~~Biconditi~~ Biconditional Statements

- To prove a theorem that is a biconditional statement of the form $p \leftrightarrow q$, we show that $p \to q$ and $q \to p$ are both true
- Sometimes iff is used as an abbreviation for "if and only if," as in "an integer n is odd iff n^2 is odd."

Week 5. Minimal HW. Proofs.

① Prove that e is irrational.

Suppose e is rational $e = \frac{p}{q}$ $p > 0$, $q > 1$ since e's not integer

$e = \sum_{k=0}^{\infty} \frac{1}{k!}$ $e_n = \sum_{k=0}^{n} \frac{1}{k!}$ Then $e - e_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots < \frac{1}{(n+1)!}\left(1 + \frac{1}{(n+1)} + \frac{1}{(n+1)^2}\right)$

$= \frac{1}{(n+1)!}\left(\frac{1}{1-\frac{1}{n+1}}\right) = \frac{1}{(n+1)!} \cdot \frac{n+1}{n+1-1} = \frac{1}{n!n}$

$n = q$ $0 < e - e_n < \frac{1}{q!q}$

$0 < q!(e - e_q) < \frac{1}{q} < 1$

$q!e = \frac{q!p}{q} = (q-1)!p \in \mathbb{Z}$

$\boxed{\text{Therefore, there exists an integer strictly between 0 and 1. Since this is absurd, e cannot be expressed as a ratio of integers.}}$

② Use a direct proof to show that every odd integer is the difference of two squares.

n = odd when $n = 2k+1$ ~~k~~ for some integers k.

$l = k+1$ $s = k$

$(k+1)^2 - k^2 = \cancel{k^2} + 2k + 1 - \cancel{k^2} = 2k+1$

$= n$

③ Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

Assume that the sum (s) of an irrational (i) and a rational (r) numbers is rational. We take $s = \frac{a}{b}$, $r = \frac{c}{d}$ where a, b, c and d are integers, and $b, d \neq 0$, then by algebra

$s + (-r) = \frac{ad - bc}{bd}$ = rational number, but $s + (-r) = r + i - r = i$ says that i is rational. We proved that s is irrational. This contradicts our hypothesis that i is irrational.

④ Prove or disprove that the product of two irrational numbers is irrational.

$\sqrt{2}$ is irrational

$\sqrt{2} \cdot \sqrt{2} = 2$ is rational. By this we disproved the proposition

⑤ Use a proof by contraposition to show that if $\overbrace{x+y \geq 2}^{p}$, where x and y are real numbers, then $\underbrace{x \geq 1 \text{ or } y \geq 1}_{q}$.

We'll prove $\neg q \to \neg p$

$$\begin{array}{r} x < 1 \\ + \; y < 1 \end{array} = x+y < 2 = \neg p$$

⑥ ~~Proo~~ Show that if n is an integer and $\overbrace{n^3+5 \text{ is odd}}^{p}$, then $\underbrace{n \text{ is even}}_{q}$ using,

a) a proof by contraposition

b) a proof by contradiction.

$\neg q$, then $\neg p$ $\quad \neg q \to \neg p$

n is odd, then n^3+5 is even. $\quad n = 2k+1$

$$(2k+1)^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2\underbrace{(4k^3+6k^2+3k+3)}_{a} = 2a$$

We proved that $\neg q \to \neg p$ is T, by that we proved that $p \to q$ by contraposition.

⑦ The barber is the one who shaves all those men who do not shave themselves. The question is, does the barber shave himself?

U – set of all men in the community

$S: U \to \{T, F\}$ $\qquad \forall x \in U: B(x) \leftrightarrow x$ is shaved by b

$\forall x \in U: (\neg S(x)) \leftrightarrow B(x)$

$B(b) \leftrightarrow S(b)$

From these it follows that: $S(b) \leftrightarrow B(b) \leftrightarrow (\neg S(b))$

$S(b) \leftrightarrow (\neg S(b))$

This is a contradiction, so the initial premises are contradictory and cannot both hold.