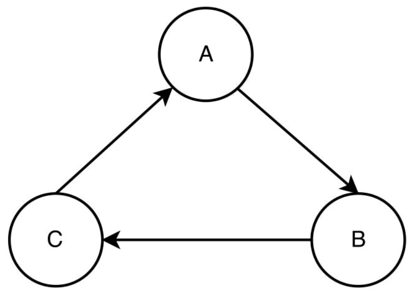

Figure 1: Network graph example

1 Approximation of random walk on time inhomogenous Markov chain

$\mathbf{A}$ is an adjacency matrix of our network which is shown in figure 1.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We can compute transition matrix $\mathbf{P}$ with respect to adjacency matrix $\mathbf{A}$ as a Page Rank transition matrix:

$$\mathbf{P} = \alpha \mathbf{A}\mathbf{D}^{-\mathbf{1}} + \frac{1-\alpha}{n} \mathbf{1} \cdot \mathbf{1}^{\mathbf{T}},$$

where matrix $\mathbf{D}$ is diagonal matrix with elements $D_{ii} = max(k_i^{out}, 1)$ where k_i^{out} is number of out-degree of each node. Factor α is the probability of following a link from node and $(1-\alpha)$ is the probability teleporting to any node. Matrix $\mathbf{1} \cdot \mathbf{1}^{\mathbf{T}}$ is square matrix with size $(n \times n)$ where each element is equal to 1.

We want to build hierarchical model in the following way. Suppose we have more than one parameter α, which implies that we have more than one transition matrix $\mathbf{P}$. We want to build hierarchical model in the following way: one step is composed of different transition matrix $\mathbf{P_i}$ which is computed with different α_i. For the first step we would take P_1 with α_1, for the second one P_2 with α_2 and so one. Last α_i is 0. Our hierarchical model with different transition matrices P_i is shown in 2.

We can write the transition matrix $\mathbf{P'}$ of the graph 2 in the following way:

$$\mathbf{P'} = \begin{bmatrix} 1-\alpha_1 & 1-\alpha_2 & 1-\alpha_3 \\ \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \end{bmatrix}$$

We want to compute different transition matrices for our adjacency matrix $\mathbf{A}$ with respect to different parameters α: $\mathbf{P_1}$ is transition matrix of $\mathbf{A}$ with

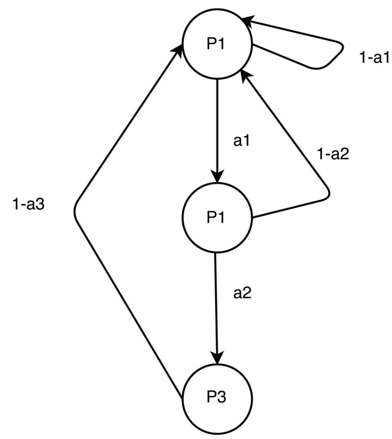

Figure 2: Hierarchical model

parameter α_1, $\mathbf{P_2}$ is transition matrix of $\mathbf{A}$ with parameter α_2 and $\mathbf{P_3}$ is transition matrix of $\mathbf{A}$ with parameter α_3. Our parameters are: $\alpha_1 = 0.8$, $\alpha_2 = 0.4$ and $\alpha_3 = 0$. If we use PageRank formula we get:

$$\mathbf{P_1} = \begin{bmatrix} 0.067 & 0.067 & 0.867 \\ 0.867 & 0.067 & 0.067 \\ 0.067 & 0.867 & 0.067 \end{bmatrix}$$

$$\mathbf{P_2} = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$$

$$\mathbf{P_2} = \begin{bmatrix} 0.333 & 0.333 & 0.333 \\ 0.333 & 0.333 & 0.333 \\ 0.333 & 0.333 & 0.333 \end{bmatrix}$$

Note that each $\mathbf{P_i}$ matrix is stochastic (sum over the columns is 1).

We can also compute stationary distribution $\pi(\mathbf{P'})$ of the graph $\mathbf{P'}$. We get:

$$\pi(\mathbf{P'}) = \begin{bmatrix} 0.47169811 & 0.37735849 & 0.1509434 \end{bmatrix} \tag{1}$$

This tells us the probabilities we will be in the state $\mathbf{P_1}$ (with probability 0.6), $\mathbf{P_2}$ (with probability 0.276) or in $\mathbf{P_3}$ (with probability 0.133).

