Continued Fractions Approximation

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Sources: CH10.15 at [GHH08]

1 Statement

Let p_n/q_n be the $(n+1)^{\text{th}}$ convergent of $\alpha \in \mathbb{R}$ where $\gcd\{p_n, q_n\} = 1$.
For $n, q \in \mathrm{N}$, if $n > 1$, $0 < q \leq q_n$, $\frac{p}{q} \neq \frac{p_n}{q_n}$, then

$$|q_n\alpha - p_n| < |q\alpha - p| \tag{1}$$

which implies

$$\left|\alpha - \frac{p_n}{q_n}\right| < \left|\alpha - \frac{p}{q}\right|$$

So the convergent p_n/q_n is the best approximation to α among rational numbers whose denominators are less than or equal to q_n.

2 Proof

At first, let's take a look at the case that p_n/q_n is the last convergent. Namely,

$$\frac{p_n}{q_n} = \alpha$$

Since $\frac{p}{q} \neq \frac{p_n}{q_n}$, $\left|\alpha - \frac{p}{q}\right| > 0$, which implies (1) is true. So, we assume p_n/q_n is not the last convergent of α.

Let x,y be numbers such that

$$x \begin{bmatrix} p_n \\ q_n \end{bmatrix} + y \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \tag{2}$$

Proceeding a couple of steps,

$$\begin{bmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = (-1) \begin{bmatrix} q_{n+1} & -p_{n+1} \\ -q_n & p_n \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

Since the values on the right hand side are integers, so are x and y. The following shows x and y are not zeros and have the opposite sign to one other. Suppose $x = 0$, from (2),

$$q = yq_{n+1}$$

But

$$0 < q \le q_n < q_{n+1} \;\Rightarrow\; 0 < y < 1$$

which is a contradiction. Hence, $x \neq 0$. Also as we suppose $y = 0$, then from (2),

$$p = xp_n,\; q = yq_n \;\Rightarrow\; \frac{p}{q} = \frac{p_n}{q_n}$$

which is a contradiction by the assumption. Hence, $y \neq 0$.
Furthermore, if we substitute q with (2) in the condition $0 < q \le q_n$,

$$0 < xq_n + yq_{n+1} \le q_n$$

If $x > 0, y > 0$, then $xq_n + yq_{n+1} > q_n$, and if $x < 0, y < 0$, then $xq_n + yq_{n+1} < 0$. Since both cases are contradictions, $xy < 0$ is true.
Now expressing $|q\alpha - p|$ with (2) gives

$$|q\alpha - p| = |x(q_n\alpha - p_n) + y(q_{n+1}\alpha - p_{n+1})|$$

From Theorem 154 at [GHH08], $q_n\alpha - p_n$ and $q_{n+1}\alpha - p_{n+1}$ have different signs to each other. Hence, $x(q_n\alpha - p_n)$ and $y(q_{n+1}\alpha - p_{n+1})$ have the same sign to each other. It follows

$$\begin{aligned}|q\alpha - p| &= |x(q_n\alpha - p_n)| + |y(q_{n+1}\alpha - p_{n+1})| \\ &> |x(q_n\alpha - p_n)| \\ &> |q_n\alpha - p_n|\end{aligned}$$

One more step makes it clearer.

$$|q_n\alpha - p_n| < |q\alpha - p| \;\Rightarrow\; q_n\left|\alpha - \frac{p_n}{q_n}\right| < q\left|\alpha - \frac{p}{q}\right|$$

Since $q \le q_n$,

$$q_n\left|\alpha - \frac{p_n}{q_n}\right| < q_n\left|\alpha - \frac{p}{q}\right| \;\Rightarrow\; \left|\alpha - \frac{p_n}{q_n}\right| < \left|\alpha - \frac{p}{q}\right|$$

References

[GHH08] Andrew Wiles G. H. Hardy, Edward M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press, 6th edition, 2008.