Image restoration for space invariant pointspread functions

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We show how to apply a truncated eigensystem expansion in the solution of image restoration problems for the case of space invariant point spread functions. The solution is obtained directly from the system of linear equations, which result from the discretization of the Fredholm integral equation of the first kind. Fast Fourier transform techniques are used in obtaining this solution. A procedure is devised to estimate the rank of the coefficient matrix that gives a best or near best solution. It is demonstrated that this algorithm compares favorably with other existing methods. Numerical results using spatially separable point spread functions are given.

I. Introduction

In the linear model, the image restoration problem is described by the Fredholm integral equation of the first kind. The discretization of this equation gives a system of linear equations of the form

$$\mathbf{g} = [H]\mathbf{f} + \mathbf{e}, \tag{1}$$

where $\mathbf{g}$ is a stacked m vector representing the known or given degraded image, $\mathbf{f}$ is a stacked n vector representing the unknown or undegraded image, and $\mathbf{e}$ is a stacked m vector representing the noise term. $[H]$ is an $m \times n$ matrix resulting from the discretization of the point spread function in the integral equation. If the known image is represented by an $I \times J$ matrix, $m = I \times J$. Also if the unknown image is represented by a $K \times L$ matrix, $n = K \times L$.

In the present work, we consider the case where matrix $[H]$ in Eq. (1) is the discretization of a space invariant point spread function (SIPSF).

Equation (1) in general is ill-posed in the sense that small changes in vector $\mathbf{g}$ may cause large changes in the solution vector $\mathbf{f}$. There are two main approaches for obtaining numerically stable and physically acceptable solutions to Eq. (1), using direct (noniterative) methods.

The first approach is demonstrated by Phillips[1] and Twomey,[2] who obtained a constrained (regularized) least-squares solution to Eq. (1) by minimizing the second derivative of the unknown vector $\mathbf{f}$. For the SIPSF case, Hunt[3] applied this technique for restoring noisy degraded images.

The second approach is demonstrated by Baker *et al.*,[4] Hanson,[5] and Varah.[6] For the case of real square symmetric matrix $[H]$, Baker *et al.*[4] obtained a least-squares solution to system (1) in terms of a truncated eigensystem expansion, that is, by discarding the terms corresponding to the smaller eigenvalues, which contribute to the large and highly oscillating components of $\mathbf{f}$.

For a general matrix $[H]$, Hanson[5] and Varah[6] obtained a least-squares solution to Eq. (1) in terms of a truncated singular value decomposition (SVD) expansion. Huang and Narendra[7] used this method for restoring noisy degraded images. A major drawback to using the SVD expansion is the high cost, in terms of the number of arithmetic operations, of computing the singular value system even for moderately small size matrices.

We consider now an efficient method of solution to Eq. (1) for the case of SIPSF. It is shown by Andrews and Hunt[8] for the case of SIPSF that matrix $[H]$ has a structure similar to that of a circulant matrix. Circulant matrices have special useful properties[9,10] as will be discussed. If $[H]$ is replaced by its approximate circulant matrix, the border regions are the only regions in the unknown image that are affected as a result. On the other hand, the obtained system of linear equations may be solved efficiently by using the fast Fourier transform (FFT) techniques (see Chap. 7 in Ref. 8).

We here pursue the approach of Baker *et al.* To

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adopt this approach, matrix $[H]$ in Eq. (1) should be square and symmetric, i.e., have real eigenvalues. If $[H]$ is not a square symmetric matrix, we premultiply Eq. (1) by $[H]^T$, where the superscript refers to the transpose. We get

$$[H]^T\mathbf{g} = [H]^T[H]\mathbf{f} + [H]^T\mathbf{e}. \tag{1'}$$

From theorem 1 in Ref. 12 or by direct substitution in Eqs. (1) and (1′) of the SVD of matrix $[H]$, the least-squares solution of Eq. (1′) is itself the least-squares solution of Eq. (1). Therefore, without loss of generality, assume that matrix $[H]$ in Eq. (1) is symmetric and thus has real eigenvalues.

