SSSS 2023 Optics Key

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Introduction

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Remember to **show all work** for partial credit. Unless otherwise specified, use appropriate **SI units** and round to **three significant digits**.

Every question here involves a laser or the interaction with light and matter, so clearly, they are all on-topic.

This is the answer key.

Off-topic Light Trivia

1. (3 points) You shoot a laser in a vacuum.

a. What is the speed of that light (give the exact value)?

b. How could that be calculated from the permittivity and permeability of free space?

c. The laser is not aimed at your eye. Can you still see the beam as it travels through space? Why or why not?

1. (Answer)

a. *The speed of light in a vacuum is defined to be $c = 299\,792\,458$ meters per second.*

b. *This speed may be calculated by $\sqrt{\frac{1}{\varepsilon_0\mu_0}}$, where ε_0 and μ_0 are the permittivity and permeability of free space, respectively.*

c. *Contrary to Star Wars and other science fiction movies, you can not see a beam of light in a vacuum with no particles to scatter light and redirect it back to you. You might see the source of the light, and you might see it after it reflects off something else, but not where the light is not interacting with anything else. (Half credit is awarded for particles scattering light.)*

2. (4 points) Now, you shine a white laser.

a. Wait, what? Is a white laser even possible? Explain, via the internal workings of a laser, why this is or is not feasible.

b. What does the color white mean in terms of the distribution of photon frequencies?

c. What is the D65 definition of white light?

d. How is white light generated by a computer?

2. (Answer)

a. *In general, typical lasers only produce monochromatic light. Lasers work by exciting a bunch of atoms, which then emit a photon of a constant wavelength when they drop back into the ground state. However, by* *choosing a particular combination of atoms, it is possible to create an effect that appears like white light, and indeed, white lasers have been demonstrated.*

b. *White is not a spectral color, so the wave has a range of frequencies in a continuum, not just one.*

c. *One definition of white light is the black-body radiation emitted at 6504 Kelvin, which closely approximates what we perceive as white.*

d. *White light in computers is generated by shining (roughly) equal intensities of red, green, and blue light, which our eyes perceive as white.*

3. (3 points) You have a lamp that emits light uniformly in all directions for one second. A wall is one meter away.

a. If you were instead two meters away from the wall but kept the lamp for a period of three seconds, what percent of the original energy would reach the wall?

b. What is this relation called?

c. There are actually a few more walls, and they collectively form a cube. What fraction of the lamps energy does each cube receive?

3. (Answer)

a. $\frac{3}{2^2} = \frac{3}{4} = 75\%$ *percent of the energy reaches the wall.*

b. *The distance in the denominator must be squared because the energy received follows an inverse-square law.*

c. *There are* 6 *walls, each receiving the same amount of energy, so each wall receives a sixth of the energy.*

4. (3 points) You shine a (harmless) laser into someone else's eyes.

a. For what range of frequencies would they see that light?

b. How do the cornea, pupil, lens, retina, rods, cones, aqueous humor, and vitreous humour interact with light entering the eye? Describe their functions as well.

4. (Answer)

a. *They can see light between 400 and 700 THz.*

b. *As light travels through the eye, it first hits the cornea, which refracts the light and protects the eye. After that, light passes through the aqueous humor, which also refracts to some degree. The light then goes through the pupil, a black dot in the middle of our eyes. Following this, it is refracted by the lens, passes through the vitreous humour (which supports the retina), and hits the retina, which captures the light and sends a signal to the brain using rods and cones, which are specialized photoreceptor cells. Rods detect general light and are sensitive at low levels of light, while cones detect color.*

5. (2 points) You spray paint a white wall so that, when illuminated by white light, it now appears red.

a. What is physically different about the wall that causes you to see the color red?

b. If you instead illuminated the white wall with red light, the wall would still appear red. How do the two ways of making the wall red differ?

