

The potential of a large dust grain in a collisionless plasma

Dogan Akpinar and George E. B. Doran

August 7, 2020

Abstract

1 Introduction

2 Background

In the case of a large dust grain with a negative equilibrium charge, we can establish regions within an infinite plasma with well-defined transitions. We firstly have the dust grain itself, followed by a positive region of space, called the sheath, usually a few electron Debye lengths in size. The electron Debye length is a characteristic length over which quasi-neutrality breaks down, defined as

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T_e}{n_0 e^2}}, \quad (1)$$

where k_B is the Boltzmann constant, T_e is the electron temperature, e is the electron charge, n_0 is the electron number density at infinity and ε_0 is the permittivity of free space. Following the sheath, we have the infinite pre-sheath where quasi-neutrality holds; quasi-neutrality is mathematically written as an approximate equality between the ion and electron densities, $Zn_i \approx n_e$.

Positive ions are continuously collected by the negative dust, so there must be a net influx of ions into the sheath to maintain the equilibrium. This established the Bohm criterion, where the speed of ions required to enter the sheath must be greater than or equal to the Bohm speed. For the cold ion case, the Bohm speed is defined as

$$c_s^{cold} = \sqrt{\frac{k_B T_e}{m_i}}, \quad (2)$$

where m_i is the ion mass and c_s^{cold} is known as the cold ion Bohm speed. However, if we consider ions with a finite temperature, the required Bohm speed becomes

$$c_s^{hot} = \sqrt{\frac{k_B(T_e + \gamma T_i)}{m_i}}, \quad (3)$$

where T_i is the ion temperature, γ is the adiabatic index and c_s^{hot} is known as the hot ion Bohm speed [1] [2].

For a large dust grain, we may consider the planar sheath limit [2]. Hence, the potential drop across the sheath for $T_i \neq 0$ is given as

$$\phi_s = \frac{k_B T_e}{2e} \ln \left[\frac{2\pi Z^2}{\mu^2} (1 + \gamma \Theta) \right], \quad (4)$$

where Z is the relative ion charge, $\mu = \sqrt{\frac{m_i}{m_e}}$ and $\Theta = \frac{T_i}{T_e}$ [1].

3 Radial motion theory (ABR)

The ABR model is a radial motion theory derived by Allen, Boyd and Reynolds. It describes the equilibrium surface potential acquired by a dust grain immersed in an infinite and stationary plasma [3].

Consider a spherical dust grain, of arbitrary radius a , immersed in this infinite plasma. Far from the surface we assume that the electron and ion densities are equal, denoted n_e and n_i respectively; this is known as quasi-neutrality. As electrons are faster than ions, it can be shown that such a dust grain will become negatively charged [4], thus ions will experience an attractive force due to the potential on the dust surface, ϕ_a . We assume that ions at infinity have no kinetic energy, hence, they move radially towards the dust grain. Therefore, it is appropriate to say that an ion at a distance r from the dust center has radial speed v_i . Using energy conservation, one can show the following,

$$\frac{1}{2} m_i v_i^2 = -Ze\phi(r), \quad (5)$$

where $\phi(r)$ is the potential at r , which vanishes as $r \rightarrow \infty$ [3].

Equation (5) then leads to an expression for the ion current, which is entirely dependant on the radial distance from the dust grain, given by

$$I_i = \frac{4\sqrt{2} n_i \pi r^2 Z^{\frac{3}{2}} e^{\frac{3}{2}} \phi_a^{\frac{1}{2}}}{m_i^{\frac{1}{2}}}. \quad (6)$$

As the potential is negative, few electrons reach the dust grain, hence, the electron density obeys a Boltzmann distribution:

$$n_e(r) = n_0 \exp\left(\frac{e\phi(r)}{k_B T_e}\right), \quad (7)$$

we further assume that only inbound electrons contribute to the electron current at the surface of the dust grain, given as

$$I_i = I_e = 4\pi a^2 n_0 e \sqrt{\frac{k_B T_e}{2\pi m_e}} \exp\left(\frac{e\phi_a}{k_B T_e}\right). \quad (8)$$

where m_e is the electron mass [3].

