Quantum Fourier Transform

Jiman Hwang

Disclaimer: This document is for self-study only and may contain false information. Source: [Dr],[Ume],[Wik]

1 Definition

A linear transform that performs Discrete Fourier Transform(DFT) on the state of qubits. Considering n-qubit system, QFT in matrix expression is

$$QFT_N = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega^{0\cdot 0} & \omega^{0\cdot 1} & \cdots & \omega^{0\cdot(N-1)} \\ \omega^{1\cdot 0} & \omega^{1\cdot 1} & \cdots & \omega^{1\cdot(N-1)} \\ \vdots & \vdots & & \vdots \\ \omega^{(N-1)\cdot 0} & \omega^{(N-1)\cdot 1} & \cdots & \omega^{(N-1)\cdot(N-1)} \end{bmatrix}$$

where

$$N = 2^n,\ \omega = e^{i2\pi/N},\ i = \sqrt{-1}$$

2 Implementation

2.1 Building a recursion

This is one of possible implementations, which doesn't reveal the world's #1 performance but it's still great compared with the classical FFT.

We use the strategy used in FFT, divide and conquer. Applying QFT to an input state $|x\rangle$,

$$QFT_N |x\rangle = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega^{0\cdot 0} & \omega^{0\cdot 1} & \cdots & \omega^{0\cdot(N-1)} \\ \omega^{1\cdot 0} & \omega^{1\cdot 1} & \cdots & \omega^{1\cdot(N-1)} \\ \vdots & \vdots & & \vdots \\ \omega^{(N-1)\cdot 0} & \omega^{(N-1)\cdot 1} & \cdots & \omega^{(N-1)\cdot(N-1)} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$$

By switching the order of elements of $|x\rangle$ and column vectors in QFT_N, we can rearrange it as follow.

$$QFT_N |x\rangle = \frac{1}{\sqrt{N}} \begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} |x_{ev}\rangle \\ |x_{od}\rangle \end{bmatrix} \tag{1}$$

where

$$A = \begin{bmatrix} \omega^{0\cdot 0} & \omega^{0\cdot 2} & \cdots & \omega^{0\cdot (N-2)} \\ \omega^{1\cdot 0} & \omega^{1\cdot 2} & \cdots & \omega^{1\cdot (N-2)} \\ \vdots & \vdots & & \vdots \\ \omega^{\left(\frac{N}{2}-1\right)\cdot 0} & \omega^{\left(\frac{N}{2}-1\right)\cdot 2} & \cdots & \omega^{\left(\frac{N}{2}-1\right)\cdot (N-2)} \end{bmatrix} \tag{2}$$

$$B = \begin{bmatrix} \omega^{\frac{N}{2}\cdot 0} & \omega^{\frac{N}{2}\cdot 2} & \cdots & \omega^{\frac{N}{2}\cdot (N-2)} \\ \omega^{\left(\frac{N}{2}+1\right)\cdot 0} & \omega^{\left(\frac{N}{2}+1\right)\cdot 2} & \cdots & \omega^{\left(\frac{N}{2}+1\right)\cdot (N-2)} \\ \vdots & \vdots & & \vdots \\ \omega^{(N-1)\cdot 0} & \omega^{(N-1)\cdot 2} & \cdots & \omega^{(N-1)\cdot (N-2)} \end{bmatrix} \tag{3}$$

$$C = \begin{bmatrix} \omega^{0\cdot 1} & \omega^{0\cdot 3} & \cdots & \omega^{0\cdot (N-1)} \\ \omega^{1\cdot 1} & \omega^{1\cdot 3} & \cdots & \omega^{1\cdot (N-1)} \\ \vdots & \vdots & & \vdots \\ \omega^{\left(\frac{N}{2}-1\right)\cdot 1} & \omega^{\left(\frac{N}{2}-1\right)\cdot 3} & \cdots & \omega^{\left(\frac{N}{2}-1\right)\cdot (N-1)} \end{bmatrix} \tag{4}$$

