1 Euclidean Vectors

Informally speaking, vectors are mathematical objects that can be added together and multiplied by real numbers:

That is, if $\mathbf{v}, \mathbf{w}$ are vectors and k is a real number, then

1. $\mathbf{v} + \mathbf{w}$ is a vector.

2. $k\mathbf{v}$ is a vector.

Some examples of vectors: polynomials, real valued functions, sequences, arrows you can draw on the number plane, etc - we don't need to focus on most these from a vector perspective for the HSC.

The vectors of focus in the HSC course are those with *direction* and *length*, also known as Euclidean Vectors.

Notation 1.1 (Component Form, Ordered Pairs, Column Notation)**.** In two/three dimensional space, we define the following perpendicular, unit vectors (vectors with magnitude equal to 1):

- $\mathbf{i}$: the unit vector in the direction of the positive x-axis.

- $\mathbf{j}$: the unit vector in the direction of the positive y-axis.

- $\mathbf{k}$: the unit vector in the direction of the positive z-axis.

Note that in two dimensions, any linear combination of $\mathbf{i}$ and $\mathbf{j}$ and in three dimensions, any linear combination of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ can represent any vector in the space.

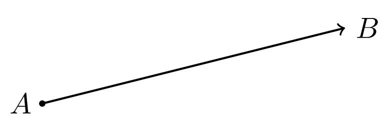
Hence, the position (a, b, c) can be represented by the following different notations:

- Component Form: $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ - Also known as vector Quaternion form.

- Ordered Pair Form: (a, b, c) - The components of the vectors are represented in the same way as the coordinate system.

- Column Notation: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ - The components of the vectors are represented in a column.

Notation 1.2 (Graphical Representation)**.** Vectors in two/three dimensional space can be represented as directed line segments. The vector $\overrightarrow{AB}$ is represented by the tail end of the line segment at A, and the head at B:



Any two vectors are equal if and only if they have equal magnitude and same direction. Hence, in a graphical representation, it does not matter where the directed line segment is drawn as long as it has the same length and direction as the vector that they represent. So, any graphical representation of a vector can be translated freely in the diagram.

2 Vector Addition and Scalar Multiplication

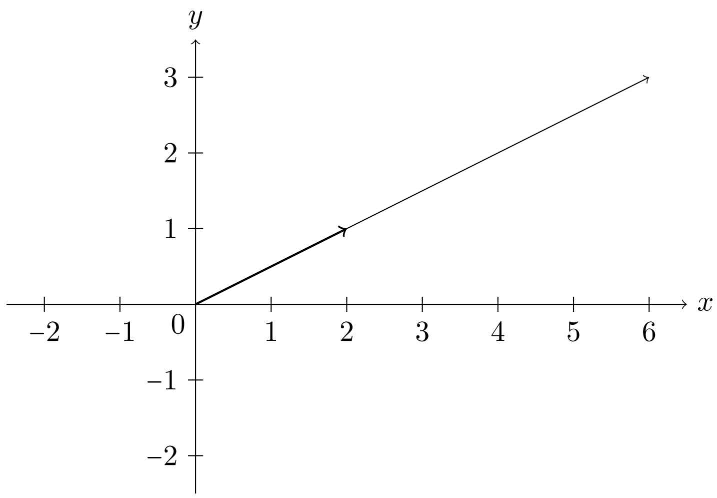
In this part, we consider vector addition and scalar multiplication algebraically and geometrically.

2.1 Scalar Multiplication

For a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $a \in \mathbb{R}$, $a\vec{v}$ is equal to $\begin{bmatrix} av_1 \\ av_2 \end{bmatrix}$.

Graphically, this is represented by the directed line segment of $\vec{v}$ lengthened by a factor of a.

