Analysis on Duffing Oscillator's Chaotic Behavior

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Abstract

In this paper the Duffing Oscillator's chaotic behavior is explained by the implementation of the Smale Horseshoe Map, and symbolic dynamics.

1 Introduction

This is the Duffing Oscillator most general form

$$\ddot{x} + \delta\dot{x} - \beta x + x^3 = \epsilon\cos(\omega t). \tag{1}$$

Physically this equation can represent an harden spring, the dynamics of a buckling beam , or the forced vibrations of a beam in an nonuniform field of two permanent magnets [1]. These applications of the the Duffing Oscillator makes it of great interest of physical scientist and engineers. However, despite fascination within the physical science community, the Duffing Oscillator is of interest of the mathematics community because of it's complex and strange dynamical behavior near it homoclinic orbits [2]. Near the unstable orbits of this equation one can find chaotic behavior which leads to unpredictable results in it's phase diagram. The Smale Horseshoe map will be used to explain and demonstrate for the result of chaos near the homoclinic orbits of this equation. An important note is that the calculation done in this is done with $\beta = 0$ since it does play a key role in the checking for Smale Horseshoes as seen in the next few sections.

2 Qualitative Analysis

Before starting exploring the Duffing Oscillator using the Smale Horse Mapping, it is important we first perform a quick qualitative analysis on the system. The results from the qualitative analysis will serve as the motivation to explore the system more in depth. First rewrite the the nonlinear ODE as a first order equations

$$\dot{x} = y \tag{2}$$

$$\dot{y} = x - x^3 - \delta x + \epsilon\cos(\omega t) \tag{3}$$

on the $(x, y) \in \mathbb{R}^2 \times S^1)$ where $S = \mathbb{R}\backslash T$ where T is the circle of length $T = 2\pi/\omega$[1]. Since the purpose of this paper is explain the use of the Smale Map in the construction of the Duffing Oscillator the calculation for solving for fixed

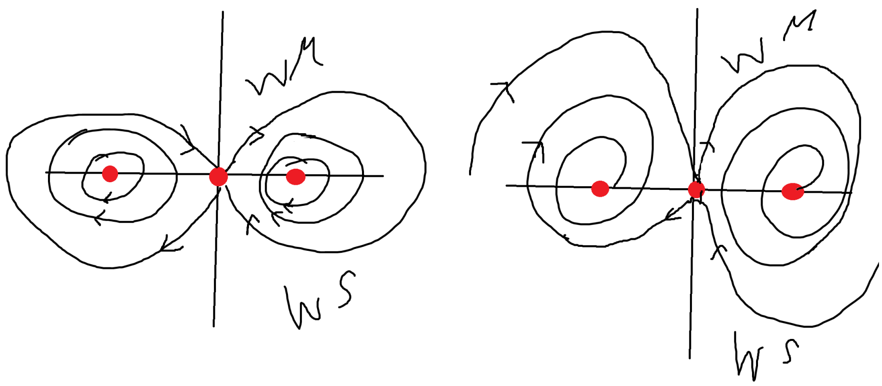

Figure 1: Here is the perturbed Duffing Oscillator to the left, while a perturbed Duffing Oscillator is on the right. Take note how the stable and unstable manifolds are labeled

point and such will be omitted. Doing the calculation shows that that for $\delta = 0$ we have two centers around the fixed point 1 and a saddle at the origin. However calculating for $\delta > 0$ shows that our centers have become sinks [5]. So saddle point of the Duffing equation contains both a unstable and stable manifold , and the centers/sinks have unstable manifolds [3]. Something important to consider is that as the ϵ parameter increases the unstable and stable manifolds start to intersect transversely creating transversal homoclinic orbits [1]. This is significant because near these intersections we observe chaos. The existence for the chaotic behavior of this system in the next section will be explained using the Smale Horseshoe Map.

3 Deeper Look At The Homoclinic Tangle

In the Poincare Map above, a strange attractor is obviously present and as well as chaotic behavior. Further more, looking at the unstable manifold and stable manifold intersection a horseshoe is created. As a rough proof observe the intersection of the stable and unstable manifolds. Let x' be the intersection point between W^μ and W^s. Since the unstable and stable manifolds are invariant then any map $T : x' \mapsto x_p$, such that x_p is a point , $x_p \in W^\mu \cap W^s$ and is in both manifolds.

Remark. The transformation T does have an inverse for the sake of this explanation.

To go further on this semi informal proof, the conditions for approaching fixed points of both manifolds must be defined.

Definition 3.1. (Fixed Point of Stable Manifold) For some 2 dimensional map T a stable manifolds approach fixed point if $T^n x \to x_p$ as $n \to \infty$.