We know that $\sum_{i=1} \pi_{\mathbf{i}} = 1$. Therefore we can combine π and matrices $\mathbf{P_i}$ to get the final transition matrix $\mathbf{P}$:

$$\mathbf{P} = \pi_{\mathbf{1}} \cdot \mathbf{P_1} + \pi_{\mathbf{2}} \cdot \mathbf{P_2} + \pi_{\mathbf{3}} \cdot \mathbf{P_3}.$$

Applying the results we already computed we get:

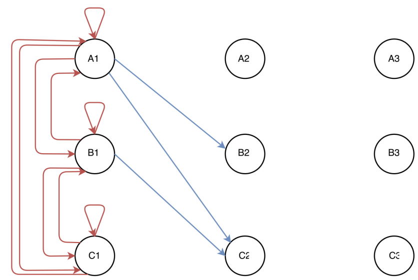

Figure 3: Transitions for the first time step

$$\mathbf{P} = \begin{bmatrix} 0.13777778 & 0.13777778 & 0.72444444 \\ 0.72444444 & 0.13777778 & 0.13777778 \\ 0.13777778 & 0.72444444 & 0.13777778 \end{bmatrix}$$

Because $\mathbf{P_i}$ are stochastic and π is stationary distribution, matrix $\mathbf{P}$ is also stochastic matrix. Matrix $\mathbf{P}$ tells us with which probabilities we will be in states A, B or C (from figure 1) if we dynamically change probabilities α_i and randomly walk on graph 2.

We will calculate stationary distribution $\pi(\mathbf{P})$ of matrix $\mathbf{P}$ as well:

$$\pi(\mathbf{P}) = \begin{bmatrix} 0.33333333 & 0.33333333 & 0.33333333 \end{bmatrix} \tag{2}$$

2 Exact solution of random walk on time inhomogenous Markov chain

In this part we have to write all the states that are possible if we take into account our two graphs on figures 2, 1. We have 3 states from graph 1 and 3 states from graph 2, which are actually time states. Altogether we have 9 states which are denoted as: $A1, B1, C1, A2, B2, C2, A3, B3, C3$. States $A1, B1, C1$ actually represents graph 1 in state P_1 from graph 2 with probability α_1. Also, same applies for all other states.

We can draw the transitions between these states (Ai, Bi, Ci). For the first time step, transitions are shown in figure 3. Blue arrows represents random surfer decision to follow links with probability $\frac{\alpha_1}{k_{out}}$. The direction is determined by graph in figure 1. Red arrows represent probability of jumping to any node in first time step: $\frac{1-\alpha_1}{3}$.

Second time step is in figure 4. Again, blue arrows represents the random surfer decision to follow links, probability is similar as before: $\frac{\alpha_2}{k_{out}}$. Red arrows again represents probability of teleportation. The direction of teleportation is determined by the graph in figure 2 (we are always jumping back to the first time step). Probabilities are similar as before: $\frac{1-\alpha_2}{3}$.

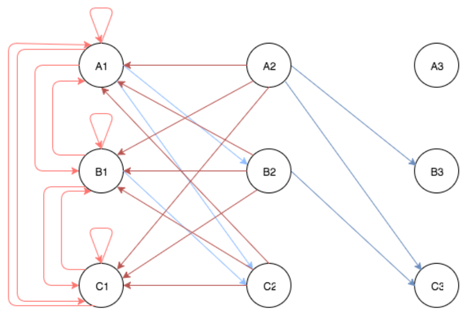

Figure 4: Transitions for the second time step

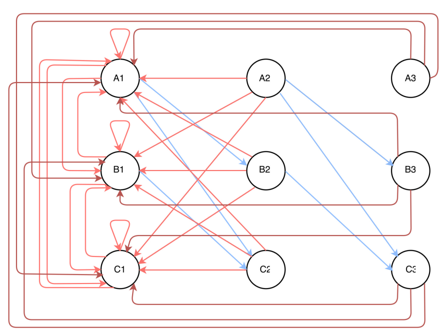

Figure 5: Transitions for the last time step

Last step consists of only teleportations back to the first time step and is shown in figure 5. The probabilities of jumping back are $\frac{1-\alpha_3}{3} = \frac{1}{3}$.