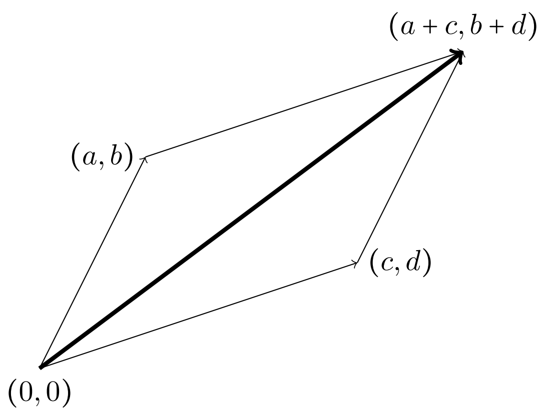
Example 2.1.1.
The vector $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$ and $3\mathbf{v}$ is shown below:



2.2 Vector Addition

$$\begin{aligned}(a\mathbf{i} + b\mathbf{j}) + (c\mathbf{i} + d\mathbf{j}) &= (a\mathbf{i} + c\mathbf{i}) + (b\mathbf{j} + d\mathbf{j}) \\ &= (a + c)\mathbf{i} + (b + d)\mathbf{j}\end{aligned}$$

Graphically, the vector sum is the diagonal of the parallelogram formed by translating the tail end of one vector to the other.



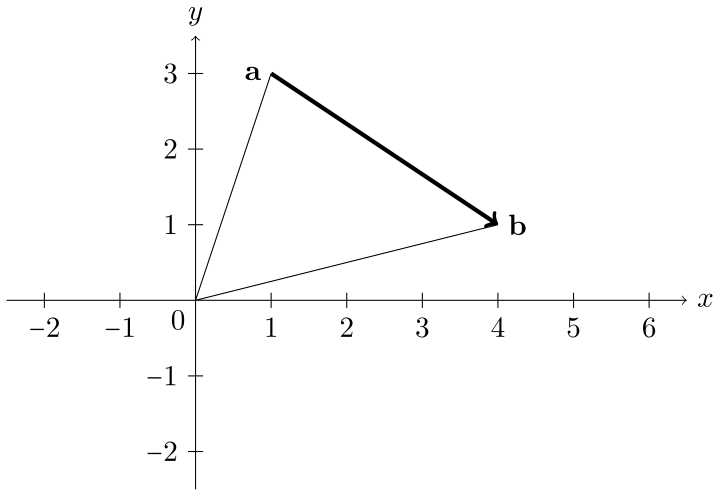
2.3 Vector Subtraction

Consider the position vectors **a** that represent the position $A = (1, 3)$ and **b** represents the position $B = (4, 1)$.

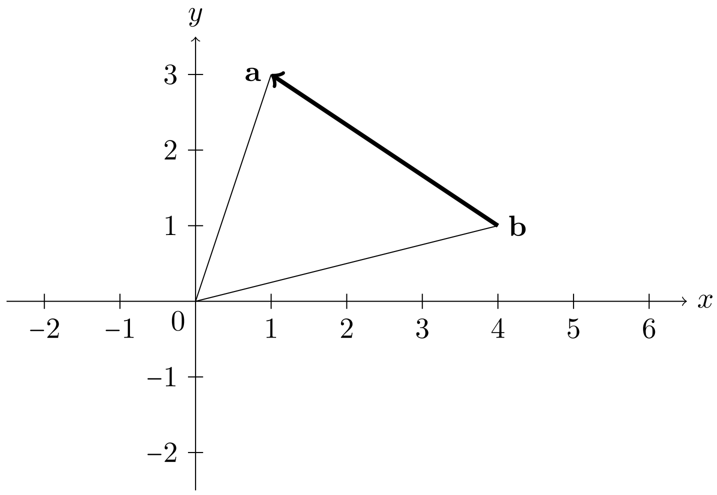
The algebraic representation of $\mathbf{b} - \mathbf{a}$ is equal to the *displacement* vector

$$\overrightarrow{AB} = (4 - 1, 1 - 3) = (3, -2)$$

This can be represented graphically by translating the directed line segment so that the tail is on A and the head is on B, and it is the off-diagonal of the parallelogram formed by the vectors:



Similarly, the geometric representation of $\mathbf{a} - \mathbf{b}$ is the displacement vector $\overrightarrow{BA}$.