5. (Answer)

a. *Your action creates a layer of spray paint on top of the wall that only reflects red light while absorbing all the other colors. Thus, only the color red is visible to your eye.*

b. *On the other hand, your second action simply adds red light while removing the white light, so you also only see red light reflected off the wall.*

6. (3 points) The sun illuminates a mist of suspended water droplets, and this forms a rainbow.

a. Does the rainbow form in front or behind the water droplets, and why?

b. Is it possible for two rainbows to be formed?

c. Over there is a snake. It can eat you even in complete darkness because of its pit organs, which can sense temperature. But it also likes rainbows. When a snake looks at a rainbow, which color appears at the top of the primary bow?

6. (Answer)

a. *Neither. The rainbow does not have a definite position. The rainbow light originates from above and is refracted and reflected inside the droplets themselves, and then shines toward you. However, the image appears to originate from behind the water droplets, as that is where the rays of rainbow light converge.*

b. *It is possible. Sometimes, two rainbows are seen because of light that refracts or reflects more than once or is scattered (e.g., diffraction) in some secondary process.*

c. *Infrared appears on the top because snakes see infrared (the question tells you that they sense temperature, so they must therefore see in the infrared).*

7. (3 points) For movies in 3D, the projector shines two beams of light intended to be viewed by different eyes. The two lenses in a pair of 3D glasses determine which beam goes to which eye. Why might the glasses prefer circular over linear polarization?

7. (Answer) *3D glasses attempt to serve different images into the two eyes, and here, the screen creates two images, polarized in different modes, and each glass only allows one mode through. If linear polarizers were used, then simply tilting one's head or leaning to the side might lead to the wrong mode because linear polarization can take on an entire unit circle of values. However, circular polarization only has two values, right and left-handed, so there is more leeway in what a person can do with their glasses.*

8. (0.5 points per part) Answer the following true or false questions.

a. Humans emit visible radiation differently from how lasers function. ______

b. Laser light can bend around objects only via gravitational lensing. ______

c. Shadows can occur when dark matter interacts with laser light. ______

d. A light year is the distance light (laser or otherwise) would travel, in standard temperature and pressure water, during one Julian year. ______

e. Laser light exhibits properties of both waves and massive particles. ______

f. The wavelength of visible laser light is huge (a few hundred millimeters), which is why our eyes are able to see it, and why other animals are sensitive to different ranges of the electromagnetic spectrum. ______

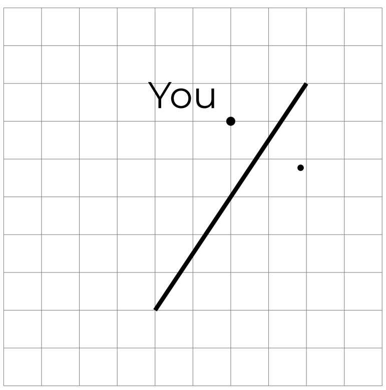
8. (Answer) ***a.*** *true,* ***b.*** *false,* ***c.*** *false,* ***d.*** *false,* ***e.*** *false,* ***f.*** *false.*

Reflection

9. (3 points) Again, you are shining a laser at a wall one meter away from you.

a. In fact, the laser light happens to be green, with $\lambda = 5 \times 10^{-7}$ meters. What phase shift would be observed between the wave sent and the wave received? Express your answer in radians.

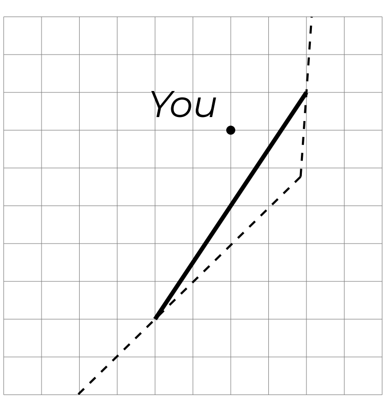
b. Now, assume the mirror is the black line in the diagram below (and you are at the point labeled You). Outline the area that you can see using the mirror. (It may be helpful to know that the second point is where you see your reflection in the mirror.)