$$D = \begin{bmatrix} \omega^{\frac{N}{2}\cdot 1} & \omega^{\frac{N}{2}\cdot 3} & \cdots & \omega^{\frac{N}{2}\cdot (N-1)} \\ \omega^{\left(\frac{N}{2}+1\right)\cdot 1} & \omega^{\left(\frac{N}{2}+1\right)\cdot 3} & \cdots & \omega^{\left(\frac{N}{2}+1\right)\cdot (N-1)} \\ \vdots & \vdots & & \vdots \\ \omega^{(N-1)\cdot 1} & \omega^{(N-1)\cdot 3} & \cdots & \omega^{(N-1)\cdot (N-1)} \end{bmatrix} \tag{5}$$

$$|x_{ev}\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_2 \\ \vdots \\ \alpha_{N-2} \end{bmatrix},\ |x_{od}\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_3 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} \tag{6}$$

We'll clean up A, B, C, and D one by one after stating some corollaries. To explain more clearly, redefine ω as follow.

$$\omega_N = \omega = e^{i2\pi/N}$$

Then, the following is readily understood.

$$\omega_N^2 = e^{i\frac{2\pi}{N}\times 2} = e^{i\frac{2\pi}{N/2}} = \omega_{N/2} \tag{7}$$

$$\omega_{N/2}^{N/2} = e^{i\frac{2\pi}{N/2}\times\frac{N}{2}} = 1 \tag{8}$$

$$\omega_N^{N/2} = e^{i\frac{2\pi}{N}\times\frac{N}{2}} = -1 \tag{9}$$

Now let's take a look at A, B, C, and D one by one. Using (7),

$$A = \begin{bmatrix} \omega_N^{0\cdot 0} & \omega_N^{0\cdot 2} & \cdots & \omega_N^{0\cdot(N-2)} \\ \omega_N^{1\cdot 0} & \omega_N^{1\cdot 2} & \cdots & \omega_N^{1\cdot(N-2)} \\ \vdots & \vdots & & \vdots \\ \omega_N^{\left(\frac{N}{2}-1\right)\cdot 0} & \omega_N^{\left(\frac{N}{2}-1\right)\cdot 2} & \cdots & \omega_N^{\left(\frac{N}{2}-1\right)\cdot(N-2)} \end{bmatrix} = \begin{bmatrix} \omega_{N/2}^{0\cdot 0} & \omega_{N/2}^{0\cdot 1} & \cdots & \omega_{N/2}^{0\cdot\left(\frac{N}{2}-1\right)} \\ \omega_{N/2}^{1\cdot 0} & \omega_{N/2}^{1\cdot 1} & \cdots & \omega_{N/2}^{1\cdot\left(\frac{N}{2}-1\right)} \\ \vdots & \vdots & & \vdots \\ \omega_{N/2}^{\left(\frac{N}{2}-1\right)\cdot 0} & \omega_{N/2}^{\left(\frac{N}{2}-1\right)\cdot 1} & \cdots & \omega_{N/2}^{\left(\frac{N}{2}-1\right)\cdot\left(\frac{N}{2}-1\right)} \end{bmatrix}$$

$$= \sqrt{\frac{N}{2}} QFT_{N/2}$$

B is reduced in a similar way.

