Gilbert-Shannon-Reeds

October 13, 2017

```
In [1]: import multiprocessing as mp
        import typing

In [2]: import matplotlib
        matplotlib.use('nbagg')
        import matplotlib.pyplot as plt
        import matplotlib.ticker
        import numpy as np
        import scipy as sp
        import sklearn.utils

In [3]: %matplotlib inline
        %load_ext autoreload
        %autoreload 2
```

From Wikipedia: > In the mathematics of shuffling playing cards, the Gilbert–Shannon–Reeds model is a probability distribution on riffle shuffle permutations that has been reported to be a good match for experimentally observed outcomes of human shuffling, and that forms the basis for a recommendation that a deck of cards should be riffled seven times in order to thoroughly randomize it. ... The deck of cards is cut into two packets... [t]hen, one card at a time is repeatedly moved from the bottom of one of the packets to the top of the shuffled deck.

Here we implement the Gilbert–Shannon–Reeds model, and verify this recommendation of seven shuffles.

Note that the functions below have `doctest` examples. To test the functions, just run `pytest` in the top level of the repository.

First, define a function to determine how many cards to split into our right hand.

```
In [4]: def get_random_number_for_right_deck(n: int, seed: int=None, ) -> int:
            """
            Return the number of cards to split into the right sub-deck.

            :param n: one above the highest number that could be returned by this
                      function.
            :param seed: optional seed for the random number generator to enable
                         deterministic behavior.
            :return: a random integer (between 1 and n-1) that represents the
                     desired number of cards.
```

```
        Examples:

        >>> get_random_number_for_right_deck(n=5, seed=0, )
        1
        """
        random = sklearn.utils.check_random_state(seed=seed, )

        return random.randint(low=1, high=n, )
```

Next, define a function to determine which hand to drop a card from.

```
In [5]: def should_drop_from_right_deck(n_left: int, n_right:int, seed: int=None, )
            """
            Determine whether we drop a card from the right or left sub-deck.

            Either `n_left` or `n_right` (or both) must be greater than zero.

            :param n_left: the number of cards in the left sub-deck.
            :param n_right: the number of cards in the right sub-deck.
            :param seed: optional seed for the random number generator to
                         enable deterministic behavior.
            :return: True if we should drop a card from the right sub-deck,
                     False otherwise.

            Examples:

            >>> should_drop_from_right_deck(n_left=32, n_right=5, seed=0, )
            True

            >>> should_drop_from_right_deck(n_left=0, n_right=5, )
            True

            >>> should_drop_from_right_deck(n_left=7, n_right=0, )
            False

            >>> should_drop_from_right_deck(n_left=0, n_right=0, )
            Traceback (most recent call last):
            ...
            ValueError: Either `n_left` or `n_right` (or both) must be greater than
            """
            if n_left > 0 and n_right > 0:
                # There are cards left in both sub-decks, so pick a
                # sub-deck at random.
                random = sklearn.utils.check_random_state(seed=seed, )
                num = random.randint(low=0, high=2, )
                boolean = (num == 0)
                return boolean
            elif n_left == 0 and n_right > 0:
```

```
            # There are no more cards in the left sub-deck, only
            # the right sub-deck, so we drop from the right sub-deck.
            return True
        elif n_left > 0 and n_right == 0:
            # There are no more cards in the right sub-deck, only
            # the left sub-deck, so we drop from the left sub-deck.
            return False
        else:
            # There are no more cards in either sub-deck.
            raise ValueError ('Either `n_left` or `n_right` '\
                              '(or both) must be greater than zero.')
```

Now we can implement the 'Gilbert–Shannon–Reeds' shuffle.

```
In [6]: def shuffle(deck: np.array, seed: int=None, ) -> np.array:
            """
            Shuffle the input 'deck' using the Gilbert-Shannon-Reeds method.

            :param seq: the input sequence of integers.
            :param seed: optional seed for the random number generator
                         to enable deterministic behavior.
            :return: A new deck containing shuffled integers from the
                     input deck.

            Examples:

            >>> shuffle(deck=np.array([0, 7, 3, 8, 4, 9, ]), seed=0, )
            array([4, 8, 3, 7, 0, 9])
            """

            # First randomly divide the 'deck' into 'left' and 'right'
            # 'sub-decks'.
            num_cards_in_deck = len(deck)
            orig_num_cards_right_deck = get_random_number_for_right_deck(
                n=num_cards_in_deck,
                seed=seed,
            )

            # By definition of get_random_number_for_right_deck():
            n_right = orig_num_cards_right_deck

            n_left = num_cards_in_deck - orig_num_cards_right_deck

            shuffled_deck = np.empty(num_cards_in_deck, dtype=int)

            # We will drop a card n times.
            for index in range(num_cards_in_deck):
                drop_from_right_deck = should_drop_from_right_deck(
                    n_left=n_left,
                    n_right=n_right,
                    seed=seed,
                )

                if drop_from_right_deck is True:
                    # Drop from the bottom of right sub-deck
                    # onto the shuffled pile.
                    shuffled_deck[index] = deck[n_right - 1]
                    n_right = n_right - 1
                else:
                    # Drop from the bottom of left sub-deck
                    # onto the shuffled pile.
                    shuffled_deck[index] = deck[
                        orig_num_cards_right_deck + n_left - 1
                    ]
                    n_left = n_left - 1

            return shuffled_deck
```