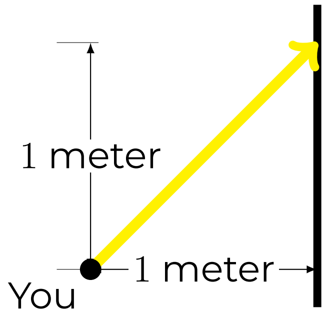
9. (Answer)

a. *The phase shift is π radians. While the light does not experience a change in phase from the path itself (the path length is a multiple of the wavelength), it still does experience a reversal in phase when it reflects off the mirror.*

b. *Let your reflection be point R. Then, if there was no mirror, all rays of light previously reflecting off the mirror to you would instead hit point R. Thus, we can draw lines from point R to the edges of the mirror, which becomes the outline.*



10. (4 points) The laser is now oriented to match the diagram below.



a. What is the angle of reflection?

b. It is also possible that the mirror is not quite smooth.

- **a.** What phenomenon would that be called, and where does the light go?
- **b.** How do we define smoothness in the first place?
(Bonus **5 points** if you can derive the criterion.)

10. (Answer)

a. *The angle of reflection would be* $45°$ *by the law of reflection.*

b.

- ***a.*** *This would be called diffuse reflection, and light would spread out instead of staying in a beam.*
- ***b.*** *Smoothness is defined by requiring that the mirror satisfy Rayleigh's Roughness Criterion* $h < \frac{\lambda}{8\cos\theta}$, *where* h *is the maximum height difference between two points,* λ *is the incident light, and* θ *is the angle of incidence.*
  - *This is derived by finding the phase difference between two reflected rays of light. If the phase difference of light in the specularly reflected direction is less than* $\frac{\pi}{2}$, *then most of the light constructively interferes.*
  - *The phase difference is the extra distance traveled by one of the rays. If the angle of incidence (and therefore the angle of reflection) is* γ, *and the height difference is* h, *then the extra distance traveled is* $\Delta\varphi = 2h\cos\gamma$, *which is found by drawing perpendiculars to the paths of light and expressing the difference in path length with trigonometry.*
  - *After multiplying by the conversion factor, the phase difference becomes* $k\Delta\varphi = 2kh\cos\gamma$, *where* $k = \frac{2\pi}{\lambda}$ *is the wavenumber of the wave.*
  - *Solving for* h *leads to the above representation.*

Color

11. (3 points) The wall is now a transparent green color.

a. If you shine a cyan light through the green glass, what color would be seen on the other side, assuming no other illumination, pure light, and a thick enough glass?

b. Why are those assumptions necessary?

c. What about shining red light through the glass?

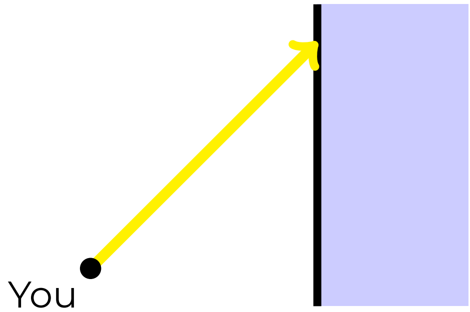
11. (Answer)

a. *The only light seen would be green, as the blue light is filtered out.*

b. *These assumptions are necessary because, without them, some other light would still be seen, tainting the color. Furthermore, not having thick glass would mean that some light might still, against the odds, pass through the glass.*

c. *However, shining red light through the glass would make it appear black because no light passes through.*

12. (3 points) If you instead shone white light through colorless glass (see the diagram below), what would happen to the colors? What order would the colors be in, and why?