$$B = \begin{bmatrix} \omega_N^{\frac{N}{2}\cdot 0} & \omega_N^{\frac{N}{2}\cdot 2} & \cdots & \omega_N^{\frac{N}{2}\cdot(N-2)} \\ \omega_N^{\left(\frac{N}{2}+1\right)\cdot 0} & \omega_N^{\left(\frac{N}{2}+1\right)\cdot 2} & \cdots & \omega_N^{\left(\frac{N}{2}+1\right)\cdot(N-2)} \\ \vdots & \vdots & & \vdots \\ \omega_N^{(N-1)\cdot 0} & \omega_N^{(N-1)\cdot 2} & \cdots & \omega_N^{(N-1)\cdot(N-2)} \end{bmatrix} = \begin{bmatrix} \omega_{N/2}^{\frac{N}{2}\cdot 0} & \omega_{N/2}^{\frac{N}{2}\cdot 1} & \cdots & \omega_{N/2}^{\frac{N}{2}\cdot\left(\frac{N}{2}-1\right)} \\ \omega_{N/2}^{\left(\frac{N}{2}+1\right)\cdot 0} & \omega_{N/2}^{\left(\frac{N}{2}+1\right)\cdot 1} & \cdots & \omega_{N/2}^{\left(\frac{N}{2}+1\right)\cdot\left(\frac{N}{2}-1\right)} \\ \vdots & \vdots & & \vdots \\ \omega_{N/2}^{(N-1)\cdot 0} & \omega_{N/2}^{(N-1)\cdot 1} & \cdots & \omega_{N/2}^{(N-1)\cdot\left(\frac{N}{2}-1\right)} \end{bmatrix}$$

Using (8),

$$B = \begin{bmatrix} \omega_{N/2}^{0\cdot 0} & \omega_{N/2}^{0\cdot 1} & \cdots & \omega_{N/2}^{0\cdot\left(\frac{N}{2}-1\right)} \\ \omega_{N/2}^{1\cdot 0} & \omega_{N/2}^{1\cdot 1} & \cdots & \omega_{N/2}^{1\cdot\left(\frac{N}{2}-1\right)} \\ \vdots & \vdots & & \vdots \\ \omega_{N/2}^{\left(\frac{N}{2}-1\right)\cdot 0} & \omega_{N/2}^{\left(\frac{N}{2}-1\right)\cdot 1} & \cdots & \omega_{N/2}^{\left(\frac{N}{2}-1\right)\cdot\left(\frac{N}{2}-1\right)} \end{bmatrix} = \sqrt{\frac{N}{2}} QFT_{N/2}$$

For C, we decompose it into two matrices.

$$C = \begin{bmatrix} \omega_N^{0\cdot 1} & \omega_N^{0\cdot 3} & \cdots & \omega_N^{0\cdot(N-1)} \\ \omega_N^{1\cdot 1} & \omega_N^{1\cdot 3} & \cdots & \omega_N^{1\cdot(N-1)} \\ \vdots & \vdots & & \vdots \\ \omega_N^{\left(\frac{N}{2}-1\right)\cdot 1} & \omega_N^{\left(\frac{N}{2}-1\right)\cdot 3} & \cdots & \omega_N^{\left(\frac{N}{2}-1\right)\cdot(N-1)} \end{bmatrix}$$

$$= \begin{bmatrix} \omega_N^{0}\omega_N^{0\cdot 0} & \omega_N^{0}\omega_N^{0\cdot 2} & \cdots & \omega_N^{0}\omega_N^{0\cdot(N-2)} \\ \omega_N^{1}\omega_N^{1\cdot 0} & \omega_N^{1}\omega_N^{1\cdot 2} & \cdots & \omega_N^{1}\omega_N^{1\cdot(N-2)} \\ \vdots & \vdots & & \vdots \\ \omega_N^{\frac{N}{2}-1}\omega_N^{\left(\frac{N}{2}-1\right)\cdot 0} & \omega_N^{\frac{N}{2}-1}\omega_N^{\left(\frac{N}{2}-1\right)\cdot 2} & \cdots & \omega_N^{\frac{N}{2}-1}\omega_N^{\left(\frac{N}{2}-1\right)\cdot(N-2)} \end{bmatrix}$$

$$= \begin{bmatrix} \omega_N^{0} & & & \\ & \omega_N^{1} & & \\ & & \ddots & \\ & & & \omega_N^{\left(\frac{N}{2}-1\right)} \end{bmatrix} \begin{bmatrix} \omega_N^{0\cdot 0} & \omega_N^{0\cdot 2} & \cdots & \omega_N^{0\cdot(N-2)} \\ \omega_N^{1\cdot 0} & \omega_N^{1\cdot 2} & \cdots & \omega_N^{1\cdot(N-2)} \\ \vdots & \vdots & & \vdots \\ \omega_N^{\left(\frac{N}{2}-1\right)\cdot 0} & \omega_N^{\left(\frac{N}{2}-1\right)\cdot 2} & \cdots & \omega_N^{\left(\frac{N}{2}-1\right)\cdot(N-2)} \end{bmatrix}$$

