Defining Random Variables

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Purpose of Presentation

The Problem

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- The Syllabus attempts to define a random variable in terms of what it tries to achieve/model but doesn't actually define what it is:
  - Define and categorise random variables
    - know that a random variable describes some aspect in a population from which samples can be drawn
    - know the difference between a discrete random variable and a continuous random variable

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  - A random variable is a variable whose possible values are outcomes of a statistical experiment or random phenomenon. (NSW Education Standards Authority, 2017, p. 73)

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- The Syllabus glossary isn't a great definition at all:
  - A random variable is a variable whose possible values are outcomes of a statistical experiment or random phenomenon. (NSW Education Standards Authority, 2017, p. 73)
- A well known textbook used in schools gives an *example* of a random variable that models tossing 5 coins rather than defines what it is.

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- Define probability concepts properly such as Random Variables.
- Analogous to being able to *compute* derivatives vs. *understand* derivatives, a deeper understanding of Random Variables enhances one's appreciation for the computations and applications.

Purpose of Presentation

(Tao, 2014, pp. 2-3)

There is a certain philosophical satisfaction in knowing *why* things work... you can certainly use things like the chain rule, L'Hôpital's rule, or integration by parts without knowing why these rules work, or whether there are exceptions to these rules. However, one can get into trouble if one applies rules without knowing where they came from and what the limits of their applicability are.

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Preliminaries

Sets

A set is a collection of distinct objects. e.g. $A = \{1, 2, a, \text{Dave}\}$, or $B = \{c, 2, \pi, \mathbb{H}, \mathfrak{g}\}$.

Recall the set operations:

- Union: $A \cup B = \{1, 2, a, c, \pi, \mathbb{H}, \mathfrak{g}, \text{Dave}\}$
- Intersection: $A \cap B = \{2\}$

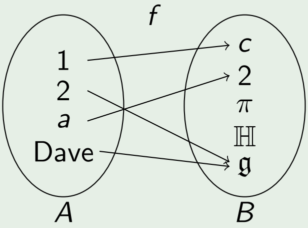
Recall that a set C is a subset of A if all elements of C are also in A, and we write: $C \subseteq A$. Example: $\{1, \text{Dave}\} \subseteq A$.

Preliminaries

Function

A function f from a set A into B, denoted $f : A \to B$ assigns to each $a \in A$ an element $f(a) = b \in B$.
The set A is called the *domain* and the set B is called the *co-domain*.
The *range* is the set $\{f(a) | a \in A\}$.

Example



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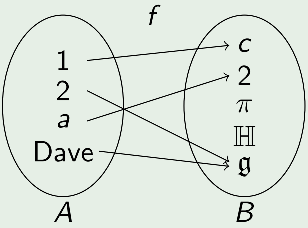

Domain: $\{1, 2, a, \text{Dave}\}$

Co-domain: $\{c, 2, \pi, \mathbb{H}, \mathfrak{g}\}$

Range: $\{c, 2, \mathfrak{g}\}$

Countable and Uncountable Sets

Two types of size (or cardinality) of sets that one encounters in mathematics are:

- Countable: elements in a countable set can be listed (enumerated), e.g the set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, any finite set.
- Uncountable: elements in an uncountable set cannot be listed (enumerated). e.g. the set of real numbers $\mathbb{R}$, the real interval $[0, 1]$.

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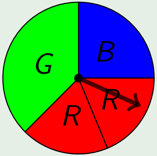
Sample Space

Sample Space (Tao, 2011, p. 192)

The Sample Space is the set of all possible states that a random system could be in, denoted Ω.

Example

Consider the coloured spinner:



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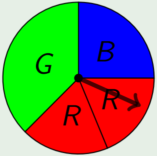

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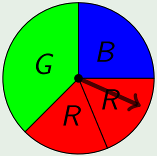

What is the sample space? $\Omega = \{B, G, R\}$
It could also be: $\Omega = [0, 2\pi)$

Events and Event Spaces

Event (Tao, 2011, p. 192)

An Event, E, is a subset of the Sample Space Ω. i.e. $E \subseteq \Omega$.

Example

The set $E = \{2, 4, 6\}$ would represent the Event of rolling even numbers on a 6-sided die with sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Event Space (Tao, 2011, p. 192)

An Event Space (also known as a sigma-field), $\mathscr{F}$, is the set of all possible Events that one can measure the probability of.

