Determinant of covariance matrix is zero.

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Problem Source: [PN]

1 Statement

Let $\mathbf{X} = [X_1, \cdots, X_n]^T$ denote a random vector, and $\mathbf{C_X}$ its covariance matrix. If $\det \mathbf{C_X} = 0$, then

$$\exists k \; X_k = \sum_{i=1, i\neq k}^{n} b_i X_i + c$$

where $b_i, c \in \mathbb{R}$ are some constants.

2 Derivation

If $\det \mathbf{C_X} = 0$, then column vectors are linearly dependent, i.e.,

$$\exists k \exists (a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n) \in \mathbb{R}^{n-1}$$

such that

$$\begin{bmatrix} \mathrm{Cov}(X_1, X_k) \\ \vdots \\ \mathrm{Cov}(X_n, X_k) \end{bmatrix} = \sum_{i=1, i\neq k}^{n} a_i \begin{bmatrix} \mathrm{Cov}(X_1, X_i) \\ \vdots \\ \mathrm{Cov}(X_n, X_i) \end{bmatrix} \tag{1}$$

The right hand side becomes

$$\sum_{i=1, i\neq k}^{n} a_i \begin{bmatrix} \mathrm{Cov}(X_1, X_i) \\ \vdots \\ \mathrm{Cov}(X_n, X_i) \end{bmatrix} = \sum_{i=1, i\neq k}^{n} \begin{bmatrix} \mathrm{Cov}(X_1, a_i X_i) \\ \vdots \\ \mathrm{Cov}(X_n, a_i X_i) \end{bmatrix} = \begin{bmatrix} \mathrm{Cov}(X_1, \sum_{i=1, i\neq k}^{n} a_i X_i) \\ \vdots \\ \mathrm{Cov}(X_n, \sum_{i=1, i\neq k}^{n} a_i X_i) \end{bmatrix} \tag{2}$$

From (1), (2),

$$\mathrm{Cov}(X_j, X_k) = \mathrm{Cov}(X_j, \sum_{i=1, i\neq k}^{n} a_i X_i) \quad \text{for } j = 1, 2, \cdots, n \tag{3}$$

Let $W = \sum_{i=1, i\neq k}^{n} a_i X_i$, and $\sigma_i^2 = \mathrm{Var}(X_i)$. From (3) for $j = k$,

$$\mathrm{Cov}(X_k, X_k) = \mathrm{Cov}(X_k, W) \rightarrow \; \sigma_k^2 = \mathrm{Cov}(X_k, W) \tag{4}$$

From (3) for $j \neq k$,

$$\begin{aligned} & \mathrm{Cov}(X_i, X_k) = \mathrm{Cov}(X_i, W) \\ \rightarrow & \mathrm{Cov}(a_i X_i, X_k) = \mathrm{Cov}(a_i X_i, W) \\ \rightarrow & \sum_{i=1, i\neq k}^{n} \mathrm{Cov}(a_i X_i, X_k) = \sum_{i=1, i\neq k}^{n} \mathrm{Cov}(a_i X_i, W) \\ \rightarrow & \mathrm{Cov}(W, X_k) = \mathrm{Cov}(W, W) \end{aligned}$$

Therefore,

$$\mathrm{Cov}(W, X_k) = \sigma_W^2 \tag{5}$$

From (4), (5),

$$\sigma_k = \sigma_W \tag{6}$$

From (4), (6),

$$\begin{aligned} & \sigma_k^2 = \mathrm{Cov}(X_k, W) \\ \rightarrow & \mathrm{Cov}(X_k, W) = \sigma_k \sigma_W \\ \rightarrow & \frac{\mathrm{Cov}(X_k, W)}{\sigma_k \sigma_W} = \rho(X_k, W) = 1 \end{aligned}$$

This states

$$X_k = \tilde{b}W + c = \sum_{i=1, i\neq k}^{n} a_i\tilde{b}X_i + c = \sum_{i=1, i\neq k}^{n} b_iX_i + c$$

where $b_i, c \in \mathbb{R}$ are some constants.

References

[PN] Hossein Pishro-Nik. www.probabilitycourse.com. http://www.probabilitycourse.com/chapter6/6_1_5_random_vectors.php.