Stirling's approximation

Jiman Hwang

July 5, 2015

This document is for self study only.

1 Stirling's approximation

1.1 Definition

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{1}$$

where $\sim$ means "asymptotically similar".

1.2 Proof

Let's start with the following term.

$$S = \ln n! = \sum_{k=1}^{n} \ln k = \sum_{k=2}^{n} \ln k \tag{2}$$

S is an approximation of I,

$$I = \int_1^n \ln x dx = n \ln n - n + 1 \tag{3}$$

From (**??**) and (**??**),

$$S - I = \ln n! - n \ln n + n - 1 \tag{4}$$

Also, using Euler-Maclaurin formula,

$$S - I = \frac{1}{2}\ln n + \sum_{k=2}^{m} \frac{B_k}{k\left(k-1\right)}\left(\frac{1}{n^{k-1}} - 1\right) + R_{m,n} \tag{5}$$

where $m = 1, 2, 3, ...$, and

$$R_{m,n} = \int_1^n \frac{P_m\left(x\right)}{mx^m} dx \tag{6}$$

From (**??**) and (**??**),

$$\ln n! - n \ln n + n - \frac{1}{2}\ln n = 1 + \sum_{k=2}^{m} \frac{B_k}{k\left(k-1\right)}\left(\frac{1}{n^{k-1}} - 1\right) + R_{m,n} \tag{7}$$

Applying a limit that n tends to ∞ on both sides,

$$\lim_{n\to\infty}\left[\ln n! - n\ln n + n - \frac{1}{2}\ln n\right] = 1 - \sum_{k=2}^{m}\frac{B_k}{k\,(k-1)} + \lim_{n\to\infty} R_{m,n} = y_m \tag{8}$$

For the last term,

$$\lim_{n\to\infty} R_{m,n} - R_{m,n} = \int_n^\infty \frac{P_m\,(x)}{mx^m}dx < \int_n^\infty \frac{dx}{x^m} = O\left(\frac{1}{n^m}\right) \tag{9}$$

Thus,

$$\lim_{n\to\infty} R_{m,n} = R_{m,n} + O\left(\frac{1}{n^m}\right) \tag{10}$$

and y_m converges. Plugging (**??**) into (**??**),

$$\ln n! - n\ln n + n - \frac{1}{2}\ln n = \sum_{k=2}^{m}\frac{B_k}{k\,(k-1)}\left(\frac{1}{n^{k-1}}\right) + R_{m,n} - \lim_{n\to\infty} R_{m,n} + y_m \tag{11}$$

Using (**??**),

$$\ln n! - n\ln n + n - \frac{1}{2}\ln n = \sum_{k=2}^{m}\frac{B_k}{k\,(k-1)}\left(\frac{1}{n^{k-1}}\right) + O\left(\frac{1}{n^m}\right) + y_m \tag{12}$$

Let $m = 1$, then

$$\ln n! - n\ln n + n - \frac{1}{2}\ln n = O\left(\frac{1}{n}\right) + y_1 \tag{13}$$

Resolving for $n!$,

$$n! = e^{O\left(\frac{1}{n}\right)+y_1}\sqrt{n}\left(\frac{n}{e}\right)^n = A\sqrt{n}\left(\frac{n}{e}\right)^n \tag{14}$$

Meanwhile, changing form of Wallis product,

$$\prod_{k=1}^{n}\left(\frac{2k}{2k-1}\frac{2k}{2k+1}\right) = \frac{2^2}{1\times 3}\times\frac{4^2}{3\times 5}\times \ldots \times \frac{(2n)^2}{(2n-1)\,(2n+1)} \tag{15}$$

$$= \frac{2^4}{(1\times 2)\,(2\times 3)}\times\frac{4^4}{(3\times 4)\,(4\times 5)}\times \ldots \tag{16}$$

$$\times\frac{(2n)^4}{[(2n-1)\,(2n)]\,[(2n)\,(2n+1)]} \tag{17}$$

$$= \frac{2^{4n}\,(n!)^4}{(2n)!\,(2n+1)!} \tag{18}$$

$$= \frac{2^{4n}}{2n+1}\frac{(n!)^4}{[(2n)!]^2} \tag{19}$$

Substituting factorials in (**??**) with (**??**),

$$\frac{2^{4n}}{2n+1}\frac{(n!)^4}{[(2n)!]^2} = \frac{2^{4n}}{2n+1}\frac{\left[A\sqrt{n}\left(\frac{n}{e}\right)^n\right]^4}{\left[A\sqrt{2n}\left(\frac{2n}{e}\right)\right]^2} \tag{20}$$

$$= \frac{A^2 n}{2\,(2n+1)} \tag{21}$$

For $n \to \infty$,

$$\lim_{n\to\infty} \frac{A^2 n}{2\left(2n+1\right)} = \frac{\pi}{2} \tag{22}$$

from Wallis product. Ans this produces

$$\lim_{n\to\infty} A = \sqrt{2\pi} \tag{23}$$

Limiting n tends to ∞ on both end sides of (**??**),

$$\lim_{n\to\infty} n! = \lim_{n\to\infty} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{24}$$

$$\Rightarrow \lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1 \tag{25}$$

$$\therefore n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{26}$$