12. (Answer) *The colors would disperse into a spectrum of spectral colors. Red would be on the bottom because it has the lowest refractive indices of all the visible colors, and that means less deviation. On the other hand, higher wavelengths such as blue or violet would be at the top.*

13. (3 points) Purple, violet, and indigo all look the same, but two of them are spectral colors, and one of them is not.

a. Identify the imposter.

b. How do spectral colors relate to lasers?

13. (Answer) *Purple is the non-spectral color, which means that a monochromatic laser could not create the color purple.*

14. (2 points) When you shine orange or green light at a band-pass filter, they go through, but if you shine indigo light, it does not go through. Next to each of the following colors, put **T** if the color is necessarily transmitted, an **A** if the color is necessarily attenuated, and an **IDK** if there is not enough information.

a. Red ______

b. Orange ______

c. Yellow ______

d. Green ______

e. Blue ______

f. Indigo ______

g. Violet ______

14. (Answer) *Each part is worth 0.5 points, except for parts **b.**, **d.**, and **f.** (award no points for those parts, regardless of the answer). **a.** IDK, **b.** T, **c.** T, **d.** T, **e.** IDK, **f.** A, **g.** A.*

15. (2 points) When you shine light through a tube of hot hydrogen gas, the wavelength 486 nanometers shows up in its emission spectra.

a. Identify that color and its energy in electronvolts.

b. What does this tell you about the composition of Hydrogen?

15. (Answer) *That wavelength corresponds to a blue (cyan is acceptable) color and has an energy of 2.551 eV (electronvolts). This means that there is a jump in the energy levels of electrons in the Hydrogen atom with an energy difference equal to 2.551 eV.*

Refraction

16. (3 points) You now shine at a block of glass. If you shine with some angle θ above the horizontal, the light goes through the glass and refracts. After the light comes out, what angle does it make to the normal?

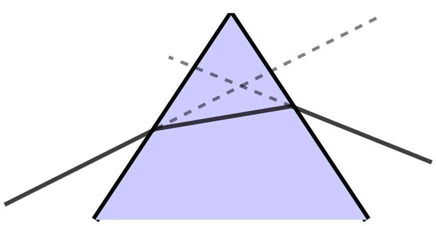
16. (Answer) *The angle is $90 - \theta$, because the refractive indices on the medium (presumably air) on both sides of the glass are the same, and anything inside does not matter. However, we now need to find the angle to the normal, so we take the complement of θ.*

17. (3 points) The glass is shaped into an isosceles triangular prism.

a. Define the angle of the prism and the angle of deviation in this context.

b. Why does the light bend upon entering the prism, and how would you quantify the bending?

c. Given the angle of the prism and the angles of incidence and refraction at the first point of incidence, how would you determine the second pair of angles, the angle of deviation, and the ratio of the refractive indices?



17. (Answer) *The angle of the prism is the angle at the top of the prism (e.g., assuming we have a two-dimensional triangular, the top angle). When light passes through the prism, it bends twice, and the sum of this bending is the angle of deviation.*

Refraction is the bending of a wave as it is transmitted due to a change in the electric and magnetic fields induced by oscillations of charged particles in that new medium. The bending is determined by Snell's Law, which states that if light travels from a medium with a refractive index of n_1 travels into n_2, and the angle of incidence is θ_1, and the angle of refraction is θ_2, then $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

Let the angle of the prism be A, the angle of deviation be d, and the ratio of refractive indices ($\frac{n_{prism}}{n_{air}}$) be n. Additionally, define the first pair of angles of incidence and reflection to be θ_{i1} and θ_{r1}, and the second pair to be θ_{i2} and $\theta r2$, respectively. By the equation $\sin\theta_{i1} = n\sin\theta_{r1}$, we can find n. From the *triangle formed by the two intersection points and the point with angle A, we have $A + (90 - \theta_{r1}) + (90 - \theta_{i2}) = 180$, so $\theta_{r1} = A - \theta_{i2}$. Thus, $n \sin\theta_{i2} = \sin\theta_{r2}$, and we can deduce that $d = \theta_{i1} - \theta_{r1} + \theta_{r2} - \theta_{i2}$.*

18. (3 points) The prism morphs into a biconcave lens with a focal length of 30 centimeters. An object is half a meter away from the pole of the lens and 10 centimeters above the axis.

