Normal Distribution

Jiman Hwang

THIS IS FOR SELF-STUDY ONLY.
Source: here

Consider throwing an arrow to the origin. Let $g(r)$ denote the pdf(Probability Density Function) that the arrow hit at a point (r, θ) in polar coordinate, (x, y) in rectangular coordinate. Also let $p(x)$ and $p(y)$ denote the pdf that the arrow hit a point $(x, ?)$ and $(?, y)$ in rectangular coordinate respectively.

Assuming x and y are independent,

$$p(x)p(y) = g(r) \tag{1}$$

The above pdfs don't vary along with $theta$. thereby

$$\begin{aligned}
&\frac{\partial}{\partial\theta}[p(x)p(y)] = \frac{\partial}{\partial\theta}g(r) \\
&\to p(x)\frac{\partial p(y)}{\partial\theta} + \frac{\partial p(x)}{\partial\theta}p(y) = 0 \\
&\to p(x)\frac{dp(y)}{dy}\frac{\partial y}{\partial\theta} + \frac{dp(x)}{dx}\frac{\partial x}{\partial\theta}p(y) = 0
\end{aligned}$$

Using the relationship between two coordinate systems; $x = r\cos\theta, y = r\sin\theta$,

$$\begin{aligned}
&p(x)p'(y)\frac{\partial y}{\partial\theta} + p'(x)\frac{\partial x}{\partial\theta}p(y) = 0 \\
&\to p(x)p'(y)(r\cos\theta) + p'(x)p(y)(-r\sin\theta) = 0 \\
&\to xp(x)p'(y) = yp'(x)p(y) \\
&\to \frac{p'(x)}{xp(x)} = \frac{p'(y)}{yp(y)} = 2k \\
&\to \frac{p'(x)}{p(x)} = 2kx \\
&\to \ln|p(x)| = kx^2 + A \\
&\to p(x) = ce^{kx^2}
\end{aligned}$$

Here, $k < 0$, otherwise, $p(x) \to \pm\infty$ as $x \to \infty$.
Let σ denote the standard deviation of $p(x)$. Then,

$$\begin{aligned}
&\int_{-\infty}^{\infty} x^2 p(x)dx = \sigma^2 \quad \because \text{The mean is 0.} \\
&\to 2\int_0^{\infty} cx^2 e^{kx^2}dx = \sigma^2 \quad \text{(even func.)} \\
&\to \int_0^{\infty} x \cdot 2cxe^{kx^2}dx = \sigma^2 \\
&\to x \cdot \frac{c}{k}e^{kx^2}\Big|_{x=0}^{x=\infty} - \int_0^{\infty}\frac{c}{k}e^{kx^2}dx = \sigma^2 \\
&\to 0 - \frac{1}{2k}\int_{-\infty}^{\infty} p(x)dx = \sigma^2 \\
&\to k = -\frac{1}{2\sigma^2}
\end{aligned}$$

To get the constant c,

$$
\begin{aligned}
&\left(\int_{-\infty}^{\infty} p(x)dx\right)^2 = 1 \\
&\to \left(\int_{-\infty}^{\infty} p(x)dx\right)\left(\int_{-\infty}^{\infty} p(y)dy\right) = 1 \\
&\to \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c^2 \exp\left\{-\frac{x^2+y^2}{2\sigma^2}\right\}dxdy = 1 \\
&\to \int_{0}^{2\pi}\int_{0}^{\infty} c^2 r \exp\left\{-\frac{r^2}{2\sigma^2}\right\}drd\theta = 1 \\
&\to 2\pi c^2(-\sigma^2)\exp\left\{-\frac{r^2}{2\sigma^2}\right\}\Bigg|_{r=0}^{r=\infty} = 1 \\
&\to c = \frac{1}{\sigma\sqrt{2\pi}}
\end{aligned}
$$

Summing up,

$$
p(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{(x-m)^2}{2\sigma^2}\right\} \tag{2}
$$

where m is the mean, and σ is the standard deviation.