It is useful to apply the following normalisations, noting that Φ is the opposite sign for simplicity:

$$\Phi = -\frac{e\phi}{k_B T_e}, \quad \rho = \frac{r}{\lambda_D}, \quad \alpha = \frac{a}{\lambda_D}, \quad J = \frac{I_i}{4\pi \lambda_D^2 n_0 e \sqrt{\frac{2k_B T_e}{m_i}}}, \quad (9)$$

where λ_D is the electron Debye length,

Poisson's law allows for the formation of a differential equation which relates the spatial variation of the potential to the difference in electron and ion densities,

$$\frac{d}{d\rho} \left(\rho^2 \frac{d\Phi}{d\rho} \right) = J Z^{-\frac{1}{2}} \Phi^{-\frac{1}{2}} - \rho^2 \exp(-\Phi). \quad (10)$$

This equation may be solved using the boundary conditions;

$$\rho \approx J^{\frac{1}{2}} Z^{-\frac{1}{4}} \Phi^{-\frac{1}{4}} \exp\left(\frac{\Phi}{2}\right), \quad (11)$$

$$\left. \frac{d\Phi}{d\rho} \right|_{\rho_b} = \frac{2\rho_b Z^{\frac{1}{2}} J^{-1} \Phi_b^{\frac{3}{2}}}{\Phi_b - \frac{1}{2}} \exp(-\Phi_b), \quad (12)$$

$$\frac{J}{\Gamma} = \frac{4Z^{\frac{1}{2}} \Phi_b^{\frac{3}{2}} (2\Phi_b - 3)(2\Phi_b + 1)}{(2\Phi_b - 1)^3}, \quad (13)$$

$$\frac{J}{\alpha^2} = \frac{\mu}{\sqrt{4\pi}} \exp(-\Phi_a), \quad (14)$$

which are formed by assuming that there exists a certain distance ρ_b , past which, quasi-neutrality applies. The potential at ρ_b is given by Φ_b , and Γ is a number much greater than unity [3]. In order to find a value for the surface potential we must solve (10), this may be achieved using a 4th order Runge-Kutta. We choose $\Gamma = 10000$ and find the roots of (13) allowing us to determine the necessary boundary conditions using (11) and (12). Hence, solving the differential equation numerically, yields the following graph of normalised dust potential as a function of normalised dust radius.

Dust surface potential variation with normalised dust grain radius - ABR

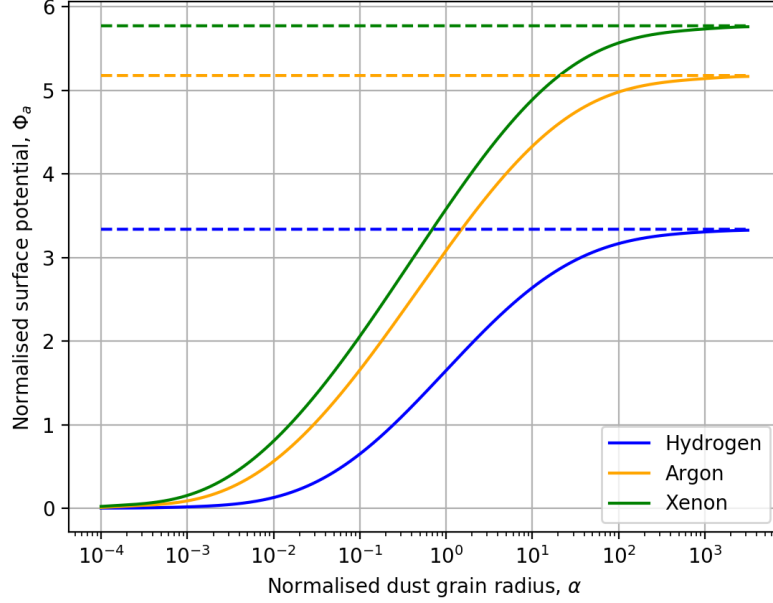


Figure 1: ABR predictions for Φ_a as a function of α for a dust grain in singly ionised Hydrogen, Argon and Xenon plasmas ($Z = 1$) [3] [4].

Thomas discusses that in the limit of $\alpha \rightarrow \infty$ the ABR potential approaches the cold planar wall limit [4], given as the following

$$\lim_{\alpha \rightarrow \infty} \Phi_a = \frac{1}{2} \ln(2\pi) - \frac{1}{2} - \ln(\mu), \quad (15)$$

where $Z = 1$ and the $-\frac{1}{2}$ is due to the potential drop across the cold ion pre-sheath, as discussed by Stangeby [1] [5]. Furthermore, one can clearly see that in the limit of $\alpha \rightarrow 0$ the ABR prediction tends to zero also [3].

