MATH 1070, Exam 1

November 16, 2009

12:00 to 12:50

Instructions

1. You may use a one page formula sheet. Formula sheets may not be shared.
2. Before you begin, enter your name in the space below.
3. Show all your work on the exam itself. If you need additional space, use the backs of the pages.
4. You may not use books or notes on the exam.

Name	

1	25	
2	25	
3	25	
4	25	
Total	100	

I.

1. Compute $T_4(f)$ and $S_4(f)$ for the integral

$$I = \int_0^1 \sqrt{x} e^x dx$$

Compute the errors $I - T_4$ and $I - S_4$.

2. The *degree of precision* of a quadrature rule is defined as follows: If the formula has zero error when integrating any polynomial of degree $\leq r$, and if the error is nonzero for some polynomial of degree $r+1$, then we say the formula has degree of precision equal to r.
Let

$$I_h = \frac{3h}{4} \left[f(0) + 3f(2h) \right].$$

What is the degree of precision of the approximation $I_h \approx \int_0^{3h} f(x)dx$?

- $f = 1 \rightarrow \int_0^{3h} 1dx = 3h \approx \frac{3h}{4}(1+3) = 3h.$
- $f = x \rightarrow \int_0^{3h} xdx = \frac{x^2}{2}\Big|_0^{3h} = \frac{9h^2}{2} \approx \frac{3h}{4}\left(0 + 3(2h)\right) = 3h = \frac{9h^2}{2}.$
- $f = x^2 \rightarrow \int_0^{3h} x^2dx = \frac{x^3}{3}\Big|_0^{3h} = \frac{3^3h^3}{3} = 3^2h^3 \approx \frac{3h}{4}\left(0^2 + 3(2h)^2\right) = 9h^3.$
- $f = x^3 \rightarrow \int_0^{3h} x^3dx = \frac{x^4}{4}\Big|_0^{3h} = \frac{3^4h^4}{4} \neq \frac{3h}{4}\left(0^3 + 3(2h)^3\right) = 36h^4.$

The degree of precision is 2.

II.

1. Consider using the trapezoidal rule T_n to estimate the integral

$$I = \int_{-1}^{1} \frac{dx}{2+x}$$

Give both a rigorous error bound for $I - T_n$ and an asymptotic error estimate $I - T_n$. Using the rigorous error bound, determine how large n should be in order that $|I - T_n| \leq 5 \times 10^{-8}$.

$$E_n^T \equiv I(f) - T_n(f) = \frac{-h^2(b-a)}{12} f''(c_n).$$

Here $h = \frac{b-a}{n} = \frac{2}{n}$, and $f(x) = (2+x)^{-1}, f'(x) = -(2+x)^{-2}, f(x) = 2(2+x)^{-3}$.

$$|E_n^T(f)| = \frac{\frac{4}{n^2}2}{12} \frac{2}{(2+c_n)^3}$$

Since $c_n \in [-1,1]$, $\frac{1}{3} \leq \frac{1}{2+c_n} \leq 1$, $\frac{1}{3^3} \leq \frac{1}{(2+c_n)^3} \leq 1$, hence

$$|E_n^T(f)| = \frac{\frac{4}{n^2}2}{12} \frac{2}{(2+c_n)^3} \leq \frac{\frac{4}{n^2}2}{12} 2 = \frac{4}{3n^2}$$

To have $E_n^T \leq 5 \times 10^{-8}$ it suffices to have

$$|E_n^T(f)| \leq \frac{4}{3n^2} \leq 5 \times 10^{-8} \text{ i.e., } n^2 \geq \frac{4}{3 \cdot 5} 10^8.$$

2. Repeat with Simpson's rule.

III.

1. Determine constants c_1 and c_2 in the formula

$$\int_0^1 f(x)dx \approx c_1 f(0) + c_2 f(1)$$

so that is exact for all polynomials of as large a degree as possible. What is the degree of precision of the formula?

- $f = 1 \rightarrow \int_0^1 1dx = 1 \approx c_1 + c_2$.
- $f = x \rightarrow \int_0^1 xdx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2} \approx c_1 \cdot 0 + c_2 \cdot 1$.

Hence $c_1 = \frac{1}{2} = c_2$.

- $f = x^2 \rightarrow \int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} \neq c_1 \cdot 0^2 + c_2 \cdot 1^2 = \frac{1}{2}$.

The degree of precision is 1.

2. To find a formula

$$\int_0^1 f(x) \ln\left(\frac{1}{x}\right) dx \approx w_1 f(x_1) + w_2 f(x_2) \equiv I_2(f)$$

which is exact for all polynomials of degree ≤ 3, set up a system of four equations with unknowns w_1, w_2, x_1, x_2. Do not solve the system. Instead, show that

$$x_1 = \tfrac{15-\sqrt{106}}{42}, \qquad x_2 = \tfrac{15+\sqrt{106}}{42}$$
$$w_1 = \tfrac{21}{\sqrt{106}}\left(x_2 - \tfrac{1}{4}\right), \quad w_2 = 1 - w_1$$

is a solution of the system.

- $f = 1 \rightarrow \int_0^1 \ln x dx = 1 \approx w_1 + w_2$
- $f = x \rightarrow \int_0^1 x(\ln 1 - \ln x)dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^1 = \frac{1}{4} \approx w_1 x_1 + w_2 x_2$.
- $f = x^2 \rightarrow \int_0^1 x^2(0 - \ln x)dx = \frac{1}{3}\left.\frac{x^3}{3}\right|_0^1 = \frac{1}{9} \approx w_1 x_1^2 + w_2 x_2^2$.
- $f = x^3 \rightarrow \int_0^1 x^3(0 - \ln x)dx = \frac{1}{4}\left.\frac{x^4}{4}\right|_0^1 = \frac{1}{16} \approx w_1 x_1^3 + w_2 x_2^3$.

IV.

1. Use the method of undetermined coefficients to derive the formula

$$f''(t) \approx Af(t+2h) + Bf(t+h) + Cf(t).$$

with the error as small as possible.

- $f(t+2h) = f(t) + f'(t)\frac{2h}{1!} + f''(t)\frac{(2h)^2}{2!}$
- $f(t+2h) = f(t) + f'(t)\frac{h}{1!} + f''(t)\frac{(h)^2}{2!}$
- $f(t) = f(t)$

$A + B + C = 0, A2h + Bh = 0, A2h^2 + B\frac{h^2}{2} = 1.$ Solve for A, B, C.

2. Find the linear least squares approximation to $f(x) = e^x$ on the interval $[0, 1]$.
Hint: Use the direct method of Example 4.7.1, but on the interval [0,1].