Assignment 5

Due: 8:30 am, 17 November 2022 (Thursday)

Course: Quantum Mechanics - 1
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Problem 1

a) Derive expressions for the Cartesian components of the angular momentum operator ($\hat{L}_x, \hat{L}_y, \hat{L}_z$) in spherical polar coordinates (r, θ, ϕ).
b) Show that $\hat{L}^2 = \hat{L}_+\hat{L}_- + \hat{L}_z^2 - \hat{L}_z$
c) Using results from a) express the raising and lowering operators, $\hat{L}_+$ and $\hat{L}_-$, in spherical polar coordinates
d) Using results from a–c) show that

$$\hat{L}^2\left(\theta,\phi\right) \quad = \quad -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial f}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi}\right]$$

Problem 2

Determine the eigenvalues and eigenfunctions of $\hat{L}^2\left(\theta,\phi\right)$ using the following steps.
Step-1: Start with

$$\hat{L}^2\left(\theta,\phi\right)\psi\left(\theta,\phi\right) \quad = \quad \lambda\psi\left(\theta,\phi\right)$$

Insert $\psi\left(\theta,\phi\right) = \Theta(\theta)\Phi(\phi)$ and simplify this equation to the form

$$f(\theta) = -g(\phi)$$

Step-2: From the above equation, we find that $f(\theta) = -g(\phi)$ for any value of ϕ. What this means is that $g(\phi)$ has to be a constant, k. Using this, we can write the above equation as two separate equations in θ and ϕ

$$\begin{aligned} f(\theta) &= -k \\ g(\phi) &= k \end{aligned}$$

Solve the ϕ-equation using the boundary condition, $\Phi(0) = \Phi(2\pi)$. For solving $g(\phi) = k$ using this boundary condition, what are the allowed values of k?

Step-3: The associated Legendre differential equation is given by

$$\frac{d}{dz}\left[p(z)\frac{d}{dz}\right]y(z)+q(z)y(z)+\lambda r(z)y(z)=0,$$

where $p(z) = (1-z^2)$, $r(z) = 1$, and $q(z) = -\frac{m^2}{1-z^2}$. Note that according to the definition, $m^2 > 0$. Express the θ-equation, $f(\theta) = k$ in the form of the associated Legendre differential equation. What is the relation between k and m?
Step-4: The associated Legendre differential equation is solvable only when the eigenvalue, λ, satisfies the condition $\lambda = l(l+1)$, where $l = 0, 1, \ldots$, and when $|m| < l$. The eigenfunction of the differential equation, $y(z)$, is a function of l and m. We denote the solutions as

$$P_l^{|m|}(z)$$

and call them the associated Legendre polynomials. Write $\Theta(\theta)$ as associated Legendre polynomials.
Step-5: The generalized orthonormal relation for the associated Legendre polynomials is given by

$$\int_{-1}^{+1} dz\, P_{l'}^{|m|} P_l^{|m|} = \frac{2}{2l+1}\frac{(l+|m|)!}{(l-|m|)!}\delta_{ll'}$$

where $\delta_{ll'}$ is the Kronecker delta. Using this information and the results from the previous steps, write the eigenfunctions of $\hat{L}^2$ in the normalized form. These eigenfunctions $\psi(\theta,\phi)$ are called as the spherical harmonics denoted by the symbol, $Y_{l,m}(\theta,\phi)$.
Step-5: Using results from the above steps, what are the eigenvalues of $\hat{L}^2$?