The relationship between chi squared and normal samples

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Source: [Lev]

1 Statement

Let $X_1, \cdots, X_n$ denote i.i.d. such that

$$X_i \sim N(\mu, \sigma^2)$$

Also define sample mean $\overline{X}$ and sample variance S^2 as

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad , \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

If random variable Y is

$$Y = \frac{n-1}{\sigma^2}S^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

then,

$$Y \sim \chi^2(n-1)$$

2 Proof

$$Y = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \sum_{i=1}^{n}\left(\frac{X_i - \overline{X}}{\sigma}\right)^2 = \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma} - \frac{\overline{X} - \mu}{\sigma}\right)^2 \tag{1}$$

$(\overline{X} - \mu)/\sigma$ becomes

$$\frac{\overline{X} - \mu}{\sigma} = \frac{1}{\sigma}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right) = \frac{1}{\sigma}\frac{1}{n}\left(\sum_{i=1}^{n} X_i - n\mu\right) = \frac{1}{n}\sum_{i=1}^{n}\frac{X_i - \mu}{\sigma} \tag{2}$$

From (1), (2),

$$Y = \sum_{i=1}^{n}(Z - \overline{Z})^2 = \sum_{i=1}^{n} Z^2 - n\overline{Z}^2 \tag{3}$$

where

$$Z_i = \frac{X_i - \mu}{\sigma} \sim N(0,1) \quad \text{i.i.d. for i=1}, \cdots, n$$
$$\overline{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i$$

Let

$$\mathbf{Z} = \begin{bmatrix} Z_1 \\ \cdots \\ Z_n \end{bmatrix}^T, \; P = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

with a_{ij}'s are determined so that P is an orthogonal matrix using Gram-Schmidt. Since P is orthogonal matrix, $P^T P = I$ where I is identity matrix. And

$$\sum_{i=1}^{n} Z^2 = \mathbf{Z}^T\mathbf{Z} = \mathbf{Z}^T I \mathbf{Z} = \mathbf{Z}^T P^T P \mathbf{Z} = \|P\mathbf{Z}\|^2 = \sum_{i=1}^{n}(\mathbf{v}_i\mathbf{Z})^2 = (\mathbf{v}_1\mathbf{Z})^2 + \sum_{i=2}^{n}(\mathbf{v}_i\mathbf{Z})^2 = n\overline{Z}^2 + \sum_{i=2}^{n} W_i^2 \tag{4}$$

From (4),

$$Y = \sum_{i=2}^{n} W_i^2 \tag{5}$$

For $i = 2, \cdots, n$, W_i is linear combination of standard normal random variables. Hence, $i = 2, \cdots, n$, W_i is a normal random variable. Let's get expectation, and variance, and verify independence.

$$\begin{aligned}
E(W_i) =& E\left(\sum_{j=1}^{n} a_{ij} Z_j\right) = \sum_{j=1}^{n} a_{ij} E(Z_j) = 0 \\
\mathrm{Var}(W_i) =& \mathrm{Var}\left(\sum_{j=1}^{n} a_{ij} Z_j\right) = \mathrm{Cov}\left(\sum_{j=1}^{n} a_{ij} Z_j \ , \ \sum_{j=1}^{n} a_{ij} Z_j\right) \\
=& \sum_{j=1}^{n} \mathrm{Var}(a_{ij} Z_j) = \sum_{j=1}^{n} a_{ij}^2 = \mathbf{v}_i^T \mathbf{v}_i = \|\mathbf{v}_i\|^2 = 1 \quad \because P \text{ is orthogonal matrix.}
\end{aligned}$$

For dependence, it is sufficient to show $\mathrm{Cov}(W_i, W_j) = 0$ for $i \neq j$ because W_i, W_j are normal random variables.

$$\begin{aligned}
\mathrm{Cov}(W_i, W_j) =& \mathrm{Cov}\left(\sum_{k=1}^{n} a_{ik} Z_k \ , \ \sum_{k=1}^{n} a_{jk} Z_k\right) = \sum_{k=1}^{n} a_{ik} a_{jk} \mathrm{Cov}(Z_k, Z_k) \\
=& \mathbf{v}_i \mathbf{v}_j^T = 0 \quad \because P \text{ is orthogonal matrix.}
\end{aligned}$$

Hence, W_i's are i.i.d. as $W_i \sim N(0, 1)$. Also from (5),

$$Y \sim \chi^2(n-1)$$

by definition of χ^2 distribution.

References

[Lev] Dr. David Levin. Distribution Theory for Normal Samples. `http://www.math.utah.edu/~levin/M5080/sampling.pdf`.