# Wallis Product

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January 4, 2015

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## 1 Wallis Product

### 1.1 Definition

$$\prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\frac{2n}{2n+1}\right)=\frac{\pi}{2} \tag{1}$$

### 1.2 Proof

Let

$$a_n=\int_0^{\pi}\sin^n xdx \tag{2}$$

where $n = 0, 1, 2, ...$, and resolving it,

$$\begin{aligned}
a_n &= \int_0^{\pi}\sin^n xdx \\
&= \int_0^{\pi}\sin^{n-1}\sin xdx \\
&= \sin^{n-1}(-\cos x)\Big|_{x=0}^{x=\pi} - \int_0^{\pi}(n-1)\sin^{n-2}(-\cos^2 x)dx \\
&= \int_0^{\pi}(n-1)\sin^{n-2}\cos^2 xdx \\
&= (n-1)\int_0^{\pi}\sin^{n-2}dx-(n-1)\int_0^{\pi}\sin^n dx \\
&= (n-1)a_{n-2}-(n-1)a_n
\end{aligned}$$

Rearranging it,

$$a_n=\frac{n-1}{n}a_{n-2} \tag{3}$$

Replacing $n$ with $2n$ and $2n+1$ respectively,

$$a_{2n}=\frac{2n-1}{2n}a_{2n-2}=\frac{2n-1}{2n}\times\frac{2n-3}{2n-2}\times\frac{1}{2}\times a_0 \tag{4}$$

$$a_{2n+1}=\frac{2n}{2n+1}a_{2n-1}=\frac{2n}{2n+1}\times\frac{2n-2}{2n-1}\times\frac{2}{3}\times a_1 \tag{5}$$

$a_0$ and $a_1$ are get easily.

$$a_0 = \int_0^\pi dx = \pi, \quad a_1 = \int_0^\pi \sin x dx = 2 \tag{6}$$

Plugging (6) into (4) and (5),

$$a_{2n} = \pi \prod_{k=1}^{n} \frac{2k-1}{2k} \tag{7}$$

$$a_{2n+1} = 2 \prod_{k=1}^{n} \frac{2k}{2k+1} \tag{8}$$

where $n = 0, 1, 2, \ldots$ .

Wallis product appears using (7) and (8).

$$\lim_{n\to\infty} \frac{a_{2n+1}}{a_{2n}} = \frac{2}{\pi} \prod_{k=1}^{\infty} \frac{2k}{2k-1} \frac{2k}{2k+1} \tag{9}$$

Meanwhile, since $0 \le \sin x \le 1$ is hold for $0 \le x \le \pi$,

$$\begin{aligned}
&\sin^{2n+1} x \le \sin^{2n} x \le \sin^{2n-1} x \quad for \quad 0 \le x \le \pi \\
&\Rightarrow a_{2n+1} \le a_{2n} \le a_{2n-1} \\
&\Rightarrow 1 \le \frac{a_{2n}}{a_{2n+1}} \le \frac{a_{2n-1}}{a_{2n+1}} = \frac{2n+1}{2n} \\
&\Rightarrow \lim_{n\to\infty} 1 \le \lim_{n\to\infty} \frac{a_{2n}}{a_{2n+1}} \le \lim_{n\to\infty} \frac{2n+1}{2n} = 1 \\
&\Rightarrow \lim_{n\to\infty} \frac{a_{2n+1}}{a_{2n}} = 1
\end{aligned}$$

$$\therefore \prod_{k=1}^{\infty} \frac{2k}{2k-1} \frac{2k}{2k+1} = \frac{\pi}{2}$$