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Generalized discrete cosine transform

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Abstract

The discrete cosine transform (DCT), introduced by Ahmed, Natarajan and Rao, has been used in many applications of digital signal processing, data compression and information hiding. There are four types of the discrete cosine transform. In simulating the discrete cosine transform, we propose a generalized discrete cosine transform with three parameters, and prove its orthogonality for some new cases. Finally, a new type of discrete cosine transform is proposed and its orthogonality is proved.
Keywords: Discrete Fourier transform, discrete sine transform, discrete cosine transform

1. Introduction

Discrete Fourier transform has been an important tool in many applications of digital signal processing, image processing and information hiding. The appearance of fast fourier transform (FFT) has greatly promoted the rapid development of the subjects above. In 1974, Ahmed, Natarajan, and Rao [1] proposed discrete cosine transform defined on real number field, it can be called DCT-II-E or DCT-III-E [5]. In 1974, Jain [3] proposed discrete fourier transform DCT-IV-E, and in 1983, Wang and Hunt [5], [6] proposed discrete cosine transform DCT-I-E. The discrete cosine transform has been used in frequency spectrum analysis, data compression, convolution computation and information hiding. Its theory and algorithms have received much attention for the last two decades [5], [6], [2], [4].

It is demonstrated that the performance of discrete cosine transform can well approximate to ideal K-L transform (Karhunen-Loeve Transform)[1]. K-L transform was proposed to dealing with a class of extensive stochastic image. After the image being transformed with K-L transform, the image restored from the result is the best approximation to the original image in the statistical sense. Moreover, for the common data model of Markov process, when the correlation coefficient $r = 1$, K-L transform is degraded to the classic DCT transform. In fact, Real-world images are neither stationary nor Markovian. They have different textures and structures, important image structures like edges, arris and lines extend over large distances in the image [4]. Therefore, different types of transformation are desirable to meet the different applications.

Actually, the discrete cosine transform (DCT-III-E) can be generalized to the unified form with parameters p, q and r, as follows:

$$X(k) = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos \frac{k(4qn + r)p\pi}{2N}, \quad k = 0, 1, \cdots, N-1.$$

There are many new meaningful transforms, such as when $(p, q, r) = (1, 1, 1)$, the new transform is orthogonal. Generally, when $\gcd(pq, N) = 1$, $\gcd(pr, 2) = 1$, where p, q and r are positive integers, the new transform is orthogonal.

In a similar way, we generalize discrete cosine transforms DCT-II-E and DCT-IV-E. Furthermore, a new type of discrete cosine transform, a new type of discrete sine transform and a new type of discrete sine-cosine transform are proposed, and their orthogonality are proved.

2. Generalized discrete cosine transform

Let $\{x(n); n = 0, 1, 2, \cdots, N\}$ be a vector of real numbers. The definitions of four common types of discrete cosine transform [5] are given as follows:

DCT-I-E:

$$X(k) = \sqrt{\frac{2}{N}} \alpha(k) \sum_{n=0}^{N} \alpha(n) x(n) \cos \frac{kn\pi}{N}, \quad k = 0, 1, \cdots, N,$$

where

$$\alpha(k) = \begin{cases} \frac{1}{\sqrt{2}} & k = 0 \text{ or } N \\ 1 & \text{else} \end{cases}$$

$$\alpha(n) = \begin{cases} \frac{1}{\sqrt{2}} & n = 0 \text{ or } N \\ 1 & \text{else} \end{cases}$$

DCT-II-E:

$$X(k) = \sum_{n=0}^{N-1} \alpha(n) x(n) \cos \frac{(2k+1)n\pi}{2N}, \quad k = 0, 1, \cdots, N-1,$$

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where $\alpha(n) = \begin{cases} \sqrt{\frac{1}{N}} & n = 0 \\ \sqrt{\frac{2}{N}} & \text{else} \end{cases}$

