

Cheatsheet Probability and Statistics

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1 Mathematical framework

1.1 Probability space

Def. 1.1. The set Ω is called the **sample space**. An element $\omega \in \Omega$ is called an **outcome** or **elementary experiment**.

Ex. 1.1. Throw of a die : $\Omega = \{1, 2, 3, 4, 5, 6\}$

Def. 1.2. A **sigma-algebra** is a subset $\mathcal{F} \subset \mathcal{P}(\Omega)$ satisfying the following properties :

P1. $\Omega \in \mathcal{F}$

P2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$: If A is an event, “not A ” is also an event.

P3. $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$: if A_1, A_2, \dots are events, then “ A_1 or A_2 or ...” is an event

Ex. 1.2. Examples of sigma-algebras for $\Omega = \{1, 2, 3, 4, 5, 6\}$:

- $\mathcal{F} = \{\emptyset, \{1, 2, 3, 4, 5, 6\}\}$
- $\mathcal{F} = \mathcal{P}(\Omega)$
- $\mathcal{F} = \{\emptyset, \{1, 2\}, \{3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\}$

Non examples of sigma-algebras for $\Omega = \{1, 2, 3, 4, 5, 6\}$:

- $\mathcal{F} = \{\{1, 2, 3, 4, 5, 6\}\}$: **P2** is not satisfied
- $\mathcal{F} = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, \{1\}, \{2, 3, 4, 5, 6\}, \Omega\}$: **P3** is not satisfied

Def. 1.3. Let Ω a sample space and \mathcal{F} a sigma-algebra. A **probability measure** on (Ω, \mathcal{F}) is a map

$$\mathbb{P} : \mathcal{F} \rightarrow [0, 1], \quad A \mapsto \mathbb{P}[A]$$

that satisfies the properties

P1. $\mathbb{P}[\Omega] = 1$

P2. (countable additivity) $\mathbb{P}[A] = \sum_{i=1}^{\infty} \mathbb{P}[A_i]$ if $A = \bigcup_{i=1}^{\infty} A_i$ (disjoint union)

Int. A probability measure is a map that associates to each event a number in $[0, 1]$

Ex. 1.3. For $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{F} = \mathcal{P}(\Omega)$, the mapping $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ defined by

$$\forall A \in \mathcal{F} \quad \mathbb{P}[A] = \frac{|A|}{6}$$

is a probability measure on (Ω, \mathcal{F}) .

Def. 1.4. Let Ω a sample space, \mathcal{F} a sigma-algebra and \mathbb{P} a probability measure. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probabilistic space**.

Int. To construct a probabilistic model, we give

- a sample space Ω : all the possible outcomes of the experiment
- a sigma-algebra $\mathcal{F} \subset \mathcal{P}(\Omega)$: the set of events
- a probability measure \mathbb{P} : gives a number in $[0, 1]$ to every event

Def. 1.5. Let $\omega \in \Omega$ (a possible outcome). Let A be an event. We say the event A **occurs** (**does not occur**) (for ω) if $\omega \in A$ ($\omega \notin A$).

1.2 Examples of probability spaces

Def. 1.6. Let Ω be a finite sample space. The **Laplace model** on Ω is the triple $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ is defined by

$$\forall A \in \mathcal{F} \quad \mathbb{P}[A] = \frac{|A|}{|\Omega|}$$

1.3 Properties of Events

Prop 1.1. (Consequences of definition 1.2). Let \mathcal{F} be a sigma-algebra on Ω . We have

P4. $\emptyset \in \mathcal{F}$

P5. $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

P6. $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$

P7. $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

Event	Graphical representation	Probab. interpretation
A^c		A does not occur
$A \cap B$		A and B occur
$A \cup B$		A or B occurs
$A \Delta B$		one and only one of A or B occurs

Figure 1: Representation of set operations

Relation	Graphical representation	Probab. interpretation
$A \subset B$		If A occurs, then B occurs
$A \cap B = \emptyset$		A and B cannot occur at the same time
$\Omega = A_1 \cup A_2 \cup A_3$ with A_1, A_2, A_3 pairwise disjoint		for each outcome ω , one and only one of the events A_1, A_2, A_3 is satisfied.

