Notes on Selected topics to accompany Sakurai's "Modern Quantum Mechanics"

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Linear vector space and Hilbert space

In quantum mechanics, we encounter kets that represent the state of a system. A ket is an abstract entity denoted by the symbol $|\cdot\rangle$. Inside the symbol, one can include any information that corresponds uniquely to the state of the system. Typically, one mentions the eigenvalue of an observable (as in $|Z+\rangle$, where $Z+$ implies that in this state, the value of the observable S_z is $+\hbar/2$) or a quantum number (as in $|n\rangle$, seen in the particle in a box or harmonic oscillator problem).

Even though the ket is an abstract quantity, one can form a vector space (denoted by $\mathbf{V}$), where each element is a ket. In order to define such a vector space, it should be possible to define the two *binary operations*, addition of kets and multiplication of a ket by a scalar.

1. a rule for addition of kets so that for any two kets, $|1\rangle$ and $|2\rangle$, their sum is defined. In other words,

$$|1\rangle + |2\rangle \in \mathbf{V}; \quad \text{if } |1\rangle, |2\rangle \in \mathbf{V}$$

2. a rule for scalar multiplication of a ket by a number c is defined

$$c|1\rangle \in \mathbf{V}; \quad \text{if } |1\rangle \in \mathbf{V}$$

It is important to note that the operations defined above (ket addition and scalar multiplication) do not mean that the corresponding states add up or anything like that. In order to avoid confusion, we could have chosen different symbols for addition and multiplication too. Individually, these two binary operations are not physically relevant, but when they are combined, one can define a linear superposition of kets that can correspond to a physical state of the system. So, properties (1) and (2) are prerequisites to define a superposition of kets.

$$|\alpha\rangle = \sum_{k=1}^{N} c_k |k\rangle.$$

Since a linear combination of basis kets, $|1\rangle \ldots |N\rangle$, is defined in the same space that contains the basis, we will call our vector space, $\in \mathbf{V}$, as linear vector space. From our knowledge of vector algebra, we already know that for a superposition to be meaningful (*i.e.* devoid of redundancy), the basis kets should be linearly independent.

In vector algebra, our basis kets are the three orthogonal unit vectors in real space. In mathematical terms, the three-dimensional vector space is defined over the real number field, $\mathbf{R}$. For application in quantum mechanics, we need the linear vector space to be of arbitrary dimension, *i.e.* N can be any integer, even much greater than 3. Further, we will also assume that the linear expansion coefficients, c_k, are complex numbers. With these two assumptions we can call the linear vector space as the Hilbert space (or the ket space). So, a Hilbert space is an N-dimensional linear vector space defined over the complex number field, $\mathbf{C}$. So, in text books a Hilbert space may be denoted by the symbol $\mathbf{H}$ or $\mathbf{C}^N$.

Hilbert space axioms

For any three kets, $|1\rangle$, $|2\rangle$, and $|3\rangle$, in $\mathbf{H}$, and two complex numbers, c_1 and c_2, the following axioms are defined

1. Addition of kets is commutative

$$|1\rangle + |2\rangle = |2\rangle + |1\rangle$$

2. Addition of kets is associative

$$|1\rangle + (|2\rangle + |3\rangle) = (|1\rangle + |2\rangle) + |3\rangle$$

3. Scalar multiplication is distributive over addition of kets

$$c\,(|1\rangle + |2\rangle) = c|1\rangle + c|2\rangle \tag{1}$$

4. Scalar multiplication is distributive over addition of scalars

$$(c_1 + c_2)\,|k\rangle = c_1|k\rangle + c_2|k\rangle$$

5. Scalar multiplication is associative in multiplication of scalars

$$c_1\,(c_2|k\rangle) = (c_1 c_2)\,|k\rangle$$

6. There is an identity element (zero ket or null ket, $|\text{null}\rangle$) for addition of kets (which is the first binary operation required to define a vector space).

$$|k\rangle + |\text{null}\rangle = |k\rangle; \, \forall |k\rangle \in \mathbf{H}$$

7. Every ket in **H** has its own inverse ket in **H** with respect to addition. The sum of a ket and its inverse gives the null ket (the identity element with respect to ket addition).

$$|k\rangle + (-|k\rangle) = |\text{null}\rangle; \, \forall |k\rangle \in \mathbf{H}$$

8. There is an identity element for scalar multiplication (the second binary operation required to define a vector space). This means that the there is a scalar $c = 1$ so that

$$1|k\rangle = |k\rangle; \, \forall |k\rangle \in \mathbf{H}$$

   An important point to note is that there is no equivalent of Axiom-2 for the identity element for scalar multiplication. What this means is that for every ket, there is no inverse ket so that the identity element for scalar multiplication 1 can be obtained. To define such an operation, we have to define the Hermitian adjoint of a ket, $|k\rangle$, called a bra, denoted by $\langle k|$.