If we define the former matrix as $W_{N/2}$, then

$$C = W_{N/2}A = \sqrt{\frac{N}{2}} W_{N/2} QFT_{N/2}$$

Similarly,

$$\begin{aligned}
D &= \begin{bmatrix} \omega_N^{\frac{N}{2}\cdot 1} & \omega_N^{\frac{N}{2}\cdot 3} & \cdots & \omega_N^{\frac{N}{2}\cdot (N-1)} \\ \omega_N^{\left(\frac{N}{2}+1\right)\cdot 1} & \omega_N^{\left(\frac{N}{2}+1\right)\cdot 3} & \cdots & \omega_N^{\left(\frac{N}{2}+1\right)\cdot (N-1)} \\ \vdots & \vdots & & \vdots \\ \omega_N^{(N-1)\cdot 1} & \omega_N^{(N-1)\cdot 3} & \cdots & \omega_N^{(N-1)\cdot (N-1)} \end{bmatrix} \\
&= \begin{bmatrix} \omega_N^{\frac{N}{2}} & & & \\ & \omega_N^{\frac{N}{2}+1} & & \\ & & \ddots & \\ & & & \omega_N^{(N-1)} \end{bmatrix} \begin{bmatrix} \omega_N^{\frac{N}{2}\cdot 0} & \omega_N^{\frac{N}{2}\cdot 2} & \cdots & \omega_N^{\frac{N}{2}\cdot (N-2)} \\ \omega_N^{\left(\frac{N}{2}+1\right)\cdot 0} & \omega_N^{\left(\frac{N}{2}+1\right)\cdot 2} & \cdots & \omega_N^{\left(\frac{N}{2}+1\right)\cdot (N-2)} \\ \vdots & \vdots & & \vdots \\ \omega_N^{(N-1)\cdot 0} & \omega_N^{(N-1)\cdot 2} & \cdots & \omega_N^{(N-1)\cdot (N-2)} \end{bmatrix}
\end{aligned}$$

Using (9), the left matrix becomes $-W_{N/2}$.

$$D = -W_{N/2}B = -\sqrt{\frac{N}{2}} W_{N/2} QFT_{N/2}$$

All together, (1) becomes

$$\begin{aligned}
QFT_N |x\rangle &= \frac{1}{\sqrt{N}} \begin{bmatrix} \sqrt{\frac{N}{2}} QFT_{N/2} & \sqrt{\frac{N}{2}} W_{N/2} QFT_{N/2} \\ \sqrt{\frac{N}{2}} QFT_{N/2} & -\sqrt{\frac{N}{2}} W_{N/2} QFT_{N/2} \end{bmatrix} \begin{bmatrix} |x_{ev}\rangle \\ |x_{od}\rangle \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} QFT_{N/2} |x_{ev}\rangle + W_{N/2} QFT_{N/2} |x_{od}\rangle \\ QFT_{N/2} |x_{ev}\rangle - W_{N/2} QFT_{N/2} |x_{od}\rangle \end{bmatrix}
\end{aligned}$$

Using tensor product,

$$\begin{aligned}
QFT_N |x\rangle = &\frac{1}{\sqrt{2}} |0\rangle \left(QFT_{N/2} |x_{ev}\rangle + W_{N/2} QFT_{N/2} |x_{od}\rangle \right) \\
&+ \frac{1}{\sqrt{2}} |1\rangle \left(QFT_{N/2} |x_{ev}\rangle - W_{N/2} QFT_{N/2} |x_{od}\rangle \right) \qquad (10)
\end{aligned}$$

(10) shows $QFT_N |x\rangle$ is divided into two sub problems $QFT_{N/2} |x_{ev}\rangle$ and $QFT_{N/2} |x_{od}\rangle$

