Central Limit Theorem(CLT)

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Source: [PNb], [PNa]

1 Statement

Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables with expected value $EX_i = \mu < \infty$ and variance $0 < \mathrm{Var}(X_i) = \sigma^2 < \infty$. Then, the random variable

$$Z_n = \frac{\overline{X} - E\overline{X}}{\sigma_{\overline{X}}} = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \quad \text{where } \overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

converges in distribution to the standard normal random variable as n goes to infinity, that is

$$\lim_{n\to\infty} Z_n \sim N(0,1)$$

Also, the sample mean $\overline{X}$ is approximately

$$\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

for $n \geq 30$.

2 Derivation

Characteristic function identifies probability distribution, therefore, we will take advantage of this property.

2.1 Characteristic function of Standard Normal Distribution

Let

$$Z \sim N(0,1)$$

Then, its characteristic function $\phi_Z(\omega)$ is

$$\begin{aligned}
\phi_Z(\omega) = E[e^{j\omega Z}] &= \int_{-\infty}^{\infty} e^{j\omega z} f_Z(z) dz \\
&= \int_{-\infty}^{\infty} e^{j\omega z} \frac{1}{2\pi} \exp\left\{ -\frac{z^2}{2} \right\} dz \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left\{ -\frac{z^2 - j2\omega z}{2} \right\} dz \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left\{ -\frac{(z-j\omega)^2 - (j\omega)^2}{2} \right\} dz \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left\{ -\frac{(z-j\omega)^2}{2} \right\} \exp\left\{ \frac{(j\omega)^2}{2} \right\} dz \\
&= \exp\left\{ -\frac{\omega^2}{2} \right\} \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left\{ -\frac{(z-j\omega)^2}{2} \right\} dz
\end{aligned}$$

Since an integral of a PDF(normal in this time) over $\mathbb{R}$ is 1

$$\phi_Z(\omega) = \exp\left\{ -\frac{\omega^2}{2} \right\} \tag{1}$$

2.2 Characteristic function of normalized sample mean

Since Z_n is normalized $\overline{X}$,

$$EZ_n = 0,\ \mathrm{Var}(Z_n) = 1$$

Getting characteristic function of Z_n,

$$\begin{aligned}\phi_{Z_n}(\omega) =& E[e^{j\omega Z_n}] \\ =& E\Bigg[\exp\left\{j\omega \frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right\}\Bigg] \\ =& E\Bigg[\exp\left\{j\omega \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}}\right\}\Bigg] \\ =& E\Bigg[\exp\left\{j\frac{\omega}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i-\mu}{\sigma}\right\}\Bigg]\end{aligned}$$

Let $Y_i = (X_i - \mu)/\sigma$, then

$$EY_i = 0, \mathrm{Var}(Y_i) = 1 \tag{2}$$

also

$$\begin{aligned}\phi_{Z_n}(\omega) =& E\Bigg[\exp\left\{j\frac{\omega}{\sqrt{n}}\sum_{i=1}^{n} Y_i\right\}\Bigg] \\ =& E\Bigg[\prod_{i=1}^{n}\exp\left\{j\frac{\omega}{\sqrt{n}}Y_i\right\}\Bigg] \\ =& \prod_{i=1}^{n} E\Bigg[\exp\left\{j\frac{\omega}{\sqrt{n}}Y_i\right\}\Bigg] && \text{since } X_i\text{'s are independent} \\ =& \left(E\Bigg[\exp\left\{j\frac{\omega}{\sqrt{n}}Y_i\right\}\Bigg]\right)^n && \text{since } X_i\text{'s are i.i.d.} \\ =& \left[\phi_{Y_i}\left(\frac{\omega}{\sqrt{n}}\right)\right]^n\end{aligned}$$

Let $a_n = [\phi_{Y_i}(\frac{\omega}{\sqrt{n}})]^n$, then

$$\ln a_n = n \ln \phi_{Y_i}\left(\frac{\omega}{\sqrt{n}}\right)$$

Let $p = \frac{1}{\sqrt{n}}$, then

$$\ln a_n = \frac{\ln \phi_{Y_i}(\omega p)}{p^2}$$

When $n \to \infty$, $p \to 0$ thus

$$\lim_{n\to\infty} \ln a_n = \lim_{p\to 0} \frac{\ln \phi_{Y_i}(\omega p)}{p^2} \tag{3}$$

Look at the numerator and denominator separately. When $p \to 0$, the denominator p^2 becomes 0, while the numerator becomes

$$\lim_{p\to 0} \ln \phi_{Y_i}(\omega p) = \lim_{p\to 0} \ln \int_{-\infty}^{\infty} e^{j\omega p y} f_{Y_i}(y)dy = \ln \int_{-\infty}^{\infty} f_{Y_i}(y)dy = \ln 1 = 0 \tag{4}$$

Since both numerator and denominator are 0, L'Hpital's rule is available on (3).

$$\lim_{n\to\infty} \ln a_n = \lim_{p\to 0} \frac{\ln \phi_{Y_i}(\omega p)}{p^2} = \lim_{p\to 0} \frac{1}{2p}\frac{\omega\phi'_{Y_i}(\omega p)}{\phi_{Y_i}(\omega p)} \tag{5}$$

Again, check the numerator and denominator. The denominator is

$$\lim_{p\to 0} 2p\phi_{Y_i}(\omega p) = 0$$

from (4). The numerator is

$$\lim_{p\to 0} \omega\phi'_{Y_i}(\omega p) = \lim_{p\to 0} \omega \int_{-\infty}^{\infty} (jy)e^{j\omega p y} f_{Y_i}(y)dy = \omega \int_{-\infty}^{\infty} (jy) f_{Y_i}(y)dy = 0$$

since $EY = 0$ from (2).

Both numerator and denominator becomes 0, so apply L'Hpital's rule again on (5).

$$\begin{aligned}
\lim_{n\to\infty} \ln a_n &= \lim_{p\to 0} \frac{1}{2p} \frac{\omega \phi'_{Y_i}(\omega p)}{\phi_{Y_i}(\omega p)} = \lim_{p\to 0} \frac{\omega^2 \phi''_{Y_i}(\omega p)}{2\phi_{Y_i}(\omega p) + 2\omega p \phi'_{Y_i}(\omega p)} \\
&= \frac{\omega^2 \phi''_{Y_i}(0)}{2} \\
&= \frac{\omega^2}{2} \left[\int_{-\infty}^{\infty} (jy)^2 e^{j\omega p y} f_{Y_i}(y) dy \right]_{p=0} \\
&= \frac{\omega^2}{2} \int_{-\infty}^{\infty} -y^2 f_{Y_i}(y) dy \\
&= -\frac{\omega^2}{2} && \text{from (2)}
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \phi_{Z_n}(\omega) = \exp\left\{ -\frac{\omega^2}{2} \right\}$$

From (1),

$$\lim_{n\to\infty} Z_n \sim N(0, 1)$$

Because $\overline{X} = \sigma_{\overline{X}} Z_n + \mu = \frac{\sigma}{\sqrt{n}} Z_n + \mu$, approximately

$$\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

for large n.

References

[PNa] Hossein Pishro-Nik. Definition. http://www.probabilitycourse.com/chapter7/7_1_2_central_limit_theorem.php.

[PNb] Hossein Pishro-Nik. Problem 9. http://www.probabilitycourse.com/chapter6/6_1_6_solved_probs.php.