Hence:

- Vector *Addition* is represented by the diagonal of the parallelogram formed by the vectors.
- Vector *Subtraction* is represented by the off-diagonal of the parallelogram formed by the vectors. Be careful with the direction of this vector.

3 Operations with Vectors

In many textbooks, the dot product is usually defined first in terms of angles and lengths, which appeals to our natural intuition about objects with direction and length. However, it is actually the existence of the dot product that defines the notion of length and angles. So firstly we define the dot product of two vectors, and then see how length and angle is derived from this.

Definition 3.1 (Dot Product)**.**
The dot product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is defined as:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Definition 3.2 (Length and Angle)**.**

- The length of a vector is the square root of the dot product of the vector by itself. i.e.
$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$
- The cosine of the angle of two unit vectors is equal to their dot product. i.e.
$$\cos \angle(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{u}}{|\mathbf{u}|} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$$

4 Parallel and Perpendicular Vectors

Definition 4.1 (Parallel Vectors)**.**
Two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if there exists a scalar $k \in \mathbb{R}$ such that:

$$\mathbf{u} = k\mathbf{v}$$

i.e. they are scalar multiples of one another.

Theorem 4.1 (Perpendicular Vectors)**.**
Two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

(Prove this!)

5 Projections

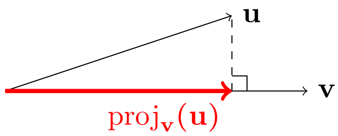
Definition 5.1 (Projection of $\mathbf{u}$ onto $\mathbf{v}$)**.**
The projection of a vector $\mathbf{u}$ onto another vector $\mathbf{v}$ is a function:

$$\text{proj}_{\mathbf{v}}(\mathbf{u})$$

such that:

1. The direction of $\text{proj}_{\mathbf{v}}(\mathbf{u})$ is in the same direction as $\mathbf{v}$.
2. The magnitude of $\text{proj}_{\mathbf{v}}(\mathbf{u})$ is equal to $|\mathbf{u}|\cos\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Graphically, it is represented by the following diagram:



The formula for $\text{proj}_{\mathbf{v}}(\mathbf{u})$ is:

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$$

Proof. The projection of $\mathbf{u}$ onto $\mathbf{v}$ can be constructed by multiplying its magnitude $|\mathbf{u}|\cos\theta$ onto the unit vector $\hat{\mathbf{v}}$:

$$\begin{aligned}
\text{proj}_{\mathbf{v}}(\mathbf{u}) &= (|\mathbf{u}|\cos\theta)\,\hat{\mathbf{v}} \\
&= (|\mathbf{u}|\cos\theta)\,\frac{\mathbf{v}}{|\mathbf{v}|} \\
&= \left(\frac{|\mathbf{u}||\mathbf{v}|\cos\theta}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} \\
&= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} \\
&= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \\
&= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}
\end{aligned}$$

□

Note: Learn the proof rather than the result! For Extension 2 students, see 2021 paper Q10 for why you should learn this proof.

6 Problem Set

1. Prove the following results about dot products:

   (a) $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2$

   (b) (Cauchy-Schwarz Inequality) $|\mathbf{u}\cdot\mathbf{v}| \le |\mathbf{u}||\mathbf{v}|$

   (c) (Triangle Inequality) $|\mathbf{u}+\mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$