2.2 Deducing (10) in an another way

Let's express $|x\rangle$ in slightly different way.

$$
\begin{aligned}
|x\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{N-2} \\ \alpha_{N-1} \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ 0 \\ \alpha_2 \\ 0 \\ \vdots \\ \alpha_{N-2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_1 \\ 0 \\ \alpha_3 \\ \vdots \\ 0 \\ \alpha_{N-1} \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_2 \\ \vdots \\ \alpha_{N-2} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha_1 \\ \alpha_3 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
= |x_{ev}\rangle\,|0\rangle + |x_{od}\rangle\,|1\rangle
\end{aligned}
$$

Let I_k be the $2^k \times 2^k$ identity matrix. If we apply $QFT_{N/2} \otimes I_1$ to $|x\rangle$, then

$$
\begin{aligned}
|\psi_1\rangle &= \left(QFT_{N/2} \otimes I_1\right)|x\rangle \\
&= \left(QFT_{N/2} \otimes I_1\right)(|x_{ev}\rangle\,|0\rangle + |x_{od}\rangle\,|1\rangle) \\
&= \left(QFT_{N/2}\,|x_{ev}\rangle\right)|0\rangle + \left(QFT_{N/2}\,|x_{od}\rangle\right)|1\rangle
\end{aligned}
$$

Applying $W_{N/2} \otimes I_1$ on the condition of the last qubit,

$$
|\psi_2\rangle = \left(QFT_{N/2}\,|x_{ev}\rangle\right)|0\rangle + \left(W_{N/2} QFT_{N/2}\,|x_{od}\rangle\right)|1\rangle
$$

Then we make superposition on the last qubit by applying $I_{n-1} \otimes H$, i.e. Hadamard gate on the last qubit.

$$
|\psi_3\rangle = QFT_{N/2}\,|x_{ev}\rangle \frac{|0\rangle + |1\rangle}{\sqrt{2}} + W_{N/2} QFT_{N/2}\,|x_{od}\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}
$$

Grouping along the last qubit,

$$
\begin{aligned}
|\psi_3\rangle = &\frac{1}{\sqrt{2}}\left(QFT_{N/2}\,|x_{ev}\rangle + W_{N/2} QFT_{N/2}\,|x_{od}\rangle\right)|0\rangle \\
&+ \frac{1}{\sqrt{2}}\left(QFT_{N/2}\,|x_{ev}\rangle - W_{N/2} QFT_{N/2}\,|x_{od}\rangle\right)|1\rangle
\end{aligned} \tag{11}
$$

Note that (11) slightly differs from (10). This can be fixed with ease as explained in the following section. For now, let A_N be a gate that does such work, i.e.

$$
\begin{aligned}
|\psi_4\rangle =& A_N\,|\psi_3\rangle \\
=& \frac{1}{\sqrt{2}}\,|0\rangle\left(QFT_{N/2}\,|x_{ev}\rangle + W_{N/2} QFT_{N/2}\,|x_{od}\rangle\right) \\
&+ \frac{1}{\sqrt{2}}\,|1\rangle\left(QFT_{N/2}\,|x_{ev}\rangle - W_{N/2} QFT_{N/2}\,|x_{od}\rangle\right)
\end{aligned}
$$

To sum up, we built the abstract structure of the circuit.

