Complex Exponentials

A can of worms we shall open!

R. Mok BSc(AdvMath), MTeach, GradCertDS

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Opening Question

Here's a few questions to warm you up for the session to see if you're awake!

1. Solve for $x \in \mathbb{N} : x^2 = 4$
   A) $x = 2$ B) $x = 4^{\frac{1}{2}}$ C) $x = \pm 2$ D) $x = \sqrt{4}$
2. Solve for $x \in \mathbb{R} : x^2 = 4$
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Answers:

1. A, B and D
2. C
3. B and C

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De Moivre's Theorem (DMT)[1]

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A triangle with side lengths a, b, c is right angled if and only if

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For all integers n

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This is to ensure the exponential of a complex number yields one value for all arguments of that complex number.

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Extending De Moivre's Theorem to Rational Exponents

If we want to extend DMT to rational exponents, this happens:

Example

$$
\begin{aligned}
1 &= (\cos 0 + i \sin 0) \text{ or } (\cos 2\pi + i \sin 2\pi) \\
1^{\frac{1}{2}} &= (\cos 0 + i \sin 0)^{\frac{1}{2}} \text{ or } (\cos 2\pi + i \sin 2\pi)^{\frac{1}{2}} \\
&= \cos 0 + i \sin 0 \text{ or } \cos \pi + i \sin \pi \\
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Exponentials of complex numbers are multi-valued functions[1].

[1] HSC uses the terminology 'one-to-many' relation.

Roots of Unity

This process is essentially how Roots of Unity are found:

Example

$$
\begin{aligned}
z^n &= 1 \\
&= (\cos(2k\pi) + i\sin(2k\pi)) \text{ for } k \in \mathbb{Z} \\
z &= \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \text{ by the extended DMT}
\end{aligned}
$$

Back to the opening question

Opening Question

Solve for $x \in \mathbb{C} : x^2 = 4$

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$\sqrt{\cdot} : \mathbb{C} \to \mathbb{C}$ is single-valued and is restricted to the principal argument.

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$\sqrt{\cdot} : \mathbb{C} \to \mathbb{C}$ is single-valued and is restricted to the principal argument.
However, the exponential is multi-valued.
What even is an exponential then? More on that later...

Complex Exponentials with Irrationals

- Rational exponents m/n divide the revolutions of arguments nicely such that their results repeat themselves with periodicity (at most) n.

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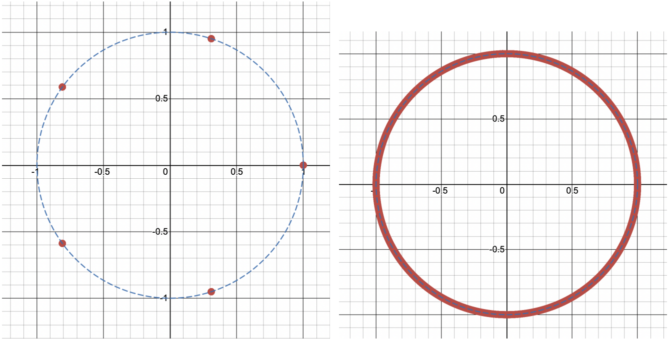

Figure: $1^{\frac{1}{5}}$ vs. irrational exponents such as 1^{π}, $(-1)^{\sqrt{2}}$, etc

Quick Recap

- De Moivre's Theorem (DMT) for integer exponents yields single values.

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- De Moivre's Theorem (DMT) for integer exponents yields single values.
- Extending DMT to rational exponents m/n yields (at most) n values.
- Extending DMT to irrational exponents yields infinite values.
- What's next? Complex exponents.

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Euler's Formula

HSC Dotpoint MEX-N1.3 introduces Euler's Formula [1] which was not in the old syllabus:

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$$e^{ix} = \cos x + i \sin x \text{ for real } x$$

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$$e^{ix} = \cos x + i \sin x \text{ for real } x$$

This opens up new avenues of discussion that were previously not possible in the old syllabus...
... such as:

De Moivre's Theorem Restated

$$(e^{ix})^n = e^{inx}$$

with the same discussions about n as in the previous section.

