W2 - HW Problems.

① Express the arithmetic series $S = 5 + 9 + 13 + \dots + 89$ in sigma notation.

$a_1 = 5 \quad a_2 = 9 \quad d = a_2 - a_1 = 9 - 5 = 4$

$a_n = a_1 + (n-1) \cdot d$

$89 = 5 + (n-1) \cdot 4$

$84 = 4(n-1)$

$n - 1 = 21$

$n = 22$

Sigma notation:

$$S = \sum_{k=1}^{22} [5 + (k-1) \cdot 4]$$

② Rewrite the arithmetic series $\sum_{k=3}^{15} (2k+1)$ to start at $k = 1$

$j = k - 2$

$$\sum_{k=3}^{15} (2k+1) = \sum_{j=1}^{13} [2(j+2)+1] = \sum_{j=1}^{13} (2j+5)$$

③ An arithmetic sequence is defined recursively by $a_1 = 12$ and $a_n = a_{n-1} + d$. If $a_{10} = 57$, find the value of d and then find a_{25}.

$a_{10} = a_1 + (10-1)d$

$a_{10} = a_1 + 9d$

$d = \frac{a_{10} - a_1}{9}$

$d = \frac{57 - 12}{9} = \frac{45}{9} = 5$

$a_{25} = 12 + (25-1) \cdot 5$

$a_{25} = 12 + 24 \cdot 5$

$a_{25} = 12 + 120$

$a_{25} = 132$

③ Find the sum of all multiples of 7 between 100 and 1000.

$a_1 = 7 \cdot 15 = 105$

$a_n = 7 \cdot 142 = 994$

$n = \frac{a_n - a_1}{d} + 1 = \frac{994 - 105}{7} + 1 = \frac{889}{7} + 1 = 127 + 1 = 128$

$S = \frac{n}{2}(a_1 + a_n) = \frac{128}{2}(105 + 994) = 64 \cdot 1099 = 70336$

⑤ Given the arithmetic series $S = \sum_{k=1}^{n} (3k+2)$, find the value of n such that $S = 2650$.

$a_1 = 3 \cdot 1 + 2 = 5$

$a_n = 3n + 2$

$S = \frac{n}{2}[5 + (3n+2)] = \frac{n}{2}(3n+7)$

$\frac{n}{2}(3n+7) = 2650$

$n(3n+7) = 5300$

$3n^2 + 7n - 5300 = 0$

$n = \frac{-7 \pm \sqrt{7^2 - 4 \cdot 3 \cdot (-5300)}}{2 \cdot 3}$

$n = \frac{-7 \pm \sqrt{49 + 63600}}{6}$

$n = \frac{-7 \pm \sqrt{63649}}{6}$

$n = \frac{-7 + 252.29}{6} \approx \frac{245.29}{6} \approx 40.88$ $\quad n = 41$

$S = \frac{41}{2}[5 + (3 \cdot 41 + 2)] = \frac{41}{2}(5 + 125) = \frac{41}{2} 130 = 2665$

Since $n = 41$ the closest solution and it's not exact solution there's no integer value of n such that $S = 2650$.

⑥ In an arithmetic sequence, the 5th term is 20, and the 15th term is 60. Show that the 10th term is the arithmetic mean of the 5th and 15th terms.

$a_5 = 20 \quad a_{15} = 60$

$a_{10} = \frac{a_5 + a_{15}}{2} \quad a_{10} = \frac{20+60}{2} = \frac{80}{2} = 4$ $\quad a_{10} = 40$ is the arithmetic mean.

⑦ A staircase has 20 steps. The first step is 5 cm high, and each subsequent step is 0,5 cm higher than the previous one. What is the total height of the staircase?

$a_1 = 5$
$d = 0{,}5$
$n = 20$

$S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$S_{20} = \frac{20}{2}\left(2 \cdot 5 + (20-1) \cdot \frac{1}{2}\right) = 10 \cdot (10 + 9.5) = 10 \cdot 19{,}5 = 195.$

⑧ An arithmetic series has a first term of 11 and a common difference of 3. Find the smallest value of n such that the sum S_n exceeds 1000.

$a_1 = 11$
$d = 3$

$S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$S_n = \frac{n}{2} \cdot (22 + 3(n-1)) = \frac{n(22 + 3n - 3)}{2} = \frac{n(3n+19)}{2}$

$\frac{n(3n+19)}{2} > 1000$

$3n^2 + 19n > 2000$

$3n^2 + 19n - 2000 > 0$

$n = \frac{-19 \pm \sqrt{19^2 - 4 \cdot 3 \cdot (-2000)}}{2 \cdot 3}$

$n = \frac{-19 \pm \sqrt{361 + 24000}}{6}$

$n = \frac{-19 \pm \sqrt{24361}}{6}$ $\sqrt{24361} = 156.03$

$n = \frac{-19 + 156{,}03}{6} \approx \frac{137.03}{6} \approx 22.84$

$n = 23$

$S_{23} = \frac{23}{2}((2 \cdot 11) + (23-1)d) = \frac{23(22 + 22 \cdot 3)}{2} = \frac{23 \cdot 88}{2} = \frac{2024}{2} = 1012$

The smallest value of n is 23. such that the $S_{23} > 1000$.

(9) $\sum_{k=3}^{12} 4\left(\frac{1}{2}\right)^k$ as a sum starting from $k=0$.

$j = k-3$

$\sum_{k=3}^{12} 4\left(\frac{1}{2}\right)^k = \sum_{j=0}^{9} 4\left(\frac{1}{2}\right)^{j+3} = \sum_{j=0}^{9} 4\left(\frac{1}{8}\right)\left(\frac{1}{2}\right)^j = \sum_{j=0}^{9}\left(\frac{1}{2}\right)^j * \frac{1}{2}$

$S = \sum_{k=0}^{9}\left(\frac{1}{2}\right)^k \cdot \frac{1}{2}$

(10) Find the 10th term of a geometric sequence if $a_2 = -6$ and $a_5 = 48$.

$a_2: \ -6 = a_1 \cdot r^2$

$a_5: \ 48 = a_1 \cdot r^4$

$\frac{48}{-6} = \frac{a_1 \cdot r^4}{a_1 \cdot r^2}$

$-8 = r^3$

$r^3 = -8$

$r = -2$

$a_1: \ -6 = a_1 \cdot (-2)^2$

$a_1 = 3$

$a_n = a_1 \cdot r^{n-1}$

$a_{10} = a_1 \cdot r^9$

$a_{10} = 3 \cdot (-2)^9$

$a_{10} = 3 \cdot (-512) = -1536$

(11) In geometric sequence, $a_4 = 54$ and $a_7 = 1458$. Find the common ratio r.

$a_4: \ 54 = a_1 \cdot r^3$

$a_7: \ 1458 = a_1 \cdot r^6$

$\frac{1458}{54} = \frac{a_1 \cdot r^6}{a_1 \cdot r^3}$

$27 = r^3$

$r^3 = 27$ $\quad r = 3$

$\cancel{r = \sqrt{378}}$

$\cancel{r \approx 19.442}$

(12) Calculate the sum of the first 15 terms of the geometric sequence where $a_1 = 8$ and $r = \frac{3}{4}$

$S_n = a_1 \frac{1-r^n}{1-r}$

$S_{15} = 8 \frac{1-\left(\frac{3}{4}\right)^{15}}{1-\frac{3}{4}} = 8 \frac{1-\left(\frac{3}{4}\right)^{15}}{\frac{1}{4}} = 32\left(1 \cdot \left(\frac{3}{4}\right)^{15}\right)$

$S_{15} \approx 32 \cdot 1 = 32$