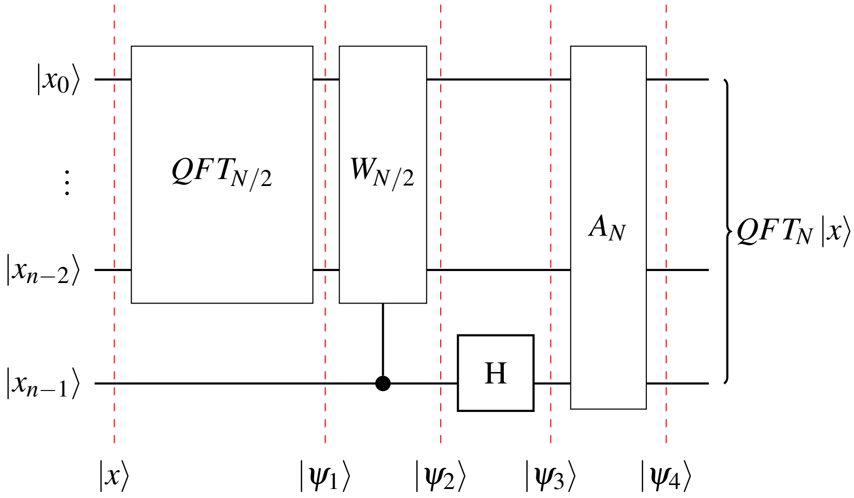

Figure 1: Incomplete quantum circuit of QFT_N

2.3 Implementing on the qubit circuit

In the previous section, we have deduced (10) by applying some gates. Among them, $W_{N/2}$ and A_N should be resolved into well-known gates.

To begin with, $W_{N/2}$ can be expressed with multiple tensor products of 2×2 gates.

$$
\begin{aligned}
W_{N/2} &= \begin{bmatrix} \omega_N^0 & & & \\ & \omega_N^1 & & \\ & & \ddots & \\ & & & \omega_N^{\frac{N}{2}-1} \end{bmatrix} = \begin{bmatrix} e^{i\frac{2\pi}{N}\cdot 0} & & & \\ & e^{i\frac{2\pi}{N}\cdot 1} & & \\ & & \ddots & \\ & & & e^{i\frac{2\pi}{N}\cdot\left(\frac{N}{2}-1\right)} \end{bmatrix} \\
&= \begin{bmatrix} 1 & \\ & e^{i\frac{2\pi}{N}\cdot\frac{N}{4}} \end{bmatrix} \otimes \begin{bmatrix} e^{i\frac{2\pi}{N}\cdot 0} & & & \\ & e^{i\frac{2\pi}{N}\cdot 1} & & \\ & & \ddots & \\ & & & e^{i\frac{2\pi}{N}\cdot\left(\frac{N}{4}-1\right)} \end{bmatrix} \\
&= R_{\pi/2} \otimes \begin{bmatrix} 1 & \\ & e^{i\frac{2\pi}{N}\cdot\frac{N}{8}} \end{bmatrix} \otimes \begin{bmatrix} e^{i\frac{2\pi}{N}\cdot 0} & & & \\ & e^{i\frac{2\pi}{N}\cdot 1} & & \\ & & \ddots & \\ & & & e^{i\frac{2\pi}{N}\cdot\left(\frac{N}{8}-1\right)} \end{bmatrix} \\
&= R_{\pi/2} \otimes R_{\pi/4} \otimes \cdots \otimes R_{\pi/(N/2)} \otimes I_1
\end{aligned}
$$

In other words, we need $n-1$ phase shift gates.

Meanwhile, judging by the order of each qubit in (10) and (11), we can simply get it done by twisting quantum channels like this.

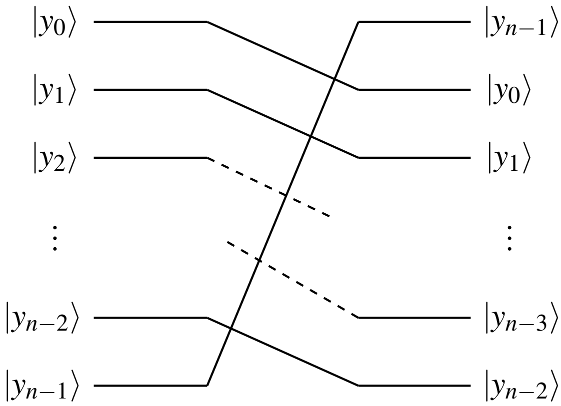

Figure 2: Permutation of quantum channels

To get the matrix expression, we ought to find a matrix A_N such that

$$A_N \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{N-1} \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_2 \\ \vdots \\ \beta_{N-2} \\ \hdashline \beta_1 \\ \beta_3 \\ \vdots \\ \beta_{N-1} \end{bmatrix}$$

To illustrate, for $N = 8$,

$$A_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

To sum up, the circuit is shown in **Fig 3**
and remember QFT_2 is equal to the Hadamard gate

$$QFT_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{12}$$

3 Number of gates

In the circuit, the number of gates is considered the performance in time. Hence, it is meaningful to take a look at how many gates are used in **Fig 3**.

