椭圆不确定集下的鲁棒对等问题推导

我们要证明在椭圆不确定集下,鲁棒对等问题可转化为如下的二阶锥规划问题:

1.

$$\min_{\widetilde{v}_{i,s} \in U_v^e} \quad \left\{ \sum_{i \in I} \sum_{s \in S_i} \widetilde{v}_{i,s} y_{i,s}
ight\} = \sum_{i \in I} \sum_{s \in S_i} ar{v}_{i,s} y_{i,s} - \Omega_v \sqrt{\sum_{i \in I} \sum_{s \in S_i} y_{i,s}^2 (\sigma_{i,s}^v)^2}$$

$$where \quad \widetilde{v}_{i,s} \in U^e_v, U^e_v = \left\{\widetilde{v}: (\widetilde{v} - ar{v})^T \Sigma_v^{-1} (\widetilde{v} - ar{v}) \leq \Omega_v^2
ight\}$$

2.

$$\max_{ ilde{c}_{i,s} \in U^e_c} \left\{ \sum_{i \in I} \sum_{s \in S_i} ilde{c}_{i,s} y_{i,s}
ight\} = \sum_{i \in I} \sum_{s \in S_i} ar{c}_{i,s} y_{i,s} + \Omega_c \sqrt{\sum_{i \in I} \sum_{s \in S_i} y_{i,s}^2 (\sigma^c_{i,s})^2}$$

where
$$\tilde{c}_{i,s} \in U_c^e, U_c^e = \left\{ \tilde{c} : (\tilde{c} - \bar{c})^T \Sigma_c^{-1} (\tilde{c} - \bar{c}) \leq \Omega_c^2 \right\}$$

Proof 1:

已知:

• 向量/矩阵求偏导:

$$rac{\partial Ax}{\partial x} = A^T \quad rac{\partial Ax}{\partial x^T} = A$$

• KKT条件:

$$\begin{aligned} & \text{min} \quad f(\mathbf{x}) \\ & \text{s.t.} \quad g_j(\mathbf{x}) = 0, \quad j = 1, \dots, m, \\ & \quad h_k(\mathbf{x}) \leq 0, \quad k = 1, \dots, p. \end{aligned}$$

定义Lagrangian 函数

$$L\left(\mathbf{x},\{\lambda_{j}\},\{\mu_{k}\}
ight)=f(\mathbf{x})+\sum_{j=1}^{m}\lambda_{j}g_{j}(\mathbf{x})+\sum_{k=1}^{p}\mu_{k}h_{k}(\mathbf{x})$$

其中 λ_j 是对应 $g_j(\mathbf{x})=0$ 的Lagrange乘数, μ_k \$是对应 $h_k(\mathbf{x})\leq 0$ 的Lagrange乘数(或称 KKT乘数)。KKT条件包括

$$egin{aligned}
abla_{\mathbf{x}} L &= \mathbf{0} \\ g_j(\mathbf{x}) &= 0, \;\; j = 1, \dots, m, \\ h_k(\mathbf{x}) &\leq 0, \\ \mu_k &\geq 0, \\ \mu_k h_k(\mathbf{x}) &= 0, \;\; k = 1, \dots, p. \end{aligned}$$

问题1的矩阵形式:

$$egin{aligned} min_{\widetilde{v}} & y_{1 imes P}^T \cdot \widetilde{v}_{P imes 1} \ & s.t. & (\widetilde{v} - ar{v})_{1 imes P}^T \cdot \Sigma_{P imes P}^{-1} \cdot (\widetilde{v} - ar{v})_{P imes 1} \leq \Omega^2 \end{aligned}$$

由KKT条件:

$$egin{align} f(\widetilde{v},\mu) &= y^T \cdot \widetilde{v} + \mu \cdot [(\widetilde{v} - \overline{v})^T \cdot \Sigma^{-1} \cdot (\widetilde{v} - \overline{v}) - \Omega^2] \ & rac{\partial f(\widetilde{v},\mu)}{\partial \widetilde{v}} = y + 2\mu \Sigma^{-1} (\widetilde{v} - \overline{v}) = 0 \quad (1) \ & \mu \cdot [(\widetilde{v} - \overline{v})^T \cdot \Sigma^{-1} \cdot (\widetilde{v} - \overline{v}) - \Omega^2] = 0 \quad (2) \ \end{cases}$$

由(1)得:

$$\widetilde{v} = \overline{v} - \frac{\Sigma}{2\mu} \cdot y$$
 (3)

带入(2):

$$(-\frac{y^T}{2\mu}) \cdot \Sigma^T \cdot \Sigma^{-1} \cdot (-\frac{\Sigma}{2\mu}) \cdot y = \frac{y^T \Sigma y}{4\mu^2} = \Omega^2 \quad (4)$$

由(4)得:

$$2\mu = \frac{\sqrt{y^T \Sigma y}}{\Omega} \quad (5)$$

把(5)带入(3):

$$\widetilde{v} = ar{v} - rac{\Omega}{\sqrt{y^T \Sigma y}} \cdot \Sigma \cdot y \quad (*)$$

所以有:

$$egin{aligned} min_{\widetilde{v}_{i,s} \in U_v^e} & \left\{ \sum_{i \in I} \sum_{s \in S_i} \widetilde{v}_{i,s} y_{i,s}
ight\} = y^T \cdot \widetilde{v} = y^T \cdot \overline{v} - \Omega \cdot rac{y^T \Sigma y}{\sqrt{y^T \Sigma y}} \ & = \sum_{i \in I} \sum_{s \in S_i} ar{v}_{i,s} y_{i,s} - \Omega_v \sqrt{\sum_{i \in I} \sum_{s \in S_i} y_{i,s}^2 (\sigma_{i,s}^v)^2} \end{aligned}$$

得证。

Proof 2:

和证明1基本一致,细微差异如下。

问题2的矩阵形式:

$$egin{aligned} min_{ ilde{c}} & y_{1 imes P}^T \cdot \widetilde{c}_{P imes 1} \ & s.t. & (\widetilde{c} - ar{c})_{1 imes P}^T \cdot \Sigma_{P imes P}^{-1} \cdot (\widetilde{c} - ar{c})_{P imes 1} \leq \Omega^2 \end{aligned}$$

由KKT条件:

$$f(\tilde{c}, \mu) = y^T \cdot \tilde{c} + \mu \cdot [\Omega^2 - (\tilde{c} - \bar{c})^T \cdot \Sigma^{-1} \cdot (\tilde{c} - \bar{c})]$$
$$\frac{\partial f(\tilde{c}, \mu)}{\partial \tilde{c}} = y - 2\mu \Sigma^{-1} (\tilde{c} - \bar{c}) = 0 \quad (1)$$
$$\mu \cdot [(\tilde{c} - \bar{c})^T \cdot \Sigma^{-1} \cdot (\tilde{c} - \bar{c}) - \Omega^2] = 0 \quad (2)$$

最终求得:

$$\widetilde{c} = \overline{c} + rac{\Omega}{\sqrt{y^T \Sigma y}} \cdot \Sigma \cdot y \quad (*)$$