Cauchy's mean value theorem

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1 Statement

Suppose two functions f and g are continueous on $[a, b]$ and differentiable on (a, b). Then, there's a number c such that

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)],\ c \in (a, b)$$

If $g(a) \neq g(b)$, then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

2 Derivation

Let

$$h(t) = f(t) - rg(t)$$

where r is a number such that $h(a) = h(b)$

That is,

$$\begin{aligned} f(a) - rg(a) &= f(b) - rg(b) \\ \rightarrow r[g(b) - g(a)] &= f(b) - f(a) \end{aligned}$$

i) $g(a) \neq g(b)$

$$r = \frac{f(b) - f(a)}{g(b) - g(a)} \tag{1}$$

By Rolle's theorem,

$$\exists c[h'(c) = 0,\ c \in (a, b)]$$

or

$$f'(c) - rg'(c) = 0$$

Plugging (1) into this,

$$\begin{aligned} \frac{f'(c)}{g'(c)} &= \frac{f(b) - f(a)}{g(b) - g(a)} \\ \rightarrow f'(c)[g(b) - g(a)] &= g'(c)[f(b) - f(a)] \end{aligned}$$

ii) $g(a) = g(b)$
By Rolle's theorem,

$$\exists c[g'(c) = 0,\ c \in (a, b)]$$

Hence,

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$$

$$\therefore f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$$