Week3 HW Problems

① Simplify the expression:

$$\log_2\left(\frac{8\sqrt{2}}{16}\right) + \log_2(32) - 2\log_2(4) =$$

$$\log_2(8\sqrt{2}) - \log_2(16) + \log_2(32) - 2\log_2(4) =$$

$$\log_2(2^3 \cdot 2^{\frac{1}{2}}) - 4 + 5 - 2 \cdot 2 =$$

$$\frac{7}{2}\log_2(2) - 3 = \frac{7}{2} \cdot 1 - 3 = \frac{1}{2}$$

② Solve for x:

$$\log_3(x-1) + \log_3(x+1) = 2$$

$$\log_3((x-1)(x+1)) = 2$$

$$\log_3(x^2-1) = 2$$

$$x^2 - 1 = 3^2 = 9$$

$$x^2 = 10$$

$$x = \sqrt{10} \qquad (x-1) \quad \sqrt{10} - 1 > 0 \ T$$

③ An initial investment of \$10,000 is made in an account that yields an annual interest rate of 6%, compounded quarterly. How many years will it take for the investment to grow to at least \$20,000?

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ Compound interest formula.

$A = \$20{,}000$

$P = \$10{,}000$

$r = 6\% = 0{,}06$

$n = 4$ times per year

t = number of years

$$A = 10{,}000\left(1 + \frac{0{,}06}{4}\right)^{4t}$$

$$\frac{20000}{10000} = (1 + 0{,}015)^{4t}$$

$$2 = (1{,}015)^{4t}$$

$$\ln(2) = \ln((1.015)^{4t})$$

$$\ln(2) = 4t\ln(1.015)$$

$$t = \frac{\ln(2)}{4\ln(1.015)} \approx \frac{0.6931}{4 \times 0.014889} \approx \frac{0.6931}{0.059556} \approx 11.64 \text{ years}$$

④ A radioactive substance decays according to the formula:

$$N(t) = N_0 e^{-kt}$$

Where N_0 is the initial amount, k is the decay constant, and t is time in years. If a sample has half-life of 5 years, find the decay constant k.

$$N(t_{\frac{1}{2}}) = \frac{N_0}{2}$$

$$\frac{N_0}{2} = N_0 e^{-kt_{\frac{1}{2}}}$$

$$\frac{1}{2} = e^{-k \cdot 5}$$

$$\ln\left(\frac{1}{2}\right) = -5k$$

$$-0.6931 = -5k$$

$$k = \frac{0.6931}{5} \approx 0.1386 / \text{per year}$$

⑤ If 100 grams of a radioactive substance decays to 70 grams in 3 hours, find the time it will take to decay to 20 grams

$$N(t) = N_0 e^{-kt} \qquad N(3) = 70 \text{grams}$$

$$70 = 100 e^{-kt}$$

$$0.7 = e^{-3k}$$

$$\ln(0.7) = -3k$$

$$-0.3567 = -3k$$

$$k = \frac{0.3567}{3} \approx 0.1189$$

find $N(t) = 20$ grams:

$$20 = 100 e^{-0.1189t}$$

$$0.2 = e^{-0.1189t}$$

$$\ln(0.2) = -0.1189t$$

$$-1.6094 = -0.1189t$$

$$t = \frac{1.6094}{0.1189} \approx 13.54 \text{ hours}$$

(6) Find the unit vector in the direction from ~~unit~~ point $A(1,2,3)$ to point $B(4,6,9)$.

$\vec{AB} = \langle 3, 4, 6 \rangle$

$|\vec{AB}| = \sqrt{3^2+4^2+6^2} = \sqrt{9+16+36} = \sqrt{61} \approx 7.81$

$\vec{u} = \frac{\vec{AB}}{|\vec{AB}|} = \left\langle \frac{3}{\sqrt{61}}, \frac{4}{\sqrt{61}}, \frac{6}{\sqrt{61}} \right\rangle$

(7) Express the vector $\vec{v} = 7\hat{i} - 2\hat{j} + 4\hat{k}$ in matrix form and find its magnitude.

$\vec{v} = \begin{bmatrix} 7 \\ -2 \\ 4 \end{bmatrix}$ $\qquad |\vec{v}| = \sqrt{7^2+(-2)^2+4^2} = \sqrt{49+4+16} = \sqrt{69} \approx 8.31$

(8) Given vectors $\vec{a} = \langle 2, -1, 3 \rangle$ and $\vec{b} = \langle -1, 4, 2 \rangle$, compute $3\vec{a} - 2\vec{b}$.

$3\vec{a} = \langle 6, -3, 9 \rangle$ $\qquad 2\vec{b} = \langle -2, 8, 4 \rangle$

$3\vec{a} - 2\vec{b} = \langle 8, -11, 5 \rangle$

(9) Find the angle between vectors $\vec{p} = \langle 1, 2, 3 \rangle$ and $\vec{q} = \langle 4, -5, 6 \rangle$.

$\vec{p} \cdot \vec{q} = 1 \cdot 4 + 2 \cdot (-5) + 3 \cdot 6 = 4 - 10 + 18 = 12$

$|\vec{p}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

$|\vec{q}| = \sqrt{4^2+(-5)^2+6^2} = \sqrt{77}$

$\cos\theta = \frac{\vec{p} \cdot \vec{q}}{|\vec{p}||\vec{q}|} = \frac{12}{\sqrt{14} \cdot \sqrt{77}} = \frac{12}{\sqrt{1{,}078}} \approx 0{,}3647$

$\theta = \arccos(0.3647) \approx 68.58°$

(10) Determine if the vectors $\vec{u} = \langle 2, -1, 4 \rangle$ and $\vec{v} = \langle -8, 4, -16 \rangle$ are orthogonal.

