Cheatsheet Probability and Statistics

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1 Mathematical framework

1.1 Probability space

Def. 1.1. The set Ω is called the **sample space**. An element $\omega \in \Omega$ is called an **outcome** or **elementary experiment**.

Ex. 1.1. *Throw of a die :* $\Omega = \{1, 2, 3, 4, 5, 6\}$

Def. 1.2. A **sigma-algebra** is a subset $\mathcal{F} \subset \mathcal{P}(\Omega)$ satisfying the following properties :

P1. $\Omega \in \mathcal{F}$

P2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$: If A is an event, "not A" is also an event.

P3. $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$: if $A_1, A_2, \ldots$ are events, then "A_1 or A_2 or ..." is an event

Ex. 1.2. *Examples of sigma-algebras for* $\Omega = \{1, 2, 3, 4, 5, 6\}$ *:*

- $\mathcal{F} = \{\emptyset, \{1, 2, 3, 4, 5, 6\}\}$
- $\mathcal{F} = \mathcal{P}(\Omega)$
- $\mathcal{F} = \{\emptyset, \{1, 2\}, \{3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\}$

Non examples of sigma-algebras for $\Omega = \{1, 2, 3, 4, 5, 6\}$ *:*

- $\mathcal{F} = \{\{1, 2, 3, 4, 5, 6\}\}$: ***P2*** *is not satisfied*
- $\mathcal{F} = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, \{1\}, \{2, 3, 4, 5, 6\}, \Omega\}$: ***P3*** *is not satisfied*

Def. 1.3. Let Ω a sample space and $\mathcal{F}$ a sigma-algebra. A **probability measure** on $(\Omega, \mathcal{F})$ is a map

$$\mathbb{P} : \mathcal{F} \to [0, 1], \quad A \mapsto \mathbb{P}[A]$$

that satisfies the properties

P1. $\mathbb{P}[\Omega] = 1$

P2. (countable additivity) $\mathbb{P}[A] = \sum_{i=1}^{\infty} \mathbb{P}[A_i]$ if $A = \bigcup_{i=1}^{\infty} A_i$ (disjoint union)

Int. A probability measure is a map that associates to each event a number in $[0, 1]$

Ex. 1.3. *For* $\Omega = \{1, 2, 3, 4, 5, 6\}$ *and* $\mathcal{F} = \mathcal{P}(\Omega)$*, the mapping* $\mathbb{P} : \mathcal{F} \to [0, 1]$ *defined by*

$$\forall A \in \mathcal{F} \quad \mathbb{P}[A] = \frac{|A|}{6}$$

is a probability measure on $(\Omega, \mathcal{F})$.

Def. 1.4. Let Ω a sample space, $\mathcal{F}$ a sigma-algebra and $\mathbb{P}$ a probability measure. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**.

Int. To construct a probabilistic model, we give

- a sample space Ω : all the possible outcomes of the experiment
- a sigma-algebra $\mathcal{F} \subset \mathcal{P}(\Omega)$: the set of events
- a probability measure $\mathbb{P}$: gives a number in $[0, 1]$ to every event

Def. 1.5. Let $\omega \in \Omega$ (a possible outcome). Let A be an event. We say the event A **occurs** (**does not occur**) (for ω) if $\omega \in A$ ($\omega \notin A$).

1.2 Examples of probability spaces

Def. 1.6. Let Ω be a finite sample space. The **Laplace model** on Ω is the triple $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is defined by

$$\forall A \in \mathcal{F} \quad P[A] = \frac{|A|}{|\Omega|}$$

1.3 Properties of Events

Prop 1.1. (Consequences of definition 1.2). Let $\mathcal{F}$ be a sigma-algebra on Ω. We have

P4. $\emptyset \in \mathcal{F}$

P5. $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

P6. $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$

P7. $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

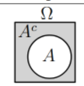
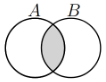
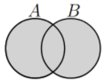
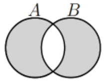
Event	Graphical representation	Probab. interpretation
A^c		A does **not** occur
$A \cap B$		A **and** B occur
$A \cup B$		A **or** B occurs
$A \Delta B$		one and only one of A or B occurs

Figure 1: Representation of set operations

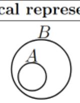
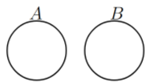
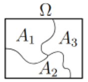
Relation	Graphical representation	Probab. interpretation
$A \subset B$		If A occurs, then B occurs
$A \cap B = \emptyset$		A and B cannot occur at the same time
$\Omega = A_1 \cup A_2 \cup A_3$ with A_1, A_2, A_3 pairwise disjoint		for each outcome ω, one and only one of the events A_1, A_2, A_3 is satisfied.

Figure 2: Representation of set relations

1.4 Properties of probability measures

Prop 1.2. (Consequences of definition 1.3). Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$.

P3. We have $\mathbb{P}[\emptyset] = 0$

P4. (**additivity**) Let $k \geq 1$, let $A_1, \ldots, A_k$ be k pairwise disjoint events, then

$$\mathbb{P}[A_1 \cup \ldots \cup A_k] = \mathbb{P}[A_1] + \ldots + \mathbb{P}[A_k]$$

P5. Let A be an event, then

$$\mathbb{P}[A^c] = 1 - \mathbb{P}[A]$$

P6. If A and B are two events (not necessarily disjoint), then

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A\ capB]$$

Prop 1.3. (**Monotonicity**). Let $A, B \in \mathcal{F}$, then

$$A \subset B \Rightarrow \mathbb{P}[A] \leq \mathbb{P}[B]$$

Prop 1.4. (**Union bound**).Let $A_1, A_2, \ldots$ be a sequence of events (not necessarily disjoint), then we have

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] \leq \sum_{i=1}^{\infty} \mathbb{P}[A_i]$$

Union bound also applies to a finite collection of events.

Prop 1.5. Let (A_n) be an increasing sequence of events (i.e. $\forall n\ A_n \subset A_{n+1}$). Then

$$\lim_{n\to\infty} \mathbb{P}[A_n] = \mathbb{P}\left[\bigcup_{n=1} \infty A_n\right].\ \textbf{increasing limit}$$

Let (B_n) be a decreasing sequence of events (i.e. $\forall n\ B_n \supset B_{n+1}$). Then

$$\lim_{n\to\infty} \mathbb{P}[B_n] = \mathbb{P}\left[\bigcap_{n=1} \infty B_n\right].\ \textbf{decreasing limit}$$

1.5 Conditional probabilities

Def. 1.7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space. Let A, B be two events with $\mathbb{P}[B] > 0$. The **conditional probability of A given B** is defined by

$$\mathbb{P}[A|B] = \frac{A \cap B}{B}$$

Ex. 1.4.