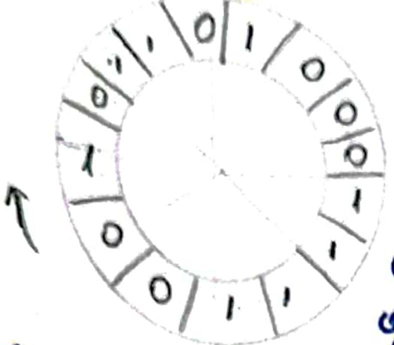
DE BRUIJN SEQUENCES

- for bits

■ The rotating drum problem



This drum has 16 segments denoted by 1 and 0.
We require that any 4 consecutive segments uniquely determine the position of the drum.
[This means that the 16 possible quadruples of consecutive 0's and 1's should be binary representations of all the integers between 0 and 15]

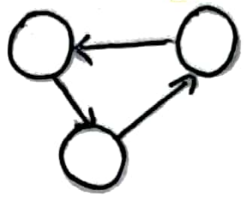
Can we do this? In how many different ways?

This is called a De Bruijn sequence (/graph)

But first, a little bit about digraphs:

A directed graph (digraph) is a graph that is composed of a set of vertices connected by directed edges (or called arcs)

each „road" is taken once



Building a De Bruijn sequence:

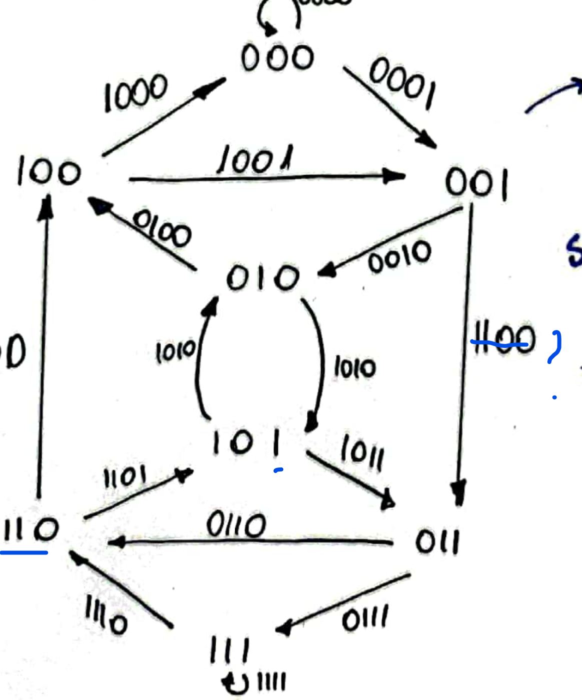
We consider a digraph (G_4) by taking all 3-tuples of 0's and 1's as vertices and joining the vertex $x_1x_2x_3$ by an arc to $x_2x_3\,0$ and $x_2x_3\,1$.

The arc $(x_1x_2x_3, x_2x_3x_4)$ is called e_j [$x_1x_2x_3x_4$ is the binary representation of the integer j]

The graph has a loop at 000 and 111

$\frac{2^{2^3}}{2^4} = \frac{2^8}{2^4} = 2^{8-4} = 2^4 = 16$ ✓ formula is valid

(G_4)



This graph is an Eulerian circuit (every vertex has in-degree 2 and out-degree 2)

Such a closed path provides us the 16-bit sequence for the drum.

Such a circular path = a De Bruijn sequence

For example:

0 011

The path 000 → 000 → 001 → 011 → 111 → 111 → 110 → 100 → 001 → 010 → 101 → 011 → 110 → 101 → 010 → → 100 → 000

corresponds to the cycle: $\boxed{0000111100101101}$

We define the graph G_m to be directed graph on $(m-1)$ tuples of 0's and 1's (It has 2^m edges)

We can define G_m^* as follows:

(i) to each edge of G_m corresponds a vertex of G_m^*

(ii) if a and b are vertices of G_m^* then there is an edge from $a \to b \iff$ the edge of G corresponding to a has as terminal end of the initial end of the edge G corresponding to b

Clearly, $G_m^* = G_{m+1}$ (we need this for induction)

(Graph theory:
(Two graphs are isomorphic if there is an edge-preserving bijective function from the vertices of G to the ones of G') — G and G'
)

Theorem 1

G_m has exactly $\frac{2^{2^{m-1}}}{2^m}$ complete cycles — proof with eigenvalues? TODO: read

TO DO :

To prove this, we'll need the following intermediate result:

Theorem 0.

Let G be a 2-in-2-out graph on m vertices with H complete cycles. Then G^* has $2^{m-1} \cdot H$ complete cycles

Proof (by induction)

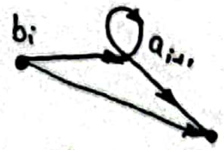
I If $m=1$ then G has one vertex p and two loops . Then $G^* = G_2$ which has one complete cycle

II Let's assume G is a connected graph. 

(Connected graphs: A connected graph is a graph that is connected in the sense of a topological space (there is a path from any point to any point in the graph)
— set of points ?)

If G has m vertices and there is a loop at every vertex, then, besides those loops G is a circuit $p_1 \to p_2 \to \dots \to p_m \to p_1$.

Let A_i be the loop $p_i \to p_i$ and B_i the arc $p_i \to p_{i+1}$



- the vertices in G^* are lowercase letters

Clearly, a cycle has two ways of going from b_i to b_{i+1}. So, G^* has 2^{m-1} complete cycles (whereas G has only one)

Using Th. 0 and induction we can easily prove Th 1

EXTENDING TO A m SIZED ALPHABET

The assumption is that we'll have exactly $\frac{m!^{m^{k-1}}}{m^k}$ de Bruijn sequences of order k.

In the previous section we proved that, for an alphabet of 2 letters we have $\frac{2^{2^{k-1}}}{2^k}$ $\left(\frac{2^{2^{k-1}}}{2^k}\right)$

Trying: proof by induction

Step 1: We already proved the formula works for the alphabet $\{0, 1\}$

Step 2:

References

„Extending De Bruijn sequences to larger alphabets" Veronica B. ductis C.

„De Bruijn sequences decoding algorithms" J. Tuliani

„A course in combinatorics" Cambridge University

„Asupra unei probleme de combinatorică" Mira-Cristiana Anisiu