Taylor Theorem

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1 Statement

Suppose $f^{(n+1)}$ exists on $[a, b]$, and $x_0 \in [a, b]$. For every $x \in [a, b]$ there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = P_n(x) + R_n(x)$$

where

$$\begin{aligned}
P_n(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\
R_n(x) &= \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}
\end{aligned}$$

2 Derivation

Given x_0 and $f(x_0)$, diffierentiable n+1 times, we will approximate $f(x)$ to a polynomial of order n by putting formula

$$f(x) = P(x) + R(x) \tag{1}$$

where $P(x)$ is a polynomial term, and $R(x)$ an error term, which satisfie

$$\begin{aligned}
P(x) =& c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots + c_n(x - x_0)^n \\
\lim_{x \to x_0} R(x) =& 0
\end{aligned}$$

Since the polynomial part has order n, we assume that the value $f^{(i)}(x)$ at $x = x_0$ has no error, where $i = 0, 1, \cdots, n$. That is,

$$R^{(i)}(x_0) = 0 \quad \text{for } i = 0, 1, \cdots, n \tag{2}$$

Differentiating both two hands side of (1) i times, and using (2),

$$f^{(i)}(x_0) = c_i(i!) \quad \text{for } i = 0, 1, \cdots, n$$

or

$$c_i = \frac{f^{(i)}(x_0)}{i!} \quad \text{for } i = 0, 1, \cdots, n \tag{3}$$

Meanwhile, let

$$F(t) = f(t) + \frac{f^{(1)}(t)}{1!}(x - t) + \frac{f^{(2)}(t)}{2!}(x - t)^2 + \cdots + \frac{f^{(n)}(t)}{n!}(x - t)^n \tag{4}$$

Differentiating with t,

$$\begin{aligned}
F'(t) =& f'(t) + \left[-\frac{f^{(1)}(t)}{0!} + \frac{f^{(2)}(t)}{1!}(x - t) \right] + \left[-\frac{f^{(2)}(t)}{1!}(x - t) + \frac{f^{(3)}(t)}{2!}(x - t)^2 \right] + \cdots \\
&+ \left[-\frac{f^{(n)}(t)}{(n-1)!}(x - t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x - t)^n \right]
\end{aligned}$$

or

$$F'(t) = \frac{f^{(n+1)}(t)}{n!}(x - t)^n \tag{5}$$

Also, by Cauchy's mean value theorem, there's a number $\xi(x)$ between x and x_0 such that

$$\frac{F'(\xi(x))}{G'(\xi(x))} = \frac{F(x) - F(x_0)}{G(x) - G(x_0)}$$

where G is an arbitrary function, differentiable between x and x_0.

Plugging (4) and (5) into this,

$$\frac{f^{(n+1)}(\xi(x))}{n!}(x-\xi(x))^n\frac{1}{G'(\xi(x))}=\frac{f(x)-P(x)}{G(x)-G(x_0)}$$

Using (1) and rearranging,

$$R(x)=\frac{f^{(n+1)}(\xi(x))}{n!}(x-\xi(x))^n\frac{G(x)-G(x_0)}{G'(\xi(x))} \tag{6}$$

Let

$$G(t)=(x-t)^{n+1} \tag{7}$$

(6) and (7) get the derivation finished.

$$R(x)=\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}$$