Complex Logarithms

Euler's Formula opens up discussion about:

$$z = |z|e^{i \arg z}$$
$$\log z = \log(|z|e^{i \arg z})$$

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Complex Logarithm [2, p. 23]

If z is a complex number $\neq 0$, then there exist complex numbers ω such that $e^{\omega} = z$ where ω is in the form $\log_e |z| + i\mathrm{Arg}z + 2n\pi i$ where $n \in \mathbb{Z}$.

Complex Exponential Equations

Example

Solve for $z \in \mathbb{C} : e^z = -1$.

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Solve for $z \in \mathbb{C} : e^z = -1$.

$$\begin{aligned} z &= \log(-1) \\ &= \log_e |-1| + i \arg(-1) \\ &= 0 + i \arg(-1) \\ &= i(2k+1)\pi \text{ for } k \in \mathbb{Z} \end{aligned}$$

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If we test any of these solutions such as $z_1 = i3\pi$:

$$e^{z_1} = e^{i3\pi} = \cos(3\pi) + i\sin(3\pi) = -1$$

Hence, $i3\pi$ is indeed a solution to $e^z = -1$.

Complex Trigonometric Equations

It can be shown using Euler's Formula that:

Trigonometric Functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Example

Solve for $z \in \mathbb{C} : \sin z = 2$

Complex Trigonometric Equations

Example

Solve for $z \in \mathbb{C} : \sin z = 2$

$$\begin{aligned}
\frac{e^{iz} + e^{-iz}}{2} &= 2 \\
e^{iz} + e^{-iz} &= 4 \\
e^{2iz} - 4e^{iz} + 1 &= 0 \\
e^{iz} &= \frac{4 \pm \sqrt{4^2 - 4(1)(1)}}{2} = 2 \pm \sqrt{3} \\
iz &= \log_e(2 \pm \sqrt{3}) + i2k\pi \text{ for } k \in \mathbb{Z} \\
z &= \frac{1}{i} \log_e(2 \pm \sqrt{3}) + 2k\pi
\end{aligned}$$

Complex Exponents

Example

Evaluate i^i.

Complex Exponents

Example

Evaluate i^i.

$$\begin{aligned} i^i &= e^{\log i^i} \\ &= e^{i \log i} \\ &= e^{i(\log_e |i| + i \arg i)} \\ &= e^{i^2 \arg i} \\ &= e^{-1(\frac{\pi}{2} + 2k\pi)} \text{ for } k \in \mathbb{Z} \\ &= \ldots, 111.3177784899\ldots, 0.2078795764\ldots, 0.0003882032039\ldots, \ldots \end{aligned}$$

More examples

Example

- Evaluate 1^i.
- Evaluate $(-1)^{\sqrt{2}}$.

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Answers:

$$1^i = e^{-2k\pi} \text{ for } k \in \mathbb{Z}$$

$$(-1)^{\sqrt{2}} = e^{\sqrt{2}i\pi(2k+1)} \text{ for } k \in \mathbb{Z}$$

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A progressive relearning at each stage

- Stage 4: A power is how many times that base multiplies itself.

$$2^3 = 2 \times 2 \times 2 = 8$$

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- Stage 5: Extend the concept of exponentials with index laws that fixes this problem.
- Stage 6: The number e is introduced in the topic of calculus. Jacob Bernoulli derived this value in 1683 when considering the compound interest problem of compounding a principal amount of $1 at an ever increasing compounding period over a year. He arrived at the discovery:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Exponential Function

e

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

if and only if

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

The proof uses concepts in Binomial Theorem and Analysis [3, pp. 64-65]. It can also be shown (using similar methods as in the proof) that:

Exponential Function [3, p. 174]

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

Plots of increasing n for $(1+\frac{x}{n})^n$

DESMOS Interactive:

`https://www.desmos.com/calculator/p4rzq24eck`

Plots of $(1+\frac{x}{n})^n$ for $x=1$ or $x=i\pi$ and differing values of n are shown below:

