$$f(p) = \max_{i=1,\dots,n} |\log a_i^T p - \log I_{des}|$$ ①

a: Show that $\exp f$ is convex on $\{p \mid a_i^T p > 0, i=1 \dots n\}$

Solution: To simplify we assume that $I_{des} = 1$

$$f(p) = \max_i |\log a_i^T p|$$

$$\{p \mid a_i^T p > 0, i = 1, 2, \dots, n\}$$

$$|\log(a_i^T p)| = \max\{\log a_i^T p, \log(1/a_i^T p)\}$$

$$\Rightarrow = \log \max\{a_i^T p, 1/a_i^T p\}$$

$$f(p) = \log \max_i \max\{a_i^T p, 1/a_i^T p\}$$

Both $a_i^T p$, $1/a_i^T p$ are convex on dom f.

$\max_i \max\{a_i^T p, 1/a_i^T p\}$ is a convex function.

$\exp f$ is convex

b: Show that the constraint 'no mor than half of the total Power is in any 10 lamps'. is convex.

Solution:

$$\sum_{i=1}^{l} P_{[i]} - 0.5 \sum_{i=1}^{m} P_i \leq 0$$

where $P_{[i]}$ is th ith largest component of P. the firs term on the left side is the Power in the l lamps with the highest Power. The second therm is one half of the total Power that is convex. Since it is the sum of $\sum_{i=1}^{l} P_{[i]}$, which is convex and a linear function

C: Show that the constraint 'no mor then half of the lamps on' is (in general) not convex.

Solution: Consider two solutions P^1 and P^2 that satisfy the constraint. in the first solutions, the first m/2 lamps are on and the rest is off (zero) in second solution the first m/2 lamps are off an the rest is on. The number of nonzero components in a convex combination of P^1 and P^2 will be m. the convex combination does not satisfy the constrain

Pr

Minimizing a function over the Probability simPlex

Find simPle necessary and sufficient conditions for $x \in R^n$ to minimize a differentiable concvex function f over the Probabiity simPlex, $\{x \mid 1^T x = 1, x \succeq 0\}$

Solution: (x) is feasible, $x \succeq 0$, $1^T x = 1$

$\nabla f(x)^T (y - x) \geq 0$ forall feasible y,

$\min \nabla f(x)_i \geq \nabla f(x)^T x$

$\Rightarrow \nabla f(x)_i \geq \nabla f(x)^T x$ for $i = 1, \dots, n \Rightarrow y \succeq 0$, $1^T y = 1$

$$\sum_{i=1}^{n} y_i \nabla f(x)_i \geq \left(\sum_{i=1}^{n} y_i\right) \nabla f(x)^T x = \nabla f(x)^T x$$

$\Rightarrow y^T \nabla f(x) \Rightarrow \nabla f(x)^T (y - x) \geq 0$

$$\min \frac{\partial f}{\partial x_i} \geq \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} \quad (*)$$

$$\Rightarrow 1^T x = 1, \; x \succeq 0 \Rightarrow \min \frac{\partial f}{\partial x_i} \leq \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} \quad (**)$$

$$\xrightarrow{(*)(**)} \min \frac{\partial f}{\partial x_i} = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}$$

$$\Rightarrow x_k > 0 \Rightarrow \frac{\partial f}{\partial x_k} = \min \frac{\partial f}{\partial x_i}$$

P_r

$$\min \frac{\max (a_i^T x + b_i)}{\min (c_i^T x + d_i)}$$

subject $Fx \leq g$

$x \in R^n$. we assume that $c_i^T x + d_i > 0$ and $\max (a_i^T x + b_i) \geq 0$

show how the Problem can be solved by solving one LP.

Solution:

$$\min_{i=1,\dots,m} (\max (a_i^T y + b_i t)$$

$$\text{sob } \min (c_i^T y + d_i t) \geq 1$$

$$Fy \leq gt$$

$$t \geq 0$$

additional variable u: $\min u$

$$\text{sub } a_i^T y + b_i t \leq u, \quad i=1,\dots,m$$

$$c_i^T y + d_i t \geq 1, \quad i=1,\dots,p$$

$$Fy \leq gt$$

$$t \geq 0$$

$t>0$, $(y,t) \neq 0$, $Fy \leq 0$ and $y \neq 0$, y is an unbounded $\{x \mid Fx \leq g\}$

