

# On Pilot Design for Wideband Sparse MIMO Channels

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## 1 Channel Model

Consider the wideband MIMO channel characterized by impulse response  $\{\mathbf{H}[l]\}_{l=0}^{L-1}$ , where  $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$  and where  $N_t, N_r, L$  denote the number of transmit antennas, receive antennas, and taps, respectively. If  $\mathbf{t}[n] \in \mathbb{C}^{N_t}$  is transmitted over the channel, then the receiver sees  $\mathbf{y}[n] = \mathbf{z}[n] + \mathbf{w}[n]$  with

$$\mathbf{z}[n] = \sum_{l=0}^{L-1} \mathbf{H}[l] \mathbf{t}[n-l] \quad (1)$$

and AWGN  $\mathbf{w}[n]$ . Likewise, if the  $N$ -symbol sequence  $\mathbf{T} \in \mathbb{C}^{N_r \times N}$  is transmitted, the receiver sees  $\mathbf{Y} = \mathbf{Z} + \mathbf{W}$  with

$$\mathbf{Z} = [\mathbf{H}[0] \ \mathbf{H}[1] \ \cdots \ \mathbf{H}[L-1]] \begin{bmatrix} \mathbf{T}\mathbf{J}_0 \\ \mathbf{T}\mathbf{J}_1 \\ \vdots \\ \mathbf{T}\mathbf{J}_{L-1} \end{bmatrix} \quad (2)$$

and AWGN  $\mathbf{W}$ , where right multiplication by  $\mathbf{J}_l$  circularly shifts the columns right by  $l$  places.

Let us now define the virtual channel coefficients

$$\mathbf{X}[l] \triangleq \mathbf{B}_{N_r}^* \mathbf{H}[l] \mathbf{B}_{N_t}, \quad (3)$$

where

$$\mathbf{B}_N \triangleq \mathbf{F}_{\sqrt{N}} \otimes \mathbf{F}_{\sqrt{N}}, \quad (4)$$

is the steering vector matrix for an  $N$ -antenna uniform square array and  $\mathbf{F}_N$  is the unitary  $N$ -DFT matrix. Note that  $\mathbf{B}_N$  is symmetric and unitary, so that

$$\mathbf{H}[l] \triangleq \mathbf{B}_{N_r} \mathbf{X}[l] \mathbf{B}_{N_t}^*, \quad (5)$$

The virtual channel coefficients  $\{\mathbf{X}[l]\}$  are expected to be sparse, which can be exploited in channel estimation. Thus, we rewrite (2) in terms of  $\{\mathbf{X}[l]\}$  as

$$\mathbf{Z} = \mathbf{B}_{N_t} \underbrace{[\mathbf{X}[0] \ \mathbf{X}[1] \ \cdots \ \mathbf{X}[L-1]]}_{\triangleq \mathbf{X}} \underbrace{\begin{bmatrix} \mathbf{B}_{N_r}^* & & \\ & \mathbf{B}_{N_r}^* & \\ & & \ddots \\ & & & \mathbf{B}_{N_r}^* \end{bmatrix}}_{= \mathbf{I}_L \otimes \mathbf{B}_{N_r}^*} \underbrace{\begin{bmatrix} \mathbf{T}\mathbf{J}_0 \\ \mathbf{T}\mathbf{J}_1 \\ \vdots \\ \mathbf{T}\mathbf{J}_{L-1} \end{bmatrix}}_{\triangleq \tilde{\mathbf{T}}}. \quad (6)$$

## 2 Pilot Design

We are interested in a pilot sequence  $\mathbf{T}$  for which:

1.  $\tilde{\mathbf{T}}$  is well conditioned,
2. there is a fast way of computing  $\mathbf{Z}$  from a given  $\mathbf{X}$ .

Towards this aim, we propose to construct  $\mathbf{T}$  such that

$$[\mathbf{T}]_{n,m} = t[\lfloor nL + mN_r/N \rfloor], \quad (7)$$

where  $N, \sqrt{N_t}, \sqrt{N_r}$  are all powers of two, and where  $\{t[n]\}_{n=0}^{N-1}$  is one of several possibilities:

- a) random BPSK,
- b) Chu sequence, i.e.,  $t[n] = \exp(j\pi n(n+1)/N)$  for  $n = 0 \dots N-1$ ,
- c) all-pass sequence, i.e.,  $\mathbf{t} = \mathbf{F}_N \mathbf{a}$  for any  $\mathbf{a}$  with  $|a_n| = 1$ .

