Bernoulli Numbers

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Chapter 1

Bernoulli Polynomials

1.1 Definition

Bernoulli polynomials $B_n(x)$ are defined as follow.

$$B_0(x) = 1 \tag{1.1}$$
$$B_n'(x) = nB_{n-1}(x) \quad for \quad n \geq 1 \tag{1.2}$$
$$\int_0^1 B_n(x)dx = 0 \quad for \quad n \geq 1 \tag{1.3}$$

1.2 Features

Note the following two features. One is

$$B_n(0) = B_n(1) \quad for \quad n \geq 0, n \neq 1 \tag{1.4}$$

Proof. Integrating from 0 to 1 along x on both side of (**??**) gives

$$\int_0^1 B_n'(x)dx = \int_0^1 nB_{n-1}(x)dx \tag{1.5}$$

and it becomes

$$B_n(1) - B_n(0) = n\int_0^1 B_{n-1}(x)dx \tag{1.6}$$

From (**??**)

$$B_n(1) - B_n(0) = 0 \quad for \quad n \geq 2 \tag{1.7}$$

(**??**) and (**??**) prove (**??**) □

The second feature is

$$B_n(0) = 0 \quad for \quad n = 3, 5, 7, ... \tag{1.8}$$

Proof. Let

$$\tilde{B}_n(x) = B_n(x + \frac{1}{2}) \quad for \quad n = 0, 1, 2, ... \tag{1.9}$$

then (**??**), (**??**), and (**??**) become

$$\tilde{B}_0(x) = 1 \tag{1.10}$$

$$\tilde{B}'_n(x) = n\tilde{B}_{n-1}(x) \quad for \quad n = 1, 2, 3, ... \tag{1.11}$$

$$\int_{-1/2}^{1/2} \tilde{B}_n(x) = 0 \quad for \quad n = 1, 2, 3, ... \tag{1.12}$$

Thus, $\tilde{B}_0(x)$ is an even function from (**??**).
Meanwhile, if $\tilde{B}_k(x)$ is an even function, then

$$\tilde{B}_k(x) = \tilde{B}_k(-x) \tag{1.13}$$

Multiplying $(k+1)$ on both side hands,

$$(k+1)\tilde{B}_k(x) = (k+1)\tilde{B}_k(-x) \tag{1.14}$$

From (**??**),

$$\tilde{B}'_{k+1}(x) = \tilde{B}'_{k+1}(-x) \tag{1.15}$$

Integrating both sides along with x gives

$$\tilde{B}_{k+1}(x) = -\tilde{B}_{k+1}(-x) + C \tag{1.16}$$

where C is a constant. Integrate both side from $-1/2$ to $1/2$ along with x gives

$$\int_{-1/2}^{1/2} \tilde{B}_{k+1}(x)dx = \int_{-1/2}^{1/2} (-\tilde{B}_{k+1}(-x) + C)dx \tag{1.17}$$

From (**??**),

$$C = 0 \tag{1.18}$$

Now (**??**) becomes

$$\tilde{B}_{k+1}(x) = -\tilde{B}_{k+1}(-x) \tag{1.19}$$

which indicates that $\tilde{B}_{k+1}(x)$ is an odd function.
Similarly, when $\tilde{B}_k(x)$ is an odd, $\tilde{B}_{k+1}(x)$ is even.
From these facts, the followings are true.

$$\tilde{B}_0(x), \tilde{B}_2(x), \tilde{B}_4(x), ... \text{ are even.}$$
$$\tilde{B}_1(x), \tilde{B}_3(x), \tilde{B}_5(x), ... \text{ are odd.}$$

From (**??**),

$$\begin{aligned}\int_{-1/2}^{1/2} \tilde{B}_n(x)dx &= \frac{1}{n+1}\int_{-1/2}^{1/2} \tilde{B}'_{n+1}(x)dx \\ &= \frac{1}{n+1}\left[\tilde{B}_{n+1}\left(\frac{1}{2}\right) - \tilde{B}_{n+1}\left(-\frac{1}{2}\right)\right] \\ &= 0\end{aligned}$$

where $n = 2, 4, 6, ...$. Since $\tilde{B}_{n+1}(x)$ is odd,

$$\tilde{B}_{n+1}\left(\frac{1}{2}\right) = \tilde{B}_{n+1}\left(-\frac{1}{2}\right) = 0 \tag{1.20}$$

□

Chapter 2

Periodic Bernoulli Polynomials

2.1 definition

A Periodic Bernoulli Polynomial $P_n(x)$ is defined as follow.

$$P_n(x) = B_n(x - \lfloor x \rfloor) \tag{2.1}$$

2.2 feature

$$P_n(0) = P_n(1) \quad for \quad n = 0, 1, 2, ... \tag{2.2}$$

Note that $B_1(0) \neq B_1(1)$

Chapter 3

Bernoulli Numbers

3.1 Definition

From these functions $B_n(x)$, Bernoulli Numbers B_n are defined as

$$B_n = B_n(0) \tag{3.1}$$

Sometimes, B_1 has two definitions.

$$B_1 = B_1(0) = -\frac{1}{2}, \quad or \quad B_1 = B_1(1) = \frac{1}{2} \tag{3.2}$$