Notes on polynomial reproduction

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Let ϕ be a scaling (refinable) function, satisfying the refinement equation

$$\phi = \sum_{k\in\mathbb{Z}} p_k \phi(2\cdot -k) \tag{1}$$

for some mask $\{p_k\}_{k\in\mathbb{Z}}$. Typically ϕ is the generating function of an MRA (if it has stable or o.n. integer translates) and can be been as the limit function of a convergent subdivision scheme.

The function ϕ is said to have polynomial reproduction of order m (degree $m-1$) if any polynomial of degree less than m can be expressed as linear combination of integer shifts of ϕ, i.e. there exist sequences $c_\ell := \{c_{k;\ell}\}_{k\in\mathbb{Z}}$ such that

$$x^\ell = \sum_{k\in\mathbb{Z}} c_k^\ell \phi(x-k), \quad \ell = 0,\dots,m-1. \tag{2}$$

The number of coefficients involved in the above formula depends on the support of the function ϕ and on the interval where x varies. Let $\operatorname{supp}\phi = [0, M]$ and $x \in [a, b]$, with $a, b \in \mathbb{Z}$. Then, $c_{k;\ell}$ is different from zero only for $a+1-M \le k \le b-1$, so that the total number of coefficients is $b-a+M-1$.

A sequence $\{p_k\}$ is said to possess the m-th order sum-rule property if m is the largest integer for which:

$$\sum_{j\in\mathbb{Z}} (2j)^\ell p_{2j} = \sum_{j\in\mathbb{Z}} (2j-1)^\ell p_{2j-1} =: \beta_\ell, \quad \ell = 0,\dots,m-1,$$

with $\beta_0 = 1$, or, equivalently, if its symbol

$$P(z) := \sum_k p_k z^k, \quad z \in \mathbb{C}\setminus\{0\}$$

factorizes as

$$P(z) = \left(\frac{z+1}{2}\right)^m R(z)$$

with $R(1) = 2$ and $R(-1) \neq 0$.

Result (see [1]): if the refinement sequence $\{p_k\}$ in (1) satisfies the sum-rule condition of order m, then ϕ reproduces polynomials up to the degree $m-1$ and the (unique) coefficient sequences c_ℓ in (2) are obtained as ([1], Cor. 5.2.1):

$$c_k^\ell = \frac{\ell!}{(m-1)!} g_\phi^{(m-1-\ell)}(k)$$

where: g_ϕ is the polynomial of degree $m-1$

$$g_\phi(x) = \sum_{j=0}^{m-1} q_j x^j$$

with coefficients q_j defined by the backward recursive formula ([1], Thm. 5.2.2)

$$\begin{cases} q_{m-1} = 1 \\ q_j = (-1)^{j+1} \displaystyle\sum_{k=j+1}^{m-1} (-1)^k \binom{k}{j} \mu_{k-j} q_k, \quad j = m-2, m-3, \dots, 0 \end{cases}$$

where μ_ℓ, $\ell = 0, \dots, m-1$, are the *discrete moments* of ϕ, i.e.

$$\mu_\ell := \sum_j j^\ell \phi(\ell)$$

which can be recursively computed as ([1], Thm. 5.1.2)

$$\begin{cases} \mu_0 = 1 \\ \mu_\ell = \dfrac{1}{2^\ell - 1} \displaystyle\sum_{j=0}^{\ell-1} \binom{\ell}{j} \beta_{\ell-j} \mu_j, \quad \ell = 1, \dots, m-1 \end{cases}$$

Figures illustrating the behaviour of such coefficients in the Daubechies and the B-spline cases (both taken of order $m = 6$).

1 Bernstein case

Proposition 1.1. *If ϕ has reproduction order m and is supported in $[0, M]$, then, for $i = 0, \dots, n$, the Bernstein polynomial $B_i^n(x)$, on the generic interval $[a, b]$, $a, b \in \mathbb{Z} \setminus \{0\}$, can be represented as*

$$B_i^n(x) = \sum_{k=-M+a+1}^{b-1} \tilde{c}_k^{i,n} \phi(x-k)$$

with

$$\tilde{c}_k^{i,n} = \binom{n}{i}\left(\frac{b}{b-a}\right)^n\left(-\frac{a}{b}\right)^i\sum_{j=0}^{i}\sum_{h=j}^{j+n-i}\binom{i}{j}\binom{n-i}{h-j}\left(\frac{b}{a}\right)^j\left(-\frac{1}{b}\right)^h c_k^h$$

where c_k^ℓ *are the coefficients related to the reproduction of the monomial of degree* h *in* (2)