DCT-III-E:

$$X(k) = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos \frac{k(2n+1)\pi}{2N}, \quad k = 0, 1, \cdots, N-1,$$

where $\alpha(k) = \begin{cases} \sqrt{\frac{1}{N}} & k = 0 \\ \sqrt{\frac{2}{N}} & \text{else} \end{cases}$

DCT-IV-E:

$$X(k) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} x(n) \cos \frac{(2k+1)(2n+1)\pi}{2N}, \quad k = 0, 1, \cdots, N-1.$$

Actually, the discrete cosine transform (DCT-III-E) can be generalized to the following unified form with parameters p, q and r:

$$X(k) = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos \frac{k(4qn+r)p\pi}{2N}, \quad k = 0, 1, \cdots, N-1. \tag{1}$$

We now prove that when $\gcd(pq, N) = 1$, $\gcd(pr, 2) = 1$, where p, q and r are positive integers, transform (1) is orthogonal.

Transform (1) can be written into matrix form:

$$X(N) = C(N) \cdot x(N)$$

where $X(N), x(N)$ are column vectors of length N,

$$C(N) = \sqrt{\tfrac{2}{N}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ \cos\frac{rp\pi}{2N} & \cos\frac{(4q+r)p\pi}{2N} & \cos\frac{(8q+r)p\pi}{2N} & \cdots & \cos\frac{[4q(N-1)+r]p\pi}{2N} \\ \cos\frac{2rp\pi}{2N} & \cos\frac{2(4q+r)p\pi}{2N} & \cos\frac{2(8q+r)p\pi}{2N} & \cdots & \cos\frac{2[4q(N-1)+r]p\pi}{2N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cos\frac{\eta r}{2N} & \cos\frac{\eta(4q+r)}{2N} & \cos\frac{\eta(8q+r)}{2N} & \cdots & \cos\frac{\eta[4q(N-1)+r]}{2N} \end{pmatrix}$$

where $\eta = (N-1)p\pi$

To prove that the new transform (1) is orthogonal, or equivalently, to prove that $C(N)$ is an orthogonal matrix. For the sake of simplicity, the coefficients $\sqrt{\frac{2}{N}}$ and $\frac{1}{\sqrt{2}}$ is omitted.

Let $0 \le k_1 < k_2 < N$, the inner product of the k_1th row and k_2th row of $C(N)$ is that

$$\begin{aligned} & \cos\frac{k_1 rp\pi}{2N} \cos\frac{k_2 rp\pi}{2N} \\ & + \cos\frac{k_1(4q+r)p\pi}{2N} \cos\frac{k_2(4q+r)p\pi}{2N} + \cdots \\ & + \cos\frac{k_1[4q(N-1)+r]p\pi}{2N} \cos\frac{k_2[4q(N-1)+r]p\pi}{2N} \\ = \; & \frac{1}{2}\{\cos\frac{(k_1-k_2)rp\pi}{2N} + \cos\frac{(k_1+k_2)rp\pi}{2N} \\ & + \cos\frac{(k_1-k_2)(4q+r)p\pi}{2N} + \cos\frac{(k_1+k_2)(4q+r)p\pi}{2N} \\ & + \cdots + \cos\frac{(k_1-k_2)[4q(N-1)+r]p\pi}{2N} \\ & + \cos\frac{(k_1+k_2)[4q(N-1)+r]p\pi}{2N}\} \end{aligned} \tag{2}$$

where the above equality follows by

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

Since $e^{ix} = \cos x + i \sin x$, we consider the real part of the following complex number:

$$\begin{aligned} & e^{i\frac{krp\pi}{2N}} + e^{i\frac{k(4q+r)p\pi}{2N}} + \cdots + e^{i\frac{k[4q(N-1)+r]p\pi}{2N}} \\ = \; & \frac{e^{i\frac{krp\pi}{2N}}}{1 - e^{i\frac{2kqp\pi}{N}}}(1 - e^{i\frac{2kqp\pi N}{N}}) \end{aligned}$$

Note that $\gcd(pq, N) = 1$ and $\gcd(pr, 2) = 1$. When $0 < k < 2N$, if $1 - e^{i\frac{2kpq\pi}{N}} = 0$, then $k = N$, the real part of

$$e^{i\frac{krp\pi}{2N}} + e^{i\frac{k(4q+r)p\pi}{2N}} + \cdots + e^{i\frac{k[4q(N-1)+r]p\pi}{2N}}$$

is zero.