The reasoning behind (7) is that the rows of  $\tilde{\mathbf{T}}$  will consist of the first  $N_t L$  circular shifts of the sequence  $\mathbf{t} \triangleq [t[0] \dots t[N-1]]$ , and thus multiplication by  $\tilde{\mathbf{T}}$  can be accomplished using fast convolution (i.e., two  $N$ -FFTs). Similarly, the operations involving  $\mathbf{B}_{N_t}$  and  $\mathbf{B}_{N_r}$  can also be accomplished using FFTs.

Figure 1 plots the singular values for one realization of  $\tilde{\mathbf{T}}$  under several choices of  $\mathbf{t}$ . For reference, it also shows the singular values for one realization of  $\tilde{\mathbf{T}}$  with i.i.d. Gaussian entries (i.e., no convolution structure). The figure shows that the convolution matrix  $\tilde{\mathbf{T}}$  constructed using random BPSK  $\mathbf{t}$  has a similar spectrum to the i.i.d. Gaussian  $\tilde{\mathbf{T}}$ , which is widely considered as an “isometry” in the field of compressive sensing. The figure also shows that the spectrum resulting from all-pass  $\mathbf{t}$  has a flat singular-value spectrum, as expected. The practical disadvantage of the all-pass sequence is that the elements of  $\mathbf{t}$  are far from constant modulus, which implies a high peak-to-average power ratio. Finally, the figure shows that the Chu sequence has a singular value spectrum close to the allpass sequence, with the advantage of constant modulus  $t[n]$ .

## 3 Fast Implementation

We now provide some details on the fast implementation of (6). Recall that, due to the construction (7), the first  $N_t$  rows in  $\tilde{\mathbf{T}}$  contain the  $\{0, L, 2L, \dots, (N_t - 1)L\}$  shifts of  $\mathbf{t}$ . Likewise, the next  $N_t$  rows in  $\tilde{\mathbf{T}}$  contain the  $\{1, L + 1, 2L + 1, \dots, (N_t - 1)L + 1\}$  shifts of  $\mathbf{t}$ , and the last  $N_t$  rows in  $\tilde{\mathbf{T}}$  contain the  $\{L - 1, 2L - 1, 3L - 1, \dots, N_t L - 1\}$  shifts of  $\mathbf{t}$ . Altogether, the rows of  $\tilde{\mathbf{T}} \in \mathbb{C}^{N_t L \times N}$  contain the  $\{0, 1, 2, \dots, N_t L\}$  shifts of  $\mathbf{t}$ . We can thus re-order the rows in  $\tilde{\mathbf{T}}$  to make  $\tilde{\mathbf{T}}$  a Toeplitz matrix, which we will denote by  $\bar{\mathbf{T}}$  in the sequel. Doing so yields

$$\mathbf{Z} = \mathbf{B}_{N_t} \mathbf{X} \begin{bmatrix} b_0 & & & & \\ & b_0 & & & \\ & & \ddots & & \\ & & & b_0 & \\ & & & & b_1 \\ & & & & & \ddots \\ & & & & & & b_{N_t-1} \\ & & & & & & & \ddots \\ & & & & & & & & b_{N_t-1} \\ & & & & & & & & & \ddots \\ & & & & & & & & & & b_{N_t-1} \end{bmatrix} \bar{\mathbf{T}} \quad (8)$$

$$= \mathbf{B}_{N_t} \underbrace{\begin{bmatrix} x_0[0] \cdots x_0[L-1] & x_1[0] \cdots x_1[L-1] & \cdots & x_{N_t-1}[0] \cdots x_{N_t-1}[L-1] \end{bmatrix}}_{\substack{\text{1st Tx antenna} \\ \triangleq \bar{\mathbf{X}}^T \quad \text{Last Tx antenna}}} (\mathbf{B}_{N_t}^* \otimes \mathbf{I}_L) \bar{\mathbf{T}} \quad (9)$$

$$= \mathbf{B}_{N_t} \left[ \bar{\mathbf{T}}^T (\mathbf{B}_{N_t}^* \otimes \mathbf{I}_L) \bar{\mathbf{X}} \right]^T, \quad (10)$$

