1 Problem Set

1. (1 point) Write down the converse of the following statement:

   If it is raining then the floor will be wet.

2. Write down the negation of the following statements:
   (a) (1 point) The city of Toronto is not in Australia.
   (b) (2 points) If the weather is good then I will be at the beach.
   (c) (2 points) Any multiple of an even number is also even.

3. (1 point) Write down the contrapositive of the following statement:

   A square has an angle sum of 360°.

4. (1 point) Write down the two converse statements equivalent to the following statement:

   A number is even if and only if it has a remainder of 0 when divided by 2.

5. (2 points) Prove that the sum of three consecutive odd integers is divisible by 3.

6. (2 points) Prove that a positive integer that ends in 325 cannot be the square of an integer.

7. (2 points) By proving the contrapositive, prove for an integer n that if $n^2 - 1$ is not divisible by 3 then n is divisible by 3.

8. (a) (2 points) Show by expanding that

   $$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \ldots + x^2 + x + 1)$$

   (b) (2 points) By proving the contrapositive, prove that if $2^n - 1$ is a prime number, then n is also prime.
   (c) (2 points) Give a counter-example of the converse.

9. (2 points) Prove that for an integer n, if $n^2 - 6n + 5$ is even then n is odd.

10. (2 points) Prove that $\cos^2(5°)$ is irrational. Hint: You are given that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

11. (2 points) Prove that if n is a composite integer, then it has a prime divisor less than or equal to $\sqrt{n}$.

12. (3 points) Prove by induction

    $$\sum_{r=1}^{n} r(3r-1) = n^2(n+1)$$

13. (4 points) A sequence is recursively defined by $a_1 = 3$ and $a_n = 2a_{n-1} - 1$ for $n > 1$. Prove by induction that $a_n = 2^n + 1$.

14. (3 points) Prove by induction for all positive integers n that $5^n + 2 \times 11^n$ is divisible by 3.

15. (3 points) Prove by induction $n! > 2^n$ for integers $n \geq 4$.

16. (a) (1 point) Prove that $x^2 + y^2 \geq 2xy$ for $x, y > 0$.

    (b) (2 points) Hence, prove if $a^2 + b^2 = c^2 + d^2 = 1$, then $ac + bd \leq 1$.

17. (3 points) By using the AM-GM inequality, show that for all positive integers n,

$$n! \leq \left(\frac{n+1}{2}\right)^n$$

18. The x and y coordinates of three points A, B and C in a number plane are all positive integers.

    (a) (2 points) Show that the area of $\triangle ABC$ is rational.

    (b) (2 points) Hence, show that $\triangle ABC$ cannot be an equilateral triangle.

19. (2 points) [Russell's Paradox]

    Let R be the set of all sets that are not members of themselves.

    Show that $R \in R$ if and only if $R \notin R$.