If $1 - e^{i\frac{2kpq\pi}{N}} \neq 0$, then

$$e^{i\frac{krp\pi}{2N}} + e^{i\frac{k(4q+r)p\pi}{2N}} + \cdots + e^{i\frac{k[4q(N-1)+r]p\pi}{2N}} = 0$$

From equality (2), when $0 \le k_1 < k_2 < N$, the inner product of the k_1th row and k_2th row of $C(N)$ is zero. We know that $C(N)$ is an orthogonal matrix.

Let $0 < k_1 < N$, the inner product of the k_1th row and itself of $C(N)$ is that

$$\begin{aligned} & \frac{2}{N}\{(\cos\frac{k_1 rp\pi}{2N})^2 + [\cos\frac{k_1(4q+r)p\pi}{2N}]^2 + \cdots \\ & + [\cos\frac{k_1[4q(N-1)+r]p\pi}{2N}]^2\} \\ = \; & 1 + \frac{1}{N}\{\cos\frac{k_1 rp\pi}{N} + \cos\frac{k_1(4q+r)p\pi}{N} \\ & + \cdots + \cos\frac{k_1[4q(N-1)+r]p\pi}{N}\} \end{aligned} \tag{3}$$

Note that

$$\begin{aligned} & e^{i\frac{krp\pi}{N}} + e^{i\frac{k(4q+r)p\pi}{N}} + \cdots + e^{i\frac{k[4q(N-1)+r]p\pi}{N}} \\ = \; & \frac{e^{i\frac{krp\pi}{N}}}{1 - e^{i\frac{4kqp\pi}{N}}}(1 - e^{i\frac{4kqp\pi N}{N}}) \end{aligned}$$

and $\gcd(pq, N) = 1$ and $\gcd(pr, 2) = 1$.

When $0 < k < N$, if $1 - e^{i\frac{4kpq\pi}{N}} = 0$, then $2k = N$, the real part of

$$e^{i\frac{krp\pi}{2N}} + e^{i\frac{k(4q+r)p\pi}{N}} + \cdots + e^{i\frac{k[4q(N-1)+r]p\pi}{N}}$$

is zero.

If $1 - e^{i\frac{4kpq\pi}{N}} \neq 0$, then

$$e^{i\frac{krp\pi}{N}} + e^{i\frac{k(4q+r)p\pi}{N}} + \cdots + e^{i\frac{k[4q(N-1)+r]p\pi}{N}} = 0$$

From equality (3), when $0 < k_1 < N$, the inner product of the k_1th row and itself of $C(N)$ is 1. We know that the product of $C(N)$ and its transpose is an identity matrix. Thus, it is easy to get the inverse transform of transform (1). We omit it here.

We remark the following three points:

(I) The transpose of $C(N)$ is an orthogonal matrix, so the transform below is also orthogonal.

$$X(k) = \sum_{n=0}^{N-1} \alpha(n) x(n) \cos \frac{n(4qk+r)p\pi}{2N}, \quad k = 0, 1, \cdots, N-1.$$

where $\gcd(pq, N) = 1$, $\gcd(pr, 2) = 1$, p, q and r are positive integers.

(II) We can obtain a generalized DCT-IV-E transform, as follows:

$$X(k) = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos \frac{(2k+1)(4qn+r)p\pi}{4N}, \quad k = 0, 1, \cdots, N-1.$$

where $\gcd(pq, N) = 1$, $\gcd(pr, 2) = 1$, p, q and r are positive integers.

(III) In a similar way, we can generalize discrete sine transforms, such as DST-II-E, DST-III-E, DST-IV-E.

