Assignment 2

Due: 2:15 pm, 21 October 2021 (Thursday)

Course: Quantum Mechanics - 1
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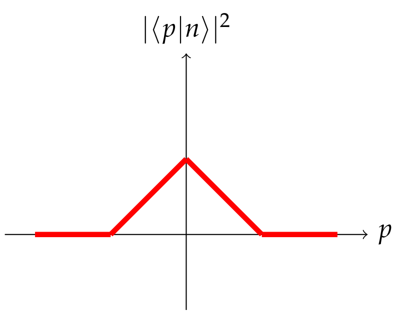
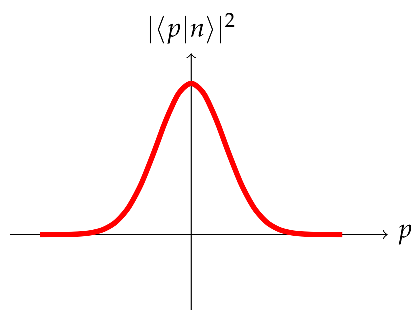
Problem 1

Determine the probability densities in the position representation, $|\psi_n(x)|^2 = |\langle x|n\rangle|^2$, and the joint uncertainty $\Delta x \Delta p$ for a one-dimensional system in a state given by ket $|n\rangle$ having the following probability densities in the momentum representation, $|\phi_n(p)|^2 = |\langle p|n\rangle|^2$. As the first step, normalize the probability densities appropriately.

a) $|\langle p|n\rangle|^2 = \begin{cases} p+B; & -A \geq p \geq 0 \\ -p+B; & 0 \geq p \geq +A \\ 0; & |p| > A \end{cases}$

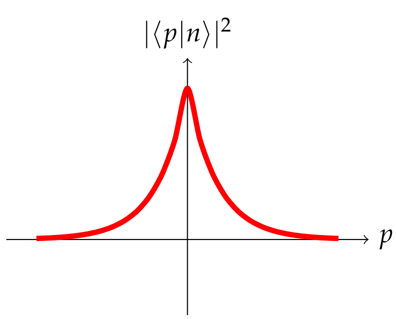
b) $|\langle p|n\rangle|^2 = \exp\left(-Ap^2\right); \quad -\infty \leq p \leq \infty$

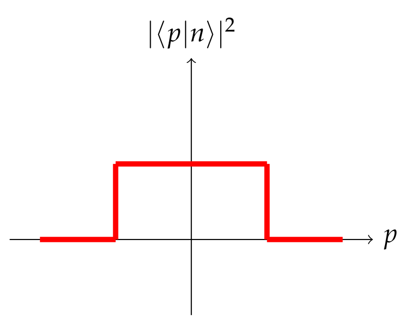




c) $|\langle p|n\rangle|^2 = \exp\left(-A|p|\right); \quad -\infty \leq p \leq \infty$

d) $|\langle p|n\rangle|^2 = \begin{cases} B; & |p| \leq A \\ 0; & |p| > A \end{cases}$





Problem 2

Prove the following:

$$
\begin{aligned}
1. \quad \langle x|\hat{p}^n|a\rangle &= (-i\hbar)^n \frac{\partial^n}{\partial x^n}\langle x|a\rangle \\
2. \quad \langle b|\hat{p}^n|a\rangle &= \int_{-\infty}^{+\infty} dx\,\langle b|x\rangle\,(-i\hbar)^n \frac{\partial^n}{\partial x^n}\langle x|a\rangle \\
3. \quad \langle p|\hat{x}^n|a\rangle &= (+i\hbar)^n \frac{\partial^n}{\partial p^n}\langle p|a\rangle \\
4. \quad \langle b|\hat{x}^n|a\rangle &= \int_{-\infty}^{+\infty} dp\,\langle b|p\rangle\,(+i\hbar)^n \frac{\partial^n}{\partial p^n}\langle p|a\rangle
\end{aligned}
$$

where n is a positive integer, $|x\rangle$ and $|p\rangle$ are position and momentum eigenkets, respectively, while $|a\rangle$ and $|b\rangle$ are kets of two arbitrary states.

Problem 3

Show that the following holds for any two Hermitian operators, $\hat{A}$ and $\hat{B}$

$$\langle\hat{A}^2\rangle\langle\hat{B}^2\rangle \geq \frac{\langle\hat{C}\rangle^2 + \langle\hat{D}\rangle^2}{4}$$

where $\hat{C} = -i[\hat{A}, \hat{B}]$ and $\hat{D} = \{\hat{A}, \hat{B}\}$.

Problem 4

We learned recently that $\langle\hat{p}\rangle = 0$ for all states of the particle-in-a-box problem. Show that in general for discrete stationary states of any quantum mechanical problem, $\langle\hat{\mathbf{p}}\rangle = \mathbf{0}$.

Problem 5

For the particle-in-a-box, derive $\langle\hat{x}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}\rangle$, and $\langle\hat{p}^2\rangle$ in both position and momentum representations for a system in $|n\rangle$.