Assignment 2

Due: 8:30 am, 29 September 2022 (Thursday)

Course: Quantum Mechanics - 1
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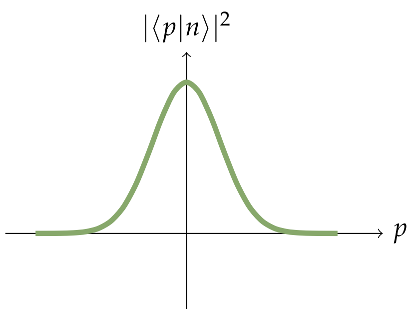
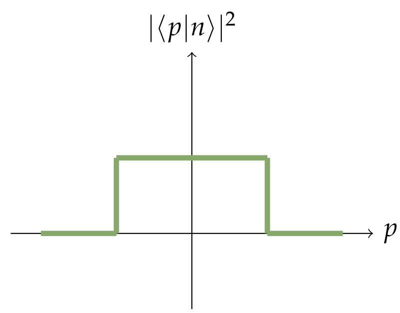
Problem 1

Determine the probability densities in the position representation, $|\psi_n(x)|^2 = |\langle x|n\rangle|^2$, and the joint uncertainty $\Delta x \Delta p$ for a one-dimensional system in a state given by ket $|n\rangle$ having the following probability densities in the momentum representation, $|\phi_n(p)|^2 = |\langle p|n\rangle|^2$. As the first step, normalize the probability densities appropriately.

a) $|\langle p|n\rangle|^2 = \exp\left(-Ap^2\right); \quad -\infty \leq p \leq \infty$

b) $|\langle p|n\rangle|^2 = \begin{cases} B; & |p| \leq A \\ 0; & |p| > A \end{cases}$





Problem 2

Prove the following relations:

$$
\begin{aligned}
1.\quad \langle x|\hat{p}^n|a\rangle &= (-i\hbar)^n \frac{\partial^n}{\partial x^n}\langle x|a\rangle \\
2.\quad \langle b|\hat{p}^n|a\rangle &= \int_{-\infty}^{+\infty} dx\, \langle b|x\rangle\, (-i\hbar)^n \frac{\partial^n}{\partial x^n}\langle x|a\rangle \\
3.\quad \langle p|\hat{x}^n|a\rangle &= (+i\hbar)^n \frac{\partial^n}{\partial p^n}\langle p|a\rangle \\
4.\quad \langle b|\hat{x}^n|a\rangle &= \int_{-\infty}^{+\infty} dp\, \langle b|p\rangle\, (+i\hbar)^n \frac{\partial^n}{\partial p^n}\langle p|a\rangle
\end{aligned}
$$

where n is a positive integer, $|x\rangle$ and $|p\rangle$ are position and momentum eigenkets, respectively, while $|a\rangle$ and $|b\rangle$ are kets of two arbitrary states.

Problem 3

For the particle-in-a-box in the n-th eigenstate, derive $\langle \hat{x} \rangle$, $\langle \hat{x}^2 \rangle$, $\langle \hat{p} \rangle$, and $\langle \hat{p}^2 \rangle$ in both position and momentum representations.

Problem 4

Suppose a system in the initial ket given as the linear combination of momentum eigenkets $|\alpha, 0\rangle = |p_0 - \delta p\rangle + 2|p_0\rangle + |p_0 + \delta p\rangle$ evolves in time. What can you say about the normalization of this ket? Derive an expression for the time-evolved wavefunction in the position representation $\psi(x,t) = \langle x|\alpha, t\rangle$. Consult a book on Waves and Oscillations, and identify carrier wave, envelope, group velocity and phase velocity in $\psi(x,t)$.

Problem 5

An operator is defined in the position representation as

$$\hat{A} = A\frac{d^2}{dx^2}\delta(x - x') + Bx^2\delta(x - x').$$

Express the same operator in the momentum representation.