

# Cheatsheet Probability and Statistics

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## 1 Mathematical framework

### 1.1 Probability space

**Def. 1.1.** The set  $\Omega$  is called the **sample space**. An element  $\omega \in \Omega$  is called an **outcome** or **elementary experiment**.

**Ex. 1.1.** Throw of a die :  $\Omega = \{1, 2, 3, 4, 5, 6\}$

**Def. 1.2.** A **sigma-algebra** is a subset  $\mathcal{F} \subset \mathcal{P}(\Omega)$  satisfying the following properties :

**P1.**  $\Omega \in \mathcal{F}$

**P2.**  $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$  : If  $A$  is an event, “not  $A$ ” is also an event.

**P3.**  $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$  : if  $A_1, A_2, \dots$  are events, then “ $A_1$  or  $A_2$  or ...” is an event

**Ex. 1.2.** Examples of sigma-algebras for  $\Omega = \{1, 2, 3, 4, 5, 6\}$  :

- $\mathcal{F} = \{\emptyset, \{1, 2, 3, 4, 5, 6\}\}$
- $\mathcal{F} = \mathcal{P}(\Omega)$
- $\mathcal{F} = \{\emptyset, \{1, 2\}, \{3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\}$

Non examples of sigma-algebras for  $\Omega = \{1, 2, 3, 4, 5, 6\}$  :

- $\mathcal{F} = \{\{1, 2, 3, 4, 5, 6\}\}$  : **P2** is not satisfied
- $\mathcal{F} = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, \{1\}, \{2, 3, 4, 5, 6\}, \Omega\}$  : **P3** is not satisfied

**Def. 1.3.** Let  $\Omega$  a sample space and  $\mathcal{F}$  a sigma-algebra. A **probability measure** on  $(\Omega, \mathcal{F})$  is a map

$$\mathbb{P} : \mathcal{F} \rightarrow [0, 1], \quad A \mapsto \mathbb{P}[A]$$

that satisfies the properties

**P1.**  $\mathbb{P}[\Omega] = 1$

**P2. (countable additivity)**  $\mathbb{P}[A] = \sum_{i=1}^{\infty} \mathbb{P}[A_i]$  if  $A = \bigcup_{i=1}^{\infty} A_i$  (disjoint union)

**Int.** A probability measure is a map that associates to each event a number in  $[0, 1]$

**Ex. 1.3.** For  $\Omega = \{1, 2, 3, 4, 5, 6\}$  and  $\mathcal{F} = \mathcal{P}(\Omega)$ , the mapping  $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$  defined by

$$\forall A \in \mathcal{F} \quad \mathbb{P}[A] = \frac{|A|}{6}$$

is a probability measure on  $(\Omega, \mathcal{F})$ .

**Def. 1.4.** Let  $\Omega$  a sample space,  $\mathcal{F}$  a sigma-algebra and  $\mathbb{P}$  a probability measure. The triple  $(\Omega, \mathcal{F}, \mathbb{P})$  is called a **probabilistic space**.

**Int.** To construct a probabilistic model, we give

- a sample space  $\Omega$  : all the possible outcomes of the experiment
- a sigma-algebra  $\mathcal{F} \subset \mathcal{P}(\Omega)$  : the set of events
- a probability measure  $\mathbb{P}$  : gives a number in  $[0, 1]$  to every event

**Def. 1.5.** Let  $\omega \in \Omega$  (a possible outcome). Let  $A$  be an event. We say the event  $A$  **occurs** (**does not occur**) (for  $\omega$ ) if  $\omega \in A$  ( $\omega \notin A$ ).