In the present work, we replace matrix $[H]$ in Eq. (1) by its approximate circulant matrix, $[H_c]$ say. This is given by Eq. (9). Then we obtain a least-squares solution $\mathbf{f}$ to Eq. (9) in terms of a truncated eigensystem expansion following the approach of Baker *et al.*[4] Our calculation is simplified by taking advantage of the special properties of the circulant matrices that enable us to use the FFT techniques. This is given in Secs. II and III. In Sec. IV, the procedure for the solution is given. In Sec. V, a practical method is described for estimating the rank of matrix $[H_c]$, which gives a best or near best solution vector $\mathbf{f}$. In Sec. VI, analysis is made for the case of spatially separable point spread functions. In Sec. VII, two computer simulation examples are presented using spatially separable point spread functions. We demonstrate in Sec. VIII by comparing the arithmetic operations counts that the present algorithm compares favorably with other existing methods for the case of SIPSF.

II. Circulant Convolution

Let $[H_c]$ be a $N \times N$ circulant matrix given by[9,10]

$$[H_c] = \begin{bmatrix} h_0 & h_1 & h_2 & \dots & h_{N-1} \\ h_{N-1} & h_0 & h_1 & \dots & h_{N-2} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ h_1 & h_2 & h_3 & \dots & h_0 \end{bmatrix} . \tag{2}$$

It is known that matrix $[H_c]$ may be factorized into

$$[H_c] = [W][D][W]^{-1}, \tag{3}$$

where $[W]$ is a $N \times N$ matrix whose columns are the eigenvectors of $[H_c]$. $[W]^{-1}$ is the inverse of $[W]$, and $[D]$ is a diagonal $N \times N$ matrix whose diagonal elements are the eigenvalues of $[H_c]$.

The kjth element of $[W]$ is given by

$$\{W\}_{kj} = \exp\left(\frac{2\pi i}{N} kj\right), \qquad k,j = 0,1,\ldots,N-1, \tag{4a}$$

and the kjth element of $[W]^{-1}$ is

$$\{W^{-1}\}_{kj} = \frac{1}{N}\exp-\left(\frac{2\pi i}{N} kj\right), \qquad k,j = 0,1,\ldots,N-1. \tag{4b}$$

The diagonal elements of $[D]$ are

$$d_k = h_0 + h_1 r_k^1 + h_2 r_k^2 + \ldots + h_{N-1} r_k^{N-1}, \qquad k = 0,1,\ldots,N-1, \tag{5}$$

where $i = (-1)^{1/2}$, and r_k is a Nth root of unity. Also by considering Eqs. (4), the diagonal elements $\{d_k\}$ in Eq. (5) are the FFT of the first row of $[H_c]$ in Eq. (2).[10]

Let $[W]^t$ be the complex conjugate transpose of $[W]$. Then from Eqs. (4a) and (4b),

$$[W]^{-1} = (1/N)[W]^t, \tag{6}$$

and Eq. (3) becomes

$$[H_c] = (1/N)[W][D][W]^t. \tag{7}$$

If matrix $[H_c]$ in Eq. (2) is symmetric so that $h_j = h_{N-j}$, $1 \le j \le (N-1)$, the eigenvalues $[d_k]$ in Eq. (5) are real and are given by

$$d_k = h_0 + h_1(r_k^1 + r_k^{N-1}) + h_2(r_k^2 + r_k^{N-2}) + \ldots + L, \qquad k = 0,1,\ldots,N-1,$$

that is,

$$d_k = h_0 + 2h_1 \cos\left(\frac{2\pi k}{N}\right) + 2h_2 \cos\left(\frac{4\pi k}{N}\right) + \ldots + L, \qquad k = 0,1,\ldots,N-1, \tag{8}$$

where $L = 2h_s \cos[(2\pi ks)/N]$ if $N = 2s + 1$, and $L = (-1)^s h_s$ when $N = 2s$.