Figure 2: Representation of set relations

1.4 Properties of probability measures

Prop 1.2. (Consequences of definition 1.3). Let \mathbb{P} be a probability measure on (Ω, \mathcal{F}) .

P3. We have $\mathbb{P}[\emptyset] = 0$

P4. (additivity) Let $k \geq 1$, let A_1, \dots, A_k be k pairwise disjoint events, then

$$\mathbb{P}[A_1 \cup \dots \cup A_k] = \mathbb{P}[A_1] + \dots + \mathbb{P}[A_k]$$

P5. Let A be an event, then

$$\mathbb{P}[A^c] = 1 - \mathbb{P}[A]$$

P6. If A and B are two events (not necessarily disjoint), then

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$$

Prop 1.3. (Monotonicity). Let $A, B \in \mathcal{F}$, then

$$A \subset B \Rightarrow \mathbb{P}[A] \leq \mathbb{P}[B]$$

Prop 1.4. (Union bound). Let A_1, A_2, \dots be a sequence of events (not necessarily disjoint), then we have

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] \leq \sum_{i=1}^{\infty} \mathbb{P}[A_i]$$

Union bound also applies to a finite collection of events.

Prop 1.5. Let (A_n) be an increasing sequence of events (i.e. $\forall n A_n \subset A_{n+1}$). Then

$$\lim_{n \rightarrow \infty} \mathbb{P}[A_n] = \mathbb{P}\left[\bigcup_{n=1}^{\infty} A_n\right]. \text{ increasing limit}$$

Let (B_n) be a decreasing sequence of events (i.e. $\forall n B_n \supset B_{n+1}$). Then

$$\lim_{n \rightarrow \infty} \mathbb{P}[B_n] = \mathbb{P}\left[\bigcap_{n=1}^{\infty} B_n\right]. \text{ decreasing limit}$$

¹i.e. $\Omega = B_1 \cup \dots \cup B_n$ and the events are pairwise disjoint.

1.5 Conditional probabilities

Def. 1.7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space. Let A, B be two events with $\mathbb{P}[B] > 0$. The **conditional probability of A given B** is defined by

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

Ex. 1.4. We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ corresponding to the throw of one die. Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. Then

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

Prop 1.6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space. Let B be an event with positive probability. Then $\mathbb{P}[\cdot | B]$ is a probability measure on Ω .

Prop 1.7. (Formula of total probability). Let B_1, \dots, B_n be a partition¹ of the sample space Ω with $\mathbb{P}[B_i] > 0$ for every $i \leq i \leq n$. Then one has

$$\forall A \in \mathcal{F} \quad \mathbb{P}[A] = \sum_{i=1}^n \mathbb{P}[A|B_i] \mathbb{P}[B_i]$$

Prop 1.8. (Bayes formula). Let $B_1, \dots, B_n \in \mathcal{F}$ be a partition of Ω with $\mathbb{P}[B_i] > 0 \forall i$. For every event A with $\mathbb{P}[A] > 0$ we have

$$\forall i = 1, \dots, n \quad \mathbb{P}[B_i|A] = \frac{\mathbb{P}[A|B_i] \mathbb{P}[B_i]}{\sum_{j=1}^n \mathbb{P}[A|B_j] \mathbb{P}[B_j]}$$

Ex. 1.5. Test to detect a disease which concerns about 1/10000 of the population. The test gives the right answer 99% of the time. If a patient has a positive test, what is the probability that he is actually sick?

We modeled the situation as $\Omega = \{0, 1\} \times \{0, 1\}$. $\mathcal{F} = \mathcal{P}(\Omega)$ and an outcome is $\omega = (\omega_1, \omega_2)$, where ω_1 is 1 if the patient is sick and ω_2 is 1 if the test is positive. Let $S = \{(1, 0), (1, 1)\}$ be the event that the patient is sick and $T = \{(0, 1), (1, 1)\}$ the event that the test is positive.