$\vec{u} \cdot \vec{v} = (2 \cdot (-8)) + ((-1) \cdot 4) + (4 \cdot (-16)) = -16 - 4 - 64 = -84$

Since $\vec{u} \cdot \vec{v} \neq 0$, ~~vec~~ the vectors are not orthogonal
(dot product)

(11) Given matrices: $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ -2 & 1 \end{bmatrix}$

Compute $2A - 3B$.

$2A = \begin{bmatrix} 4 & -2 \\ 0 & 6 \end{bmatrix}$ $3B = \begin{bmatrix} 12 & 15 \\ -6 & 3 \end{bmatrix}$ $2A - 3B = \begin{bmatrix} -8 & -17 \\ 6 & 3 \end{bmatrix}$

(12) Compute the product:

$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ $E = CD$

$E_{1,1} = 1 \cdot 5 + 2 \cdot 7 = 19$ $E_{2,1} = 3 \cdot 5 + 4 \cdot 7 = 43$

$E = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

$E_{1,2} = 1 \cdot 6 + 1 \cdot 8 = 22$ $E_{2,2} = 3 \cdot 6 + 4 \cdot 8 = 50$

(13) Use Gaussian elimination to solve the system:

$$\begin{cases} x + y + z = 6 \\ 2x - y + 3z = 14 \\ -3x + 2y - 2z = -10 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 3 & 14 \\ -3 & 2 & -2 & -10 \end{array}\right]$$

Step 1: Use R_1 to eliminate x from R_2 and R_3:

$R_2 = R_2 - 2R_1$:

$2 - (2 \cdot 1) = 0$

$-1 - (2 \cdot 1) = -3$

$3 - (2 \cdot 1) = 1$

$14 - (2 \cdot 6) = 2$

$R_3 = R_3 + 3R_1$:

$-3 + (3 \cdot 1) = 0$

$2 + (3 \cdot 1) = 5$

$-2 + (3 \cdot 1) = 1$

$-10 + (3 \cdot 6) = 8$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & 1 & 2 \\ 0 & 5 & 1 & 8 \end{array}\right]$$

Step 2: Use R_2 to eliminate y from R_3:

Multiply R_2 by $\frac{5}{-3}$ and add to R_3:

$$R_3 = R_3 + \left(\frac{5}{-3} R_2\right)$$

Compute:

$R_3[2]$: $5+\left(\frac{5}{\cancel{-3}}\cdot(\cancel{-3})\right)=10$

$R_3[3]$: $1+\left(\frac{5}{-3}\cdot 1\right)=1-\frac{5}{3}=\frac{3}{3}-\frac{5}{3}=-\frac{2}{3}$

$R_4[4]$: $8+\left(\frac{5}{-3}\cdot 2\right)=8-\frac{10}{3}=\frac{24-10}{3}=\frac{14}{3}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -\frac{2}{3} & \frac{14}{3} \end{array}\right]$$

Step 3: Solve for z:

$$-\frac{2}{3}z=\frac{14}{3}$$

$$z=\frac{14}{3}:\left(-\frac{2}{3}\right)=\frac{\cancel{14}^{7}}{\cancel{3}}\cdot\left(-\frac{\cancel{3}}{\cancel{2}}\right)=-7 \quad z=-7$$

Step 4: Back-substitute z into R_2:

$$-3y+1\cdot 7=2$$

$$-3y=9$$

$$y=-3$$

Step 5: Back-substitute z and y into R_1:

$$x+(-3)+(-7)=6$$

$$x-10=6$$

$$x=16$$

(14) Find the reduced-row-echelon form of the matrix:

$$B = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Step 1: Eliminate the -1 in R_1 using R_3:

- $R_1 = R_1 + R_3 \times 1$: $\quad -1 + (1 \cdot 1) = 0$

$\quad 0 + ((-1) \cdot 1) = -1$

Updated $R_1 = [1, 2, 0, -1]$

Step 2: Eliminate the 3 in $R_2[3]$

- $R_2 = R_2 - 3R_3$: $\quad 3 - (3 \cdot 1) = 0$

$\quad 5 - (3 \cdot (-1)) = 8$

Updated $R_2 = [0, 1, 0, 8]$

Step 3: Eliminate the 2 in $R_1[2]$:

- $R_1 = R_1 - 2R_2$: $\quad 2 - (2 \cdot 1) = 0$

$\quad -1 - (2 \cdot 8) = -17$

Updated $R_1 = [1, 0, 0, -17]$

$$B = \begin{bmatrix} 1 & 0 & 0 & -17 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

(15) Given matrix $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, find A^{-1} using row operations.

$[A \mid I]$: $\left[\begin{matrix} 2 & 1 \\ 5 & 3 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right]$

Transform A into I while performing the same operations on I to obtain A^{-1}.

Step 1: Make $a_{11} = 1$

- $R_1 = R_1 : 2$: $\quad R_1 = \left[1, \frac{1}{2} \middle| \frac{1}{2}, 0\right]$

Step 2: Eliminate a_{21}:

- $R_2 = R_2 - 5R_1$ $\quad R_2 = \left[0, \frac{1}{2} \middle| -\frac{5}{2}, 1\right]$

Step 3: Make $a_{22} = 1$

$\cdot R_2 = R_2 \cdot 2 :$ $\quad R_2 = [0, 1 \mid -5, 2]$

Step 4: Eliminate a_{12}:

$\cdot R_1 = R_1 - (\frac{1}{2} R_2)$ $\quad R_1 = [1, 0 \mid 3, -1]$

$$[A \mid I] = \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right] A^{-1}$$