Finally, we run some experiments to confirm the recommendation of seven shuffles for a deck of 52 cards.

```
In [7]: num_cards = 52
        max_num_shuffles = 20
        num_decks = 10000

        # Shuffling the cards using a uniform probability
        # distribution results in the same expected frequency
        # for each card in each deck position.
        uniform_rel_freqs = np.full(
            shape=[num_cards, num_cards],
            fill_value=1./num_cards,
        )

In [16]: def calculate_differences(
             num_shuffles: int
             ) -> typing.Tuple[np.float64, np.float64, np.float64,]:
             """
             Calculate differences between observed and uniform distributions.

             :param The number of times to shuffle the deck each time.
             :return Three metrics for differences between the
                     observed and uniform relative frequencies.
             """
             shuffled_decks = np.empty(shape=[num_decks, num_cards], )

             # First create a random deck.
```

```
          orig_deck = np.array(range(num_cards))
          np.random.shuffle(orig_deck)

          for i in range(num_decks):
              # Now shuffle this deck using the Gilbert-Shannon-Reeds method.
              new_deck = orig_deck
              for j in range(num_shuffles):
                  new_deck = shuffle(new_deck)

              shuffled_decks[i] = new_deck

          # Calculate the relative frequencies of each card in each position.
          rel_freqs = np.empty(shape=[num_cards, num_cards], )

          for i in range(num_cards):
              col = shuffled_decks[:, i]

              # Make sure that each card appears at least once in this
              # position, by first adding the entire deck, and then
              # subtracting 1 from the total counts of each card in
              # this position.
              col = np.append(col, orig_deck)
              col_freqs = sp.stats.itemfreq(col)[:, 1]
              col_freqs = col_freqs - 1
              rel_freqs[i] = col_freqs / num_decks

          # Here I use three metrics for differences between the
          # observed and uniform relative frequencies:
          # * The sum of the squared element-wise differences,
          # * The relative information entropy, and
          # * The Kolmogorov-Smirnov statistic.
          sum_squared = np.sum(np.square(np.subtract(uniform_rel_freqs, rel_freq
          entropy = sp.stats.entropy(rel_freqs.flatten(), uniform_rel_freqs.flat
          kstest = sp.stats.kstest(rel_freqs.flatten(), 'uniform').statistic

          return sum_squared, entropy, kstest
```

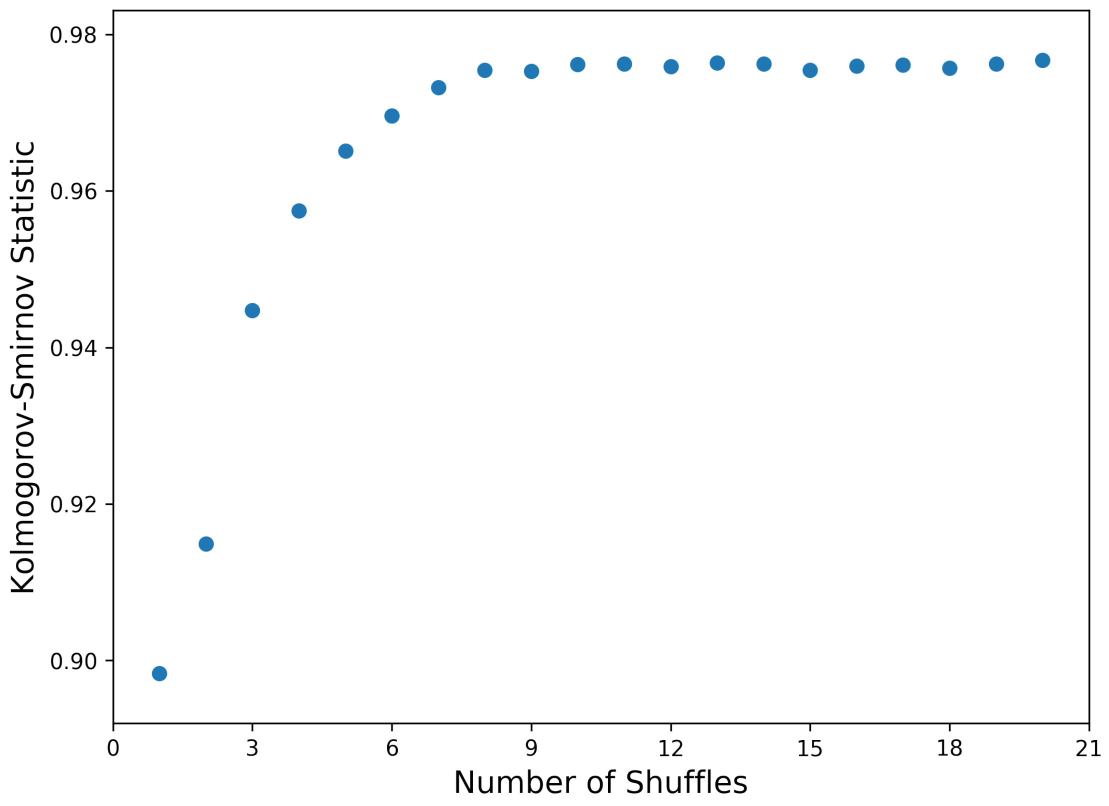
```
In [17]: # Now run the experiment using all our CPUs!

         with mp.Pool(mp.cpu_count() - 2) as p:
             results = p.map(calculate_differences, range(1, max_num_shuffles+1))
             results = np.array(results)
```

```
In [18]: sums_squared = results[:, 0]
         entropies = results[:, 1]
         kstests = results[:, 2]
```