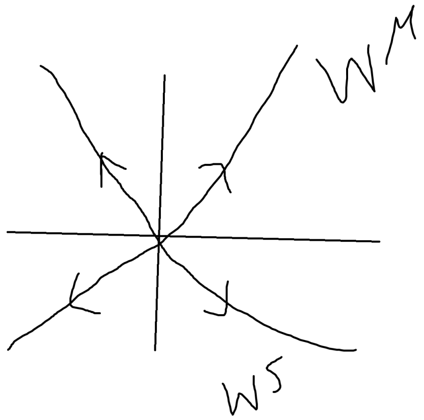

Figure 2: Here is a close up on the intersection of invariant manifolds of the Duffing Equation

Definition 3.2. (Fixed Point of Unstable Manifold) For some 2 dimensional map T an unstable manifold approach fixed point if $T^{-n}x \to x_p$ as $n \to \infty$.

By definition 3.1 and 3.2, their must be an infinite amount of points to approach x_p, and consequently it leads to an infinite amount of intersected points by the invariant manifolds. So geometrically as $T^n(x')$ approach the fixed point, then consequently W^μ must stretch along the direction the unstable direction such that it does not intersect itself. This is transformation is significant because it is the Smale Horseshoe mapping! Consider a square B and place it at x_p. Observe the mapping $T^n(B)$ as W^μ deforms and stretch it takes the square and maps it looks similar to the Smale Horseshoe, and when performing the inverse map $T^b(B)$ the square if deformed to stretch like a rectangle. Thus combing these mapping ends up giving the Smale Horseshoe map. This then roughly proves that the Smale Horseshoe appears by the homoclinic tangle [6].

Proposition 1. A transverse homoclinic point will led to the creation of a horseshoe.

Knowing this fact, the tools and conditions necessary to prove chaos near by the transverse homoclinic point. Expanding this idea further, the following theorem confirms this thought and adds more structure.

Theorem 3.1 (The Birkhoff-Smale Homoclinic Theorem)**.** *Suppose that a diffeomorphism $P : M \to M$, where M is an n-dimensional manifold, has a hyperbolic fixed point x_p, with a stable $W^s(x_p)$ and unstable $W^\mu(x_p)$ manifold that intersect transversely at some point $x_0 \neq x_p$*

$$W^s \perp W^\mu$$

where $dim(W^s)+dim(W^\mu) = n$, then M contains a hyperbolic set Λ , invariant under P, on which P is topologically conjugate to a shift on finitely many symbols.

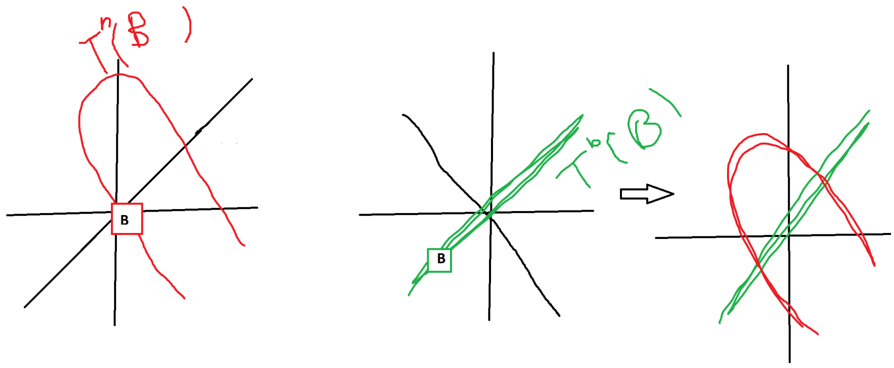

Figure 3: Here is the geometric representation on the applied mapping of T and T inverse on the square B, and the combined mapping of the two.

By this theorem, and the lemma in the next section, the existence of horseshoes in the Duffing Oscillator phase will be proven.

4 Existence of Smale's Horseshoe in Duffings Oscillator

Although last section proved the existence of the horse shoes at the Duffing Oscillator's homoclinic transverse intersection using a non formal geometric argument, by the following lemma and the last theorem a complete proof can be made.

Lemma 4.1 (Birnir)**.** *The stable and unstable manifolds of the Poincaré Map intersect transversely if and only if the Melnikov function*

$$M(t_0) = \int_{-\infty}^{\infty} f(x_0(t-t_0)) \wedge g(x_0(t-t_0), t)dt$$

has simple zeros.