From the hypotheses, we have

$$\mathbb{P}[S] = \frac{1}{10000}, \quad \mathbb{P}[T|S] = \frac{99}{100}, \quad \mathbb{P}[T|S^c] = \frac{1}{100}$$

By applying the Bayes formula to the partition $\Omega = S \cup S^c$, we obtain

$$\mathbb{P}[S|T] = \frac{\mathbb{P}[T|S] \mathbb{P}[S]}{\mathbb{P}[T|S] \mathbb{P}[S] + \mathbb{P}[T|S^c] \mathbb{P}[S^c]} \simeq 0.0098$$

1.6 Independence

Def. 1.8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Two events A and B are said to be **independent** if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \mathbb{P}[B]$$

A is independent of B iff A is independent of B^c .

If $\mathbb{P}[A] \in \{0, 1\}$, then A is independent of every event.

If A is independent with itself (i.e. $\mathbb{P}[A \cap A] = \mathbb{P}[A]^2$), then $\mathbb{P}[A] \in \{0, 1\}$.

Prop 1.9. Let $A, B \in \mathcal{F}$ be two events with $\mathbb{P}[A], \mathbb{P}[B] > 0$. Then the following are equivalent :

- $\mathbb{P}[A \cap B] = \mathbb{P}[A] \mathbb{P}[B]$: **A and B are independent**
- $\mathbb{P}[A|B] = \mathbb{P}[A]$: **the occurrence of B has no influence on A**
- $\mathbb{P}[B|A] = \mathbb{P}[B]$: **the occurrence of A has no influence on B**

Def. 1.9. Let I be an arbitrary set of indices. A collection of events $(A_i)_{i \in I}$ is said to be **independent** if

$$\forall J \subset I \text{ finite} \quad \mathbb{P}\left[\bigcap_{j \in J} A_j\right] = \prod_{j \in J} \mathbb{P}[A_j]$$

Int. Three events A, B and C are independent if the following 4 equations are satisfied :

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \mathbb{P}[B]$$

$$\mathbb{P}[A \cap C] = \mathbb{P}[A] \mathbb{P}[C]$$

$$\mathbb{P}[B \cap C] = \mathbb{P}[B] \mathbb{P}[C]$$

$$\mathbb{P}[A \cap B \cap C] = \mathbb{P}[A] \mathbb{P}[B] \mathbb{P}[C]$$

2 Random variables and distribution functions

2.1 Abstract definition

Def. 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A **random variable** (r.v.) is a map $X : \Omega \rightarrow \mathbb{R}$ s.t.

$$\forall a \in \mathbb{R} \quad \{\omega \in \Omega : X(\omega) \leq a\} \in \mathcal{F}$$

Ex. 2.1. We throw a fair die. The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$ and we consider the Laplace model $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose we gamble on the outcome in such a way that our profit is -1 if the outcome is 1, 2 or 3; 0 if the outcome is 4 and 2 if the outcome is 5 or 6. Our profit can be represented by the mapping X defined by

$$\forall \omega \in \Omega \quad X(\omega) = \begin{cases} -1 & \text{if } \omega = 1, 2, 3, \\ 0 & \text{if } \omega = 4, \\ 2 & \text{if } \omega = 5, 6 \end{cases}$$

Since $\mathcal{F} = \mathcal{P}(\Omega)$, we have $\{\omega : X(\omega) \leq a\} \in \mathcal{F}$ for every a . Therefore, X is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

Def. 2.2. When events are defined in terms of random variable, we omit the dependence in ω . E.g. for $a \leq b$ we write

$$\{X \leq a\} = \{\omega \in \Omega : X(\omega) \leq a\}$$

$$\{a < X \leq b\} = \{\omega \in \Omega : a < X(\omega) \leq b\}$$

$$\{X \in \mathbb{Z}\} = \{\omega \in \Omega : X(\omega) \in \mathbb{Z}\}$$

When consider the probability of events as above, we omit the brackets

$$\mathbb{P}[X \leq a] = \mathbb{P}[\{X \leq a\}] = \mathbb{P}[\{\omega \in \Omega : X(\omega) \leq a\}]$$

