Complex Exponentials

A can of worms we shall open!

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Opening Question

Here's a few questions to warm you up for the session to see if you're awake!

1. Solve for $x \in \mathbb{N} : x^2 = 4$
   A) $x = 2$ B) $x = 4^{\frac{1}{2}}$ C) $x = \pm 2$ D) $x = \sqrt{4}$
2. Solve for $x \in \mathbb{R} : x^2 = 4$
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Answers:

1. A, B and D
2. C
3. B and C

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De Moivre's Theorem (DMT) (**syllabus**)

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A triangle with side lengths a, b, c is right angled if and only if

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This is to ensure the exponential of a complex number yields one value for all arguments of that complex number.

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Extending De Moivre's Theorem to Rational Exponents

If we want to extend DMT to rational exponents, this happens:

Example

$$\begin{aligned}
1 &= (\cos 0 + i \sin 0) \text{ or } (\cos 2\pi + i \sin 2\pi) \\
1^{\frac{1}{2}} &= (\cos 0 + i \sin 0)^{\frac{1}{2}} \text{ or } (\cos 2\pi + i \sin 2\pi)^{\frac{1}{2}} \\
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$$

Exponentials of complex numbers are multi-valued functions[1].

[1]HSC uses the terminology 'one-to-many' relation.

Roots of Unity

This process is essentially how Roots of Unity are found:

Example

$$
\begin{aligned}
z^n &= 1 \\
&= (\cos(2k\pi) + i\sin(2k\pi)) \text{ for } k \in \mathbb{Z} \\
z &= \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \text{ by the extended DMT}
\end{aligned}
$$

Back to the opening question

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Solve for $x \in \mathbb{C} : x^2 = 4$

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$\sqrt{\cdot} : \mathbb{C} \to \mathbb{C}$ is single-valued and is restricted to the principal argument.

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However, the exponential is multi-valued.

What even is an exponential then? More on that later...

Complex Exponentials with Irrationals

- Rational exponents m/n divide the revolutions of arguments nicely such that their results repeat themselves with periodicity (at most) n.

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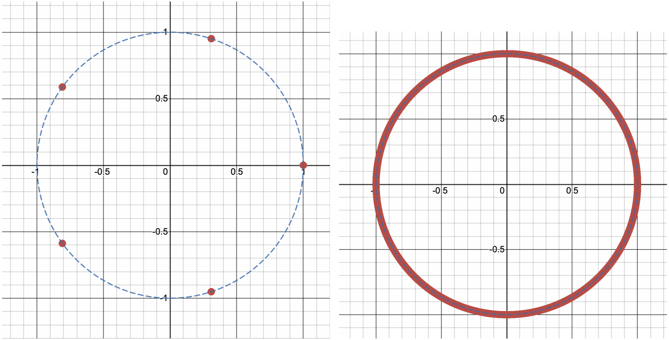

Figure: $1^{\frac{1}{5}}$ vs. irrational exponents such as 1^{π}, $(-1)^{\sqrt{2}}$, etc

Quick Recap

- De Moivre's Theorem (DMT) for integer exponents yields single values.

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- De Moivre's Theorem (DMT) for integer exponents yields single values.
- Extending DMT to rational exponents m/n yields (at most) n values.
- Extending DMT to irrational exponents yields infinite values.
- What's next? Complex exponents.

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Euler's Formula

HSC Dotpoint MEX-N1.3 introduces Euler's Formula (**syllabus**) which was not in the old syllabus:

Euler's Formula

$$e^{ix} = \cos x + i \sin x \text{ for real } x$$

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... such as:

De Moivre's Theorem Restated

$$(e^{ix})^n = e^{inx}$$

with the same discussions about n as in the previous section.

Complex Logarithms

Euler's Formula opens up discussion about:

$$z = |z|e^{i \arg z}$$
$$\log z = \log(|z|e^{i \arg z})$$

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(Principal) Complex Logarithm (daners)

$$\mathrm{Log} z = \log_e |z| + i\mathrm{Arg} z$$

Complex Exponential Equations

Example

Solve for $z \in \mathbb{C} : e^z = 1$.

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$$\begin{aligned} z &= \log(1) \\ &= \log_e |1| + i \arg(1) \\ &= 0 + i \arg(1) \\ &= i2k\pi \text{ for } k \in \mathbb{Z} \end{aligned}$$

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If we test any of these solutions such as $z_1 = i2\pi$:

$$e^{z_1} = e^{i2\pi} = 1$$

Hence, $i2\pi$ is indeed a solution to $e^z = 1$.

Complex Trigonometric Equations

It can be shown using Euler's Formula that:

Trigonometric Functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Complex Trigonometric Equations

Example

Solve for $z \in \mathbb{C} : \sin z = 2$

Complex Trigonometric Equations

Example

Solve for $z \in \mathbb{C} : \sin z = 2$

$$\begin{aligned} \frac{e^{iz} + e^{-iz}}{2} &= 2 \\ e^{iz} + e^{-iz} &= 4 \\ e^{2iz} - 4e^{iz} + 1 &= 0 \end{aligned}$$

$$\begin{aligned} e^{iz} &= \frac{4 \pm \sqrt{4^2 - 4(1)(1)}}{2} = 2 \pm \sqrt{3} \\ iz &= \log_e(2 \pm \sqrt{3}) + i2k\pi \text{ for } k \in \mathbb{Z} \\ z &= \frac{1}{i} \log_e(2 \pm \sqrt{3}) + 2k\pi \end{aligned}$$

Complex Exponents

Example

Evaluate i^i.