It is interesting to observe that, for a symmetric circulant matrix $[H_c]$, for $N = 2s + 1$, $(N-1)$ of the eigenvalues occurs in pairs, that is, $d_j = d_{N-j}$, $j = 1,2,\ldots, s$. Also for $N = 2s$, $(N-2)$ of the eigenvalues occurs in pairs, i.e., $d_j = d_{N-j}$, $j = 1,2,\ldots, s-1$. The corresponding eigenvectors of each two equal eigenvalues are the complex conjugate of one another. For this reason, in Sec. III, we reduce the rank of matrix $[H_c]$ by two at a time by replacing the two smallest eigenvalues in absolute value by zeros, or we reduce the rank by one if d_s (for even N) is the smallest eigenvalue. The reduction of the rank is done in this way so that the obtained matrix is real.

III. Truncated Eigensystem Expansion

Assume from now on that Eq. (1) is replaced by

$$\mathbf{g} = [H_c]\mathbf{f} + \mathbf{e}, \tag{9}$$

where $[H_c]$ is a square symmetric circulant matrix of proper size, $N \times N$ say, and each $\mathbf{g}$, $\mathbf{f}$, and $\mathbf{e}$ is a N vector. The parameter N should be a power of 2 so that we may use the radix-2 FFT techniques (see example 2 in Sec. VII).

Following the approach of Baker *et al.*,[4] matrix $[H_c]$ will be approximated by another circulant matrix of smaller rank, matrix $[\overline{H}_c]$ of rank r say. This is done as follows. We retain in $[D]$ in Eq. (7) the largest r diagonal elements in absolute value $r \le N$ and replace the remaining $(N - r)$ diagonal elements by zeros. The rank r will be estimated in Sec. V.

Therefore instead we solve the equation

$$\mathbf{g} = [\tilde{H}_c]\mathbf{f} + \boldsymbol{\rho}, \tag{10}$$

where $\boldsymbol{\rho}$ is the residual vector, and matrix $[\overline{H}_c]$ is given by

$$[\overline{H}_c] = (1/N)[W][\overline{D}][W]^t. \tag{11}$$

$[\overline{D}]$ in Eq. (11) is itself matrix $[D]$ in Eq. (7) with its $(N - r)$ smallest diagonal elements in absolute value replaced by zeros.

The least-squares solution of Eq. (10) is given by

$$\hat{\mathbf{f}} = [\overline{H}_c]^+ \mathbf{g}, \tag{12}$$

where $[\overline{H}_c]^+$ is the Moore-Penrose pseudo-inverse[11] of matrix $[\overline{H}_c]$.

Let $w(k)$ and $w^t(k)$ be the kth columns of $[W]$ and of $[W]^t$, respectively. Each of the columns $\{w(k)\}$ and $\{w^t(k)\}$ are mutually orthogonal. Also $\|w(k)\|_2^2 = N$, and $\|w^t(k)\|_2^2 = N$, $k = 0,1,\ldots,(N-1)$. Hence from Eq. (11) the pseudo-inverse $[\overline{H}_c]^+$ is given by[11,12]

$$[\overline{H}_c]^+ = (1/N)[W][S][W]^t, \tag{13}$$

where $[S] = [\overline{D}]^+$, and $[\overline{D}]^+$ is the pseudo-inverse of matrix $[\overline{D}]$. $[S]$ is a diagonal matrix whose diagonal elements are $s_k = 1/d_k$ if $d_k \neq 0$, and $s_k = 0$ if $d_k = 0$, $k = 0,1,\ldots,N-1$. In practice, we take $s_k = 1/d_k$ if $|d_k| > Eps$ and $s_k = 0$ if $|d_k| \leq Eps$, where Eps is a specified tolerance. For the IBM 370 computer, where the round-off level for single precision calculation is about 10^{-6}, Eps is usually taken 10^{-4}. Hence from Eqs. (6) and (13), the least squares solution (12) of Eq. (10) is given by[12]

$$\hat{\mathbf{f}} = [W][S][W]^{-1}\mathbf{g}. \tag{14}$$

To further reduce the rank of matrix $[\overline{H}_c]$, we simply replace the smallest in absolute value, two (or one) diagonal elements of $[\overline{D}]$ at hand by zeros, that is, we replace the largest in absolute value, two (or one) diagonal elements of $[S]$ by zeros (see the last paragraph in Sec. II).