To use the Melnikov method the Duffing Oscillator equation must be but into a first order system

$$\dot{x} = f(x) + \epsilon g(x,t), x \in \mathbb{R}^2, t \in \mathbb{R} \tag{4}$$

and assuming the perturbation g is period in time and the unperturbed system has a hyperbolic stationary solution. Rewrite the Duffing Oscillator in this form and using Lemma 4.1, then the next big theorem can be proved.

Theorem 4.2 (Existence of Smale's Horse in Duffing Oscillator Phase)**.** *For the damped and driven Duffing Oscillator*

$$\ddot{x} + \delta\dot{x} - \beta x + x^3 = \epsilon\cos(\omega t)$$

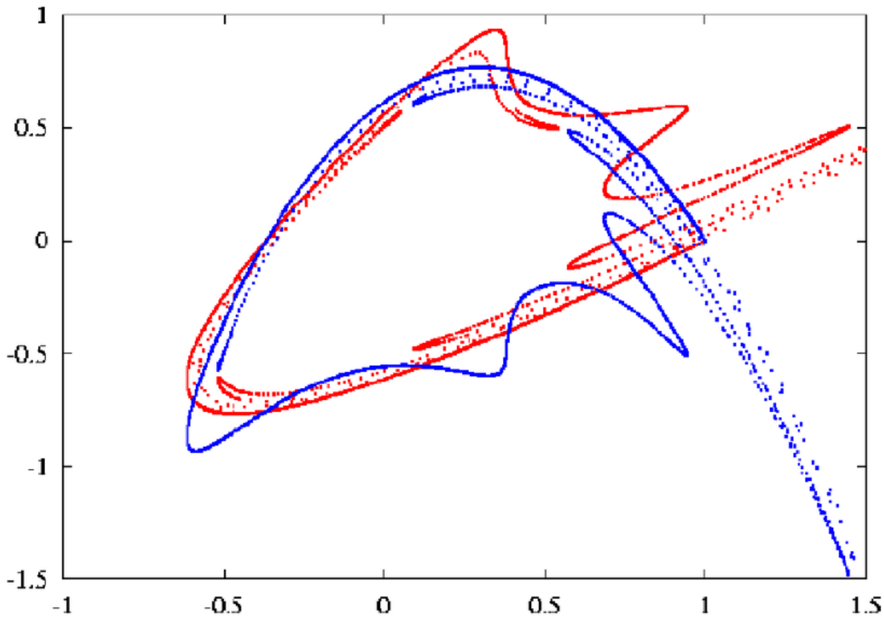

Figure 4: Here is the image of the homoclinic tangle of the perturbed Duffing Oscillator. By the finds, around the homoclinic transverse intersection there exist various horseshoes near by. (Image retrieved Ralph Abraham [7])

there exist a Smale Horseshoe in its phase space if $\frac{\epsilon}{\delta} > \frac{4cosh(\frac{\pi}{2})}{\sqrt{32}\pi}$.

Proof. Consider the driven and driven Duffing Oscillator

$$\ddot{x} + \delta\dot{x} - \beta x + x^3 = \epsilon\cos(\omega t)$$

Rewrite this equation as a first order system of ODEs

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x - x^3 \end{pmatrix} + \begin{pmatrix} y \\ \delta\dot{x} + \epsilon\cos(\omega t) \end{pmatrix}$$

Notice how this is already in the form given in equation 4 needed in Lemma 4.1, and that $g(x, t)$ is periodic. Afterwards the homoclinic orbit is found and we define f and g. Then the wedge product between the two are taken to which complex analysis tells us that simple zeros only exist only if

$$\frac{\epsilon}{\delta} > \frac{4cosh(\frac{\pi}{2})}{\sqrt{32}\pi}$$

□

This is an important result as without satisfying this certain feature then it would not be possible to have Smale's Horseshoe anywhere in the phase diagram. Take note how having a dampening $\delta = 0$ cannot satisfy this Theorem 4.2. This fact would translate to the fact that it is not possible to have a homoclinic connection in undamped Duffing Oscillator!

Corollary 4.2.1. *The Duffing Oscillator with $\delta = 0$ cannot have Smale's Horseshoe in the phase space.*

Corollary 4.2.2. *The Duffing Oscillator with $\delta = 0$ cannot have a homoclinic connection.*

These results will play an important role for checking the chaotic behavior of the Duffing Oscillator.

5 Existence of Chaos in Duffing Oscillator

The driven and dampened Duffing Oscillator exhibits very strange behavior that appears near the transverse homoclinic intersection. One can notice that it appears to look completely random, and confusing. However, this phenomena can be explained through the use of symbolic dynamics and a few properties of the Smale Horseshoe map.