We can finally write our transition matrix $\mathbf{T}$ for this example:

$$\mathbf{T} = \begin{bmatrix} \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2 & 0 \\ \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} \\ \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} \\ 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 \\ 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 \\ 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 \\ 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can calculate stationary distribution of matrix $\mathbf{T}$:

$$\pi(\mathbf{T}) = \begin{bmatrix} 0.15723 & 0.125786 & 0.050314 & 0.15723 & 0.125786 & 0.050314 & 0.15723 & 0.125786 & 0.050314 \end{bmatrix}$$

Stationary distribution $\pi_{\mathbf{T}}$ of matrix $\mathbf{T}$ tells us, that we will be in state $A1$ with probability 0.1572327, in state $A2$ with probability 0.18867925, in state $A3$ with probability 0.05031447 and so on. States $A1, B1, C1$ can be represented as matrix $\mathbf{P_1}$ with probability α_1 (what we already have calculated in previous section), states $A2, B2, C2$ can be represented with matrix $\mathbf{P_2}$ and lastly, states $A3, B3, C3$ can be represented with matrix $\mathbf{P_3}$.

We can therefore write our stationary distribution in such a way, that we merge the states that are under same matrix P_i, meaning:

$$\pi(\mathbf{P'}*) = \begin{bmatrix} \pi(P_1) & \pi(P_2) & \pi(P_3) \end{bmatrix} = \begin{bmatrix} \pi(A1 + B1 + C1) & \pi(A2 + B2 + C2) & \pi(A3 + B3 + C3) \end{bmatrix}$$

$$\pi(\mathbf{P'}*) = \begin{bmatrix} 0.471698113208 & 0.377358490566 & 0.150943396226 \end{bmatrix}$$

We can see that we get the same results as in equation 1 in the previous section.

We can also sum up stationary distributions of nodes:

$$\pi(\mathbf{P}*) = \begin{bmatrix} \pi(A1 + A2 + A3), \pi(B1 + B2 + B3), \pi(C1 + C2 + C3) \end{bmatrix}$$

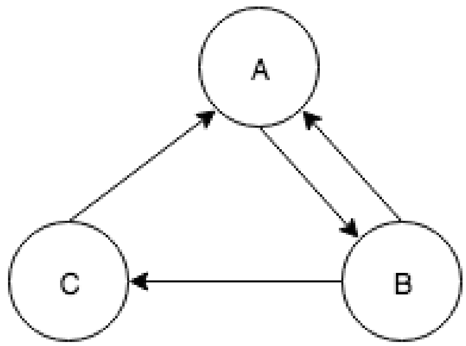

Figure 6: Network graph example

$$\pi(\mathbf{P}*) = \begin{bmatrix}0.33 & 0.33 & 0.33\end{bmatrix}$$

We can see that computed stationary distribution is the same as the stationary distribution $\pi(P)$ in equation 2.

3 More examples

The first network example in figure 1 was really simple. Lets take a look at more complex examples. Note that the probabilities are the same as before.

3.1 First example

The network is illustrated in figure 6. We can write adjacency matrix $\mathbf{A}$ for this example:

$$\mathbf{A} = \begin{bmatrix}0 & 1 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$$

We will use same transition matrix $\mathbf{P'}$ (in figure 2). Therefore the stationary distribution of this transition matrix $\mathbf{P'}$ is the same as in previous example:

$$\pi(\mathbf{P'}) = \begin{bmatrix}0.47169811 & 0.37735849 & 0.1509434\end{bmatrix} \tag{3}$$

However the network is not the same as before, therefore our transition matrix $\mathbf{T}$ will be different.

$$\mathbf{T} = \begin{bmatrix} \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} \\ 0 & 0 & 0 & \frac{\alpha_1}{2} & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha_2}{2} & 0 & 0 & \alpha_2 & 0 \\ \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} \\ \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} \\ 0 & 0 & 0 & \frac{\alpha_1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha_2}{2} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Plugging in the values, we get:

$$\mathbf{T} = \begin{bmatrix} 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0.4 & 0 \\ 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 \\ 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Stationary distribution $\pi(\mathbf{T})$ of matrix $\mathbf{T}$ is:

$$\pi(\mathbf{T}) = \begin{bmatrix} 0.15723 & 0.188679 & 0.050314 & 0.15723 & 0.125786 & 0.07547 & 0.15723 & 0.062893 & 0.025157 \end{bmatrix}$$