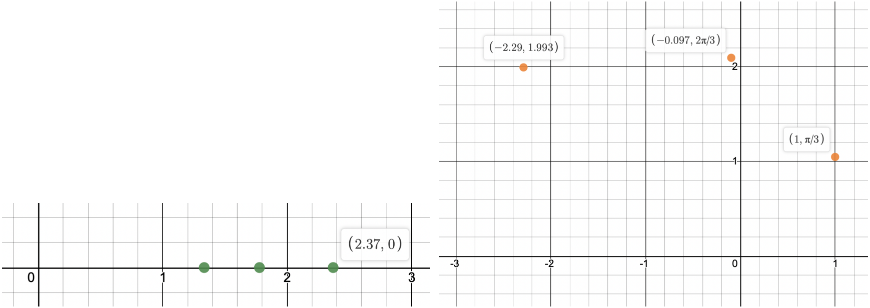

Figure: $(1+\frac{x}{3})^n$ for $n=1,2,3$

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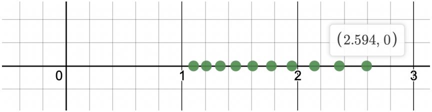

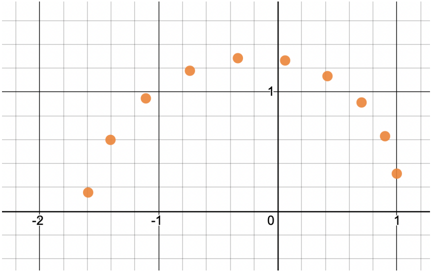

Figure: $(1+\frac{x}{10})^n$ for $n=1,2,\ldots,10$

Plots of increasing n for $(1+\frac{x}{n})^n$

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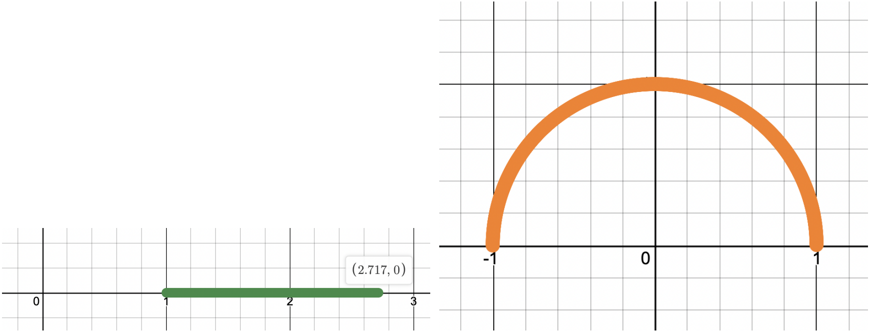

Figure: $(1+\frac{x}{1000})^n$ for $n = 1, 2, \ldots, 1000$

What is an exponential?

What does a^b really mean?

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a^b is equivalent to $\exp(b \log a)$ [4, p. 46].
Ahlfors [4] adopts the convention that if a is restricted to positive numbers, $\log a$ shall be real (unless the contrary is stated) and a^b has a single value. Otherwise, $\log a$ is the complex logarithm and a^b has in general infinitely many values.

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We already need, and use, this definition in deriving Calculus results such as:

$$\begin{aligned} y &= a^x \\ &= e^{\log_e(a^x)} \\ &= e^{x \log_e(a)} \\ \frac{dy}{dx} &= \log_e(a) e^{x \log_e(a)} = \log_e(a) a^x \end{aligned}$$

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Bibliography

[1] NSW Education Standards Authority, *Mathematics extension 2 stage 6 syllabus*, 2017.

[2] T. M. Apostol, *Mathematical Analysis*, 2nd ed. Addison-Wesley, 1974, p. 23, ISBN: 0-201-00288-4.

[3] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed. McGraw-Hill, Inc., 1976, pp. 64–65, 178, ISBN: 0-07-054235-X.

[4] L. V. Ahlfors, *Complex Analysis*, 3rd ed. McGraw-Hill, Inc., 1979, p. 46, ISBN: 0-07-000657-1.

Source files and PDF for this presentation can be found on: `https://github.com/moksifu/complex_exponentials` under the MIT Licence.