IV. Procedure of Solution

The following steps are followed in the solution of Eq. (10):

(a) The diagonal elements of matrix $[D]$ in Eq. (5) are calculated once and for all. These are the FFT of the first row of matrix $[H_c]$ in Eq. (2) and are given by Eq. (8) for symmetric $[H_c]$. The diagonal elements of $[S]$ are then calculated and ordered in an increasing manner.

(b) Vector $\mathbf{u} = [W]^{-1}\mathbf{g}$ in Eq. (14) is calculated only once. Again it is the FFT of vector $\mathbf{g}$.

(c) The vector $\mathbf{v} = [S]\mathbf{u}$ is calculated.

(d) Finally, $\mathbf{f} = [W]\mathbf{v}$ is calculated. This is also a FFT.

Assume that we have obtained the solution vector $\hat{\mathbf{f}}$ for a large value of rank r. If $\hat{\mathbf{f}}$ is not smooth enough, we decrease the rank of matrix $[\overline{H}_c]$ and obtain a new solution. This is done by replacing by zeros, the largest in absolute value, two (or one) diagonal elements of $[S]$ and proceeding to step (c). In effect, since $[S]$ is a diagonal matrix, we replace by zeros the elements of vector $\mathbf{u}$ in (c), which correspond to the largest two (or one) diagonal elements of $[S]$. We repeat this process until the obtained solution vector is smooth enough.

In Sec. V, a procedure is described for determining the rank of matrix $[\overline{H}_c]$, which gives a best or near best restored image.

V. Optimum Value of the Rank

The approaches of Phillips,[1] Twomey,[2] and Baker *et al.*[4] are not complete. In the former, a regularization parameter needs to be estimated. In the approach of Baker *et al.*, we need to determine the rank of matrix $[\overline{H}_c]$, which gives a best or near best solution vector $\hat{\mathbf{f}}$.

Hunt[3] described a practical procedure for estimating the regularization parameter. His method is based on knowledge of the unbiased estimate of the variance $S^2(e)$ and of the mean $\mu(e)$ of the noise vector $\mathbf{e}$ in Eq. (9). Using our notation, the relation between $S^2(e)$, $\mu(e)$, and $\mathbf{e}$ derived by Hunt[3] is

$$\mathbf{e}^2 = (N-1)S^2(e) + \mu^2(e). \tag{15}$$

It is assumed that $S^2(e)$ and $\mu^2(e)$ are known, and thus $\mathbf{e}^2$ is estimated.

In Hunt's method, the regularization parameter is increased or decreased so that the square of the residual for the obtained restored image equals $\mathbf{e}^2$.

In our method, for a sufficiently large value of the rank r of matrix $[\overline{H}_c]$, the elements of the solution vector $\hat{\mathbf{f}}$ have large values and are highly oscillating. Reducing the rank of $[\overline{H}_c]$ has a dampening effect on the solution $\hat{\mathbf{f}}$. In this section we describe an analogous method for estimating the best rank of matrix $[\overline{H}_c]$. It is summarized as follows.

If the calculated solution $\hat{\mathbf{f}}$ of Eq. (10) equals the ideal solution $\mathbf{f}$ of Eq. (9), $\rho^2 = \mathbf{e}^2$, where ρ is defined in Eq. (10). If $\rho^2 > \mathbf{e}^2$, we decrease the rank of $[\overline{H}_c]$ and vice versa. This method works well for the cases of low and moderate blur (see the examples for Figs. 2 and 4 of Sec. VII). For the case of severe distortion, we found out that noise persists in the restored images, and some measure should be taken to get rid of it.

Our procedure for estimating a best rank for $[\overline{H}_c]$ for the cases of low and moderate distortion is as follows.

Calculate the solution vectors of Eq. (10) for two consecutive values of the rank of $[\overline{H}_c]$, r_1, and r_2, say, $r_2 < r_1$. We start with a sufficiently large value of r_1. Let the two solutions be $\hat{\mathbf{f}}_1$ and $\hat{\mathbf{f}}_2$, respectively.