2.2 Distribution function

Def. 2.3. Let X be a random variable on a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$. The **distribution function of X** is the function $F_X : \mathbb{R} \rightarrow [0, 1]$ defined by

$$\forall a \in \mathbb{R} \quad F_X(a) = \mathbb{P}[X \leq a]$$

Ex. 2.2. Same example with the die. Let X be the random variable defined as above. For $a \in \mathbb{R}$ we have

$$F_X(a) = \begin{cases} 0 & \text{if } a < -1, \\ 1/2 & \text{if } -1 \leq a < 0, \\ 2/3 & \text{if } 0 \leq a < 2, \\ 1 & \text{if } a \geq 2 \end{cases}$$

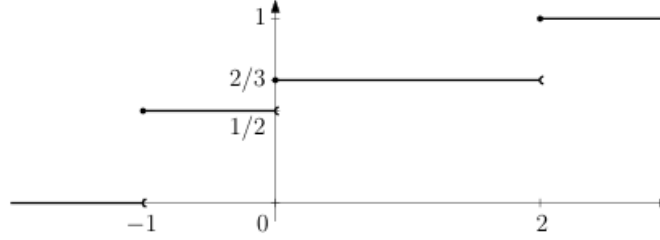


Figure 3: Graph of the distribution function F_X

Prop 2.1. (Basic identity). Let $a < b$ be two real numbers. Then

$$\mathbb{P}[a < X \leq b] = F(b) - F(a)$$

Prop 2.2. Let X be a r.v. on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The distribution function $F = F_X : \mathbb{R} \rightarrow [0, 1]$ of X satisfies the following properties :

- i. F is nondecreasing
- ii. F is right continuous²
- iii. $\lim_{a \rightarrow -\infty} F(a) = 0$ and $\lim_{a \rightarrow \infty} F(a) = 1$

2.3 Independence

Def. 2.4. Let X_1, \dots, X_n be n random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that they are **independent** if $\forall x_1, \dots, x_n \in \mathbb{R} \quad \mathbb{P}[X_1 \leq x_1, \dots, X_n \leq x_n] = \mathbb{P}[X_1 \leq x_1] \dots \mathbb{P}[X_n \leq x_n]$.

One can show that X_1, \dots, X_n are independent iff $\forall I_1 \subset \mathbb{R}, \dots, I_n \subset \mathbb{R}$ intervals $\{X_1 \in I_1\}, \dots, \{X_n \in I_n\}$ are independent.

Prop 2.3. (Grouping). Let X_1, \dots, X_n be n independent r.v. Let $1 \leq i_1 < i_2 < \dots < i_k \leq n$ be some indices and ϕ_1, \dots, ϕ_k some functions. Then $Y_1 = \phi_1(X_{i_1}, \dots, X_{i_1}), Y_2 = \phi_2(X_{i_1+1}, \dots, X_{i_2}), \dots, Y_k = \phi_k(X_{i_{k-1}+1}, \dots, X_{i_k})$ are independent.

Def. 2.5. An infinite sequence X_1, X_2, \dots of random variables is said to be

- **independent** if X_1, \dots, X_n are independent for every n
- **independent and identically distributed** (i.i.d) if they are independent and they have the same distribution function, i.e. $\forall i, j \quad F_{X_i} = F_{X_j}$.

2.4 Transformation of random variables

We can create r.v. from other r.v. on the same probability space. For example, consider $Z_1 = \exp(X_1), Z_2 = X_1 + X_2$. Not to forget : r.v. are maps $\Omega \rightarrow \mathbb{R}$.

We can work with r.v. as if they were real numbers with the following notation :

Def. 2.6. If X is a r.v. and $\phi : \mathbb{R} \rightarrow \mathbb{R}$, then we write

$$\phi(X) := \phi \circ X$$

to $\phi(X)$ a new mapping $\Omega \rightarrow \mathbb{R}$.

We also consider function of several variables. If X_1, \dots, X_n are n r.v. and $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$, then we write

$$\phi(X_1, \dots, X_n) := \phi \circ (X_1, \dots, X_n)$$

2.5 Construction of random variables

Def. 2.7. Let $p \in [0, 1]$. A r.v. X is said to be a **Bernoulli r.v. with parameter p** if

$$\mathbb{P}[X = 0] = 1 - p \quad \text{and} \quad \mathbb{P}[X = 1] = p$$

In this case, we write $X \sim \text{Ber}(p)$.