Lemma 5.1 (Birnir)**.** *The Horseshoe map $f : \Lambda \to \Lambda$ is topologically conjugate to a shift on two symbols.*

Corollary 5.1.1 (Birnir)**.** *There are infinitely many periodic orbits for all periods in Λ.*

Corollary 5.1.2 (Birnir)**.** *Λ contains chaotic orbits topologically conjugate to random flips of a coin*

Corollary 5.1.3 (Birnir)**.** *Λ is uncountable and it contains uncountably many aperiodic orbits.*

With the given information now the Duffing Oscillator will be shown to exhibit having chaotic behavior. The Smale Horseshoe map is topologically to a shift on two symbols. However by the Birkhoff-Smale Homoclinic Theorem, the morphing of the unstable and stable manifolds are topologically conjugate to a shift on finitely many symbols. Recall by Propostion 1 that the actions of the morphing of the manifolds implies that $T^{n+b}(B)$, the combined mapping of the T and T inverse on the square B, actually creates the Smale Horse map. Thus, the transformation on W^{μ} and W^{s} is the Smale Horse map. This then implies that T is topologically conjugate to a shift on two symbols. More important the mapping T must have some Λ such that it contains infinitely many periodic orbits for all periods, and most important contains chaotic orbits topologically conjugate to random flips of a coin. So the periodic orbits near the homoclinic connection of the Duffing Oscillator are chaotic!

Remark. A more solid proof can be made by directly using symbolic dynamics to prove the chaotic behavior. This is done by having points on the invariant manifolds represented then performing the Smale Horseshoe mapping and relating it to the shift in these points within the bifinte series.

6 Conclusion

The Duffing Oscillator experiences various bizarre phenomenas as the unstable and stable manifolds intersect each. Through the analysis of using the Smale

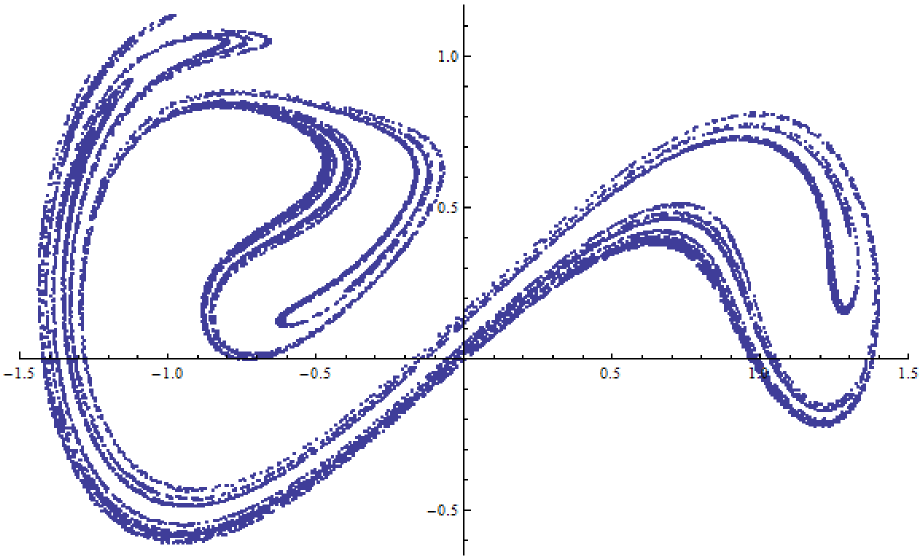

Figure 5: Here is a picture of the strange attractor of the perturbed Duffing Oscillator. This a good example of topological mixing done by repeated Smale Horseshoe Mapping. (Image retrieved from Wolfram)

Horseshoe it revealed that horse shoes appear for certain values of ϵ and δ. Likewise the existence of the homoclinic intersection depends on these damping and driving.So depending on the damping and driving, it can shoe when the system will exhibit chaotic behavior. While various other statements were made, these by far are the most important pieces of information to get.

7 Last Comments

Overall, I hope that this has been good introduction to the chaotic dynamics of the Duffing Equation. I tried my best to explain things with my limited understanding of the symbolic dynamics, and I wish it made sense. I got a good amount of influence and help from the references listed down below, along with lecture notes. I tried my best to cite correctly so if it does not meet standard please tell me!Finally I could not figure out to give Ralph Abraham, so since the link to homoclinic tangle was confusing to cite so I will put it here at the end as a remark.

Remark. Here is where the image 4 came from.

$$https://www.researchgate.net/profile/Ralph_Abraham$$

References

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[5] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer, Springer 1983.

[6] O.Knill *HORSE SHOES AND HOMOCLINIC TANGLE.*

$http : //www.math.harvard.edu/archive/118r_spring_05/handouts/horseshoe.pdf$