4 Modified orbital motion limited (MOML)

Orbital motion limited (OML) is known to model the potential on a small spherical dust grain immersed in an infinite and collisionless plasma, it does so by considering energy and angular momentum conservations of ions along side a critical grazing incidence. OML considers an equilibrium of ion and electron currents at the dust surface, $I_i = I_e$, while simultaneously invoking quasi-neutrality. Hence, the standard result acquired from OML is the following

$$\frac{\sqrt{\Theta}}{\mu} \left(1 - \frac{Z}{\Theta} \Phi_a \right) \approx \exp(\Phi_a). \quad (16)$$

Using available OM data, Willis discusses that any error in OML is negligible for small dust grains, he further proposes an upper radius limit for OML which is Θ dependant [2]. Furthermore, it should be noted that OML guarantees the existence of absorption radii, $r_A > a$, such that any ion within r_A approaching the dust grain will be collected [4].

In order to model a large dust grain, we must slightly change our approach to the problem. We now apply OML to the boundary between the sheath and pre-sheath, this establishes the assumption that as $\alpha \rightarrow \infty$ any ion that enters the sheath will be collected by the dust grain. For such dust grains, of large radii, the majority of absorption radii occur within the sheath, hence, applying OML in this way eliminates most of the inaccuracies introduced by absorption radii and ensures the validity of MOML. However, it will be shown later that the validity of MOML in fact breaks down for small Θ [4].

Replacing Φ_a on the left hand side in (16) with Φ_s physically amounts to saying that as $\alpha \rightarrow \infty$, all ions that enter the sheath are absorbed. Considering an equilibrium of electron and ion currents at the sheath edge while simultaneously invoking quasi-neutrality and the expression for the hot ion Bohm speed, (3), one acquires the following relationship between Φ_s and Φ_a

$$\Phi_s \approx \Phi_a - \frac{1}{2} \ln \left[\frac{2\pi Z^2}{\mu^2} (1 + \gamma\Theta) \right], \quad (17)$$

where the second term is the normalised sheath potential drop, (4). Substituting (17) into the modified (16) and manipulating in terms of the principle branch of the Lambert W function, W_0 , yields

$$\Phi_a \approx \frac{\Theta}{Z} - W_0 \left(\sqrt{2\pi\Theta(1 + \gamma\Theta)} \exp \left(\frac{\Theta}{Z} \right) \right) + \frac{1}{2} \ln \left[\frac{2\pi Z^2}{\mu^2} (1 + \gamma\Theta) \right]. \quad (18)$$

Willis compares the MOML solution with simulated data ran by PIC and concludes that $\gamma = \frac{5}{3}$ seems to produce the most appropriate predictions [2]. Hence, $\gamma = \frac{5}{3}$ is chosen as the default value for our investigation. It is worth noting that from (Fig. 2) we see that for extreme Θ values, the choice of γ has very little affect on the predicted potential; the same can be said for intermediate values of Θ as there is very little difference in the predicted potentials for each γ value.

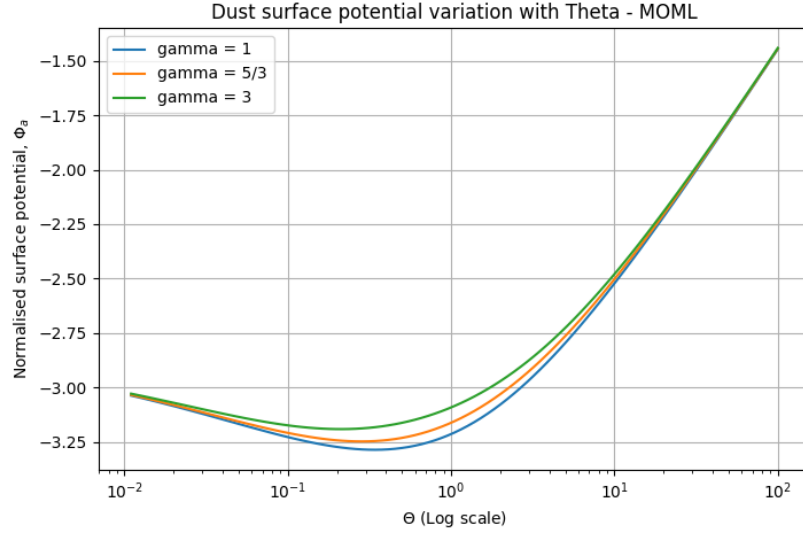


Figure 2: The MOML prediction for Φ_a as a function of Θ , for a hydrogenic ($\mu \approx 43$) and singly ionised ($Z = 1$) plasma with different values of γ [4].

