Spectral theorem

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1 Statement

If A is a n-by-n matrix, then

$$A = A^T \quad \Leftrightarrow \quad \exists P, \exists D(A = PDP^{-1} = PDP^T) \tag{1}$$

where P is the orthogonal matrix($P^T = P^{-1}$) which consists of eigenvectors of A, and $D = \mathbf{diag}(\lambda_1, \cdots, \lambda_n)$ where $\lambda_1, \cdots, \lambda_n$ are corresponding eigenvalues of A.

2 Derivation

2.1 $A = A^T \leftarrow \exists P, \exists D(A = PDP^{-1} = PDP^T)$

$$A = PDP^T = (PD^T P^T)^T = (PDP^T)^T = A^T$$

2.2 $A = A^T \quad \rightarrow \quad \exists P, \exists D(A = PDP^{-1} = PDP^T)$

Use induction.

2.2.1 for $n = 1$

Let $A = a, P = \mathbf{v} = 1, D = a$. Then,

$$A\mathbf{v} = a\mathbf{v} \;\rightarrow\; a = a$$

Thus, a is an eigenvalue of A and $\mathbf{v}$ is the corresponding eigenvector. Also,

$$P^T = P^{-1} = 1 \text{ and } A = PDP^{-1} = PDP^T = a$$

This proves (1) for $n = 1$.

2.2.2 $(n-1) \rightarrow (n)$

Assume (1) is held when A is a $(n-1)$-by-$(n-1)$ matrix where $n > 1$.
Let λ_1 and $\mathbf{v}_1$ are a pair of eigenvalue and egienvector of A, i.e.,

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$$

Also let $\|\mathbf{v}_1\| = 1$.
And find $\mathbf{v}_2, \cdots, \mathbf{v}_n$ using Gram-Schmidt process such that

$$\mathbf{v}_i \bullet \mathbf{v}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

where $i, j \in \{1, \cdots, n\}$.

If we define matrix V as the following,

$$V = \begin{bmatrix} \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

Then, matrix $V^T AV$ is a $(n-1) \times (n-1$ matrix and there exist an orthogonal matrix P and diagonal matrix D such that

$$V^T AV = PDP^T = PDP^{-1}$$

by the assumption.

Let $U = \begin{bmatrix} v_1 \ VP \end{bmatrix}$, then

$$\begin{aligned} U^T U &= \begin{bmatrix} \mathbf{v}_1^T \\ P^T V^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & VP \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{v}_1^T \mathbf{v}_1 & \mathbf{v}_1^T VP \\ P^T V^T \mathbf{v}_1 & P^T V^T VP \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{v}_1^T \mathbf{v}_1 & (\mathbf{v}_1^T V) P \\ P^T (V^T \mathbf{v}_1) & P^T (V^T V) P \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & P^T P \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & I_{n-1} \end{bmatrix} \\ &= I_n \end{aligned}$$

This asserts U is an orthogonal matrix.

Also,

$$\begin{aligned} U^T A U &= \begin{bmatrix} \mathbf{v}_1^T \\ P^T V^T \end{bmatrix} A \begin{bmatrix} \mathbf{v}_1 & VP \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{v}_1^T A \mathbf{v}_1 & \mathbf{v}_1^T A V P \\ P^T V^T A \mathbf{v}_1 & P^T V^T A V P \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{v}_1^T (A \mathbf{v}_1) & (A \mathbf{v}_1)^T V P \\ P^T V^T (A \mathbf{v}_1) & P^T (V^T A V) P \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 \mathbf{v}_1^T \mathbf{v}_1 & \lambda_1 \mathbf{v}_1^T V P \\ \lambda_1 P^T V^T \mathbf{v}_1 & P^T (P D P^T) P \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix} \\ &= D_1 \end{aligned}$$

Since D is a diagonal matrix, so is D_1. Using $U^T = U^{-1}$, the above equation is led to

$$A = U D_1 U^T = U D_1 U^{-1} \tag{2}$$

This finishes the proof by [Lay11].

References

[Lay11] D.C. Lay. *Linear Algebra and Its Applications*, page 396. Pearson College Division, 2011.