$\Rightarrow$ if $t > 0$ for all feasible y, t

$$\min t \max (a_i^T (y/t) + b_i)$$

$$\text{sub } \min (c_i^T (y/t) + d_i) \geq 1/t$$

$$F(y/t) \leq g$$

$$t \geq 0$$

P_E

$$\min t \max (a_i^T(y/t)+b_i)$$

$$\text{sub } \min (c_i^T(y/t)+d_i) = 1/t$$

$$F(y/t) \leq g$$

$$t \geq 0$$

$\max (a_i^T(y/t)+b_i) \geq 0$ if $F(y/t) \leq g$, choosing t such

$$\min (c_i^T(y/t)+d_i) = 1/t$$

for the optimal t in the cost function

$$\min \frac{\max (a_i^T(y/t)+b_i)}{\min (c_i^T(y/t)+d_i)}$$

$$\text{sub } F(y/t) \leq g$$

$$t \geq 0$$

this is the Problem of the assignment with $x = y/t$

P_0

Show that $X = B^T A^{-1} B$ solves the SDP

$$\min \operatorname{tr} X$$

$$\text{sub} \begin{bmatrix} A & B \\ B^T & X \end{bmatrix} \succeq 0$$

with variable $X \in S^n$, where $A \in S^m_{++}$ and $B \in R^{m \times n}$ are given conclude that $\operatorname{tr}(B^T A^{-1} B)$ is a Convex function of (A, B), for A Positive definite.

Solution: $X \succeq B^T A^{-1} B \Rightarrow \operatorname{tr} X \geq \operatorname{tr}(B^T A^{-1} B)$ for all feasible X

if $X = B^T A^{-1} B$, it is optimal.

$$\operatorname{tr}(B^T A^{-1} B) = \inf_X F(X, A, B)$$

$$F(X, A, B) = \operatorname{tr} X$$

with domain

$$\operatorname{dom} F = \left\{ (X, A, B) \in S^m \times S^m \times S^m \,\middle|\, A \succ 0, \begin{bmatrix} A & B \\ B^T & X \end{bmatrix} \succeq 0 \right\}$$

F is convex, jointly in A, B, X. therefore it's infimum over X, which is $\operatorname{tr}(B^T A^{-1} B)$ is convex in A, B.

$$G(A,B) = A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}$$

show $X = G(A,B)$ Solve th SDP

$$\max \operatorname{tr} X$$
$$\text{sub} \begin{bmatrix} A & X \\ X & B \end{bmatrix} \succeq 0$$

$X \in S^m$, $A \in S^m_{++}$, $B \in S^m_{++}$ are given

if U and V are Positive semidefinite with $U \preceq V$ then $U^{1/2} \preceq V^{1/2}$.

Solution:

$$X A^{-1} X \preceq B$$

$$(A^{-1/2} X A^{-1/2})^2 = A^{-1/2} X A^{-1} X A^{-1/2} \preceq A^{-1/2} B A^{-1/2}$$

$$A^{-1/2} X A^{-1/2} \preceq (A^{-1/2} B A^{-1/2})^{1/2}$$

$$X \preceq A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}$$

$\Rightarrow X \preceq G(A,B)$, $\operatorname{tr} X \leq \operatorname{tr} G(A,B)$

$$X A^{-1} X = A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2} A^{-1} A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}$$

$$= A^{1/2}(A^{-1/2} B A^{-1/2}) A^{1/2}$$

$$= B$$

$\Rightarrow X = G(A,B)$ is feasible

Pv

Suppose that $f: R^n \to R$ is nonnegative and convex and $g: R^n \to R$ is positive and concave. Show that the function f^2/g, with domain $\mathrm{dom}\, f \cap \mathrm{dom}\, g$. is convex

Solution: assume that $n=1$, $m \in [0.1]$.
x and y be in the domains of f and g.

define $Z = mX + (1-m)y$

$$f(z) \leq m f(x) + (1-m) f(y)$$

$$f(z)^2 \leq (m f(x) + (1-m) f(y))^2$$

$$g(z) \geq m g(x) + (1-m) g(y)$$

$$\frac{f(z)^2}{g(z)} \leq \frac{(m f(x) + (1-m) f(y))^2}{m g(x) + (1-m) g(y)}$$

$$\frac{(m f(x) + (1-m) f(y))^2}{m g(x) + (1-m) g(y)} \leq \frac{m f(x)^2}{g(x)} + \frac{(1-m) f(y)^2}{g(y)}$$

