Minimum Mean Squared Error(MMSE) of an Estimator

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Source: [PN]

1 Statement

Let $\hat{X} = g(Y)$ be an estimator of the random variable X, given that we have observed the random variable Y. The Mean Squared Error(MSE) of this estimator is defined as

$$E[(X - \hat{X})^2] = E[(X - g(Y))^2]. \tag{1}$$

The Minimum Mean Squared Error(MMSE) estimator of X,

$$\hat{X}_M = E[X|Y], \tag{2}$$

has the lowest MSE among all possible estimators.

2 Proof

Let $h(X, \hat{X}) = E[(X - \hat{X})^2]$, then

$$\frac{\partial h}{\partial \hat{X}} = E[-2(X - \hat{X})]$$

If h is minimum at $\hat{X}_M$, then

$$\left.\frac{\partial h}{\partial \hat{X}}\right|_{\hat{X}=\hat{X}_M} = E[-2(X - \hat{X}_M)] = 0 \quad \Rightarrow \quad E[\hat{X}_M] = E[X] \tag{3}$$

Here's my thought. Whenever $\hat{X}_M$ satisfies (3), it is MMSE. Since we have Y from observation, let's **define an MMSE** regarding Y. From (3),

$$E[\hat{X}_M] = E[X] = E[E[X|Y]] \tag{4}$$

If we define $\hat{X}_M = E[X|Y]$, then (4) is true, hence $\hat{X}_M$ becomes an MMSE. I guess $\hat{X}_M = E[X|Y]$ is not the only MMSE.

References

[PN] Hossein Pishro-Nik. Mean Squared Error (MSE). `http://www.probabilitycourse.com/chapter9/9_1_5_mean_squared_error_MSE.php`.