Young's inequality and Hölder's Inequality for probability

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Source: [wik], [PN]

1 Statement

1.1 Young's inequality

For $\alpha, \beta, p, q \in \mathbb{R}$ with $\alpha, \beta \geq 0, p, q > 0$, the following is true.

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$

1.2 Hölder's Inequality for probability

For random variables X, Y, and $p, q \in \mathbb{R}$ with $p, q > 0$, the following is true.

$$E\left[|XY|\right] \leq E\Big[|X|^p\Big]^{\frac{1}{p}} E\Big[|Y|^q\Big]^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

2 Derivation

2.1 Young's inequality

Since $f(x) = \ln x$ is a concave function, for $0 < a \leq b$,

$$t \ln a + (1-t) \ln b \leq \ln(ta + (1-t)b) \tag{1}$$

where $0 \leq t \leq 1$. The left hand side becomes

$$t \ln a + (1-t) \ln b = \ln a^t b^{1-t}$$

Then (1) is

$$a^t b^{1-t} \leq ta + (1-t)b$$

Here, let $a^t = \alpha, b^{1-t} = \beta$, then

$$\alpha\beta \leq t\alpha^{1/t} + (1-t)\beta^{1/(1-t)}$$

Let $1/t = p, 1/(1-t) = q$, then

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q} \tag{2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Equality holds iff

$$a = b \Leftrightarrow \alpha^p = \beta^q$$

Moreover, (2) is held even when $\alpha\beta = 0$.

2.2 Hölder's Inequality for probability

Put $\alpha = U, \beta = V$ and apply expectation on (2), then

$$E|UV| \leq \frac{E|U|^p}{p} + \frac{E|V|^q}{q}.$$

Choose $U = \frac{|X|}{(E|X|^p)^{\frac{1}{p}}}$, $V = \frac{|Y|}{(E|Y|^q)^{\frac{1}{q}}}$. Then

$$\frac{E|XY|}{(E|X|^p)^{\frac{1}{p}} (E|Y|^q)^{\frac{1}{q}}} \leq \frac{E|X|^p}{pE|X|^p} + \frac{E|Y|^q}{qE|Y|^q} = \frac{1}{p} + \frac{1}{q} = 1$$

Hence,

$$E|XY| = (E|X|^p)^{\frac{1}{p}} (E|Y|^q)^{\frac{1}{q}}$$

References

[PN] Hossein Pishro-Nik. problem 7. http://www.probabilitycourse.com/chapter6/6_2_6_solved6_2.php.

[wik] wiki. Young's inequality. https://en.wikipedia.org/wiki/Young%27s_inequality#Standard_version_for_conjugate_H.C3.B6lder_exponents.