Prop 2.4. (Existence theorem of Kolmogorov). There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an infinite sequence of r.v. X_1, X_2, \dots (on this probability space) that is an iid sequence of Bernoulli r.v. with parameter $1/2$.

²i.e. $F(a) = \lim_{h \downarrow 0} F(a + h)$ for every $a \in \mathbb{R}$

Prop 2.5. A r.v. U is said to be **uniform r.v. in $[0, 1]$** if its distribution function is equal to

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

In this case, we write $U \sim \mathcal{U}([0, 1])$.

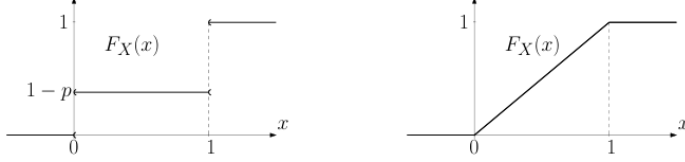


Figure 4: Left: distribution function of a Bernoulli r.v. with parameter p . Right: distribution function of a uniform r.v. in $[0, 1]$.

Prop 2.6. The mapping $Y : \Omega \rightarrow [0, 1]$ defined by $Y(\omega) = \sum_{n=1}^{\infty} 2^{-n} X_n(\omega)$ is a uniform r.v. in $[0, 1]$.

Def. 2.8. The **generalized inverse** of F^3 is the mapping $F^{-1} : (0, 1) \rightarrow \mathbb{R}$ defined by

$$\forall \alpha \in (0, 1) \quad F^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}$$

Int. By definition of the infimum and using right continuity of F , we have $\forall x \in \mathbb{R}$ and $\forall \alpha \in (0, 1)$

$$(F^{-1}(\alpha) \leq x) \iff (\alpha \leq F(x))$$

Prop 2.7. (Inverse transform sampling). Let $F : \mathbb{R} \rightarrow [0, 1]$.⁴ Let U be a uniform r.v. in $[0, 1]$. Then the r.v. $X = F^{-1}(U)$ has distribution $F_X = F$.

Prop 2.8. Let F_1, F_2, \dots be a sequence of functions $\mathbb{R} \rightarrow [0, 1]$.⁵ Then there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of independent r.v. X_1, X_2, \dots on this probability space s.t.

- for every i X_i has distribution function F_i (i.e. $\forall x \mathbb{P}[X_i \leq x] = F_i(x)$)
- X_1, X_2, \dots are independent

3 Discrete and continuous r.v.

3.1 Discontinuity / continuity points of F

Prop 3.1. Let $X : \Omega \rightarrow \mathbb{R}$ be a r.v. with distribution function F . Then for every a in \mathbb{R} we have

$$\mathbb{P}[X = a] = F(a) - F(a-)$$

where $F(a-) := \lim_{h \downarrow 0} F(a - h)$.

Int. Fix $a \in \mathbb{R}$

→ If F is not continuous at a point $a \in \mathbb{R}$, then the “jump size” $F(a) - F(a-)$ is equal to the probability that $X = a$

→ If F is continuous at a point $a \in \mathbb{R}$, then $\mathbb{P}[X = a] = 0$

3.2 Almost sure events

Def. 3.1. Let $A \in \mathcal{F}$ be an event. We say that A occurs **almost surely (a.s.)** if $\mathbb{P}[A] = 1$.

3.3 Discrete random variables

Def. 3.2. A r.v. $X : \Omega \rightarrow \mathbb{R}$ is said to be **discrete** if there exists some set $W \subset \mathbb{R}$ finite or countable s.t. $X \in W$ a.s..

Def. 3.3. Let X be a discrete r.v. taking some values in some finite or countable set $W \subset \mathbb{R}$. The **distribution of X** is the sequence of numbers $(p(x))_{x \in W}$ defined by

$$\forall x \in W \quad p(x) := \mathbb{P}[X = x]$$

Prop 3.2. The distribution $(p(x))_{x \in W}$ of a discrete r.v. satisfies $\sum_{x \in W} p(x) = 1$.

