 2, and 3 show the simulated results obtained using Matlab R2017b software.

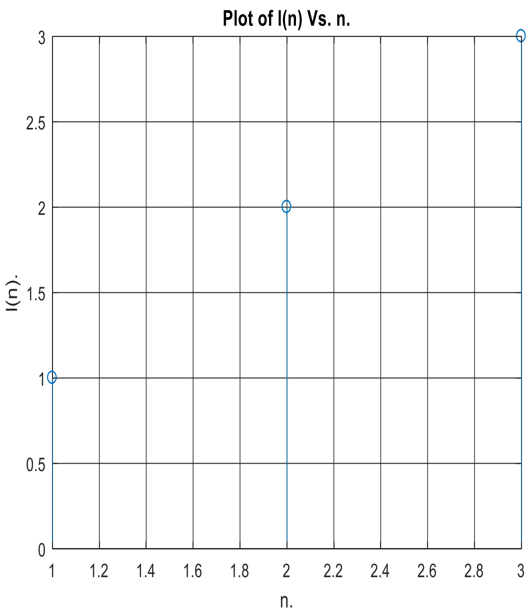

Fig. 1. Plot of the discrete-time signal $l(n) = \{1,2,3\}$ versus n.

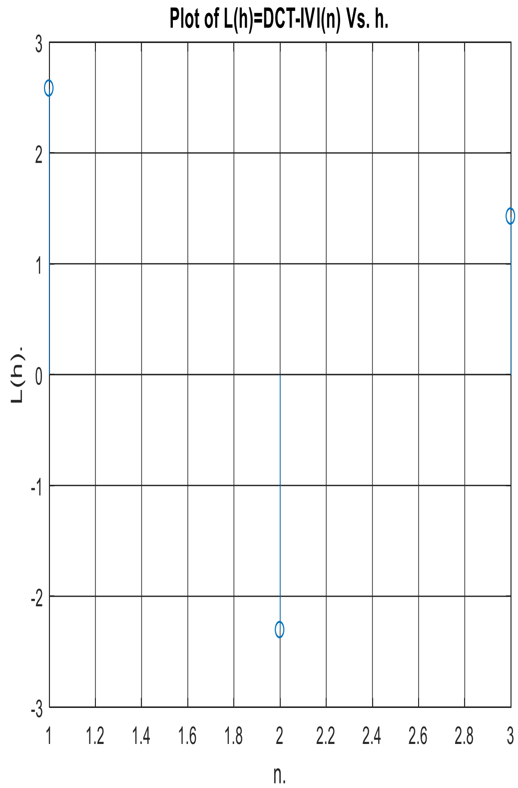

Fig. 2. Plot of the $L(h) = DCT_{IV}[l(n)]$ versus h.

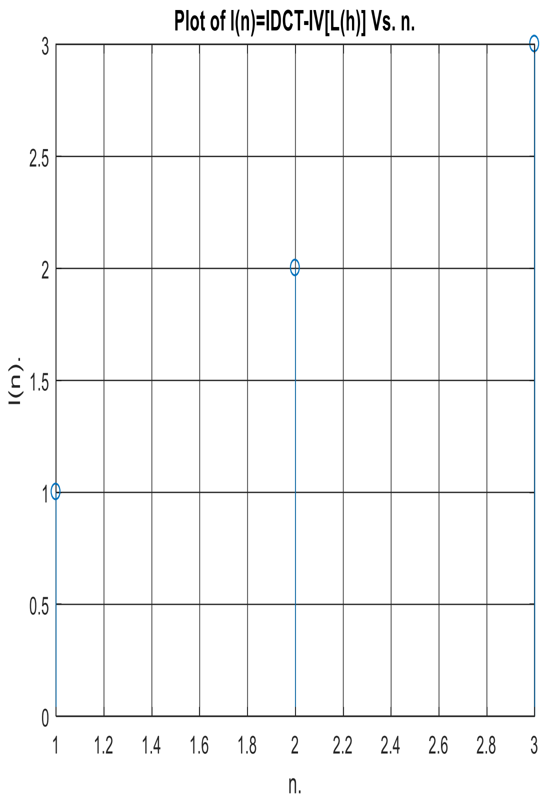

Fig. 3. Plot of $l(n) = IDCT_{IV}[L(h)] = \{1,2,3\}$ versus n.

VI. CONCLUSION

A derivation of the Duality Property for DCT-IV has been given in this paper. It is a common practice in Signal Theory where one encounters signals having the same shape in the temporal and frequency domains. If a signal whose DCT-IV exists is known, then, for a signal having the same structure in the frequency domain, this Duality Property can be easily applied for inverting it, thus, avoiding the mathematical labour involved in actually evaluating the inverse. This leads to the saving of computation time and outlay. This property can be used in discrete-time signal processing.

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