- **a.** What is the power of the lens in diopters? Answer to 20 significant figures.
- **b.** Where is the image formed? What is its magnification? Is it real or virtual? Is it upright or inverted?
- **c.** How would you derive the thin lens and magnification equations through ray diagrams?
- **d.** Is that biconcave lens used in glasses for myopic or hyperopic people, and why?

18. (Answer) *The power is given by the reciprocal of the focal length, so it is $\frac{1}{0.3 \textit{ meters}} = 3.\underbrace{333\cdots33}_{19\ \textit{3's}}$ diopters (significant figures are required).*

By the thin lens equation, $\frac{1}{f} = \frac{1}{o} + \frac{1}{i}$, so the distance to the image is 75 centimeters, a virtual image, implying it is also upright. Furthermore, by the magnification equation, the image is 15 centimeters high.

Both are derived through similar right triangles, formed by a ray parallel to the axis and the ray straight through the center of the pole of the lens. Let the height of the object be h_o, the height of the image be h_i, the focal length be f, the distance to the object be o, and the distance to the image be i. For the magnification equation, we use h_o (measured at the object) and o as one triangle, and h_i and i as the second triangle. For the thin lens equation, we use h_o (measured at the pole) and f (to the right of the lens) as one triangle, and h_i and $i - f$ as the second triangle. Solving, we get the thin lens equation too.

A biconcave lens will diverge light, allowing light in the eyes of a myopic person (which converges in front of the retina) to focus farther back instead.

19. (3 points) You shine a laser into an optical system. This optical system can be described by one of the terms catoptric, catadioptric, or dioptric. Give an example and a brief etymology for each.

19. (Answer) *A catoptric system uses solely mirrors and is based on the Greek word meaning reflection (or mirror). An example is a convex rearview* *mirror. A dioptric system uses solely lenses and is based on the Greek word meaning binocular (or refraction). An example is a simple binocular. A catadioptric system combines elements of both. Examples include certain complicated telescopes.*

20. (3 points) A laser in one medium is shone at a second, transparent medium. Normally, light can refract, but when the angle of incidence is greater than the critical angle, the light is too oblique for refraction to occur. This phenomenon is known as total internal reflection.

- **a.** Explain why there could ever be a case where refraction is impossible, and using this, derive the value of the critical angle.
- **b.** Explain why certain cases of frustrated total internal reflection show comparisons to quantum tunneling.

20. (Answer)

- ***a.*** *The angle of refraction is* $\theta_r = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right)$. *But* $\arcsin$ *is not defined for inputs greater than* 1, *and if* n_1 *were sufficiently greater than* n_2, *that might occur. Specifically, the critical angle is when* $\frac{n_1}{n_2}\sin\theta_i = 1$, *so it must be* $\sin^{-1}\frac{n_2}{n_1}$.
- ***b.*** *Light only TIRs for certain media. So let's say we have a medium, and light currently TIRs into a second medium. Well, if we put a third medium so that they exactly touch, and the third medium has a high enough refractive index such that no TIR occurs, then the light will simply transmit. So there's an intermediate phase if the two media are only slightly separated by a few wavelengths. Then, even though the light is supposed to TIR because it first hits the low refractive index medium, it can sense that there's a third medium close by and still be transmitted. This is like quantum tunneling, where a particle can sense that there is a place of low potential just beyond a barrier and go through it. Indeed, both involve exponential decay probabilities.*

21. (2 points) The sun illuminates a sign saying *Objects in Mirror Are Closer Than They Appear*. What does it mean, and aside from in the rear-view mirrors of vehicles, where else might this label make sense?

