Derivation of wave equation

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Disclaimer: This document is for self-study only and may contain false information.
Inspired from [Kis]

1 Introduction

We consider a simple wave that propagates in one direction with constant velocity v.

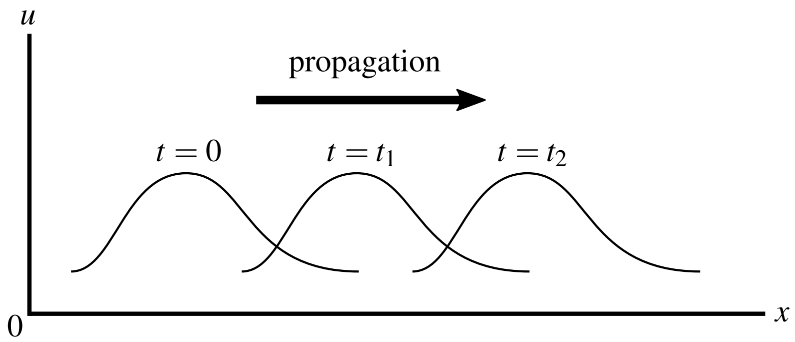

Figure 1: Propagation of wave

This wave is described by function $u : (\mathbb{R},\mathbb{R}) \to \mathbb{R}$ such that

$$\forall x,t \quad |u(x,t)| < \infty, \quad u(x,t) = u(x - vt, 0) \tag{1}$$

Define the initial image

$$h(x) \stackrel{\text{def}}{=} u(x,0) = u(x + vt, t)$$

2 First order differential equation

To get an explicit form of u, differentiate (1) with respect to x and t respectively.

$$\begin{aligned}\text{with } x : \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x} u(x - vt, 0) = h'(x - vt) \\ \text{with } t : \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} u(x - vt, 0) = -vh'(x - vt)\end{aligned}$$

Combining them,

$$\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x}$$

Assuming $u(x,t) = f(x)\, g(t)$, the above equation becomes

$$\begin{aligned} f(x)\, g'(t) &= -v f'(x)\, g(t) \\ \Rightarrow -v \frac{f'(x)}{f(x)} &= \frac{g'(t)}{g(t)} = (\text{const}) \stackrel{\text{def}}{=} k \end{aligned}$$

Separating variables,

$$f'(x) = -\frac{k}{v} f(x), \quad g'(t) = kg(t) \quad \text{for some constants } c_1, c_2 \in \mathbb{C}$$

Hence,

$$u(x,t) = ce^{k(t-x/v)} \quad \text{for some } c \in \mathbb{C}$$

However, this can't be justified because if $k \in \mathbb{R}$, then $u(x,0) \to \pm\infty$ as $x \to \pm\infty$. Also if $k \in \mathbb{C}, k \notin \mathbb{R}$, then $\exists x_0\; u(x_0,0) \notin \mathbb{R}$. Both cases are contradictions.

3 Wave equation

To obtain an explicit form of u, differentiate (1) with respect to x and t twice respectively.

$$\begin{aligned} \text{with } x: \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2}{\partial x^2} u(x - vt, 0) = h''(x - vt) \\ \text{with } t: \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2}{\partial t^2} u(x - vt, 0) = v^2 h''(x - vt) \end{aligned}$$

Combining them,

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

Check out the solution at Sec 12.3 in [Kre].

4 Thought: Extension

Extending to N dimensions gives

$$\frac{\partial^2 u}{\partial t^2} = \sum_{j=1}^{N} \sum_{k=1}^{N} v_j v_k \frac{\partial^2 u}{\partial x_j \partial x_k}$$

But they say the answer for three dimensions [Rif] is

$$\frac{\partial^2 u}{\partial t^2} = v^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right)$$

To deduce it,

$$\forall j \neq k \quad \frac{\partial^2 u}{\partial x_j \partial x_k} = 0$$

must be true, which seems impossible for general case. From where I got wrong?

References

[Kis] Joe Kiskis. Wave equation. `http://kiskis.physics.ucdavis.edu/landau/phy9hc_03/wave.pdf`.

[Kre] Erwin Kreyszig. *Advanced Engineering Mathematics*. The name of the publisher, 10 edition.

[Rif] D. M. Riffe. 3d wave equation and plane waves / 3d differential operators. `http://www.physics.usu.edu/riffe/3750/Lecture%2018.pdf`.