Periodicity of squared root of natural numbers

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Sources: myself

1 Statement

For $n \in \mathbb{N}$, if $\sqrt{n}$ is irrational, then $\sqrt{n}$ is periodic when it is expressed as continued fractions(CH 10 at [GHH08]). That is, if

$$\sqrt{n} = [x_0, x_1, \cdots]$$

then

$$\exists i \exists p \quad x_j = x_{j+p},\ p > 0 \quad \text{for } j = i, i+1, \cdots \tag{1}$$

2 Proof

First of all, we posit $\sqrt{n}$ has infinite continued fractions. Suppose for finite natural number N,

$$\sqrt{n} = [x_0, x_1, \cdots, x_N]$$

where x_i are all integers. However, if we put the right hand side into fractional form, p/q where p and q are integers, then it contradicts the assumption. Thus, $\sqrt{n}$ has infinite continued fractions.

Dealing with infinite terms with a finite ones might be beneficial. Let

$$\sqrt{n} = [x_0, \cdots, x_{i-1}, \omega_i]$$

where

$$\omega_i = [x_i, x_{i+1}, \cdots] \tag{2}$$

By equation (10.5.1) at [GHH08]

$$\sqrt{n} = \frac{\omega_i p_{i-1} + p_{i-2}}{\omega_i q_{i-1} + q_{i-2}}$$

or,

$$\omega_i = \frac{(p_{i-2}q_{i-1} - p_{i-1}q_{i-2})\sqrt{n} + (p_{i-1}p_{i-2} - q_{i-1}q_{i-2}n)}{q_{i-1}^2 n - p_{i-1}^2}$$

By Theorem 150 in [GHH08],

$$\omega_i = \frac{\sqrt{n} + (-1)^{i-1}(p_{i-1}p_{i-2} - q_{i-1}q_{i-2}n)}{(-1)^{i-1}\left(q_{i-1}^2 n - p_{i-1}^2\right)}$$

Now, let

$$a_i \stackrel{\text{def}}{=} (-1)^{i-1}(p_{i-1}p_{i-2} - q_{i-1}q_{i-2}n)$$
$$b_i \stackrel{\text{def}}{=} (-1)^{i-1}\left(q_{i-1}^2 n - p_{i-1}^2\right)$$

Then

$$\sqrt{n} = \left[x_0, \cdots, x_{i-1}, \frac{\sqrt{n} + a_i}{b_i}\right]$$

where

$$\frac{\sqrt{n} + a_i}{b_i} = [x_i, x_{i+1}, \cdots], \{a_i, b_i\} \subset \mathbb{Z} \tag{3}$$

It follows

$$x_i = \left\lfloor \frac{\sqrt{n} + a_i}{b_i} \right\rfloor \tag{4}$$

If permutation (a_i, b_i) periodic in sequence $\{a_i\}, \{b_i\}$ from some index, then (1) is held. To do that, we first prove the following lemma using induction.

$$\forall j \quad 0 < b_j < 2\sqrt{n} \tag{5}$$

For $j = 0$, $b_j = 1$. Since the least irrational number of $\sqrt{n}$ is when $n = 2$, (5) is held. Now suppose

$$0 < b_i < 2\sqrt{n} \tag{6}$$

and we ought to prove this statement is true for $i+1$ instead of i. From (3),

$$\frac{\sqrt{n} + a_{i+1}}{b_{i+1}} = \frac{1}{\frac{\sqrt{n}+a_i}{b_i} - x_i} = \frac{\sqrt{n} - (a_i - b_i x_i)}{\left[n - (a_i - b_i x_i)^2\right]/b_i}$$

It follows

$$a_{i+1} = b_i x_i - a_i \tag{7}$$

$$b_{i+1} = \frac{n - a_{i+1}^2}{b_i} \tag{8}$$

Also from (4),

$$\frac{\sqrt{n}+a_i}{b_i} = x_i + \alpha_i \quad (0 < \alpha_i < 1)$$

Rearranging,

$$\sqrt{n} - \alpha_i b_i = b_i x_i - a_i$$

Using (7) on the right hand side, and squaring on both sides gives,

$$n - 2\alpha_i b_i \sqrt{n} + \alpha_i^2 b_i^2 = a_{i+1}^2$$

Using (8),

$$b_i b_{i+1} = 2\alpha_i b_i \sqrt{n} - \alpha_i^2 b_i^2$$

or

$$b_i = \frac{2\alpha_i \sqrt{n} - b_{i+1}}{\alpha_i^2}$$

Putting it into (6),

$$0 < \frac{2\alpha_i \sqrt{n} - b_{i+1}}{\alpha_i^2} < 2\sqrt{n}$$

Expressing for b_{i+1},

$$2\alpha_i \sqrt{n}(1-\alpha_i) < b_{i+1} < 2\alpha_i \sqrt{n}$$

Considering the range of α_i, it follows

$$0 < b_{i+1} < 2\sqrt{n}$$

Therefore, (6) is held for all i's. Now rearrange (8) as follow.

$$a_{i+1}^2 + b_i b_{i+1} = n$$

Because a_{i+1}, b_i, b_{i+1} are integers and b_i, b_{i+1} are positive,

$$\left\{a_{i+1}^2, b_i, b_{i+1}\right\} \subset \{0, 1, \cdots, n\}$$

following

$$\left|\left\{(a_{i+1}^2, b_i, b_{i+1}) \;:\; i = 0, 1, \cdots\right\}\right| < \infty$$

narrowing it down,

$$|\{(a_i, b_i) \;:\; i = 0, 1, \cdots\}| < \infty$$

Thus, as $i \to \infty$,

$$\exists i \exists p \quad a_i = a_{i+p}, b_i = b_{i+p}, 0 < p$$

by Pigeon hole's principle(CH4 in [Bru09]). Furthermore, from (7) and (8),

$$\exists j \quad a_j = a_{j+p}, b_j = b_{j+p} \quad \text{for } i, i+1, \cdots$$

Finally by (4),

$$x_j = x_{j+p} \quad \text{for } j = i, i+1, \cdots$$

References

[Bru09] Richard A. Brualdi. *An Introduction to the Theory of Numbers*. Pearson, 5th edition, 2009.

[GHH08] Andrew Wiles G. H. Hardy, Edward M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press, 6th edition, 2008.