Notes on Selected topics to accompany Sakurai's "Modern Quantum Mechanics"

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Linear vector space and Hilbert space

In quantum mechanics, we encounter kets that represent the state of a system. A ket is an abstract entity denoted by the symbol $|\cdot\rangle$. Inside the symbol, one can include any information that corresponds uniquely to the state of the system. Typically, one mentions the eigenvalue of an observable (as in $|Z+\rangle$, where $Z+$ implies that in this state, the value of the observable S_z is $+\hbar/2$) or a quantum number (as in $|n\rangle$, seen in the particle in a box or harmonic oscillator problem).

Even though the ket is an abstract quantity, one can form a vector space (denoted by $\mathbf{V}$), where each element is a ket. In order to define such a vector space, it should be possible to define

1. a rule for addition of kets so that for any two kets, $|a\rangle$ and $|b\rangle$, their sum is defined. In other words,

$$|a\rangle + |b\rangle \in \mathbf{V}; \quad \text{if } |a\rangle, |b\rangle \in \mathbf{V}$$

2. a rule for scalar multiplication of a ket by a number c is defined

$$c|a\rangle \in \mathbf{V}; \quad \text{if } |a\rangle \in \mathbf{V}$$

Individually, these two properties are not physically relevant, but when they are combined, one can define a linear superposition of kets that can correspond to a physical state of the system. So, properties (1) and (2) are prerequisites to define a superposition of kets.

$$|\alpha\rangle = \sum_{k=1}^{N} c_k |k\rangle.$$

Since a linear combination of basis kets, $|1\rangle \ldots |N\rangle$, is defined in the same space that contains the basis, we will call our vector space, $\in \mathbf{V}$, as linear vector space. From our knowledge of vector algebra, we already know that for a superposition to be meaningful (*i.e.* devoid of redundancy), the basis kets should be linearly independent.

In vector algebra, our basis kets are the three orthogonal unit vectors in real space. For application in quantum mechanics, we need the linear vector space to be of arbitrary dimension, *i.e.* N can be any integer, even much greater than 3. Further, we will also assume that the linear expansion coefficients, c_k, are complex numbers. With these two assumptions we can call the linear vector space as the Hilbert space.