2. Determination of the electrostatic potential [100 points]

Aim of this task is to determine the electrostatic potential $V(x,y)$ in a 2D square domain $[0,1] \times [0,1]$ that satisfies Laplace's equation:

$$\nabla^2 V(x,y) = 0,$$

subject to the following boundary conditions:

- Outer boundaries (all four sides): $V = 0$.
- A circular conductor of radius $r = 0.2$ centered at $(0.5, 0.5)$ is held at $V = 1$. Inside the conductor, $V = 1$ is fixed and excluded from updates.

Please perform the following tasks:

1. Discretize the domain on an $N \times N$ uniform grid (suggested $N = 101$ so $h = 1/(N-1)$). But also try and compare a smaller and a larger value of N.
2. Implement three iterative solvers: Jacobi, Gauss–Seidel, and SOR.
3. Use the infinity norm of the potential change $\max_{i,j} |\phi_{i,j}^{(k+1)} - \phi_{i,j}^{(k)}|$ as a convergence criterion with tolerance $\mathrm{tol} = 10^{-6}$.
4. For SOR, test several values of ω (e.g. 1.0, 1.2, 1.5, the theoretical estimate, and near 2.0) and find which gives the fastest convergence.
5. Plot at least (for different values of N):
   (a) the final potential (equipotential contours),
   (b) the electric field vectors $\mathbf{E} = -\nabla V$,
   (c) convergence history (residual or max-change vs iteration) for the three methods.
6. Report the number of iterations and elapsed CPU time for each method and ω; discuss the results.
7. What happens if the potential at the outer boundaries at all four sides is changed to $V = 1$. Plot the solution using the SOR approach.