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Continuing with the running example of the spinner on the previous slide, the Event Space would be:

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- $\mathscr{F} = \{\{\}, \{B\}, \{G\}, \{R\}, \{B, G\}, \{B, R\}, \{G, R\}, \{B, G, R\}\}$

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Continuing with the running example of the spinner on the previous slide, the Event Space would be:

- $\mathscr{F} = \{\{\}, \{B\}, \{G\}, \{R\}, \{B, G\}, \{B, R\}, \{G, R\}, \{B, G, R\}\}$
- What would $\mathscr{F}$ be if the Sample Space was $[0, 2\pi)$? (Not in the scope of today's presentation - open for questions at the end of the presentation)

Probability Measure

Probability Measure (Daners, 2012, p. 10)

The probability measure of an Event E, denoted $P(E)$, is a function $P : \mathscr{F} \to [0, 1]$, and:

- For all $n \in \mathbb{N}$, if the events $E_1, E_2, \ldots, E_n \in \mathscr{F}$ are disjoint (i.e. they all have empty intersection), then $P(E_1 \cup E_2 \cup \ldots \cup E_n) = P(E_1) + P(E_2) + \ldots + P(E_n)$.
- $P(\{\}) = 0$
- $P(\Omega) = 1$

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- $P(\{\}) = 0$
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Example

Continuing with the running example of the spinner with sample space $\Omega = \{B, G, R\}$, we can measure the probability of the Events with the following function:
$P(\{\}) = 0$, $P(\{B\}) = \frac{1}{4}$, $P(\{G\}) = \frac{3}{8}$, $P(\{R\}) = \frac{3}{8}$. The probability of the other events can be found using the first property listed above.

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- Countable and Uncountable sets
- Sample Space Ω
- Event Space $\mathscr{F}$
- Probability Measure $P : \mathscr{F} \to [0, 1]$
- Note: Many references call the triple $(\Omega, \mathscr{F}, P)$ a Probability Space.

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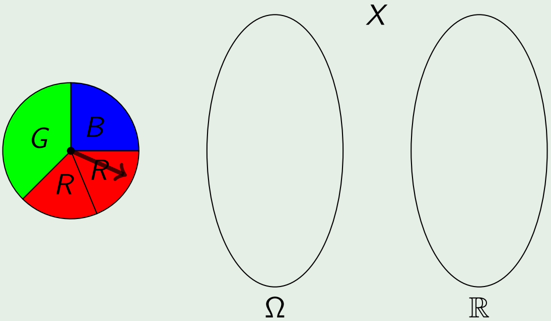
- $[X < a] = \{\omega \in \Omega | X(\omega) < a\}$
- $[a < X < b] = \{\omega \in \Omega | a < X(\omega) < b\}$, etc
- More generally: $[X \in S] = \{\omega \in \Omega | X(\omega) \in S\}$

Example of Discrete Random Variable

Example



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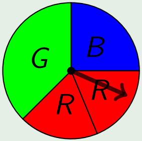

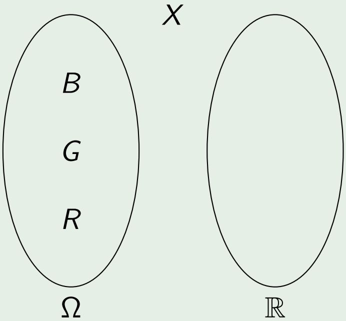

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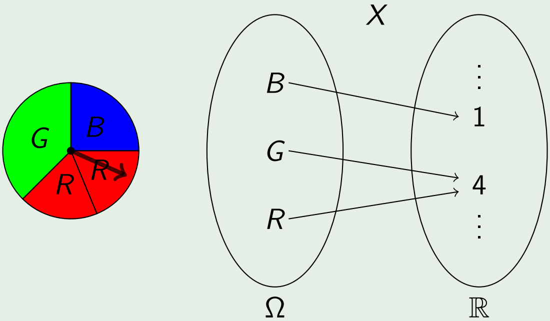

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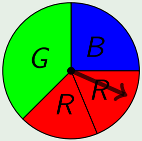

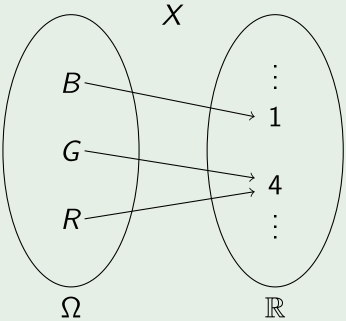

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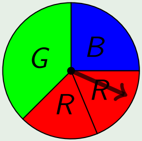

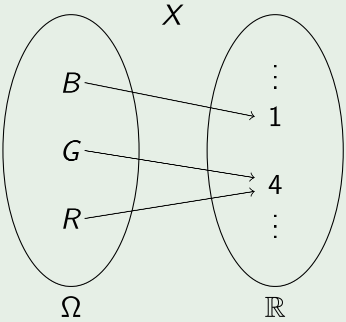

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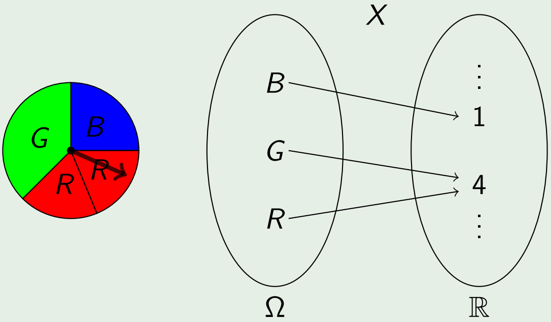

