Research Memorandum

January 30, 2018

Accelerated Alternating Direction Method of Multipliers for Frictional Contact [*]

Yoshihiro Kanno [†]

1 Algorithm Coulomb

The previous algorithm can be reformulated as well. As the upshot, the fast ADMM (Algorithm **??**) for solving problem (**??**) is formally stated in Algorithm 1.

Algorithm 1 Fast ADMM for problem (**??**).

Require: $\tilde{\boldsymbol{u}}^0 = \hat{\boldsymbol{u}}^0$, $\boldsymbol{\zeta}^0 = \hat{\boldsymbol{\zeta}}^0$, $\tau_0 = 1$, and $\rho > 0$.

1: **for** $k = 0, 1, 2, \dots$ **do**

2: $\quad \boldsymbol{v}^{k+1}$ solves $\Big[M + \rho H^\top H\Big]\boldsymbol{v} = -\boldsymbol{f} + \rho H^\top(\tilde{\boldsymbol{u}}^k - \boldsymbol{b}(s) - \hat{\boldsymbol{\zeta}}^k)$

3: $\quad \tilde{\boldsymbol{u}}^{k+1} := \Pi_{K^*_{e,\mu}}(H\boldsymbol{v}^{k+1} + \hat{\boldsymbol{\zeta}}^k + \boldsymbol{b}(\boldsymbol{s}))$

4: $\quad \boldsymbol{\zeta}^{k+1} := \hat{\boldsymbol{\zeta}}^k + H\boldsymbol{v}^{k+1} - \tilde{\boldsymbol{u}}^{k+1} + \boldsymbol{b}(s)$

5: $\quad \tau_{k+1} := \dfrac{1}{2}\Big(1 + \sqrt{1 + 4\tau_k^2}\Big)$

6: $\quad \hat{\boldsymbol{u}}^{k+1} := \hat{\boldsymbol{u}}^k + \dfrac{\tau_k - 1}{\tau_{k+1}}(\tilde{\boldsymbol{u}}^{k+1} - \tilde{\boldsymbol{u}}^k)$

7: $\quad \hat{\boldsymbol{\zeta}}^{k+1} := \hat{\boldsymbol{\zeta}}^k + \dfrac{\tau_k - 1}{\tau_{k+1}}(\boldsymbol{\zeta}^{k+1} - \boldsymbol{\zeta}^k)$

8: **end for**

In step 2 of Algorithm 1, the coefficient matrix of the system of linear equations is common to all the iterations. Hence, we carry out the Cholesky factorization only at the first iteration (i.e., $k = 0$); at the following iterations, we can compute $\boldsymbol{v}^{k+1}$ only with the back-substitutions. The projection in step 3 can be computed explicitly by using the formula in (7). The computations in the other steps are only matrix-vector products and vector additions. Therefore, the computational cost required for one iteration is small, even for a large-scale problem.

In a similar manner, we can apply Algorithm **??** to problem (**??**), as formally stated in Algorithm 2.

- To solve the friction problem, parameter $\boldsymbol{s}$ in problem (**??**) should be updated. This corresponds to updating $\boldsymbol{b}_i$ and d_i in problem (**??**). One obvious way is that, once we solve problem (**??**) with fixed $\boldsymbol{b}_i$ and d_i by the fast ADMM (with restart), then update $\boldsymbol{b}_i$ and d_i by using the obtained solution, and repeat this procedure. Another possibility is to update $\boldsymbol{b}_i$ and d_i at each iteration of the fast ADMM (with restart). Intuitively, the latter saves the total computational cost, but stability of the algorithm in this case is not clear.

[*] `/doc/contact/admm/`

[†] Laboratory for Future Interdisciplinary Research of Science and Technology, Institute of Innovative Research, Tokyo Institute of Technology, Nagatsuta 4259, Yokohama 226-8503, Japan. E-mail: `kanno.y.af@m.titech.ac.jp`.

Algorithm 2 Fast ADMM for problem with restart (**??**).

Require: $\tilde{\boldsymbol{u}}^0 = \hat{\boldsymbol{u}}^0$, $\boldsymbol{\zeta}^0 = \hat{\boldsymbol{\zeta}}^0$, $\tau_0 = 1$, and $\rho > 0$.