2. If $|\mathbf{a}| = 6$, $|\mathbf{b}| = 7$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$, find:

   (a) $\mathbf{a}\cdot\mathbf{b}$

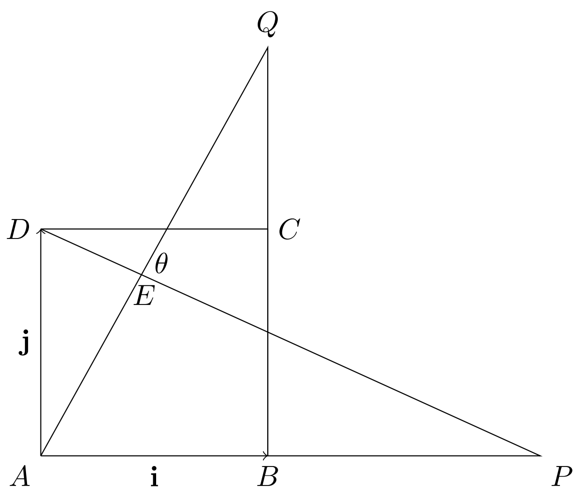
   (b) $(\mathbf{a}+2\mathbf{b})\cdot(3\mathbf{a}-\mathbf{b})$

   (c) $|\mathbf{a}+\mathbf{b}|$

3. If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are unit vectors, find the angle between $\mathbf{u}$ and $\mathbf{v}$.

4. $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{p}$ are vectors such that $\mathbf{a}\cdot\mathbf{c} = 3$, $\mathbf{b}\cdot\mathbf{c} = 1$ and $\mathbf{p} = \mathbf{a} + k\mathbf{b}$. If $\mathbf{p}$ is perpendicular to $\mathbf{c}$, find the value of k.

5. In the figure below, $ABCD$ is a square with $\overrightarrow{AB} = \mathbf{i}$ and $\overrightarrow{AD} = \mathbf{j}$. P and Q are respectively points on AB and BC produced with $BP = k$ and $CQ = m$. AQ and DP intersect at E and $\angle QEP = \theta$.



   (a) By calculating $\overrightarrow{AQ}\cdot\overrightarrow{DP}$, find $\cos\theta$ in terms of m and k.

   It is given that $\dfrac{DE}{EP} = \dfrac{1}{4}$.

(b) Express $\overrightarrow{AE}$ in terms of k.

(c) Let $\dfrac{AE}{AQ} = r$. Express $\overrightarrow{AE}$ in terms of r and m.

(d) If $\theta = 90°$, use the above results to find the values of m, k and r.

6. A, B and C are points on a plane such that $\overrightarrow{OA} = 3\mathbf{i} - \mathbf{j}$, $\overrightarrow{BC} = 7\mathbf{i} + \mathbf{j}$, and $\overrightarrow{OC} = x\mathbf{i} + y\mathbf{j}$, where O is the origin.

   (a) Find $\overrightarrow{CA}$, $\overrightarrow{OB}$ and $\overrightarrow{AB}$ in terms of x, y, $\mathbf{i}$ and $\mathbf{j}$.

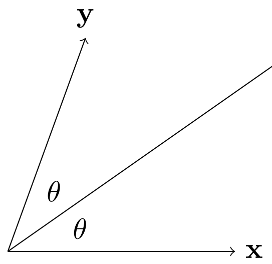
   (b) Given that $\overrightarrow{AB} \cdot \overrightarrow{BC} = 4\overrightarrow{BC} \cdot \overrightarrow{CA}$, show that $y = 30 - 7x$.

   (c) Furthermore, if $|\overrightarrow{BC}| = \sqrt{5}|\overrightarrow{CA}|$ and $x > 0$, $y > 0$,
   i. find x and y.

   ii. show that CA is perpendicular to AB.

   iii. show that O lies on AB.

7. In the figure below, $\mathbf{x}$ and $\mathbf{y}$ are unit vectors, each of which makes an angle θ with the vector $\mathbf{z}$, where $\theta \in \left(0, \frac{\pi}{2}\right)$.



   (a) Show that $\mathbf{x} \cdot \mathbf{z} = \mathbf{y} \cdot \mathbf{z}$.