21. (Answer) *Rear-view mirrors are convex, meaning that the images of objects appear smaller and farther away — they smush the entire road behind the vehicle into one small mirror image. Convex mirrors also appear in security mirrors or near blind-turns so that people can see around corners.*

22. (4 points) You shine a light toward a concave mirror, and it happens to have a focal length.

a. Is this focal length positive or negative?

b. How do you define the focal length of a parabola?

c. Can any other shapes have a focal length?

22. (Answer)

a. *Being a concave mirror, it would need to have a positive focal length.*

b. *The focal length of a parabola is the distance from the focus to the vertex of the parabola. The focus is the point where all rays parallel to the axis of the parabola will meet after reflecting off of the parabola.*

c. *No other shape has a focus, and so no other shape has a focal length. However, spheres can approximate parabolas and therefore can also be said to have a focal length.*

23. (1 point) If the radius of curvature of a mirror in a specific orientation is -0.2 meters, find the focal length of that mirror after it is rotated π radians in a direction perpendicular to the axis of the mirror.

23. (Answer) 10 *centimeters.*

24. (1 point per part) For each of the following, write the definition or associated equation.

a. Thin lens

b. Thin lens equation

c. Aberration

d. Positive meniscus

e. Diverging mirror

f. Magnification equation

g. Cartesian Sign Convention

h. Lensmaker's Formula

i. Nodal point

j. Surface vertex

24. (Answer)

a. Thin lens *— A lens that is approximated to have zero thickness, applying to lenses that are thin relative to the radius of curvature*

b. Thin lens equation *— The equation $\frac{1}{i} = \frac{1}{f} + \frac{1}{o}$, where i is the distance to the image, f is the focal length, and o is the object distance (other versions of the equation are also acceptable); the equation only applies to thin lenses.*

c. Aberration *— When something goes wrong with the image (e.g., chromatic aberration)*

d. Positive meniscus *— A convex lens with one radius of curvature being positive, and the other negative, but the positive one being smaller than the negative one*

e. Diverging lens *— A concave lens, which diverges light (sends them away from each other)*

f. Magnification equation *— The equation $\frac{i}{o} = -\frac{h_i}{h_o}$, where h_i and h_o are the heights of the image and object, resspectively*

g. Cartesian Sign Convention *— A convention on the signs of distances in optical diagrams; light travels from the left to the right, and any distance measured rightward is positive*

h. Lensmaker's Formula *— The equation*

$$P = \frac{n - n_0}{n_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right),$$

where P is the power of a lens, n is the refractive index of the material of the lens, n_0 is the refractive index of the material surrounding the lens, R_1 is the radius of curvature of the right edge of the lens, and R_2 is the radius of curvature of the left edge of the lens

i. Nodal point *— There are two nodal points in a lens, and if light is shot at one nodal point, it will exit appearing as if it were shot at the other nodal point*

j. Surface vertex *— The two points where the refractive surfaces cross the optical axis*

Optical Devices

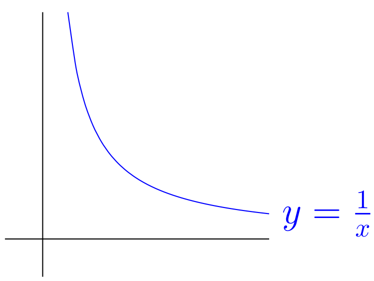
25. (3 points) To illuminate a specimen, a microscope shines a light from below the specimen. Describe the path of the light after, and name the parts of the microscope it interacts with.

25. (Answer) *The light first travels through the specimen. Some light is absorbed or scattered, which allows the details of the specimen to be observed (this part is important; if a response skips straight to the objective lens, dock points). The light is then magnified through the objective lens, which then travels to the back of the microscope toward the eyepiece. The eyepiece further magnifies the image and projects it toward the retina of the observer's eye.*

26. (3 points) A rangefinder or telemeter is an optical device that measures the distance from an observer to a certain target. One type of rangefinder is known as a laser rangefinder, which shoots a laser and hopes it comes back.

a. Describe why a laser rangefinder might fail (both practically and physically).

b. A stadiametric rangefinder is an alternative, and a hint at its operation is shown below. Fully explain how it works.



c. Based on its name, describe how a stereoscopic rangefinder works. Remember that a stereoscope is a device that creates an apparently three-dimensional image by showing different images to the two eyes.