(a) Each element $\hat{f}_1^{(i)}$ of the calculated vector $\hat{\mathbf{f}}_1$ should be within practical limits. For example, each should satisfy the following inequalities: $-0.3f_{\max} \leq \hat{f}_1^{(i)} \leq 1.3f_{\max}$, $i = 1,2,\ldots,N$, where $f_{\max}$ is the maximum pixel value in the real image. Usually $f_{\max} = 255$.

If the above inequalities are not satisfied, go to (c); otherwise go to (b).

(b) Calculate the residuals ρ_1 and ρ_2 of Eq. (10), which correspond to the solutions $\hat{\mathbf{f}}_1$ and $\hat{\mathbf{f}}_2$, respectively. If $|\rho_1^2 - \mathbf{e}^2| < |\rho_2^2 - \mathbf{e}^2|$, rank r_1 is accepted, and the solution of the problem is $\mathbf{f} = \mathbf{f}_1$; otherwise go to (c).

(c) Choose a further smaller rank $r_3 < r_2$. Replace r_1 by r_2, r_2 by r_3, and go to (a).

In order not to lose too much information from the image $[G]$, a lower limit for the rank of matrix $[\overline{H}_c]$ should be set. This depends on the comparative size of the eigenvalues of matrix $[H_c]$ (see the solved example in Ref. 14).

When a solution $\hat{\mathbf{f}}$ is found, we examine its components $\hat{f}^{(i)}$, $i = 1,2,\ldots,N$. If $\hat{f}^{(i)} < 0$, we set $\hat{f}^{(i)} = 0$, and if $\hat{f}^{(i)} > f_{\max}$, we set $\hat{f}^{(i)} = f_{\max}$.

The value of ρ may be calculated in an economical manner as follows. From Eqs. (14) and (11), $[\overline{H}_c]\hat{f} = [W][\overline{D}][S][W]^{-1}g$, or

$$[\overline{H}_c]\hat{\mathbf{f}} = [W][\overline{I}][W]^{-1}\mathbf{g}, \tag{16}$$

where from the definitions of matrices $[\overline{D}]$ and $[S]$, $[\overline{I}]$ is a diagonal matrix where each diagonal element is either one or zero. Hence in Eq. (16), to get $[\overline{I}][W]^{-1}\mathbf{g}$, the elements of vector $\mathbf{u} = [W]^{-1}\mathbf{g}$ are multiplied either by one or zero according to the value of the diagonal elements of $[\overline{I}]$. Let vector $\overline{\mathbf{u}} = [\overline{I}]\mathbf{u}$. Then $[W]\overline{\mathbf{u}}$ is obtained as a FFT, and ρ is calculated from Eq. (10).

For the case of severe blur, the procedure is much simpler. We reduce the rank of matrix $[\overline{H}_c]$ too much. We do that by giving larger value to the parameter *Eps* defined following Eq. (13). We replace by zero every eigenvalue of matrix $[\overline{H}_c]$ whose absolute value is less than *Eps*. If the noise still persists in the restored image, the rank is further reduced by further increasing the value of *Eps*.

VI. Separable Point Spread Functions

To facilitate the understanding of the present method for practical problems, in this section analysis is made for the case where the point spread function is spatially separable, that is, matrix $[H]$ in Eq. (1) may be written in the form

$$[H] = [A] \otimes [B],$$

where $\otimes$ is the Kronecker product operator.[9] $[A]$ and $[B]$ are matrices of dimensions $I \times K$ and $J \times L$, respectively (see Ref. 8, p. 70 or Ref. 13, p. 202). In this case Eq. (1) may be written in the form[9]

$$[G] = [A][F][B]^T + [E], \tag{17}$$

where $[G]$, $[A]$, $[F]$, $[B]$, and $[E]$ are matrices of respective dimensions $I \times J$, $I \times K$, $K \times L$, $J \times L$, and $I \times J$. Without loss of generality, we assume that matrices $[A]$ and $[B]$ each are square symmetric matrices. If not, we premultiply Eq. (17) by $[A]^T$ and or postmultiply it by $[B]$. Equation (17) is then approximated by replacing $[A]$ and $[B]$, respectively, by their circulant matrices $[A_c]$ and $[B_c]$. We get