   (b) Let $\mathbf{z} = m\mathbf{x} + n\mathbf{y}$ for some constants m and n. By expressing $\mathbf{x} \cdot \mathbf{z}$ and $\mathbf{y} \cdot \mathbf{z}$ in terms of m, n and θ, show that $m = n$.

8. A bug initially at the origin moves in a two-dimensional plane according to the velocity vector $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ ms^{-1}.

   (a) Calculate the speed of the bug.

   (b) After two seconds, a gust of wind blows on the bug with velocity $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ ms^{-1} while the bug attempts to move with the same velocity.
   Find the position of the bug after another 3 seconds.

9. Prove that the midpoints of the sides of a quadrilateral join to form a parallelogram.

10. Prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

11. In a triangle ABC, M and N are the midpoints of AB and AC respectively.
Prove that $MN \parallel BC$ and $MN = \frac{1}{2}BC$.
(Note: in the old HSC syllabus, students were required to memorise this result and the reason: "line joining midpoints of two sides of a triangle is parallel to the third side and half its length".)

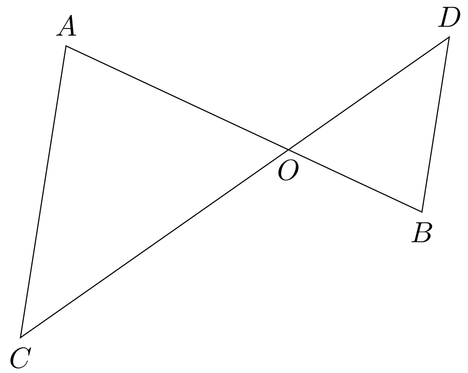
12. Prove that an angle in a semicircle is a right angle.

13. In a circle with centre O, X is the midpoint of some chord MN. Prove that $OX \perp MN$.

14. In a quadrilateral $ABCD$, M and N are the midpoints of AD and BC respectively.
Show that $\overrightarrow{AB} + \overrightarrow{DC} = 2\overrightarrow{MN}$.

15. In a quadrilateral $ABCD$, E and F are the midpoints of AB and CD respectively.
G is the midpoint of EF.
Prove that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} = \mathbf{0}$.

16. In the figure below, the lines AB and CD intersect at O. If $\dfrac{AO}{OB} = \dfrac{CO}{OD}$, prove by using vectors that $AC \parallel DB$.



17. In a parallelogram $ABCD$, E and F are two points on the diagonal AC such that $AE = FC$.
Prove that $BFDE$ is a parallelogram.

18. Given two vectors $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ and $\mathbf{v} = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}$, where $\theta > \phi$:

(a) Show that $\mathbf{u}$ and $\mathbf{v}$ are unit vectors.

(b) By considering the dot product, prove that $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$.

19. In the standard coordinate system, the vector $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$ means the vector $\mathbf{v}$ is the sum of 3 multiples of $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and 5 multiples of $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The position of this vector is given by the coordinate $(3, 5)$ in this system.

    Let the vectors $\mathbf{e_1} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{e_2} = -2\mathbf{i} + 4\mathbf{j}$.

    We wish to form a new coordinate system with $\mathbf{e_1}$ and $\mathbf{e_2}$, i.e. vectors will be written in the form $a_1\mathbf{e_1} + a_2\mathbf{e_2}$ in this new coordinate system for some real constants a_1, a_2. In this new coordinate system, let the position of the vector be notated by $(a_1, a_2)'$.

    (a) Convert $(6, 3)$ in the standard coordinate system to the new coordinate system given by $\mathbf{e_1}$ and $\mathbf{e_2}$.

    (b) Convert $(13, 37)$ in the standard coordinate system to the new coordinate system. (Hint: use the concept of vector projections.)

    (c) Consider the vectors $\mathbf{v} = 2\mathbf{e_1} + 3\mathbf{e_2}$ and $\mathbf{w} = -5\mathbf{e_1} + 7\mathbf{e_2}$.
    Find $\mathbf{v} \cdot \mathbf{w}$.