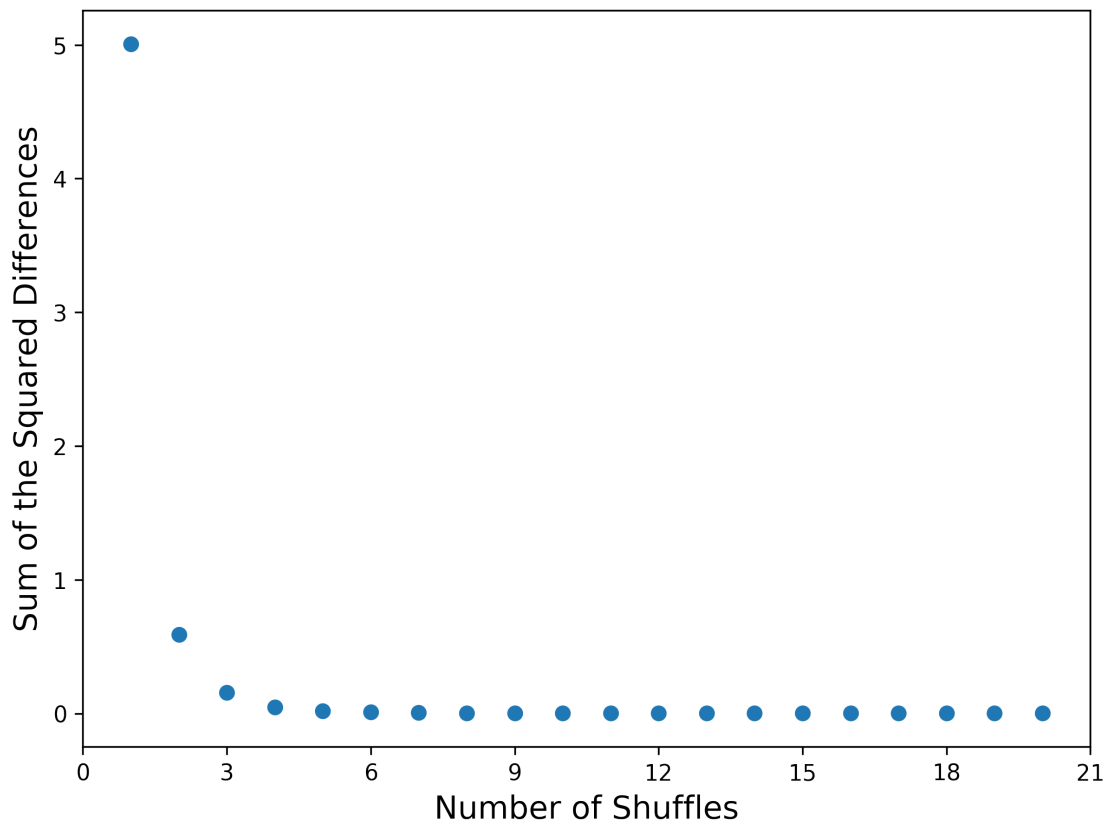
The KS statistics are of most use here. You can see how the statistic approaches its maximum value around num_shuffles = 7.

```
In [19]: fs = 14
```