$$[G] = [A_c][F][B_c]^T + [E]. \tag{17'}$$

Let $[X] = [F][B_c]^T$. Then Eq. (17′) is given by

$$[G] = [A_c][X] + [E]. \tag{18}$$

Let $\mathbf{g}_j$, $\mathbf{x}_j$, and $\mathbf{e}_j$ denote the jth column of $[G]$, $[X]$, and $[E]$, respectively. Then from Eq. (18) we have

$$\mathbf{g}_j = [\overline{A}_c]\mathbf{x}_j + \rho_j, \qquad j = 1,\ldots,J. \tag{19}$$

The J equations (19) are then solved one by one in such a way that the obtained solution is smooth enough. This is done by using the techniques described in Secs. IV and V, that is, $x_j = [\overline{A}_c]^+ g_j$, where the rank of matrix $[\overline{A}_c]$ may differ for different j.

Once matrix $[X]$ is obtained, the equation

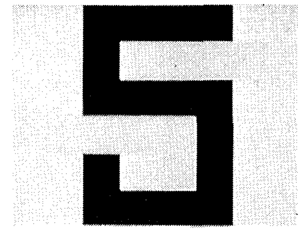


Fig. 1. The 16×16 matrix representing character 5. Each point inside the character is given pixel value 7, and each point outside is given value 0.

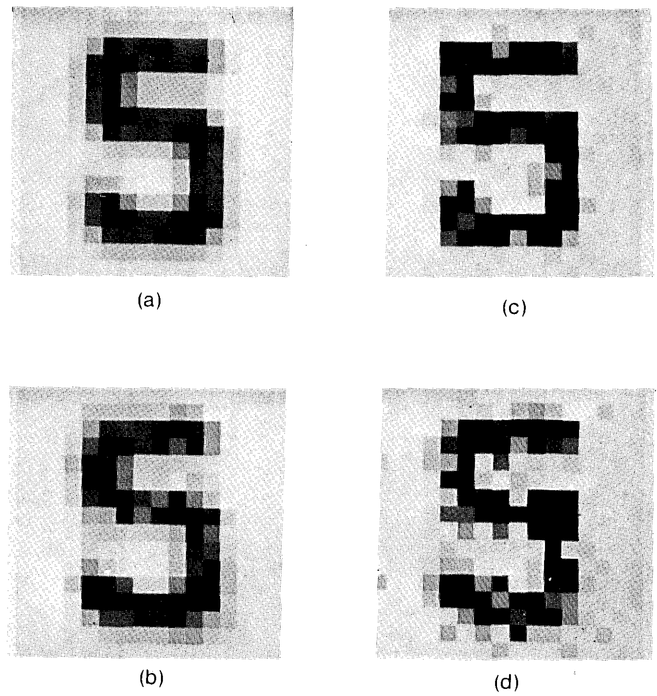


Fig. 2. (a) Smeared image with additive Gaussian noise, mean = 0, standard deviation = 0.1, SNR $\approx$ 920; (b) smeared image with additive Gaussian noise, mean = 0, standard deviation = 0.5, SNR $\approx$ 37; (c) restored for (a); (d) restored for (b).

$$[\overline{B}_c][F]^T = [X]^T + [R] \tag{20}$$

is established, where $[R]$ in Eq. (20) has a meaning similar to that of Eq. (18). Solution $[F]$ is then calculated in the same manner.[14]

For severely blurred images, the rank of matrix $[\overline{A}_c]$ is the same for all j in Eq. (19). The same is true for the rank of matrix $[\overline{B}_c]$.