for $k = 0, 1, 2, \ldots$ **do**
- $\boldsymbol{v}^{k+1}$ solves $\left[M + \rho H^\top H\right]\boldsymbol{v} = -\boldsymbol{f} + \rho H^\top(\tilde{\boldsymbol{u}}^k - \boldsymbol{b}(s) - \hat{\boldsymbol{\zeta}}^k)$
- $\tilde{\boldsymbol{u}}^{k+1} := \Pi_{K^*_{e,\mu}}(H\boldsymbol{v}^{k+1} + \hat{\boldsymbol{\zeta}}^k + \boldsymbol{b}(\boldsymbol{s}))$
- $\boldsymbol{\zeta}^{k+1} := \hat{\boldsymbol{\zeta}}^k + H\boldsymbol{v}^{k+1} - \tilde{\boldsymbol{u}}^{k+1} + \boldsymbol{b}(s)$
- **if** $e_k < \eta e_{k-1}$ **then**
  - $\tau_{k+1} := \frac{1}{2}\left(1 + \sqrt{1 + 4\tau_k^2}\right)$
  - $\hat{\boldsymbol{u}}^{k+1} := \hat{\boldsymbol{u}}^k + \frac{\tau_k - 1}{\tau_{k+1}}(\tilde{\boldsymbol{u}}^{k+1} - \tilde{\boldsymbol{u}}^k)$
  - $\hat{\boldsymbol{\zeta}}^{k+1} := \hat{\boldsymbol{\zeta}}^k + \frac{\tau_k - 1}{\tau_{k+1}}(\boldsymbol{\zeta}^{k+1} - \boldsymbol{\zeta}^k)$
- **else**
  - $\tau_{k+1} := 1$
  - $\hat{\boldsymbol{u}}^{k+1} := \tilde{\boldsymbol{u}}^k$
  - $\hat{\boldsymbol{\zeta}}^{k+1} := \boldsymbol{\zeta}^k$
  - $e_k \leftarrow e_{k-1}/\eta$
- **end if**

end for

2 Projection onto second order cone

$$\Pi_{K_{e,\mu}}(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \|\boldsymbol{x_2}\| \le -\frac{1}{\mu}x_1, \\ \boldsymbol{x} & \text{if } \|\boldsymbol{x_2}\| \le \mu x_1, \\ \dfrac{1}{1+\mu^2}\left(x_1 + \mu\|\boldsymbol{x}_2\|\right)\begin{bmatrix} 1 \\ \mu\boldsymbol{x}_2/\|\boldsymbol{x}_2\| \end{bmatrix} & \text{if } -\mu\|\boldsymbol{x_2}\| < x_1 < \frac{1}{\mu}\|\boldsymbol{x_2}\|, \end{cases} \tag{1}$$

$$\Pi_{K^*_{e,\mu}}(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \|\boldsymbol{x_2}\| \le -\mu x_1, \\ \boldsymbol{x} & \text{if } \|\boldsymbol{x_2}\| \le \frac{1}{\mu}x_1, \\ \dfrac{\mu^2}{1+\mu^2}\left(x_1 + \dfrac{1}{\mu}\|\boldsymbol{x}_2\|\right)\begin{bmatrix} 1 \\ \frac{1}{\mu}\boldsymbol{x}_2/\|\boldsymbol{x}_2\| \end{bmatrix} & \text{if } -\frac{1}{\mu}\|\boldsymbol{x_2}\| < x_1 < \mu\|\boldsymbol{x_2}\|, \end{cases} \tag{2}$$

2.1 Projection onto Coulomb's friction cone

For vector $\boldsymbol{x} = (x_1, \boldsymbol{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, its spectral factorization with respect to $K_{e,\mu}$ is defined by [8]

$$\boldsymbol{x} = \lambda_1\boldsymbol{u}^1 + \lambda_2\boldsymbol{u}^2. \tag{3}$$

Here, λ_1, $\lambda_2 \in \mathbb{R}$ are the spectral values given by

$$\lambda_i = x_1 + (-1)^i\,\mu^{(-1)^i}\|\boldsymbol{x_2}\|, \tag{4}$$

and $\boldsymbol{u}^1$, $\boldsymbol{u}^2 \in \mathbb{R}^n$ are the spectral vectors given by

$$\boldsymbol{u}^i = \begin{cases} \dfrac{1}{1+\mu^2} \begin{bmatrix} \mu^{2(2-i)} \\ (-1)^i \, \mu \boldsymbol{x}_2 / \|\boldsymbol{x}_2\| \end{bmatrix} & \text{if } \boldsymbol{x}_2 \neq \boldsymbol{0}, \\ \dfrac{1}{1+\mu^2} \begin{bmatrix} \mu^{2(2-i)} \\ (-1)^i \, \mu \boldsymbol{\omega} \end{bmatrix} & \text{if } \boldsymbol{x}_2 = \boldsymbol{0}, \end{cases} \tag{5}$$

with $\boldsymbol{\omega} \in \mathbb{R}^{n-1}$ satisfying $\|\boldsymbol{\omega}\| = 1$.
For $\boldsymbol{x} \in \mathbb{R}^n$, let $\Pi_{K_{e,\mu}}(\boldsymbol{x}) \in \mathbb{R}^n$ denote the projection of $\boldsymbol{x}$ onto $K_{e,\mu}$, i.e.,

$$\Pi_{K_{e,\mu}}(\boldsymbol{x}) = \arg\min\{\|\boldsymbol{x}' - \boldsymbol{x}\| \mid \boldsymbol{x}' \in K_{e,\mu}\}. \tag{6}$$

This can be computed explicitly as [4]

$$\Pi_{K_{e,\mu}}(\boldsymbol{x}) = \max\{0, \lambda_1\}\boldsymbol{u}^1 + \max\{0, \lambda_2\}\boldsymbol{u}^2. \tag{7}$$

