# Fourier Transform

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## 1 Definition

There are many definitions for Fourier Transform. Among them, let me introduce a popular one. Let $f(t)$ be a piecewise continuous function. Then the Fourier Transform of $f(t)$ is

$$\hat{f} \stackrel{\text{def}}{=} \mathscr{F}\{f\} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\nu t} dt$$

where $i = \sqrt{-1}$. Also, inverse one is

$$f(t) = \mathscr{F}^{-1}\left\{\hat{f}\right\} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \hat{f}(\nu)\, e^{-i2\pi\nu t} d\nu$$

Note that if $t$ is time, then $\nu$ is frequency.

## 2 Derivation

If $f_p(x) \in \mathbb{C}(-\infty,\infty)$ satisfies $\forall t f_p(t) = f_p(t-p)$, then the Fourier Series of $f_p(t)$ is

$$f_p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\omega_n t + b_n \sin\omega_n t\right) \tag{1}$$

where

$$\begin{aligned} a_n &= \frac{2}{p}\int_0^p f_p(x)\cos\omega_n x dx && (2) \\ b_n &= \frac{2}{p}\int_0^p f_p(x)\sin\omega_n x dx \\ \omega_n &= \frac{2\pi n}{p} && (3) \end{aligned}$$

Let $g_n(t)$ be

$$g_n(t) = a_n\cos\omega_n t + b_n\sin\omega_n t$$

By replacing $a_n$ and $b_n$,

$$
\begin{aligned}
g_n(t) =& \cos\omega_n t \times \frac{2}{p}\int_0^p f_p(x)\cos\omega_n x dx + \sin\omega_n t \times \frac{2}{p}\int_0^p f_p(x)\sin\omega_n x dx \\
=& \frac{2}{p}\int_0^p f_p(x)\left(\cos\omega_n t\cos\omega_n x + \sin\omega_n t\sin\omega_n x\right)dx \\
=& \frac{2}{p}\int_0^p f_p(x)\cos\left(\omega_n t - \omega_n x\right)dx \qquad (4)
\end{aligned}
$$

Since $g_n(t) = g_{-n}(t)$, (1) becomes

$$
f_p(t) = \frac{a_0}{2} + \frac{1}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} g_n(t)
$$

Replacing $a_0$ with (2) and $g_n(t)$ with (4),

$$
\begin{aligned}
f_p(t) =& \frac{1}{p}\int_0^p f_p(x)\,dx + \frac{1}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{2}{p}\int_0^p f_p(x)\cos\left(\omega_n t - \omega_n x\right)dx \\
=& \sum_{n=-\infty}^{\infty}\frac{1}{p}\int_0^p f_p(x)\cos\left(\omega_n t - \omega_n x\right)dx \qquad (5)
\end{aligned}
$$

Note that the following is true.

$$
h(t) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty}\frac{1}{p}\int_0^p f_p(x)\,i\sin\left(\omega_n t - \omega_n x\right)dx = 0 \qquad (6)
$$

$$
\begin{aligned}
\because \sum_{n=-\infty}^{\infty}\sin\left(\omega_n t - \omega_n x\right) =& \sum_{n=-\infty}^{-1}\sin\left(\omega_n t - \omega_n x\right) + \sin\left(\omega_0 t - \omega_0 x\right) + \sum_{n=1}^{\infty}\sin\left(\omega_n t - \omega_n x\right) \\
=& \sum_{n=1}^{\infty}\sin\left(\omega_{-n} t - \omega_{-n} x\right) + 0 + \sum_{n=1}^{\infty}\sin\left(\omega_n t - \omega_n x\right) \\
=& -\sum_{n=1}^{\infty}\sin\left(\omega_n t - \omega_n x\right) + \sum_{n=1}^{\infty}\sin\left(\omega_n t - \omega_n x\right) \\
=& 0
\end{aligned}
$$

By adding up (5) and (6),

$$
\begin{aligned}
f_p(t) =& \sum_{n=-\infty}^{\infty}\frac{1}{p}\int_0^p f_p(x)\left[\cos\left(\omega_n t - \omega_n x\right) + i\sin\left(\omega_n t - \omega_n x\right)\right]dx \\
=& \sum_{n=-\infty}^{\infty}\frac{1}{p}\int_0^p f_p(x)\,e^{i(\omega_n t - \omega_n x)}dx \\
=& \sum_{n=-\infty}^{\infty}\frac{1}{p}\int_{-p/2}^{p/2} f_p(x)\,e^{i(\omega_n t - \omega_n x)}dx \quad \because \text{the integrand is periodic by } p \qquad (7)
\end{aligned}
$$

Now, it's time to get rid of periodicity by extending one phase of $f_p(t)$. let

$$f(t) \stackrel{\text{def}}{=} \lim_{p\to\infty} f_p(t)$$
$$\Delta \stackrel{\text{def}}{=} \frac{2\pi}{p} \quad \text{or} \quad \frac{1}{p} = \frac{\Delta}{2\pi}$$

From (3),

$$\omega_n = n\Delta$$

Replacing (7) with the above terms,

$$f_p(t) = \sum_{n=-\infty}^{\infty} \frac{\Delta}{2\pi} \int_{-p/2}^{p/2} f_p(x)\, e^{i(\omega_n t - \omega_n x)} dx$$

As $p \to \infty$

$$\Delta \to d\omega,\ \omega_n \to \omega,\ \sum_{n=-\infty}^{\infty} \to \int_{-\infty}^{\infty}$$

Thus,

$$f(t) = \lim_{p\to\infty} f_p(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{i(\omega t - \omega x)} dx d\omega \tag{8}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x} dx d\omega \tag{9}$$

Inspired from the fact that $f(x)$ appears again on the right hand side with repeating structure, we may define one definition of FT in terms of angular velocity $\omega$ (if $t$ is time).

$$\hat{f}(\omega) \stackrel{\text{def}}{=} \mathscr{F}\{f\} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt$$
$$f(\omega) \stackrel{\text{def}}{=} \mathscr{F}^{-1}\{\hat{f}\} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t} d\omega$$

Note that $\frac{1}{\sqrt{2\pi}}$ is for normalization.

To define FT in terms of frequency $\nu$, substitute $\omega$ with $2\pi\nu$. Then (9) becomes

$$f(t) = \int_{-\infty}^{\infty} e^{i2\pi\nu t} \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\nu x} dx d\nu \tag{10}$$

Finally, we define FT in terms of frequency.

$$\hat{f}(\nu) \stackrel{\text{def}}{=} \mathscr{F}\{f\} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\nu t} dt$$
$$f(\nu) \stackrel{\text{def}}{=} \mathscr{F}^{-1}\{\hat{f}\} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \hat{f}(\nu)\, e^{i2\pi\nu t} d\nu$$

# References

[Erw11] Erwin Kreyszig. *Advanced Engineering Mathematics*, page 522. Wiley, 10th edition, 2011.

[Wei] Weisstein, Eric W. Fourier transform. `http://mathworld.wolfram.com/FourierTransform.html`.