3. A new type of discrete cosine transform

Let $\{x(n); n = 0, 1, 2, \cdots, N-1\}$ be a vector of real numbers. We define a new form of discrete cosine transform, as follows:

$$X(k) = \sqrt{\frac{4}{2N-1}} \alpha(k) \sum_{n=0}^{N-1} \alpha(n) x(n) \cos \frac{(2k+1)(2n+1)\pi}{2N-1}, \quad k = 0, 1, \cdots, N-1, \tag{4}$$

where

$$\alpha(k) = \begin{cases} \frac{1}{\sqrt{2}} & k = N-1 \\ 1 & \text{else} \end{cases}$$

$$\alpha(n) = \begin{cases} \frac{1}{\sqrt{2}} & n = N-1 \\ 1 & \text{else} \end{cases}$$

The transform above can be written into matrix form:

$$X(N) = C(N) \cdot x(N)$$

where $X(N)$, $x(N)$ are column vectors of length N,

$$C(N) = \sqrt{\frac{4}{2N-1}} \begin{pmatrix} \cos\frac{\pi}{2N-1} & \cos\frac{3\pi}{2N-1} & \cdots & \cos\frac{(2N-3)\pi}{2N-1} & -\frac{1}{\sqrt{2}} \\ \cos\frac{3\pi}{2N-1} & \cos\frac{9\pi}{2N-1} & \cdots & \cos\frac{3(2N-3)\pi}{2N} & -\frac{1}{\sqrt{2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cos\frac{(2N-3)\pi}{2N-1} & \cos\frac{(2N-3)3\pi}{2N-1} & \cdots & \cos\frac{(2N-3)(2N-3)\pi}{2N-1} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \cdots & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}$$

For the sake of simplicity, the coefficient $\sqrt{\frac{4}{2N-1}}$ is omitted in the following discussions. Let $0 \leq k_1 < k_2 < N-1$, the inner product of the k_1th row and k_2th row of $C(N)$ is that

$$\begin{aligned} & \cos\frac{(2k_1+1)\pi}{2N-1} \cos\frac{(2k_2+1)\pi}{2N-1} \\ & + \cos\frac{(2k_1+1)3\pi}{2N-1} \cos\frac{(2k_2+1)3\pi}{2N-1} + \cdots \\ & + \cos\frac{(2k_1+1)(2N-3)\pi}{2N-1} \cos\frac{(2k_2+1)(2N-3)\pi}{2N-1} + \frac{1}{2} \\ = \; & \frac{1}{2}[\cos\frac{2(k_1-k_2)\pi}{2N-1} + \cos\frac{2(k_1+k_2+1)\pi}{2N-1} \\ & + \cos\frac{2(k_1-k_2)3\pi}{2N-1} + \cos\frac{2(k_1+k_2+1)3\pi}{2N-1} + \cdots \\ & + \cos\frac{2(k_1-k_2)(2N-3)\pi}{2N-1} \\ & + \cos\frac{2(k_1+k_2+1)(2N-3)\pi}{2N-1}] + \frac{1}{2} \end{aligned} \tag{5}$$

Since $e^{ix} = \cos x + i \sin x$, we consider the real part of the following complex number:

$$\begin{aligned} & e^{i\frac{2k\pi}{2N-1}} + e^{i\frac{2k3\pi}{2N-1}} + \cdots + e^{i\frac{2k(2N-3)\pi}{2N-1}} \\ = \; & \frac{e^{i\frac{2k\pi}{2N-1}}}{1 - e^{i\frac{4k\pi}{2N-1}}} (1 - e^{i\frac{4k(N-1)\pi}{2N-1}}) \end{aligned}$$

When $0 < k < 2N-1, 1 - e^{i\frac{4k\pi}{2N-1}} \neq 0$.

