Growth of a population

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Description

In a small town the population is P_0 at the beginning of a year. The population regularly increases by *percent*% per year and moreover *aug* new inhabitants per year come to live in the town. How many years N does the town need to see its population greater or equal to P_N inhabitants? Arguments P_0, P_N, *percent* and *aug* are known.

Derivation

First, lets denote

$$r = \left(1 + \frac{percent}{100}\right)$$

To can use the following formulas to find the population at Nth year:

$$\begin{aligned}
P_1 &= rP_0 + aug \\
P_2 &= rP_1 + aug \Rightarrow P_2 = r^2P_0 + (r+1)aug \\
P_3 &= rP_2 + aug \Rightarrow P_3 = r^3P_0 + (r^2+r+1)aug \\
P_4 &= rP_3 + aug \Rightarrow P_4 = r^4P_0 + (r^3+r^2+r+1)aug \\
&\ldots \\
P_N &= r^NP_0 + \sum_{i=0}^{N-1} r^i aug \qquad (1)
\end{aligned}$$

We have two cases we need to handle:

1. when growth percent is equal to 0, i.e. $percent = 0$;
2. when growth percent is not equal to 0, i.e. $percent \neq 0$.

When $percent = 0$, then

$$P_N = P_0 + N \cdot aug.$$

When $percent \neq 0$, then we notice, that for variable aug a geometric series [1] forms. We know, that when $r \neq 1$, then

$$\sum_{i=0}^{N-1} r^i aug = aug \frac{1 - r^N}{1 - r}. \tag{2}$$

Since r is always $r > 1$ in our case, then we can write (2) as

$$aug \frac{r^N - 1}{r - 1}. \tag{3}$$

Substituting (2) and (3) into (1) we get

$$\begin{aligned}
P_N &= r^N P_0 + aug \frac{r^N - 1}{r - 1} \\
P_N &= \frac{r^N (r-1) P_0 + aug \cdot r^N - aug}{r - 1} \\
P_N &= \frac{r^N \big[(r-1) P_0 + aug \big] - aug}{r - 1} \\
&\Rightarrow r^N = \left. \frac{(r-1) P_N + aug}{(r-1) P_0 + aug} \right| \log \\
&\Rightarrow N \log(r) = \log \left(\frac{(r-1) P_N + aug}{(r-1) P_0 + aug} \right) \\
&\Rightarrow N = \frac{\log \left(\frac{(r-1) P_N + aug}{(r-1) P_0 + aug} \right)}{\log(r)}. \quad \square
\end{aligned}$$

Cites

1. https://en.wikipedia.org/wiki/Geometric_series