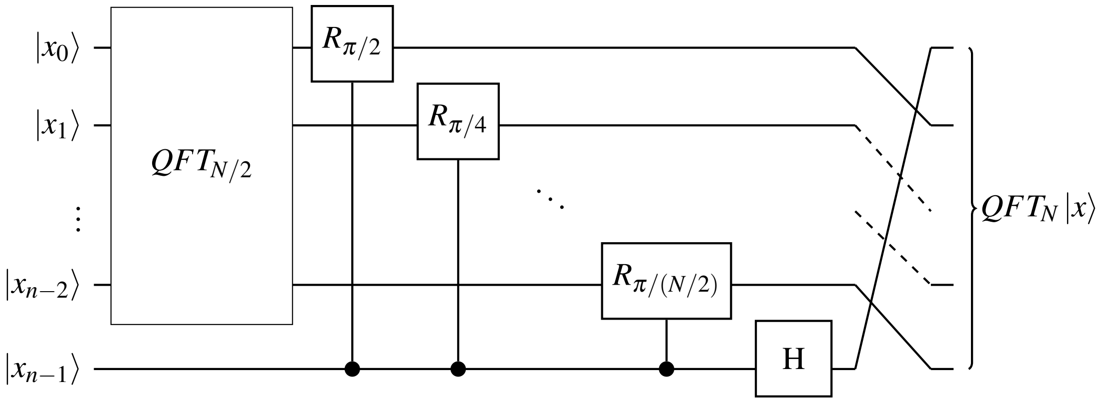

Figure 3: Complete quantum circuit of QFT_N

Let $G(n)$ be the number of gates in QFT_{2^n}. Then,

$$\begin{aligned} G(n) &= (\text{\# of gates in } QFT_{2^{n-1}}) + (\text{\# of gates of phase shifts}) + (\text{one hadamard}) \\ &= G(n-1) + (n-1) + 1 = G(n-1) + n \quad \text{for } n = 1, 2, \cdots \end{aligned}$$

and the initial condition is

$$G(1) = 1$$

from (12). Solving this recurrence, we earn

$$G(n) = \frac{n(n+1)}{2}$$

or

$$G(n) = \frac{1}{2}(\lg N)(\lg N + 1) = O\left(\lg^2 N\right)$$

Comparing classical FFT, which runs at speed $O(N \lg N)$, this is exponential improvement.

4 Rationale

4.1 Some contradictions against other material

There are plenty of materials that explain QFT. However, in my point of view, they seem to be wrong. For instance, the resultant circuit of QFT_8 is shown in Fig 1 in [Ron]. We posit this results different from what it should make. Using Matlab code (Sec 5), we simulated both Fig 3 and Fig 1 in [Ron].

For $N = 8$ and input $|x\rangle = \begin{bmatrix} 0 & 0 & 1/\sqrt{2} & 0 & 0 & 0 & 0 & 1/\sqrt{2} \end{bmatrix}^T$, the correct answer and the result from Fig 3 were the same as

$$\begin{bmatrix} 0.5000+0.0000i \\ 0.1768+0.0732i \\ -0.2500-0.2500i \\ -0.1768-0.4268i \\ 0.0000+0.0000i \\ -0.1768+0.4268i \\ -0.2500+0.2500i \\ 0.1768-0.0732i \end{bmatrix}$$

but one from Fig 1 in [Ron] was

$$\begin{bmatrix} 0.5000+0.0000i \\ 0.0000+0.0000i \\ -0.2500-0.2500i \\ -0.0000-0.3536i \\ 0.0000+0.0000i \\ -0.3536+0.3536i \\ -0.2500+0.2500i \\ 0.3536-0.0000i \end{bmatrix}$$

Many materials insist the method like in Fig 1 in [Ron]. I wonder if they tried one specific example like this.