Note that

$$\begin{aligned} & e^{i\frac{4k\pi(N-1)}{2N-1}} \\ = \; & \cos(2k\pi - \frac{2k\pi}{2N-1}) + i \sin(2k\pi - \frac{2k\pi}{2N+1}) \\ = \; & \cos(\frac{2k\pi}{2N-1}) - i \sin(\frac{2k\pi}{2N-1}) \end{aligned}$$

By setting $\alpha = \frac{2k\pi}{2N-1}$, we have

$$\begin{aligned} & \frac{e^{i\frac{2k\pi}{2N-1}}}{1 - e^{i\frac{4k\pi}{2N-1}}} \\ = \; & \frac{\cos\alpha + i \sin\alpha}{1 - \cos 2\alpha - i \sin 2\alpha} \\ = \; & \frac{\cos\alpha + i \sin\alpha}{2 \sin\alpha(\sin\alpha - i \cos\alpha)} \\ = \; & \frac{(\cos\alpha + i \sin\alpha)(\sin\alpha + i \cos\alpha)}{2 \sin\alpha(\sin\alpha - \cos\alpha)(\sin\alpha + i \cos\alpha)} \\ = \; & \frac{i}{2 \sin\alpha} \end{aligned}$$

Thus, the real part of

$$e^{i\frac{2k\pi}{2N-1}} + e^{i\frac{2k3\pi}{2N-1}} + \cdots + e^{i\frac{2k(2N-3)\pi}{2N-1}}$$

is $\frac{i}{2\sin(\alpha)} i \sin(\alpha) = -\frac{1}{2}$.

From equality (5), for $0 \leq k_1 < k_2 < N-1$, the inner product of the k_1th row and k_2th row of $C(N)$ is

$$\frac{1}{2}(-\frac{1}{2}-\frac{1}{2})+\frac{1}{2}=0$$

By setting $\alpha = \frac{(2k+1)\pi}{2N-1}$, we have

$$\begin{aligned} & e^{i\frac{(2k+1)\pi}{2N-1}} + e^{i\frac{(2k+1)3\pi}{2N-1}} + \cdots + e^{i\frac{(2k+1)(2N-3)\pi}{2N-1}} \\ = & \frac{e^{i\frac{(2k+1)\pi}{2N-1}}}{1-e^{i\frac{2(2k+1)\pi}{2N-1}}}(1-e^{i\frac{2(2k+1)(N-1)\pi}{2N-1}}) \\ = & \frac{i}{2\sin\alpha}(1+\cos\alpha - i\sin\alpha) \\ = & \frac{1}{2}+\frac{i}{2\sin\alpha}(1+\cos\alpha) \end{aligned}$$

Now, for $0 \leq k_1 < N-1$, the inner product of the k_1th row and $(N-1)$st row of $C(N)$ is

$$\frac{1}{2}(-\frac{1}{\sqrt{2}})+(-\frac{1}{2})(-\frac{1}{\sqrt{2}})=0$$

Thus, $C(N)$ is an orthogonal matrix. Equivalently, transform (4) is orthogonal.

It is easy to know that the product of $C(N)$ and its transpose is an identity matrix. Thus, we can get the inverse transform of transform (4). We omit it here.

Similarly, we can obtain a new form of discrete sine transform as follows:

$$X(k) = \sqrt{\frac{4}{2N+1}} \sum_{n=0}^{N-1} x(n) \sin \frac{(2k+1)(2n+1)\pi}{2N+1}, \quad k = 0, 1, \cdots, N-1.$$

And a new form of discrete sine-cosine transform as follows:

$$X(k) = \sqrt{\frac{2}{2N+1}} \sum_{n=0}^{2N} x(n) \cos \frac{(2k+1)(2n+1)\pi}{2N+1}, \quad k = 0, 1, \cdots, N-1;$$

$$X(N) = -\sqrt{\frac{1}{2N+1}} \sum_{n=0}^{2N} x(n);$$

$$X(N+1+k) = \sqrt{\frac{2}{2N+1}} \sum_{n=0}^{2N} x(n) \sin \frac{(2k+1)(2n+1)\pi}{2N+1}, \quad k = 0, 1, \cdots, N-1.$$

The orthogonality of above transforms follows from an analysis similar to that of new discrete cosine transform (4).

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