Again, we can sum up nodes which are generated under the same probability:

$$\pi(\mathbf{P}'*) = \begin{bmatrix} \pi(P_1) & \pi(P_2) & \pi(P_3) \end{bmatrix} = \begin{bmatrix} \pi(A1+B1+C1) & \pi(A2+B2+C2) & \pi(A3+B3+C3) \end{bmatrix}$$

$$\pi(\mathbf{P}'*) = \begin{bmatrix} 0.471698113208 & 0.377358490566 & 0.150943396226 \end{bmatrix}$$

We can see that stationary distribution is again the same as stationary distribution of transition matrix $\mathbf{P}'$ (equation 3.

We can also sum up stationary distribution by nodes (A1+A2+A3) etc.:

$$\pi = \begin{bmatrix} \pi(A1+A2+A3), \pi(B1+B2+B3), \pi(C1+C2+C3) \end{bmatrix}$$

$$\pi(\mathbf{P}*) = \begin{bmatrix} 0.396226415094 & 0.358490566038 & 0.245283018868 \end{bmatrix} \tag{4}$$

In order to verify the result from equation 4, we have to compute matrix $\mathbf{P}$ and its stationary distribution.

$$\mathbf{P} = \begin{bmatrix} 0.13777778 & 0.43111111 & 0.72444444 \\ 0.72444444 & 0.13777778 & 0.13777778 \\ 0.13777778 & 0.43111111 & 0.13777778 \end{bmatrix}$$

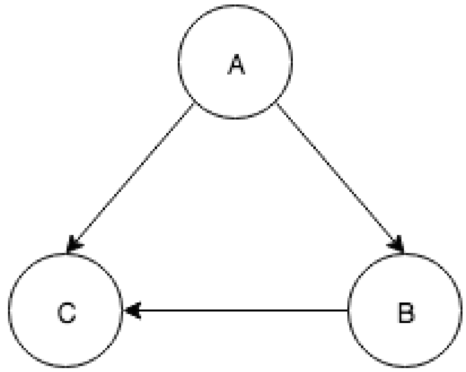

Figure 7: Network graph example

Stationary distribution $\pi(P)$ of matrix P is:

$$\pi(\mathbf{P}) = \begin{bmatrix} 0.38612593 & 0.36122376 & 0.2526503 \end{bmatrix} \tag{5}$$

The results from equations 5 and 4 are almost the same.

3.2 Second example

Second example is in figure 7. Note that node C does not have any outlinks.

We can write adjacency matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Again, we will use the same transition matrix **P'** (figure 2) and the stationary distribution is the same:

$$\pi(\mathbf{P'}) = \begin{bmatrix} 0.47169811 & 0.37735849 & 0.1509434 \end{bmatrix} \tag{6}$$

The transition matrix $\mathbf{T}$ is different:

$$\mathbf{T} = \begin{bmatrix}
\frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{alpha_1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{alpha_2}{3} & 0 \\
\frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} \\
\frac{\alpha_1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{alpha_1}{3} & 0 & 0 \\
0 & \frac{\alpha_2}{2} & 0 & 0 & 0 & 0 & 0 & \frac{alpha_2}{3} & 0 \\
\frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} & \frac{1-\alpha_1}{3} & \frac{1-\alpha_2}{3} & \frac{1-\alpha_3}{3} \\
\frac{\alpha_1}{2} & 0 & 0 & \alpha_1 & 0 & 0 & \frac{alpha_1}{3} & 0 & 0 \\
0 & \frac{\alpha_2}{2} & 0 & 0 & \alpha_2 & 0 & 0 & \frac{alpha_2}{3} & 0
\end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.26666667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.13333333 & 0 \\ 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 \\ 0.4 & 0 & 0 & 0 & 0 & 0 & 0.26666667 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0.13333333 & 0 \\ 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 & 0.06666667 & 0.2 & 0.33333333 \\ 0.4 & 0 & 0 & 0.8 & 0 & 0 & 0.26666667 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0.4 & 0 & 0 & 0.13333333 & 0 \end{bmatrix}$$