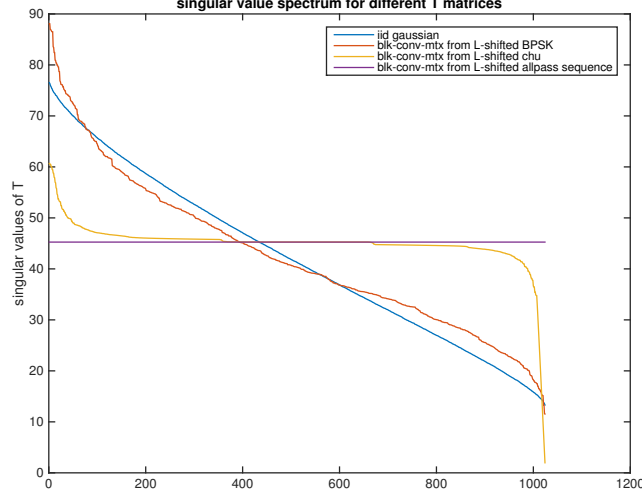


Figure 1: Singular values of  $\tilde{\mathbf{T}}$  for various constructions of  $\mathbf{t}$ .

where  $\mathbf{b}_i$  is the  $i$ th column of  $\mathbf{B}_{N_r}^*$  and  $\mathbf{x}_i[l]$  is the  $i$ th column of  $\mathbf{X}[l]$ . For the last equality above, we exploited the fact  $(\mathbf{A} \otimes \mathbf{B})^\top = \mathbf{A}^\top \otimes \mathbf{B}^\top$  and the symmetry of  $\mathbf{B}_{N_r}^*$ .

The fast implementation of (10) is now visible. First, for any  $\mathbf{v} \in \mathbb{C}^{LN_t}$ , notice that

$$(\mathbf{B}_{N_t}^* \otimes \mathbf{I}_L)\mathbf{v} = \text{vec}(\mathbf{V}\mathbf{B}_{N_t}^*) = \text{vec}([\mathbf{B}_{N_t}\mathbf{V}^H]^\text{H}), \quad (11)$$

where  $\mathbf{V} \in \mathbb{C}^{L \times N_t}$  is the column-wise matricization of  $\mathbf{v}$ . Furthermore, for any  $\mathbf{u} \in \mathbb{C}^{N_t}$ , notice that

$$\mathbf{B}_{N_t}\mathbf{u} = (\mathbf{F}_{\sqrt{N_t}} \otimes \mathbf{F}_{\sqrt{N_t}})\mathbf{u} = \text{vec}(\mathbf{F}_{\sqrt{N_t}}\mathbf{U}\mathbf{F}_{\sqrt{N_t}}), \quad (12)$$

where  $\mathbf{U} \in \mathbb{C}^{\sqrt{N_t} \times \sqrt{N_t}}$  is the column-wise matricization of  $\mathbf{u}$ . Thus the multiplication  $(\mathbf{B}_{N_t}^* \otimes \mathbf{I}_L)\bar{\mathbf{X}}$  in (10) can be accomplished using  $\sqrt{N_t}$ -point FFTs. The result of that multiplication is then left-multiplied by  $\bar{\mathbf{T}}^\top$ , which can be performed via fast convolution using  $N$ -point FFTs. And the result of that multiplication is then left-multiplied by  $\mathbf{B}_{N_t}$ , which can be performed as in (12) using  $\sqrt{N_t}$ -point FFTs. [Should count how many total FFTs of each length.]

If the above complexity is still too high, it can be reduced by choosing  $\mathbf{t} = [1, 0, 0, \dots, 0]$ , which turns  $\bar{\mathbf{T}}$  into an  $N_t L \times N$  identity matrix. The cost is a drastic reduction in SNR. Still, it may be worth investigating as an option.