Pa

Show that the following functions $f: R^n \to R$ are convex

a) $f(x) = -\exp(-g(x))$ where $g: R^n \to R$ has a convex domain and satisfies

$$\begin{bmatrix} \nabla^2 g(x) & \nabla g(x) \\ \nabla g(x)^T & 1 \end{bmatrix} \succeq 0$$

for $x \in \operatorname{dom} g$

b) The function $f(x) = \max\{\|AP_x - b\| \mid P \text{ is a Permutation matrix}\}$

with $A \in R^{m \times n}$, $b \in R^m$

Solution: a) the gradient and Hessian of f are

$$\nabla f(x) = e^{-g(x)} \nabla g(x)$$

$$\nabla^2 f(x) = e^{-g(x)} \nabla^2 g(x) - e^{-g(x)} \nabla g(x) \nabla g(x)^T$$

$$= e^{-g(x)} (\nabla^2 g(x) - \nabla g(x) \nabla g(x)^T)$$

$$\succeq 0$$

b) f is the maximum of convex function $\|AP_x - b\|$, Parameterized by P.

Show that a function $f: \mathbb{R} \to \mathbb{R}$ is convex if and only if dom f is convex and

$$\det \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ f(x) & f(y) & f(z) \end{bmatrix} \geq 0$$

for all $x, y, z \in$ dom f with $x < y < z$.

Solution:

$$\det \begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ f(t_1) & f(t_2) & f(t_3) \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ f(t_1) & f(t_2) & f(t_3) \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ t_1 & t_2 - t_1 & t_3 - t_2 \\ f(t_1) & f(t_2) - f(t_1) & f(t_3) - f(t_2) \end{bmatrix}$$

$$= (t_2 - t_1)(f(t_3) - f(t_2)) - (t_3 - t_2)(f(t_2) - f(t_1))$$

This is nonnegative if and only if

$$\frac{t_3 - t_1}{(t_2 - t_1)(t_3 - t_2)} f(t_2) \leq \frac{1}{t_2 - t_1} f(t_1) + \frac{1}{t_3 - t_2} f(t_3)$$

$$f(m t_1 + (1-m) t_3) \leq m f(t_1) + (1-m) f(t_3)$$

$$m = \frac{t_2 - t_1}{t_3 - t_1}, \quad 1 - m = \frac{t_3 - t_2}{t_3 - t_1}$$

$$\max \sum_{i=1}^{n} \log \left(\frac{\sum_{j=1}^{n} B_{ij} P_j}{\sum_{j=1}^{n} B_{ij} P_j - P_i} \right)$$

$$\text{sub } \sum_{i=1}^{n} P_i = 1$$

$$P_i \geq 0 \,,\ i = 1, \dots, n$$

$B \in R^{n \times n}$, $B = A + v 1^T$, $B_{ij} = A_{ij} + v_i$, $i, j = 1, \dots, n$

B is nonsingular and $B^{-1} = I - C$

with $C_{ij} \geq 0$ Use $y = Bp$ as variables.

Solution: $y = Bp$, $p = B^{-1} y = y - Cy$

cost function reduces to

$$\sum_{i=1}^{n} \log (y_i / (Cy)_i) = \log \prod_i (y_i / (Cy)_i)$$

$$y_i \geq (Cy)_i \,,\ i = 1, \dots, n$$

$$1^T p = 1 \,,\ \prod_i y_i = 1$$

This results in GP. $\min \prod_i (Cy)_i$

$$\text{sub } y_i \geq (Cy)_i \,,\ i = 1, \dots, n$$

$$\prod_i y_i = 1.$$

Q 12.11 book

$$\sigma_i^2 + \sum_{j\neq i} G_{ij} P_j,$$

$$\gamma_i = \frac{G_{ii} P_i}{\sigma_i^2 + \sum_{j\neq i} G_{ij} P_j}$$

$$U(P) = \sum_{i=1}^{n} \log R_i$$

Solution. First we show that $f(x) = \log\log(1+e^x)$ is concave.

$$f'(x) = \frac{1}{(\log(1+e^x))(1+e^{-x})}$$

$$f''(x) = \frac{1}{\log(1+e^x)(1+e^{-x})^2}\left(\frac{-1}{\log(1+e^x)} + \frac{1}{e^x}\right)$$

the first term is Positive. The second is negative since $e^x \geq \log(1+e^x)$ which follows from $\log(1+u) \leq u$

$$\log R_i = \log\log(1+\gamma_i) = \log\log(1+e^{\log \gamma_i})$$

$$\log \gamma_i = \log\left(\frac{G_{ii} e^{z_i}}{N_i + \sum_{j\neq i} G_{ij} e^{z_j}}\right)$$

$$= \log G_{ii} + z_i - \log\left(N_i + \sum_{j\neq i} G_{ij} e^{z_j}\right)$$

P_{12}