Prop 3.3. Let X be a discrete r.v. with values in a finite or countable set W almost surely, and distribution p . Then the distribution function of X is given by

$$\forall x \in \mathbb{R} \quad F_X(x) = \sum_{\substack{y \leq x \\ y \in W}} p(y)$$

Int. W = {positions of the jumps of F_X },
 $p(x)$ = “height of the jump” at $x \in W$.

3.4 Examples of discrete random variables

The simplest (non constant) r.v. is the Bernoulli r.v. defined in definition 2.7.

Def. 3.4. Let $0 \leq p \leq 1$, let $n \in \mathbb{N}$. A r.v. X is said to be a **binomial r.v. with parameters n and p** if it takes values in $W = \{0, \dots, n\}$ and

$$\forall k \in \{0, \dots, n\} \quad \mathbb{P}[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

In that case we write $X \sim \text{Bin}(n, p)$.

Prop 3.4. (Sum of independent Bernoulli and binomial). Let $0 \leq p \leq 1$, let $n \in \mathbb{N}$. Let X_1, \dots, X_n be n independent Bernoulli r.v. with parameter p . Then

$$S_n := X_1 + \dots + X_n$$

is a binomial r.v. with parameter n and p .

Int. In particular, the distribution $\text{Bin}(1, p)$ is the same as the distribution $\text{Ber}(p)$. One can also check that if $X \sim \text{Bin}(m, p)$ and $Y \sim \text{Bin}(n, p)$ and X, Y are independent, then $X + Y \sim \text{Bin}(m + n, p)$.

Def. 3.5. Let $0 \leq p \leq 1$. A r.v. X is said to be a **geometric r.v. with parameter p** if it takes values in $W = \mathbb{N} \setminus \{0\}$ and

$$\forall k \in \mathbb{N} \quad \mathbb{P}[X = k] = (1-p)^{k-1} \cdot p$$

In this case, we write $X \sim \text{Geom}(p)$.

³satisfying prop. 2.2

⁴See footnote 3

⁵See footnote 3

Prop 3.5. Let X_1, X_2, \dots be a sequence of infinitely many independent Bernoulli r.v. with parameter p . Then

$$T := \min\{n \geq 1 : X_n = 1\}$$

is a geometric r.v. with parameter p .

Prop 3.6. (Absence of memory of the geometric distribution). Let $T \sim \text{Geom}(p)$ for some $0 < p < 1$. Then

$$\forall n \geq 0 \forall k \geq 1 \quad \mathbb{P}[T \geq n+k | T > n] = \mathbb{P}[T \geq k]$$

Def. 3.6. Let $\lambda > 0$ be a positive real number. A r.v. X is said to be a **Poisson r.v. with parameter λ** if it takes values in $W = \mathbb{N}$ and

$$\forall k \in \mathbb{N} \quad \mathbb{P}[X = k] = \frac{\lambda^k}{k!} e^{-\lambda}$$

In this case, we write $X \sim \text{Poisson}(\lambda)$.

Prop 3.7. (Poisson approximation of the binomial). Let $\lambda > 0$. For every $n \geq 1$, consider a r.v. $X_n \sim \text{Bin}(n, \frac{\lambda}{n})$. Then

$$\forall k \in \mathbb{N} \quad \lim_{n \rightarrow \infty} \mathbb{P}[X_n = k] = \mathbb{P}[N = k]$$

where N is a Poisson r.v. with parameter λ .

3.5 Continuous random variables

Def. 3.7. A r.v. $X : \Omega \rightarrow \mathbb{R}$ is said to be **continuous** if its distribution function F_X can be written as

$$F_X(a) = \int_{-\infty}^a f(x) dx \quad \text{for all } a \in \mathbb{R}$$

for some nonnegative function $f : \mathbb{R} \rightarrow \mathbb{R}_+$, called the **density** of X .

Int. $f(x) dx$ represents the probability that X takes a value in the infinitesimal interval $[x, x + dx]$.

Prop 3.8. The density f of a r.v. satisfies $\int_{-\infty}^{+\infty} f(x) dx = 1$.