VII. Numerical Results

A program for the present algorithm is written in Fortran IV and was tested on the IBM 370/3032 computer. Two examples are considered. In each example, the image $[G]$ is obtained by blurring an ideal image $[F]$ and adding noise to the blurred image. We took the signal to noise ratio as

$$\text{SNR} = \text{variance of ideal image/variance of noise.}$$

First is the numerical example solved in Ref. 7 for character 5. The point spread function is spatially separable, and matrix $[H]$ is given by $[H] = [A] \otimes [A]$, where A is a 16×16 tridiagonal matrix, with each diagonal, subdiagonal, and superdiagonal element equaling 1/3. In Refs. 7 and 14, matrix $[A]$ is an 8×8 tridiagonal matrix of the same structure as $[A]$. The largest and smallest eigenvalues of $[A_c]$ are, respectively, 1.0 and 0.078211. This computer simulation example is shown in Figs. 1 and 2.

The second example is the classical image of the GIRL. The results are in Figs. 3–6. To start, a portion of the image of the GIRL is decimated by taking every second pixel every second line. This is given by matrix $[F]$ and is shown in Fig. 3. $[F]$ is an 103×64 matrix. To be able to use the FFT techniques, matrices $[F]$ and $[G]$ are extended by zeros so that each is an 128×64 matrix. The smeared noisy image $[G]$ is obtained from

Fig. 3. The 103×64 matrix representing a portion of the image of the GIRL decimated by taking every second pixel every second line.

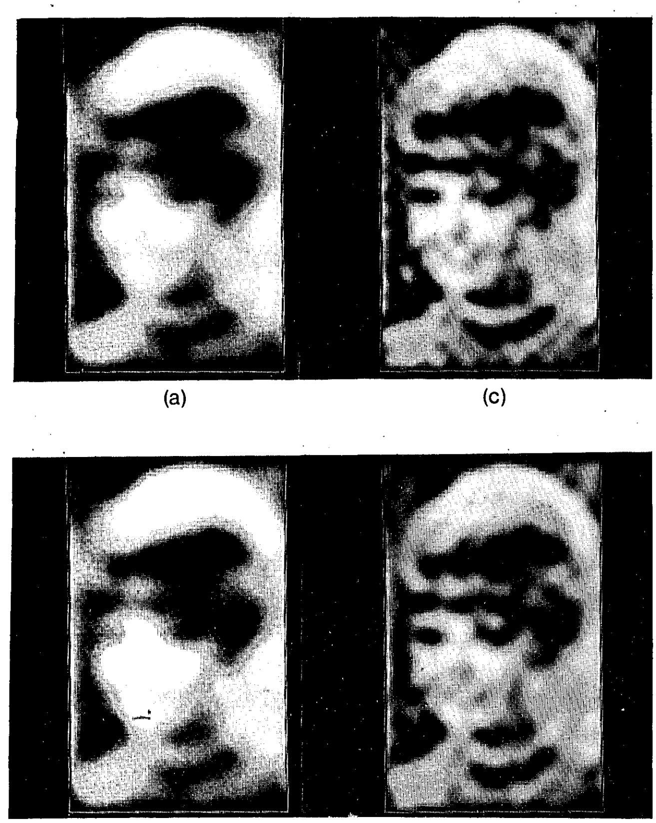
(a) (b)

Fig. 5. Restoration from severe SIPSF blur ($K = 4$). (a) Blurred image with no noise, $K = 4, L = 15$. (b) Restored for (a), rank of $[\overline{A}_c]$ = 118, and rank of $[\overline{B}_c] = 60$.

(a) (c)

(b) (d)

Fig. 4. Restoration from moderate SIPSF blur ($K = 1$). (a) Blurred noisy image, $K = 1, L = 5$, with additive Gaussian noise, mean = 0, standard deviation = 1, SNR $\approx$ 1750; (b) blurred noisy image, $K = 1, L = 5$, with additive Gaussian noise, mean = 0, standard deviation = 3, SNR $\approx$ 195; (c) restored for (a); (d) restored for (b).



(a) (c)

(b) (d)

Fig. 6. Restoration from severe SIPSF blur ($K = 4$). (a) Blurred noisy image, $K = 4, L = 15$, with additive Gaussian noise, mean = 0, standard deviation = 0.5, SNR $\approx$ 7000; (b) blurred noisy image, $K = 4, L = 15$, with additive Gaussian noise, mean = 0, standard deviation = 1, SNR $\approx$ 1750; (c) restored for (a), rank of $[\overline{A}_c] = 35$, and rank of $[\overline{B}_c] = 17$; (d) restored for (b), rank of $[\overline{A}_c] = 31$, and rank of $[\overline{B}_c] = 15$.