### 1.2 Examples of probability spaces

**Def. 1.6.** Let  $\Omega$  be a finite sample space. The **Laplace model** on  $\Omega$  is the triple  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\mathcal{F} = \mathcal{P}(\Omega)$  and  $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$  is defined by

$$\forall A \in \mathcal{F} \quad \mathbb{P}[A] = \frac{|A|}{|\Omega|}$$

### 1.3 Properties of Events

**Prop 1.1.** (Consequences of definition 1.2). Let  $\mathcal{F}$  be a sigma-algebra on  $\Omega$ . We have

**P4.**  $\emptyset \in \mathcal{F}$

**P5.**  $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

**P6.**  $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$

**P7.**  $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

Event	Graphical representation	Probab. interpretation
$A^c$		$A$ does <b>not</b> occur
$A \cap B$		$A$ and $B$ occur
$A \cup B$		$A$ or $B$ occurs
$A \Delta B$		one and only one of $A$ or $B$ occurs

Figure 1: Representation of set operations

Relation	Graphical representation	Probab. interpretation
$A \subset B$		If $A$ occurs, then $B$ occurs
$A \cap B = \emptyset$		$A$ and $B$ cannot occur at the same time
$\Omega = A_1 \cup A_2 \cup A_3$ with $A_1, A_2, A_3$ pairwise disjoint		for each outcome $\omega$ , one and only one of the events $A_1, A_2, A_3$ is satisfied.

Figure 2: Representation of set relations

## 1.4 Properties of probability measures

**Prop 1.2.** (Consequences of definition 1.3). Let  $\mathbb{P}$  be a probability measure on  $(\Omega, \mathcal{F})$ .

**P3.** We have  $\mathbb{P}[\emptyset] = 0$

**P4.** (**additivity**) Let  $k \geq 1$ , let  $A_1, \dots, A_k$  be  $k$  pairwise disjoint events, then

$$\mathbb{P}[A_1 \cup \dots \cup A_k] = \mathbb{P}[A_1] + \dots + \mathbb{P}[A_k]$$

**P5.** Let  $A$  be an event, then

$$\mathbb{P}[A^c] = 1 - \mathbb{P}[A]$$

**P6.** If  $A$  and  $B$  are two events (not necessarily disjoint), then

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$$

**Prop 1.3.** (**Monotonicity**). Let  $A, B \in \mathcal{F}$ , then

$$A \subset B \Rightarrow \mathbb{P}[A] \leq \mathbb{P}[B]$$

**Prop 1.4.** (**Union bound**). Let  $A_1, A_2, \dots$  be a sequence of events (not necessarily disjoint), then we have

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] \leq \sum_{i=1}^{\infty} \mathbb{P}[A_i]$$

Union bound also applies to a finite collection of events.

**Prop 1.5.** Let  $(A_n)$  be an increasing sequence of events (i.e.  $\forall n A_n \subset A_{n+1}$ ). Then

$$\lim_{n \rightarrow \infty} \mathbb{P}[A_n] = \mathbb{P}\left[\bigcup_{n=1}^{\infty} A_n\right]. \text{ **increasing limit**}$$

Let  $(B_n)$  be a decreasing sequence of events (i.e.  $\forall n B_n \supset B_{n+1}$ ). Then

$$\lim_{n \rightarrow \infty} \mathbb{P}[B_n] = \mathbb{P}\left[\bigcap_{n=1}^{\infty} B_n\right]. \text{ **decreasing limit**}$$

<sup>1</sup>i.e.  $\Omega = B_1 \cup \dots \cup B_n$  and the events are pairwise disjoint.

## 1.5 Conditional probabilities

**Def. 1.7.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be some probability space. Let  $A, B$  be two events with  $\mathbb{P}[B] > 0$ . The **conditional probability of  $A$  given  $B$**  is defined by

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

**Ex. 1.4.** We consider the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  corresponding to the throw of one die. Let  $A = \{1, 2, 3\}$  and  $B = \{2, 4, 6\}$ . Then

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

**Prop 1.6.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be some probability space. Let  $B$  be an event with positive probability. Then  $\mathbb{P}[\cdot | B]$  is a probability measure on  $\Omega$ .