Because our network consisted of one node (C) that does not have any outlinks, we have to consider this when creating the matrix T. We only need to take care of the 7th and 8th column of the matrix T because they do not sum to 1 if we do not take care of the dangling node C. Therefore we can imagine that from node $C1$ we can get to nodes $A2$, $B2$ and $C2$ with equal probability of $\frac{alpha_1}{3}$ (7th column). Same goes for the 8th column: we can get from node $C2$ to nodes $A3$, $B3$ and $C3$ with equal probability of $\frac{alpha_2}{3}$.

Stationary distribution of the matrix $\mathbf{T}$ is:

$$\pi = \begin{bmatrix} 0.1572 & 0.0419 & 0.0307 & 0.157 & 0.1048 & 0.039 & 0.1572 & 0.2306 & 0.081 \end{bmatrix}$$

Again, we can sum up nodes which are generated under the same probability:

$$\pi = \begin{bmatrix} \pi(P_1) & \pi(P_2) & \pi(P_3) \end{bmatrix} = \begin{bmatrix} \pi(A1 + B1 + C1) & \pi(A2 + B2 + C2) & \pi(A3 + B3 + C3) \end{bmatrix}$$

$$\pi = \begin{bmatrix} 0.47169811 & 0.37735849 & 0.1509434 \end{bmatrix}$$

The probabilities are almost the same as the stationary distribution of $\mathbf{P}'$ in equation 6.

We can also sum up stationary distribution by nodes (A1+A2+A3) etc.:

$$\pi = \begin{bmatrix} \pi(A1 + A2 + A3), \pi(B1 + B2 + B3), \pi(C1 + C2 + C3) \end{bmatrix}$$

$$\pi = \begin{bmatrix} 0.22990915 & 0.30118798 & 0.46890287 \end{bmatrix} \tag{7}$$

In order to verify the result from equation 7, we have to compute matrix $\mathbf{P}$ and its stationary distribution.

$$\mathbf{P} = \begin{bmatrix} 0.13777778 & 0.13777778 & 0.33333333 \\ 0.43111111 & 0.13777778 & 0.33333333 \\ 0.43111111 & 0.72444444 & 0.33333333 \end{bmatrix}$$

Corresponding stationary distribution of matrix $\mathbf{P}$:

$$[\pi(\mathbf{P}) = \begin{bmatrix} 0.2383134 & 0.3012641 & 0.4604225 \end{bmatrix} \tag{8}$$

We can see that the result from equation 8 is almost equal to the one in equation 7.

4 Generating transition matrix $\mathbf{T}$

Before we actually write down the transition matrix $\mathbf{T}$. However it is possible to generate it from adjacency matrix $\mathbf{A}$ and transition matrix $\mathbf{P'}$.

In order to generalize it - to work with dangling nodes as well - we have to first make a small change to matrix $\mathbf{A}$. We first sum up all the columns of matrix $\mathbf{A}$. We replace those columns that sum to 0 with columns full of ones. Everything else stays the same.

In order to generate matrix $\mathbf{T}$, we have to first "normalize" adjacency matrix $\mathbf{A}$ (to take care of the nodes that have more than one outgoing link). Then we calculate the Kronecker product of $\mathbf{A_{norm}}$ and $\mathbf{P'}$:

$$\mathbf{T'} = \mathbf{A_{norm}} \otimes \mathbf{P'}.$$

Also we know that the first (0-th) row, and all i-th rows that are by modulo of number of transition probabilities α 0, contains only teleportation probabilities. We can replace those rows by teleportation probabilities and this is already our matrix $\mathbf{T}$.

In order to verify our method, we calculate the stationary distribution of matrix $\mathbf{T}$ and sum it up in two ways as before (one way: group all same nodes from $\mathbf{A}$ together, another way: group all nodes from $\mathbf{P'}$ together) and compare it to stationary distributions of $\mathbf{P'}$ and $\mathbf{P}$.

This method works (there are around 1% deviations in simple examples, which could be due to numeric precision).