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- $[X > -1] = \{B, G, R\}$

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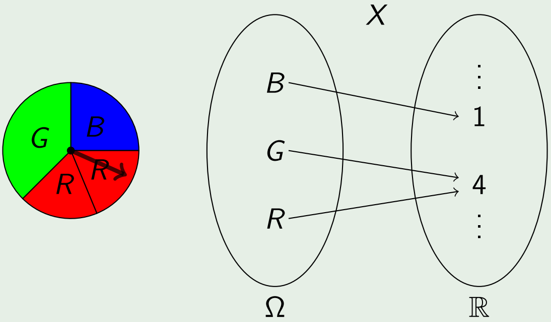

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- $[1 < X \leq 3] = [-\sqrt{3}, -1) \cup (1, \sqrt{3}]$

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- In the example of the spinner, let's say spinning a B awards us with \$1 and spinning a R or G awards us with \$4.
- We don't actually care about getting B, G or R - we care about the money!

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$P(X = 4) = P(\{G, R\}) = \frac{3}{8} + \frac{3}{8} = \frac{3}{4}$.

Probability Distributions

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Example

As with the spinner, we can measure the probability of the Random Variable with:

k	1	4	Everything else (usually not written)
$P(X = k)$	$\frac{1}{4}$	$\frac{3}{4}$	0

Probability Distributions

Continuous Random Variable Probabilities

A probability measure for a Continuous Random Variable X can be defined by:

$$P(a < X < b) = \int_a^b f(x)\, dx$$

where $f(x)$ is called the *probability density function*.

Note: $P(X = k) = 0$ for any $k \in \mathbb{R}$

Cumulative Distribution Function

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The Cumulative Distribution Function $F(x)$ of a continuous random variable X with probability density function $f(x)$ is

$$F(x) = P(X < x) = \int_{-\infty}^{x} f(x)\, dx$$

It accumulates the total of probabilities in the distribution.

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This allows us to find quantiles easily. For example the median can be found by solving for k in $F(k) = 0.5$.
Note also that by the Fundamental Theorem of Calculus, differentiating the cumulative distribution function $F(x)$ will yield the probability density function $f(x)$.

Expectation and Variance

Often researchers are interested in finding the average (mean) result of a random phenomenon and the spread of the results.

Expectation

For Discrete Random Variables:

$$\mu = E(X) = \sum_{x \in X} xP(X = x)$$

For Continuous Random Variables (Note: NOT IN SYLLABUS):

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)\, dx$$

Note that if a and b are constants:

$$E(a + bX) = a + bE(X)$$

Variance

Variance

$$Var(X) = E((X - \mu)^2)$$

By expanding and simplifying, we can find a simpler formula:

$$\begin{aligned} Var(X) &= E(X^2 - 2X\mu + \mu^2) \\ &= E(X^2) - E(2X\mu) + E(\mu^2) \\ &= E(X^2) - 2\mu E(X) + \mu^2 = E(X^2) - 2\mu^2 + \mu^2 \\ &= E(X^2) - \mu^2 = E(X^2) - E(X)^2 \end{aligned}$$

Standard Deviation

Often to *standardise* the squared units of the Variance to units (not squared), the standard deviation is used:

$$\sigma = \sqrt{Var(X)}$$

Important Considerations

- If a and b are constants: $Var(a + bX) = b^2 Var(X)$

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- Why $E((X - \mu)^2)$ and not $E(X - \mu)$ to measure spread?
- Why not $E(|X - \mu|)$?
- What is the link between mean and variance in Random Variables with the mean and variance used in Statistics?

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- Covered in HSC Science Extension
  - t Distribution
  - χ^2 Distribution
  - F Distribution
- Not covered in any HSC syllabus but really cool for exploration
  - Poisson Distribution

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Double Dice

Two 6-sided dice are thrown and the result on the top face are added.

1. What is the sample space?
2. How can we define a random variable here that is appropriate to the scenario?
3. What subset of the sample space does $[X < \pi]$ represent?
4. Find $P(X < \pi)$.

A Geometric Distribution Example

Let X be a random variable that represents the number of tosses of a four-sided die until a 3 is thrown.

1. What is the sample space?
2. How can we define a random variable that represents the scenario?
3. Find $P(X = 1)$, $P(X = 2)$, $P(X = x)$.
4. Find $E(X)$.

Independent Binomial Probability

John tosses a coin 4 times and independently Paul tosses a coin 4 times.

1. How can we define the random variables for John and Paul's results?
2. What distribution do these random variables have?
3. In random variable notation, how can we represent the probability that they toss the same number of heads?
4. Find the probability that they toss the same number of heads.

Note: for numbers higher than 4, you may need to prove binomial identities such as $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{k}$

Distribution Functions

A continuous random variable is distributed according to the probability density function $f(x)$ such that:

$$f(x) = \begin{cases} kx^2 + 1 \text{ if } 0 \leq x \leq 3 \\ 0 \text{ otherwise} \end{cases}$$

where k is some constant.

1. Find the value of k.
2. Find the cumulative distribution function.

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Source files and PDF for this presentation can be found on: https://github.com/moksifu/aisnsw_focus_day_2021 under the MIT Licence.