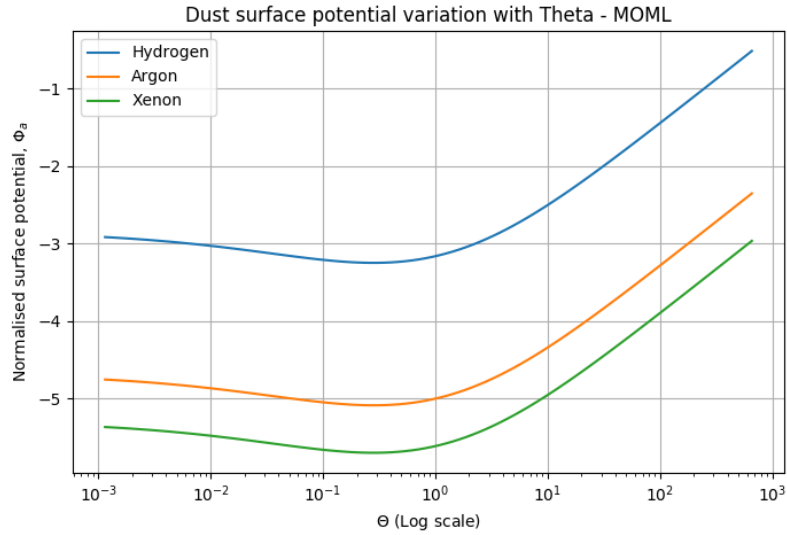


Figure 3: Normalised surface potential as a function of Θ for singly ionised ($Z = 1$) Hydrogen, Argon and Xenon plasmas according on the MOML prediction, plotted on a log-linear scale. The normalised surface potential progressively becomes more negative as the ion species becomes heavier [4].

5 SCEPTIC numerical fit

6 Comparison of MOML and ABR with SCEPTIC data

7 Flowing sheath approximation

8 Conclusion

9 References and Acknowledgements

- [1] P. C. Stangeby, *The Plasma Sheath*. Boston, MA: Springer US, 1986, pp. 41–97. [Online]. Available: https://doi.org/10.1007/978-1-4757-0067-1_3
- [2] C. T. N. Willis, “Dust in stationary and flowing plasmas,” Physics PhD Thesis, Imperial College London, March 2012.
- [3] K. R. V. and A. J. E., “The floating potential of spherical probes and dust grains. part 1. radial motion theory,” *Journal of Plasma Physics*, vol. 67.4, pp. 243–50, 2002.
- [4] D. M. Thomas, “Theory and simulation of the charging of dust in plasmas,” Physics PhD Thesis, Imperial College London, March 2016.
- [5] P. Stangeby, *The Plasma Boundary of Magnetic Fusion Devices*, ser. Series in Plasma Physics and Fluid Dynamics. Taylor & Francis, 2000. [Online]. Available: <https://books.google.co.uk/books?id=qOliQgAACAAJ>

10 Appendix

10.1 Symbol dictionary

e	Electron charge
ε_0	Permittivity of free space
k_B	Boltzmann's constant
a	Dust radius
α	Normalised dust radius
a	Subscript indicating a quantity at the dust grain surface
r	Distance from the centre of the dust grain
ρ	Normalised distance from the centre of the dust grain
λ_D	Debye length
m_j	Mass
n_j	Density
T_j	Temperature
I_j	Current
j	Subscript indicating a plasma particle
i	Subscript indicating an ion quantity
e	Subscript indicating an electron quantity
0	Subscript indicating an electron quantity at infinity
μ	Root mass ratio
Θ	Ratio of ion to electron temperature
γ	Heat capacity ratio
u	Flow velocity
v	Normalised flow velocity
Γ	ABR correction factor
Z	Ion charge number
Q	Dust grain charge
ϕ	Electric potential
Φ	Normalised electric potential