Eq. (17), where matrix $[A]$ is an 128×128 matrix, and $[B]$ is a 64×64 matrix.

In this example, we use a truncated Gaussian point spread function. The elements of the band of matrix $[A]$ are given by

$$a_{ij} = C \exp[-(i-j)^2/K^2], \tag{21}$$

where C is an amplitude scaling constant, and K is a blur-spread constant. By truncated, we mean the width of the band is L, that is, in Eq. (21), $a_{ij} = 0$ for $(i-j)^2 > \theta K^2$, where we took $\theta = 6$. Some authors take $\theta = 3$. For $K = 1$, we took $L = 5$, and for $K = 4$, we took $L = 15$. Matrix $[B_c]$ in Eq. (20) has the same structure as matrix $[A_c]$.

For $K = 1$, the largest and smallest eigenvalues of $[A_c]$ and $[B_c]$ are, respectively, 1.0 and 0.16959, that is, $[A_c]$ and $[B_c]$ are well-conditioned matrices. Figure 4 indicates that the restoration for this case is successful.

For $K = 4$, the largest and smallest eigenvalues of $[A_c]$ and $[B_c]$ are, respectively, 1.0 and 2.0325×10^{-4}. In this last case $[A_c]$ and $[B_c]$ are very ill-conditioned. For Figs. 5 and 6, the rank of matrix $[\overline{A}_c]$ is the same for all the J equations (19). Also the rank of $[\overline{B}_c]$ is the same for all the K equations (20). For Fig. 6, the given values of the ranks of matrices $[\overline{A}_c]$ and $[\overline{B}_c]$ show that for severely blurred images, the presence of even a small amount of noise can be very obstructive to the restoration of such images.

VIII. Comments

For the sake of comparing the present method with other methods for the SIPSF case, we assume that $[F]$ and $[G]$ are each $N \times N$ matrices.

Again remember that the eigenvalue of $[A_c]$ and $[B_c]$ are calculated only once, and each needs one FFT. The calculation of the diagonal elements of matrix $[S]$ needs a maximum of N divisions.

In the present method, the least-squares solution is obtained directly from Eq. (19) for each left-hand side vector $\mathbf{g}_j$, $j = 1,2,\ldots,N$, and then from Eq. (20). It is known that each FFT needs $2N \cdot \log_2(N)$ multiplications. Hence, with our method the initial least-squares solution of Eq. (19) requires $(6N^2 + 2N)\log_2(N) + N^2 + N$ multiplications/divisions (M/D). Let p be the average number of iterations to get a final solution for each left-hand side of Eq. (19). Let p denote the same for the solutions of Eq. (20). Again assume that p in the present method is comparable with the corresponding number of iterations p_1 of the method of Hunt.[3] Therefore, from Secs. IV and V, the total number of M/D for the solution of the restoration problem using the present method is $(12N^2 + 4N)\log_2(N) + 2N^2 + 2N + 8pN^2\log_2(N)$ (M/D).

We have two methods with which to compare the present algorithm: Hunt[3] and Reddi.[15] Reddi's method is essentially Hunt's method but with a computationally improved scheme. The following estimates are quoted from Reddi.[15] For a 2-D $N \times N$ image we have the following.

The least-squares solution in Hunt's method requires $16N^2\log_2(N) + 5.5N^2 + p_1[4N^2\log_2(N) + 4N^2]$ (M/D). The least-squares solution in Reddi's method requires $16N^2\log_2(N) + 7.5N^2 + p_2(2N^2)$ (M/D).

We must remember that, for preventing the wrap-around effect in Hunt's method, the parameter N could be much larger than the actual dimension of the image, that is, much larger than N in our method. Hence, because of this and the simplicity of our method, it compares favorably with these methods for the case of SIPSF.

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