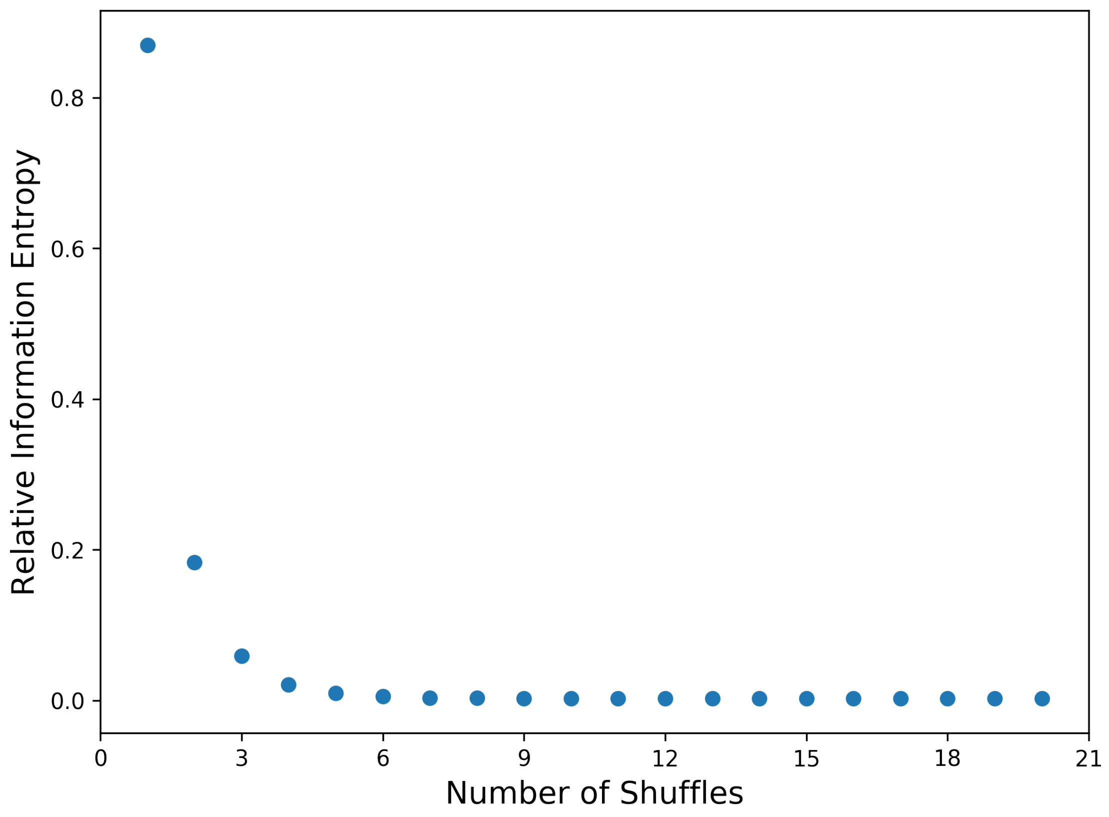
```
In [20]: fig, ax = plt.subplots(figsize=(8, 6), dpi=300)
         ax.scatter(range(1, max_num_shuffles + 1), kstests, );
         ax.xaxis.set_major_locator(matplotlib.ticker.MaxNLocator(integer=True))
         ax.set_xlabel('Number of Shuffles', fontsize=fs, )
         ax.set_ylabel('Kolmogorov-Smirnov Statistic', fontsize=fs, )
         ax.set_xlim([0, max_num_shuffles + 1])
         plt.show();
```



```
In [21]: fig, ax = plt.subplots(figsize=(8, 6), dpi=300)
         ax.scatter(range(1, max_num_shuffles + 1), sums_squared, );
         ax.xaxis.set_major_locator(matplotlib.ticker.MaxNLocator(integer=True))
         ax.set_xlabel('Number of Shuffles', fontsize=fs, )
         ax.set_ylabel('Sum of the Squared Differences', fontsize=fs, )
         ax.set_xlim([0, max_num_shuffles + 1])
         plt.show();
```



```
In [22]: fig, ax = plt.subplots(figsize=(8, 6), dpi=300)
         ax.scatter(range(1, max_num_shuffles + 1), entropies, );
         ax.xaxis.set_major_locator(matplotlib.ticker.MaxNLocator(integer=True))
         ax.set_xlabel('Number of Shuffles', fontsize=fs, )
         ax.set_ylabel('Relative Information Entropy', fontsize=fs, )
         ax.set_xlim([0, max_num_shuffles + 1])
         plt.show();
```


Relative Information Entropy
0.0
0.2
0.4
0.6
0.8
Number of Shuffles
0
3
6
9
12
15
18
21

```
In [ ]:
```