Proof. We first derive a transformation formula for expressing Bernstein polynomials on a generic interval $[a, b]$ in terms of monomials:

$$\begin{aligned} B_i^n(x) &= \binom{n}{i}\frac{1}{(b-a)^n}(x-a)^i(b-x)^{n-i} \\ &= \binom{n}{i}\frac{1}{(b-a)^n}\sum_{j=0}^{i}\binom{i}{j}x^j(-a)^{i-j}\sum_{\ell=0}^{n-i}\binom{n-i}{\ell}(-x)^\ell b^{n-i-\ell} \\ &= \binom{n}{i}\left(\frac{b}{b-a}\right)^n\left(-\frac{a}{b}\right)^i\sum_{j=0}^{i}\sum_{\ell=0}^{n-i}\binom{i}{j}\binom{n-i}{\ell}\left(-\frac{1}{a}\right)^j\left(-\frac{1}{b}\right)^\ell x^{j+\ell} \\ &= \binom{n}{i}\left(\frac{b}{b-a}\right)^n\left(-\frac{a}{b}\right)^i\sum_{j=0}^{i}\sum_{h=j}^{j+n-i}\binom{i}{j}\binom{n-i}{h-j}\left(-\frac{1}{a}\right)^j\left(-\frac{1}{b}\right)^{h-j} x^h \end{aligned}$$

Since, from (2), $x^h = \sum_{k=-M+a+1}^{b-1} c_k^h \phi(x-k)$, the statement follows. $\square$

Remark 1.2. *If either* $a = 0, b > 0$ *or* $b = 0, a < 0$*, we have, respectively,*

$$\tilde{c}_k^{i,n} = \sum_{h=i}^{n}\binom{n}{h}\binom{h}{i}(-1)^{h-i}b^{-h}c_k^h$$

$$\tilde{c}_k^{i,n} = \sum_{h=n-i}^{n}\binom{n}{i}\binom{i}{h-n+i}(-1)^{n-i}(-a)^{-h}c_k^h$$

The following figures illustrate the different behaviour of the Bernstein coefficients with respect to the monomial coefficients. They are much smaller! But this has to be formally proved.

References

[1] C. Chui and J. De Villiers, Wavelet subdivision methods: GEMS for rendering curves and surfaces, CRC Press, 2011

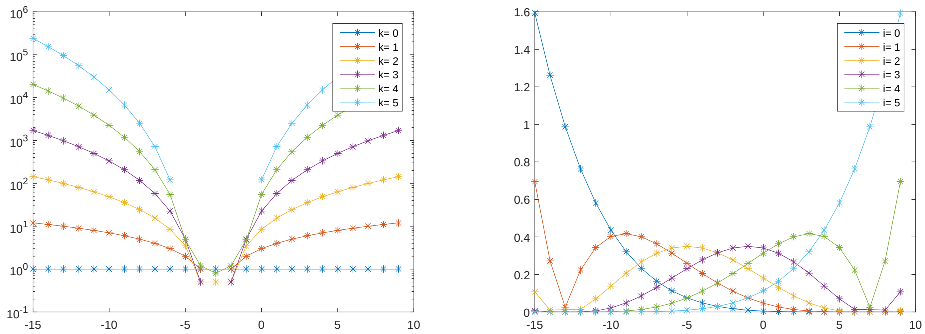

Figure 1: Coefficients reproducing monomials (left) and Bernstein polynomials (right) up to the degree 5. The scaling function ϕ is the B-spline of order $m = 6$ (support $[0, 6]$). The interval considered here is $[a, b] = [-10, 10]$.

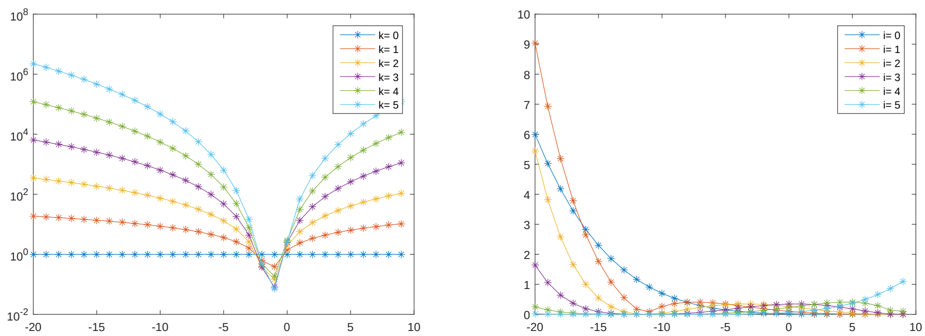

Figure 2: Coefficients reproducing monomials (left) and Bernstein polynomials (right) up to the degree 5. The scaling function ϕ is the Daubechies function of order $m = 6$ (support $[0, 12]$). The interval considered here is $[a, b] = [-10, 10]$.