Dimensionality Reduction

Question 1: Note: In this question, all columns will be written in their transposed form, as rows, to make the typography simpler. Matrix M has three rows and three columns, and the columns form an orthonormal basis. One of the columns is [2/7,3/7,6/7], and another is [6/7, 2/7, -3/7]. Let the third column be [x,y,z]. Since the length of the vector [x,y,z] must be 1, there is a constraint that $x^2+y^2+z^2 = 1$. However, there are other constraints, and these other constraints can be used to deduce facts about the ratios among x, y, and z. Compute these ratios.

---> Let C1 be [2/7,3/7,6/7], C2 be [6/7, 2/7, -3/7] and C3 be [x, y, z]
The dot product of any two columns must be zero.
C1.C2 = (2/7 * 6/7) + (3/7 * 2/7) + (6/7 * -3/7) = 0
C2.C3 = (6/7 * x) + (2/7 * y) + (-3/7 * z) = 0 → 6x +2y -3z = 0 – Eq 1
C3.C1 = (x * 2/7) + (y * 3/7) + (z * 6/7) = 0 → 2x + 3y + 6z = 0 – Eq 2
2 * Eq 1 + Eq 2 → 12x + 4y -6z + 2x + 3y +6z = 0 → 14x + 7y = 0 → y = -2x
3 * Eq 2 – Eq 1 → 6x + 9y + 18z – 6x – 2y + 3z = 0 → 7y + 21z = 0 → y = -3z

Question 2: Find the eigenvalues and eigenvectors of the following matrix:

2	3
3	10

You should assume the first component of an eigenvector is 1. Then, find out One eigenvalue and One eigenvector.

---> Let the given matrix be A = 2 3   and the eigen vector be of the form 1
3 10   e
Ax = λx ➔ 2 3   * 1 = λ * 1 → 2 + 3e = λ and 3 + 10e = λe → 3 + 10e = (2 + 3e)e
3 10   e   e
3e2 – 8e + 3 = 0 → e = 3, -1/3
The eigen vectors are 1 and 1
3   -1/3
The eigen values are 2 + 3e = λ → λ = 2 + 3*3 = 11 and λ = 2 + 3*(-1/3) = 1

Question 3: Suppose [1,3,4,5,7] is an eigenvector of some matrix. What is the unit eigenvector in the same direction? Find out the components of the unit eigenvector.

----> Given the eigen vector of some matrix be M = [1,3,4,5,7]
To get the unit eigen vector of given matrix, we need to divide each component by square root of sum of squares in the same direction.
Sum of squares = 12 + 32 + 42 + 52 +72 = 100 and its square root is 10

Unit Eigen Vector = [1/10,3/10,4/10,5/10,7/10]

Question 4: Suppose we have three points in a two dimensional space: (1,1), (2,2), and (3,4). We want to perform PCA on these points, so we construct a 2-by-2 matrix, call it N, whose eigenvectors are the directions that best represent these three points. Construct the matrix N and identify, its elements.

The given three points in a 2- D space are (1,1), (2,2), and (3,4).
We should construct a matrix whose rows correspond to points and columns correspond
to dimensions of the space.
Then the matrix will be M =

$$M = \begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{matrix} \qquad M^T M = \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{matrix} * \begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{matrix} = \begin{matrix} 14 & 17 \\ 17 & 21 \end{matrix}$$

Question 5: Consider the diagonal matrix M =

1	0	0
0	2	0
0	0	0

Compute its Moore-Penrose pseudoinverse.

Moore-Penrose pseudoinverse means the matrix having diagonal elements replaced by 1 and divided by corresponding elements of given matrix and the other elements will be zero. Moore-Penrose pseudoinverse of given matrix is

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{matrix}$$

Question 6: When we perform a CUR dcomposition of a matrix, we select rows and columns by using a particular probability distribution for the rows and another for the columns. Here is a matrix that we wish to decompose:

1	2	3
4	5	6
7	8	9
10	11	12

Calculate the probability distribution for the rows.

---> Probability with which we choose now = (sum of squares of elements in the rows)/(sum of squares of elements in the matrix)

Sum of squares of elements in the matrix = 12*13*25/6 = 3900/6 = 650

P(R1) = (12 + 22 + 32)/650 = 14/650 = 0.02

P(R2) = (42 + 52 + 62) /650 = 77/650 = 0.12

P(R3) = (72 + 82 + 92)/650 = 194/650 = 0.298

P(R4) = (102 + 112 + 122)/650 = 365/650 = 0.56