Complex Exponents

Example

Evaluate i^i.

$$\begin{aligned} i^i &= e^{\log i^i} \\ &= e^{i \log i} \\ &= e^{i(\log_e |i| + i \arg i)} \\ &= e^{i^2 \arg i} \\ &= e^{-1(\frac{\pi}{2} + 2k\pi)} \text{ for } k \in \mathbb{Z} \\ &= \ldots, 111.3177784899\ldots, 0.2078795764\ldots, 0.0003882032039\ldots, \ldots \end{aligned}$$

More examples

Example

- Evaluate 1^i.
- Evaluate $(-1)^{\sqrt{2}}$.

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Answers:

$$1^i = e^{-2k\pi} \text{ for } k \in \mathbb{Z}$$
$$(-1)^{\sqrt{2}} = e^{\sqrt{2}i\pi(2k+1)} \text{ for } k \in \mathbb{Z}$$

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A progressive relearning at each stage

- Stage 4: A power is how many times that base multiplies itself.

$$2^3 = 2 \times 2 \times 2 = 8$$

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- Stage 5: Extend the concept of exponentials with index laws that fixes this problem.
- Stage 6: The number e is introduced in the topic of calculus. Jacob Bernoulli derived this value in 1683 when considering the compound interest problem of compounding a principal amount of \$1 at an ever increasing compounding period over a year. He arrived at the discovery:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

e

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

if and only if

$$e = \sum_{k=0}^{\infty}\frac{1}{k!}$$

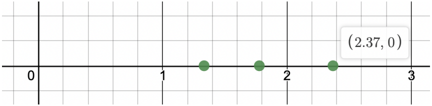
The proof uses concepts in Binomial Theorem and Analysis (**rudin**).
It can also be shown (using similar methods as in the proof) that:

Exponential Function (rudin)

$$\exp(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

Plots of increasing n for $(1+\frac{x}{n})^n$

DESMOS Interactive: https://www.desmos.com/calculator/p4rzq24eck
Plots of $(1+\frac{x}{n})^n$ for $x=1$ or $x=i\pi$ and differing values of n are shown below:



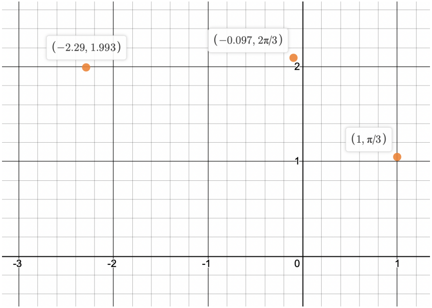


Figure: $(1+\frac{x}{3})^n$ for $n=1,2,3$

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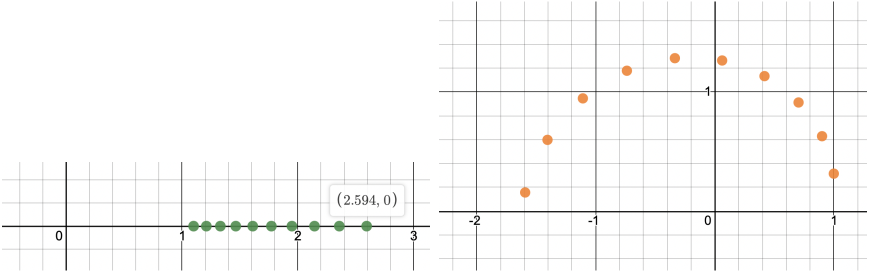

Figure: $(1+\frac{x}{10})^n$ for $n = 1, 2, \ldots, 10$

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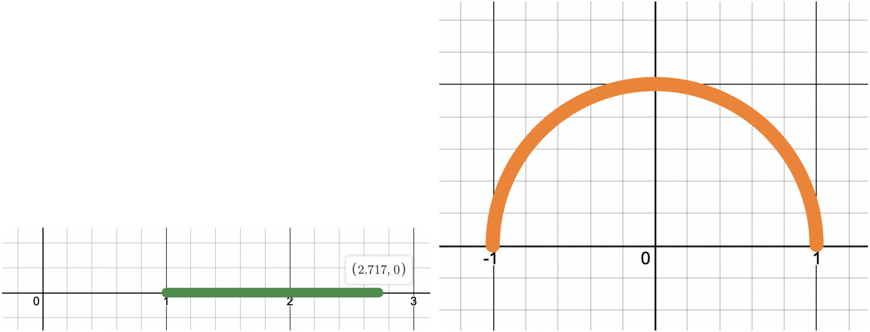

Figure: $(1+\frac{x}{1000})^n$ for $n=1,2,\ldots,1000$

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