**Prop 1.7.** (**Formula of total probability**). Let  $B_1, \dots, B_n$  be a partition<sup>1</sup> of the sample space  $\Omega$  with  $\mathbb{P}[B_i] > 0$  for every  $i \leq n$ . Then one has

$$\forall A \in \mathcal{F} \quad \mathbb{P}[A] = \sum_{i=1}^n \mathbb{P}[A|B_i] \mathbb{P}[B_i]$$

**Prop 1.8.** (**Bayes formula**). Let  $B_1, \dots, B_n \in \mathcal{F}$  be a partition of  $\Omega$  with  $\mathbb{P}[B_i] > 0 \forall i$ . For every event  $A$  with  $\mathbb{P}[A] > 0$  we have

$$\forall i = 1, \dots, n \quad \mathbb{P}[B_i|A] = \frac{\mathbb{P}[A|B_i] \mathbb{P}[B_i]}{\sum_{j=1}^n \mathbb{P}[A|B_j] \mathbb{P}[B_j]}$$

**Ex. 1.5.** Test to detect a disease which concerns about 1/10000 of the population. The test gives the right answer 99% of the time. If a patient has a positive test, what is the probability that he is actually sick?

We modeled the situation as  $\Omega = \{0, 1\} \times \{0, 1\}$ .  $\mathcal{F} = \mathcal{P}(\Omega)$  and an outcome is  $\omega = (\omega_1, \omega_2)$ , where  $\omega_1$  is 1 if the patient is sick and  $\omega_2$  is 1 if the test is positive. Let  $S = \{(1, 0), (1, 1)\}$  be the event that the patient is sick and  $T = \{(0, 1), (1, 1)\}$  the event that the test is positive.

From the hypotheses, we have

$$\mathbb{P}[S] = \frac{1}{10000}, \quad \mathbb{P}[T|S] = \frac{99}{100}, \quad \mathbb{P}[T|S^c] = \frac{1}{100}$$

By applying the Bayes formula to the partition  $\Omega = S \cup S^c$ , we obtain

$$\mathbb{P}[S|T] = \frac{\mathbb{P}[T|S] \mathbb{P}[S]}{\mathbb{P}[T|S] \mathbb{P}[S] + \mathbb{P}[T|S^c] \mathbb{P}[S^c]} \simeq 0.0098$$

## 1.6 Independence

**Def. 1.8.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Two events  $A$  and  $B$  are said to be **independent** if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \mathbb{P}[B]$$

$A$  is independent of  $B$  iff  $A$  is independent of  $B^c$ .

If  $\mathbb{P}[A] \in \{0, 1\}$ , then  $A$  is independent of every event.

If  $A$  is independent with itself (i.e.  $\mathbb{P}[A \cap A] = \mathbb{P}[A]^2$ ), then  $\mathbb{P}[A] \in \{0, 1\}$ .

**Prop 1.9.** Let  $A, B \in \mathcal{F}$  be two events with  $\mathbb{P}[A], \mathbb{P}[B] > 0$ . Then the following are equivalent :

- $\mathbb{P}[A \cap B] = \mathbb{P}[A] \mathbb{P}[B]$  : **A and B are independent**
- $\mathbb{P}[A|B] = \mathbb{P}[A]$  : **the occurrence of B has no influence on A**
- $\mathbb{P}[B|A] = \mathbb{P}[B]$  : **the occurrence of A has no influence on B**

**Def. 1.9.** Let  $I$  be an arbitrary set of indices. A collection of events  $(A_i)_{i \in I}$  is said to be **independent** if

$$\forall J \subset I \text{ finite} \quad \mathbb{P}\left[\bigcap_{j \in J} A_j\right] = \prod_{j \in J} \mathbb{P}[A_j]$$

**Int.** Three events  $A, B$  and  $C$  are independent if the following 4 equations are satisfied :

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \mathbb{P}[B]$$

$$\mathbb{P}[A \cap C] = \mathbb{P}[A] \mathbb{P}[C]$$

$$\mathbb{P}[B \cap C] = \mathbb{P}[B] \mathbb{P}[C]$$

$$\mathbb{P}[A \cap B \cap C] = \mathbb{P}[A] \mathbb{P}[B] \mathbb{P}[C]$$

2	Random variables and distribution functions	
2.1	Abstract definition	