Euler-Maclaurin formula

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Chapter 1

Euler-Maclaurin formula

1.1 Definition

Let n and m are integers, function $f(x)$ is analytic from m to n,and I is the following.

$$I = \int_m^n f(x)dx \tag{1.1}$$

and its approximation S is

$$S = \sum_{k=m+1}^{n} f(k) \tag{1.2}$$

The following Euler-Maclaurin formula describes the relationship between S and I.

$$S - I = \sum_{k=1}^{s} (-1)^k \frac{B_k}{k!} [f^{(k-1)}(b) - f^{(k-1)}(a)] + R_m \tag{1.3}$$

where $s = 1, 2, 3, ...$, B_k is kth Bernoulli Number(but, $B_1 = -1/2$), and

$$R_s = \int_m^n (-1)^{s+1} \frac{P_s(x)}{s!} f^{(s)}(x)dx \tag{1.4}$$

where $P_s(x)$ is sth Periodic Bernoulli Polynomial.

1.2 Proof

Let k is an integer and consider the following integral.

$$\int_k^{k+1} f(x)dx \tag{1.5}$$

For x that varies from k to $k+1$,

$$P_1'(x) = P_0(x) = 1 \tag{1.6}$$

Rewriting (**??**), and processing,

$$\begin{aligned}\int_k^{k+1} f(x)dx = \int_k^{k+1} P_1'(x)f(x)dx &= [P_1(x)f(x)]_k^{k+1} - \int_k^{k+1} P_1(x)f'(x)dx \\ &= \lim_{x\to(k+1)-} P_1(x)f(x) - \lim_{x\to k+} P_1(x)f(x) - \int_k^{k+1} P_1(x)f'(x)dx \\ &= \frac{1}{2}[f(k)+f(k+1)] - \int_k^{k+1} P_1(x)f'(x)dx\end{aligned}$$

Summing for $k = m, m+1, ..., n-1$ on both sides,

$$\begin{aligned}\int_m^n f(x)dx &= \frac{1}{2}f(m) + f(m+1) + f(m+2) + ... + f(n-1) + \frac{1}{2}f(n) \\ &\quad - \int_m^n P_1(x)f'(x)dx \\ &= \sum_{k=m+1}^{n} f(k) + \frac{1}{2}f(m) - \frac{1}{2}f(n) - \int_m^n P_1(x)f'(x)dx\end{aligned}$$

Rewriting using S and I, and rearranging,

$$S - I = \frac{1}{2}[f(n) - f(m)] + \int_m^n P_1(x)f'(x)dx \tag{1.7}$$

The last term will be resolved recursively. In order to do that, let's consider the following term.

$$\int_m^n \frac{P_t(x)}{t!} f^{(t)}(x)dx \tag{1.8}$$

where t is a natural number. To resolve (**??**), let k be an integer,

$$\begin{aligned}\int_k^{k+1} \frac{P_v(x)}{v!} f^{(v)}(x)dx &= \int_k^{k+1} \frac{P_{v+1}'(x)}{(v+1)!} f^{(v)}(x)dx \\ &= \left[\frac{P_{v+1}(x)}{(v+1)!} f^{(v)}(x)\right]_{x=k}^{x=k+1} - \int_k^{k+1} \frac{P_{v+1}(x)}{(v+1)!} f^{(v+1)}(x)dx \\ &= \frac{B_{v+1}}{(v+1)!}[f^{(v)}(k+1) - f^{(v)}(k)] - \int_k^{k+1} \frac{P_{v+1}(x)}{(v+1)!} f^{(v+1)}(x)dx\end{aligned}$$

Summing for $k = m, m+1, ..., n-1$,

$$\int_m^n \frac{P_v(x)}{v!} f^{(v)}(x)dx = \frac{B_{v+1}}{(v+1)!}[f^{(v)}(n) - f^{(v)}(m)] - \int_m^n \frac{P_{v+1}(x)}{(v+1)!} f^{(v+1)}(x)dx \tag{1.9}$$

Using (**??**), the last term in (**??**) is resolved recursively

$$\begin{aligned}S - I &= \frac{1}{2}[f(n) - f(m)] + \frac{B_2}{2!}[f^{(1)}(n) - f^{(1)}(m)] - \frac{B_3}{3!}[f^{(2)}(n) - f^{(2)}(m)] \\ &\quad + -... + (-1)^s \frac{B_s}{s!}[f^{(s-1)}(n) - f^{(s-1)}(m)] \\ &\quad + \int_m^n (-1)^{s+1} \frac{P_s}{s!} f^{(s)}(x)dx \\ &= \sum_{k=1}^{s} (-1)^k \frac{B_k}{k!}[f^{(k-1)}(n) - f^{(k-1)}(m)] + \int_m^n (-1)^{s+1} \frac{P_s}{s!} f^{(s)}(x)dx\end{aligned}$$

where $m = 1, 2, 3, ...$