4.2 Definition of ω

The definitions of ω in DFT and QFT are different,

$$\omega = e^{-\frac{2\pi}{N}} \text{ and } \omega = e^{\frac{2\pi}{N}}$$

respectively. I haven't found any clue of this difference. Maybe there are couple of applications in Quantum Computation that benefits from the latter definition.

5 QFT in Matlab

```
function res = test()
    clc;    clear;

    % system definition
    n=3;    N=2^n;

    % input
%     x = rand(N,1) + 1i * rand(N,1);
```

```
%      x = x/sqrt(sum(abs(x).^2));
    x = [0 0 1/sqrt(2) 0 0 0 0 1/sqrt(2)]';

    % the correct answer
    answer = QFT(x)

    % recursive method in document
    recursive = QFT_recur(x)

    % method in reference
    if n==3
        reference = QFT_reference(x)
    end
end

function y = QFT(x)
    Q_N = genQFT(size(x,1));
    y = Q_N*x;
end

function y = QFT_recur(x)
    N = size(x,1);
    if N == 1
        y = x;
    else
        x_ev = x(1:2:N);
        x_od = x(2:2:N);
        y_ev = QFT_recur(x_ev);
        y_od = QFT_recur(x_od);
        w_N = exp(1i*2*pi/N);
        W_N2 = diag(w_N.^(0:N/2-1));
        A = y_ev + W_N2*y_od;
        B = y_ev - W_N2*y_od;
        y = 1/sqrt(2)*( kron(A,[1;0]) + kron(B,[0;1]) );
        R = genROT(N);
        y = R*y;
    end
end

function y = QFT_reference(x)
    H = 1/sqrt(2)*[1 1;1 -1];
    G1 = kron(kron(H,eye(2)),eye(2));
    G2 = eye(8);    G2(7,7) = 1i;    G2(8,8) = 1i;
    G3 = eye(8);    G3(7,7) = exp(1i*pi()/4);    G3(8,8) =
       exp(1i*pi()/4);
```

```
    G4 = kron(kron(eye(2),H),eye(2));
    G5 = eye(8);    G5(4,4) = 1i;    G5(8,8) = 1i;
    G6 = kron(kron(eye(2),eye(2)),H);
    G7 = zeros(8);    G7(1,1) = 1;    G7(5,2) = 1;
    G7(3,3) = 1;    G7(7,4) = 1;    G7(2,5) = 1;
    G7(6,6) = 1;    G7(4,7) = 1;    G7(8,8) = 1;

    y = G7*G6*G5*G4*G3*G2*G1*x;
end

function Q = genQFT(N)
    Q = zeros(N);
    w = exp(1i*2*pi/N);
    for k1 = 1:N
        for k2 = 1:N
            Q(k1,k2) = w^((k1-1) * (k2-1));
        end
    end
    Q = Q ./sqrt(N);
end

function R = genROT(N)
    R = zeros(N);
    for k1 = 1:N
        R(k1, floor( (k1-1)/(N/2) ) + mod(k1-1,N/2)*2+1 )
          = 1;
    end
end
```

Code 1: code for simulation

References

[Dr] Dr Iain Styles. Lecture 6: The Quantum Fourier Transform. http://www.cs.bham.ac.uk/internal/courses/intro-mqc/current/lecture06_handout.pdf.

[Ron] Ronald de Wolf. The classical and quantum fourier transform. http://homepages.cwi.nl/~rdewolf/qfourierintro.pdf.

[Ume] Umesh Vazirani. Chapter 5: Quantum Fourier Transform. https://d37djvu3ytnwxt.cloudfront.net/c4x/BerkeleyX/CS-191x/asset/chap5.pdf.

[Wik] Wikipedia. Quantum Fourier transform . https://en.wikipedia.org/wiki/Quantum_Fourier_transform.