26. (Answer)

a. (Answer) *Obviously, a laser rangefinder requires batteries. The path of light might also not reflect off the target or be deflected by some other obstacle or wind or refraction. Furthermore, this does not take into account the curvature of the Earth, where you can see the target from refraction, but the target is actually over the horizon. Finally, the beam might spread out or travel at different speeds if the medium is not a perfect vacuum.*

b. (Answer) *The stadiametric rangefinder works by similar triangles. These generally contain a series of markings in a hyperbolic shape (e.g., the positive portion of the curve $\frac{1}{x}$) with known radial distances from a baseline. Then, given the height of a distant object, its distance may be determined by how much radial distance it takes up. This device also works backward, e.g., finding a height based on an object's distance.*

c. (Answer) *A stereoscopic rangefinder is composed of two eyepieces (e.g., with binoculars). They (lenses) are angled such that a point a certain distance away will converge perfectly (when this convergent point is away from the target, double images of the target will be seen). The angle is then adjusted until there are no double images, and then the degree of rotation will determine the distance.*

27. (2 points) MRI, CT, and PET scans all use electromagnetic waves to image the body.

a. What ranges of the electromagnetic spectrum are used, and why?

b. Specific to CT and PET scans, what is being measured and why does that give information about the position of bones and other body parts?

27. (Answer)

a. *Radio waves, X-rays, and gamma rays are used because they are uniquely able to penetrate the body.*

b. *For CT and PET scans, bones contain calcium, which attenuates X-rays and gamma rays well, unlike other internal organs.*

28. (1 point) Is a true one-way mirror, which reflects light in one direction, but transmits light in the other direction, impossible, without an external energy source controlling the mirror? Why or why not?

28. (Answer) *It is impossible. If the temperature on the transmitting side is low, but the temperature on the reflecting side is high, then the cold side would get even colder, while the hot side even hotter. This decreases the entropy of the system. Since there is no external energy source to compensate for this loss of entropy, this would violate the second law of thermodynamics.*

Math

29. (1 point) A 50-watt monochromatic light source emits 10^{20} photons every second. What is the frequency of those photons?

29. (Answer) 755 *THz.*

30. (2 points) You have a cool infinity mirror setup. The mirror in front of you is currently 5 meters away, and the mirror behind you is currently 15 meters away. If you are walking at a speed of one meter per second forward, at what speed (possibly negative) are you and the 25th closest reflection of you approaching each other?

30. (Answer) 2 *meters per second.*

31. (6 points) You shine a light at a wall with two small holes, and behind the wall is a screen. But instead of seeing two bright spots on the screen, you see a whole bunch of bright spots, all aligned along a line.

a. What is causing these bright spots?

b. What is the distance between any two consecutive bright spots in terms of any necessary variables?

c. What approximations did you use in creating your formula (list as many as you can)?

31. (Answer)

a. *The bright spots are caused by interference of the two beams of light because light can exhibit wave properties.*

b. *If the wavelength of light is λ, the distance between the wall and the screen is D, and the distance between the two slits is d, then the distance between any two consecutive bright spots is $\frac{\lambda D}{d}$.*

c. *We used many approximations.*

- *The light must be monochromatic.*
- *The screen must be parallel to the wall.*
- *The slits must have a size similar to the wavelength of the light.*
- *The distance between the slits must be small compared to the distance between the wall and the screen.*
- *The small angle approximation was used to derive the formula.*