Prop 3.9. Let X be a r.v. Assume the distribution function F_X is continuous and piecewise \mathcal{C}^1 , i.e. that there exist $x_0 = -\infty < x_1 < \dots < x_{n-1} < x_n = +\infty$ s.t. F_X is \mathcal{C}^1 on every interval (x_i, x_{i+1}) . Then X is a continuous r.v. and a density f can be constructed by defining

$$\forall x \in (x_i, x_{i+1}) \quad f(x) = F'_X(x)$$

and setting arbitrary values at x_1, \dots, x_{n-1} .

3.6 Examples of continuous random variables

Def. 3.8. A continuous r.v. X is said to be **uniform in $[a, b]$** if its density is equal to

$$f_{a,b}(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b], \\ 0 & x \notin [a, b] \end{cases}$$

In this case, we write $X \sim \mathcal{U}([a, b])$.

Def. 3.9. A continuous r.v. T is said to be **exponential with parameter $\lambda > 0$** if its density is equal to

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & x < 0 \end{cases}$$

In this case, we write $T \sim \text{Exp}(\lambda)$.

Def. 3.10. A continuous r.v. X is said to be **normal with parameters m and $\sigma^2 > 0$** if its density is equal to

$$f_{m,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

In this case, we write $X \sim \mathcal{N}(m, \sigma^2)$.

4 Expectation

4.1 Expectation for general r.v.

Def. 4.1. Let $X : \Omega \rightarrow \mathbb{R}_+$ be a r.v. with nonnegative values. The **expectation** of X is defined as

$$\mathbb{E}[X] = \int_0^\infty (1 - F_X(x)) dx$$

Prop 4.1. Let X be a nonnegative r.v. Then we have $\mathbb{E}[X] \geq 0$, with equality iff $X = 0$ almost surely.

Def. 4.2. Let X be a r.v. If $\mathbb{E}[|X|] < \infty$, then the expectation of X is defined by $\mathbb{E}[X] = \mathbb{E}[X_+] - \mathbb{E}[X_-]$, where X_+ and X_- are the positive and negative parts of X defined by $X_+(\omega) = \begin{cases} X(\omega) & \text{if } X(\omega) \geq 0, \\ 0 & \text{if } X(\omega) < 0, \end{cases}$ and $X_-(\omega) =$

$$\begin{cases} -X(\omega) & \text{if } X(\omega) \leq 0, \\ 0 & \text{if } X(\omega) > 0. \end{cases}$$

4.2 Expectation of a discrete r.v.

Prop 4.2. Let $X : \Omega \rightarrow \mathbb{R}$ be a discrete r.v. with values in W (finite or countable) almost surely. We have

$$\mathbb{E}[X] = \sum_{x \in W} x \cdot \mathbb{P}[X = x]$$

provided the sum is well defined.

Prop 4.3. Let $X : \Omega \rightarrow \mathbb{R}$ be a discrete r.v. with values in W (finite or countable) almost surely. For every $\phi : \mathbb{R} \rightarrow \mathbb{R}$, we have

$$\mathbb{E}[\phi(X)] = \sum_{x \in W} \phi(x) \cdot \mathbb{P}[X = x]$$

provided the sum is well defined.

4.3 Expectation of a continuous r.v.

Prop 4.4. Let X be a continuous r.v. with density f . Then we have

$$\mathbb{E}[X] = \int_{-\infty}^\infty x \cdot f(x) dx$$

provided the integral is well defined.

Prop 4.5. Let X be a continuous r.v. with density f . Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be s.t. $\phi(X)$ is a r.v. Then we have

$$\mathbb{E}[\phi(X)] = \int_{-\infty}^\infty \phi(x) f(x) dx$$

provided the integral is well defined.

4.4 Calculus

Prop 4.6. (Linearity of the expectation). Let $X, Y : \Omega \rightarrow \mathbb{R}$ be r.v.'s, let $\lambda \in \mathbb{R}$. Provided the expectations are well defined, we have

1. $\mathbb{E}[\lambda \cdot X] = \lambda \cdot \mathbb{E}[X]$
2. $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

Prop 4.7. Let X, Y be two r.v. If X and Y are independent, then $\mathbb{E}[XY] = \mathbb{E}[X] \mathbb{E}[Y]$.

