$\sin(nx)$ and $\cos(nx)$ in terms of $\sin(x)$ and $\cos(x)$

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1 Statement

For $m \in \mathbb{Z}$, $\sin(nx)$ and $\cos(nx)$ are expressed as

$$\begin{bmatrix}\sin(nx)\\ \cos(nx)\end{bmatrix} = \begin{bmatrix}\cos(x) & \sin(x)\\ -\sin(x) & \cos(x)\end{bmatrix}^{n-1}\begin{bmatrix}\sin(x)\\ \cos(x)\end{bmatrix}$$

or

$$\sin((2n+1)x) = \sum_{k=0}^{n}\binom{2n+1}{2k+1}(-1)^k \sin^{2k+1}x \cdot \cos^{2n-2k}x$$
$$\cos((2n+1)x) = \sum_{k=0}^{n}\binom{2n+1}{2k}(-1)^k \sin^{2k}x \cdot \cos^{2n-2k+1}x$$

for $n \geq 0$, and

$$\sin(2nx) = \sum_{k=0}^{n-1}\binom{2n}{2k+1}(-1)^k \sin^{2k+1}x \cdot \cos^{2n-2k-1}x$$
$$\cos(2nx) = \sum_{k=0}^{n}\binom{2n}{2k}(-1)^k \sin^{2k}x \cdot \cos^{2n-2k}x$$

for $n \geq 1$.

2 Derivation

2.1 Matrix form

Observe

$$\sin(nx) = \sin((n-1)x + x) = \sin((n-1)x)\cos x + \cos((n-1)x)\sin x$$

Similarly,

$$\cos(nx) = \cos((n-1)x + x) = \cos((n-1)x)\cos x - \sin((n-1)x)\sin x$$

Together,

$$\begin{bmatrix} \sin(nx) \\ \cos(nx) \end{bmatrix} = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix} \begin{bmatrix} \sin((n-1)x) \\ \cos((n-1)x) \end{bmatrix}$$

Using recursive structure,

$$\begin{bmatrix} \sin(nx) \\ \cos(nx) \end{bmatrix} = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}^{n-1} \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix} \tag{1}$$

2.2 General form

In this section, we solve (1). Let

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = X\sin x + I\cos x$$

Observe that

$$X^2 = -I \tag{2}$$

On the other hand, (1) becomes

$$\begin{bmatrix} \sin(nx) \\ \cos(nx) \end{bmatrix} = A^{n-1} \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix} \tag{3}$$

Expanding A^{n-1},

$$A^{n-1} = \sum_{r=0}^{n-1} \binom{n-1}{r} \left(\sin^r x \cdot \cos^{n-r-1} x\right) X^r \tag{4}$$

We have two cases: n is odd, or n is even. Let's take a look at them one by one.

2.2.1 n is odd

Let $n = 2m+1$ where $m = 0, 1, \cdots$, then (4) becomes

$$\begin{aligned} A^{2m} = &\sum_{k=0}^{m} \binom{2m}{2k} (-1)^k \left(\sin^{2k} x \cdot \cos^{2m-2k} x\right) I \\ &+ \sum_{k=1}^{m} \binom{2m}{2k-1} (-1)^{k-1} \left(\sin^{2k-1} x \cdot \cos^{2m-2k+1} x\right) X \end{aligned} \tag{5}$$

by separating odd and even terms and using (2), or

$$\begin{aligned} A^{2m} = &\sum_{k=0}^{m} \binom{2m}{2k} (-1)^k \left(\sin^{2k} x \cdot \cos^{2m-2k} x\right) I \\ &+ \sum_{k=0}^{m-1} \binom{2m}{2k+1} (-1)^k \left(\sin^{2k+1} x \cdot \cos^{2m-2k-1} x\right) X \end{aligned} \tag{6}$$

by alternatively representing the second summation. Using (6), $\sin(nx)$ in (3) becomes

$$\begin{aligned}\sin((2m+1)x) &= \sum_{k=0}^{m}\binom{2m}{2k}(-1)^k\left(\sin^{2k+1}x\cdot\cos^{2m-2k}x\right)\\ &\quad+\sum_{k=0}^{m-1}\binom{2m}{2k+1}(-1)^k\left(\sin^{2k+1}x\cdot\cos^{2m-2k}x\right)\\ &= \sum_{k=0}^{m}\binom{2m+1}{2k+1}(-1)^k\left(\sin^{2k+1}x\cdot\cos^{2m-2k}x\right)\end{aligned}$$

Similarly, using (5), $\cos(nx)$ in (3) becomes,

$$\cos((2m+1)x) = \sum_{k=0}^{m}\binom{2m+1}{2k}(-1)^k\sin^{2k}x\cdot\cos^{2m-2k+1}x$$

2.2.2 *n* is even

This time, $n = 2m$ where $m = 1, 2, \cdots$. All procedures are similar with the above case. Then (4) becomes

$$\begin{aligned}A^{2m-1} &= \sum_{k=0}^{m-1}\binom{2m-1}{2k}(-1)^k\left(\sin^{2k}x\cdot\cos^{2m-2k-1}x\right)I\\ &\quad+\sum_{k=1}^{m}\binom{2m-1}{2k-1}(-1)^{k-1}\left(\sin^{2k-1}x\cdot\cos^{2m-2k}x\right)X\end{aligned} \tag{7}$$

or

$$\begin{aligned}A^{2m-1} &= \sum_{k=0}^{m-1}\binom{2m-1}{2k}(-1)^k\left(\sin^{2k}x\cdot\cos^{2m-2k-1}x\right)I\\ &\quad+\sum_{k=0}^{m-1}\binom{2m-1}{2k+1}(-1)^k\left(\sin^{2k+1}x\cdot\cos^{2m-2k-2}x\right)X\end{aligned} \tag{8}$$

Applying the above equations result for $\sin(2mx)$ and $\cos(2mx)$.