32. (6 points) A common "paradox" in polarization is known as the three-polarizer paradox. You shine unpolarized light of intensity I_0 through three linear polarizers, with axes of angles $0, 45,$ and $90°$ to the horizontal (in that order).

a. What is the resulting intensity in terms of I_0?

b. What is the resulting intensity if the middle polarizer ($45°$) is removed?

c. Why is this considered a paradox?

d. We can generalize this result to the n-polarizer paradox. There, n polarizers have angles equally spaced from 0 to $90°$ inclusive. What is the least integer value of n such that the resulting intensity is greater than $\frac{I_0}{3}$?

e. As n approaches infinity, what does the resulting intensity approach? Justification via calculus is not necessary, but its inclusion would be helpful. (Hint: for calculus approaches, use the double angle formula $\cos 2\theta = 2\cos^2\theta - 1$ and later the Taylor expansion $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$.)

32. (Answer)

a. (Answer) $\frac{I_0}{8}$. *The intensity is halved by all three polarizers, and* $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

b. (Answer) 0. *The first and third polarizers are otherwise perpendicular, and so no energy is transmitted. Light comes through the first polarizer, and although half is attenuated, the other half becomes polarized along its axis. This means that the light is then perpendicular to the second polarizer, and this clearly means complete attenuation.*

c. (Answer) *Intuitively, adding more filters should decrease the intensity.*

d. (Answer) *We build a function $f(n)$ to describe the ratio of the resulting intensity over I_0.*

$$f(n) = \frac{1}{2}\left(\cos^2\left(\frac{\pi}{2}\frac{1}{n-1}\right)\right)^{n-1}.$$

The angular difference between two adjacent polarizers is $\frac{\pi}{2}\frac{1}{n-1}$, and by Malus's Law, the intensity drop is the square of the cosine of that difference. We then take this to the power of $n-1$, because each of the last $n-1$ filters decreases the intensity by this much, while the first one halves the intensity regardless. Now, we can check this for a few small values of n, and see that $f(7) \approx 0.330 < \frac{1}{3} < f(8) \approx 0.350$. Thus, 8 polarizers are necessary.

e. (Answer) *First, we should think that the multiplicative effect of the later filters should approach* 1*. While there are infinitely many of them, each only attenuates a tiny fraction of incident light. Furthermore, by the paradox itself, it is believable that the intensity should be a strictly increasing function. But we can prove this with calculus too. Restating the double angle formula, we have for any* θ*, that*

$$\cos^2\theta = \frac{1+\cos 2\theta}{2},$$

so our function is equivalent to

$$\frac{1}{2}\left(\frac{1+\cos\frac{\pi}{n-1}}{2}\right)^{n-1}.$$

Now, note that the Taylor Expansion of $\cos\theta$ *is*

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots.$$

We can, for sake of simplicity, let $x = n-1$*, to find*

$$\frac{1}{2}\left(\frac{1+\cos\frac{\pi}{x}}{2}\right)^{x},$$

which seems cleaner (this is valid because both approach ∞*. Then, expanding* $\cos\frac{\pi}{x}$*, our limit is*

$$\lim_{x\to\infty}\frac{1}{2}\left(\frac{1+1-\frac{\pi^2}{2!x^2}+\frac{\pi^4}{4!x^4}-\frac{\pi^6}{6!x^6}+\cdots}{2}\right)^{x},$$

or equivalently,

$$\lim_{x\to\infty}\frac{1}{2}\left(1+\frac{-\frac{\pi^2}{2!x^2}+\frac{\pi^4}{4!x^4}-\frac{\pi^6}{6!x^6}+\cdots}{2}\right)^{x}.$$

Assuming x *to be an integer, the only terms in the full expansion that do not contain an* x *in the denominator is when we select the* 1 *term every time. Thus, we have*

$$\lim_{x\to\infty}\frac{1}{2}\left(1+O\left(\frac{1}{x^2}\right)\right) = \frac{1}{2}.$$