4.5 Tailsum formulas

Prop 4.8. (Tailsum formula for nonnegative r.v.'s). Let X be a r.v., s.t. $X \geq 0$ almost surely. Then we have $\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X > x] dx$.

Prop 4.9. (Tailsum formula for discrete r.v.'s). Let X be a discrete r.v. taking values in $\mathbb{N} = \{0, 1, 2, \dots\}$. Then $\mathbb{E}[X] = \sum_{n=1}^\infty \mathbb{P}[X \geq n]$.

4.6 Characterizations via expectations

Prop 4.10. Let X be a r.v. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ s.t. $\int_{-\infty}^{+\infty} f(x)dx = 1$. then the following are equivalent :

- i. X is continuous with density f ,
- ii. For every function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ piecewise continuous, bounded : $\mathbb{E}[\phi(X)] = \int_{-\infty}^\infty \phi(x)f(x)dx$

Prop 4.11. Let X, Y be two discrete r.v.'s. Then the following are equivalent

- i. X, Y are independent
- ii. For every $\phi : \mathbb{R} \rightarrow \mathbb{R}, \psi : \mathbb{R} \rightarrow \mathbb{R}$ piecewise continuous, bounded : $\mathbb{E}[\phi(X)\psi(Y)] = \mathbb{E}[\phi(X)]\mathbb{E}[\psi(Y)]$.

Prop 4.12. Let X_1, \dots, X_n be n r.v.'s. Then the following are equivalent

- i. X_1, \dots, X_n are independent

- ii. For every $\phi_1 : \mathbb{R} \rightarrow \mathbb{R}, \dots, \phi_n : \mathbb{R} \rightarrow \mathbb{R}$ piecewise continuous, bounded : $\mathbb{E}[\phi_1(X_1) \cdots \phi_n(X_n)] = \mathbb{E}[\phi_1(X_1)] \cdots \mathbb{E}[\phi_n(X_n)]$.

4.7 Inequalities

Prop 4.13. (Monotonicity). Let X, Y be two r.v.'s s.t. $X \leq Y$ a.s. Then $\mathbb{E}[X] \leq \mathbb{E}[Y]$, provided the two expectations are well defined.

Prop 4.14. (Markov's inequality). Let X be a nonnegative r.v. Then for every $a > 0$, we have

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$$

Prop 4.15. (Jensen's inequality). Let X be a r.v. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. If $\mathbb{E}[\phi(X)]$ and $\mathbb{E}[X]$ are well defined, then

$$\phi(\mathbb{E}[X]) \leq \mathbb{E}[\phi(X)]$$

4.8 Variance

Def. 4.3. Let X be a variable s.t. $\mathbb{E}[X^2] < \infty$. The **variance of X** is defined by

$$\sigma_X^2 = \mathbb{E}[(X - m)^2], \quad \text{where } m = \mathbb{E}[X]$$

The square root σ_X of the variance is called the **standard deviation of X** .

Prop 4.16. Let X be a r.v. s.t. $\mathbb{E}[X^2] < \infty$. Then for every $a \geq 0$ we have

$$\mathbb{P}[|X - m| \geq a] \leq \frac{\sigma_X^2}{a^2}, \quad \text{where } m = \mathbb{E}[X]$$

Prop 4.17. (Basic properties of the variance).

1. Let X be a r.v. with $\mathbb{E}[X^2] < \infty$. Then $\sigma_X^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.
2. Let X be a r.v. with $\mathbb{E}[X^2] < \infty$, let $\lambda \in \mathbb{R}$. Then $\sigma_{\lambda X}^2 = \lambda^2 \cdot \sigma_X^2$.
3. Let X_1, \dots, X_n be n pairwise independent r.v.'s and $S = X_1 + \dots + X_n$. Then $\sigma_S^2 = \sigma_{X_1}^2 + \dots + \sigma_{X_n}^2$.

4.9 Covariance

Def. 4.4. Let X, Y be two r.v.'s. Assume that $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$ (finite second moment). We define the **covariance between X and Y** as

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Int. With X and Y independent : $\text{Cov}(X, Y) = 0$.