Number of nodes of given height in a complete k-ary tree

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1 Statement

Given a complete k-ary tree T consisting of n nodes, let x_i denote the number of nodes of height i in T. Then,

$$x_0 = \left\lfloor \frac{k-1}{k} n + \frac{1}{k} \right\rfloor$$
$$x_i = \left\lfloor \frac{k-1}{k} (n - x_0 - x_1 - \cdots - x_{i-1}) + \frac{1}{k} \right\rfloor \quad \text{for } i = 1, 2, \cdots, (\text{height of } T)$$

2 Derivation

2.1 Main idea

1. Get the number of leaves by means of counting the number of edges in two different ways.
2. Get the number of nodes of height i using the fact that it still remains a complete k-ary tree after extracting all leaves.

2.2 Number of leaves

Given a complete k-ary tree T consisting of n nodes, the number of edges, e, is

$$e = n - 1 \tag{1}$$

as each node is connected to its parent but the root.

On the other hand, e is expressed in an another way. Let x_i denote the number of nodes of height i in T. Among $(n - x_0)$ internal nodes, at most one has less than k children and the others have k children respectively. This is true because if more than one have less than k children respectively, then T is not a complete tree. Therefore,

$$k(n - x_0 - 1) + 0 < e \leq k(n - x_0 - 1) + k$$

From (1),

$$k(n-x_0-1) < n-1 \leq k(n-x_0)$$

Solving for x_0,

$$\frac{k-1}{k}n+\frac{1}{k}-1 < x_0 \leq \frac{k-1}{k}n+\frac{1}{k}$$

or

$$x_0 = \left\lfloor \frac{k-1}{k}n+\frac{1}{k} \right\rfloor \tag{2}$$

2.3 Number of nodes of height i

We begin with a lemma.

$$\boxed{\text{Let } T' \text{ denote the graph } T \text{ except the leaves and edges connected to them. Then } T' \text{ is a complete } k\text{-ary tree.}} \tag{3}$$

Proof. Consider building a complete k-ary tree from empty one. We follow one of these rules at each insertion.

(i) If the last level is full, put the new node into the leftmost position in the new level.

(ii) Otherwise, put the new node on the right of the lastly added one.

At each step, the tree is complete. Now suppose we got T in this manner. Note that the leaves are inserted lastly. Since every tree is complete during the build process, withdrawing nodes in reverse order also gives a complete k-ary tree. □

Taking T' and T in (3), we contend an another lemma.

$$\boxed{\text{If } L' \text{ is the set of leaves in } T' \text{ and } H_1 \text{ is the set of nodes in } T \text{ of height 1, then } L' = H_1.} \tag{4}$$

Proof. Let L denote the leaves in T as shown in Fig 1.

$\forall\, u \in L'$ is not a leaf in T, so (height of u in T) ≥ 1. Furthermore, if (height of u in T) $>$ 1, then $\exists\, v \in L$ of height 1, contradiction. Hence (height of u in T) $= 1$, and

$$L' \subset H_1 \tag{5}$$

Meanwhile, in order to get T' from T, we extract all nodes of height 0 in T. As $\forall\, u \in H_1$ has a child or children of height 0, u becomes a leaf in T'. Therefore,

$$H_1 \subset L' \tag{6}$$

From (5) and (6), $L' = H_1'$ □

Based on (3) and (4), we get x_1 by counting the number of leaves in T'. In (2), replace x_0 with x_1, n with $n-x_0$

$$x_1 = \left\lfloor \frac{k-1}{k}(n-x_0)+\frac{1}{k} \right\rfloor$$

In the similar way, the general form is given as

$$x_i = \left\lfloor \frac{k-1}{k}(n-x_0-x_1-\cdots-x_{i-1})+\frac{1}{k} \right\rfloor \quad \text{for } i = 1, 2, \cdots, (\text{height of } T)$$

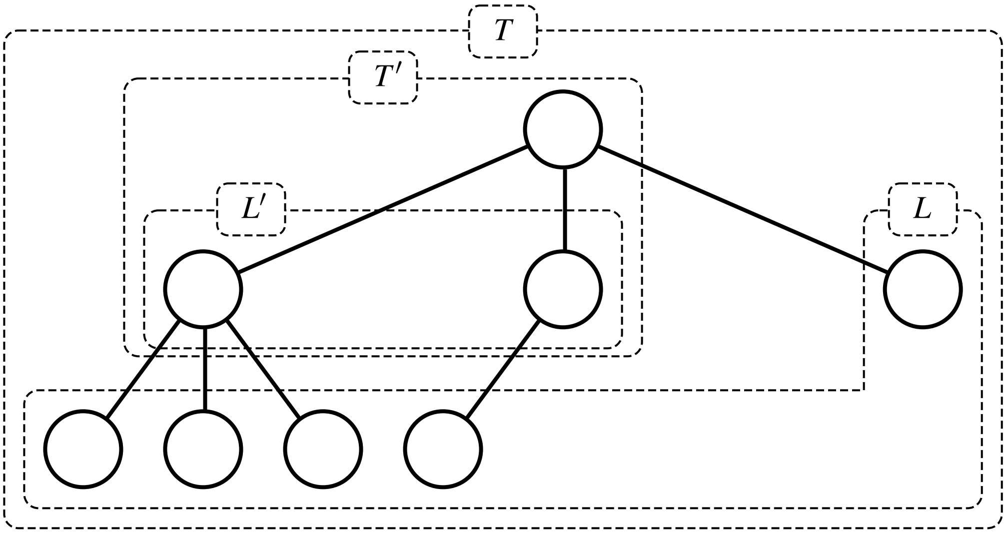

Figure 1: an example complete 3-ary tree