Therefore the projection of $\boldsymbol{x}$ onto $K_{e,\mu}$ could be written as follows

$$\Pi_{K_{e,\mu}}(\boldsymbol{x}) = \begin{cases} 0 & \text{if } -\boldsymbol{x} \in K^*_{e,\mu} \to \lambda_i \leq 0 \\ \boldsymbol{x} & \text{if } \boldsymbol{x} \in K_{e,\mu} \to \lambda_i \geq 0 \\ \dfrac{(x_1 + \mu\|\boldsymbol{x}_2\|)}{1+\mu^2} \begin{bmatrix} 1 \\ \mu \boldsymbol{x}_2 / \|\boldsymbol{x}_2\| \end{bmatrix} & \text{if } -\boldsymbol{x} \notin K^*_{e,\mu} \wedge \boldsymbol{x} \notin K_{e,\mu} \to \lambda_1 < 0 \wedge \lambda_2 > 0 \end{cases} \tag{8}$$

Now, it is easy to see that the dual of $K_{e,\mu}$ is also a second-order cone

$$K^*_{e,\mu} = K_{e,\frac{1}{\mu}} = \{(x_1, \boldsymbol{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|\boldsymbol{x}_2\| \leq \frac{1}{\mu} x_1\} \tag{9}$$

Consequently, the projection of $\boldsymbol{x}$ onto $K^*_{e,\mu}$ could be written as follows

$$\Pi_{K^*_{e,\mu}}(\boldsymbol{x}) = \begin{cases} 0 & \text{if } -\boldsymbol{x} \in K_{e,\mu} \to \lambda^*_i \leq 0 \\ \boldsymbol{x} & \text{if } \boldsymbol{x} \in K^*_{e,\mu} \to \lambda^*_i \geq 0 \\ \dfrac{\mu^2 \left(x_1 + \frac{1}{\mu}\|\boldsymbol{x}_2\|\right)}{1+\mu^2} \begin{bmatrix} 1 \\ \frac{1}{\mu} \boldsymbol{x}_2 / \|\boldsymbol{x}_2\| \end{bmatrix} & \text{if } -\boldsymbol{x} \notin K_{e,\mu} \wedge \boldsymbol{x} \notin K^*_{e,\mu} \to \lambda^*_1 < 0 \wedge \lambda^*_2 > 0, \end{cases} \tag{10}$$

where $\lambda^*_i = x_1 + (-1)^i \, \mu^{(-1)^{i+1}} \|\boldsymbol{x}_2\|$.

References

[1] V. Acary, F. Cadoux, C. Lemaréchal, J. Malick: A formulation of the linear discrete Coulomb friction problem via convex optimization. *ZAMM*, **91**, 155–175 (2011).

[2] V. Acary, F. Cadoux: Applications of an existence result for the Coulomb friction problem. In: G. E. Stavroulakis (ed.), *Recent Advances in Contact Mechanics*, pp. 45–66, Springer-Verlag, Berlin (2013).

[3] S. Boyd, N. Parikh, E. Chu, B. Peleato, J. Eckstein: Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends in Machine Learning*, **3**, 1–122 (2010).

[4] M. Fukushima, Z.-Q. Luo, P. Tseng: Smoothing functions for second-order-cone complementarity problems. *SIAM Journal on optimization*, **12**, 436–460 (2001).

[5] P. Giselsson, S. Boyd: Diagonal scaling in Douglas–Rachford splitting and ADMM. *The 53rd IEEE Conference on Decision and Control*, 5033–5039, Los Angeles (2014).

[6] T. Goldstein, B. O'Donoghue, S. Setzer, R. Baraniuk: Fast alternating direction optimization methods. *SIAM Journal on Imaging Sciences*, **7**, 1588–1623 (2014).

[7] W. Hager, C. Ngo, M. Yashtini, H.-C. Zhang: An alternating direction approximate Newton algorithm for ill-conditioned inverse problems with application to parallel MRI. *Journal of the Operations Research Society of China*, **3**, 139–162 (2015).

[8] S. Hayashi, N. Yamashita, M. Fukushima: A combined smoothing and regularization method for monotone second-order cone complementarity problems. *SIAM Journal on Optimization*, **15**, 593-615 (2005).

[9] J. M. C. Martins, A. Pinto da Costa: Stability of finite-dimensional nonlinear elastic systems with unilateral contact and friction. *International Journal of Solids and Structures*, **37**, 2519–2564 (2000).

[10] J. M. C. Martins, A. Pinto da Costa, F. M. F. Simões: Some notes on friction and instabilities. In: J. A. C. Martins, M. Raous (eds.), *Friction and Instabilities*, pp. 65–136, Springer-Verlag, Wien (2002).

[11] A. Pinto da Costa, J. A. C. Martins, I. N. Figueiredo, J. J. Júdice: The directional instability problem in systems with frictional contacts. *Computer Methods in Applied Mechanics and Engineering*, **193**, 357–384 (2004).

Contents