椭圆不确定集下的鲁棒对等问题推导

我们要证明在椭圆不确定集下，鲁棒对等问题可转化为如下的二阶锥规划问题：

1.

$$\min_{\tilde{v}_{i,s}\in U_v^e}\left\{\sum_{i\in I}\sum_{s\in S_i}\tilde{v}_{i,s}y_{i,s}\right\}=\sum_{i\in I}\sum_{s\in S_i}\bar{v}_{i,s}y_{i,s}-\Omega_v\sqrt{\sum_{i\in I}\sum_{s\in S_i}y_{i,s}^2(\sigma_{i,s}^v)^2}$$

$$where\quad \tilde{v}_{i,s}\in U_v^e, U_v^e=\left\{\tilde{v}:(\tilde{v}-\bar{v})^T\Sigma_v^{-1}(\tilde{v}-\bar{v})\le\Omega_v^2\right\}$$

2.

$$\max_{\tilde{c}_{i,s}\in U_c^e}\left\{\sum_{i\in I}\sum_{s\in S_i}\tilde{c}_{i,s}y_{i,s}\right\}=\sum_{i\in I}\sum_{s\in S_i}\bar{c}_{i,s}y_{i,s}+\Omega_c\sqrt{\sum_{i\in I}\sum_{s\in S_i}y_{i,s}^2(\sigma_{i,s}^c)^2}$$

$$where\quad \tilde{c}_{i,s}\in U_c^e, U_c^e=\left\{\tilde{c}:(\tilde{c}-\bar{c})^T\Sigma_c^{-1}(\tilde{c}-\bar{c})\le\Omega_c^2\right\}$$

Proof 1:

已知：

- 向量/矩阵求偏导：

$$\frac{\partial Ax}{\partial x}=A^T\quad \frac{\partial Ax}{\partial x^T}=A$$

- KKT条件：

$$
\begin{aligned}
\min \quad & f(\mathbf{x}) \\
\text{s.t.} \quad & g_j(\mathbf{x}) = 0, \quad j = 1, \ldots, m, \\
& h_k(\mathbf{x}) \le 0, \quad k = 1, \ldots, p.
\end{aligned}
$$

定义Lagrangian 函数

$$L\left(\mathbf{x}, \{\lambda_j\}, \{\mu_k\}\right) = f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x}) + \sum_{k=1}^{p} \mu_k h_k(\mathbf{x})$$

其中 λ_j 是对应 $g_j(\mathbf{x}) = 0$ 的Lagrange乘数， μ_k $是对应 $h_k(\mathbf{x}) \le 0$ 的Lagrange乘数(或称KKT乘数)。KKT条件包括

$$
\begin{aligned}
\nabla_{\mathbf{x}} L &= \mathbf{0} \\
g_j(\mathbf{x}) &= 0, \ \ j = 1, \ldots, m, \\
h_k(\mathbf{x}) &\le 0, \\
\mu_k &\ge 0, \\
\mu_k h_k(\mathbf{x}) &= 0, \ \ k = 1, \ldots, p.
\end{aligned}
$$

问题1的矩阵形式：

$$min_{\widetilde{v}} \quad y_{1\times P}^{T} \cdot \widetilde{v}_{P\times 1}$$

$$s.t. \quad (\widetilde{v} - \bar{v})_{1\times P}^{T} \cdot \Sigma_{P\times P}^{-1} \cdot (\widetilde{v} - \bar{v})_{P\times 1} \le \Omega^2$$

由KKT条件：

$$f(\widetilde{v}, \mu) = y^T \cdot \widetilde{v} + \mu \cdot [(\widetilde{v} - \bar{v})^T \cdot \Sigma^{-1} \cdot (\widetilde{v} - \bar{v}) - \Omega^2]$$

$$\frac{\partial f(\widetilde{v}, \mu)}{\partial \widetilde{v}} = y + 2\mu\Sigma^{-1}(\widetilde{v} - \bar{v}) = 0 \quad (1)$$

$$\mu \cdot [(\widetilde{v} - \bar{v})^T \cdot \Sigma^{-1} \cdot (\widetilde{v} - \bar{v}) - \Omega^2] = 0 \quad (2)$$

由（1）得：

$$\widetilde{v} = \bar{v} - \frac{\Sigma}{2\mu} \cdot y \quad (3)$$

带入（2）：

$$(-\frac{y^T}{2\mu}) \cdot \Sigma^T \cdot \Sigma^{-1} \cdot (-\frac{\Sigma}{2\mu}) \cdot y = \frac{y^T \Sigma y}{4\mu^2} = \Omega^2 \quad (4)$$

由（4）得：

$$2\mu = \frac{\sqrt{y^T \Sigma y}}{\Omega} \quad (5)$$

把（5）带入（3）：

$$\widetilde{v} = \bar{v} - \frac{\Omega}{\sqrt{y^T \Sigma y}} \cdot \Sigma \cdot y \quad (*)$$

所以有：

$$min_{\widetilde{v}_{i,s} \in U_v^e} \quad \left\{ \sum_{i \in I} \sum_{s \in S_i} \widetilde{v}_{i,s} y_{i,s} \right\} = y^T \cdot \widetilde{v} = y^T \cdot \bar{v} - \Omega \cdot \frac{y^T \Sigma y}{\sqrt{y^T \Sigma y}}$$

$$= \sum_{i \in I} \sum_{s \in S_i} \bar{v}_{i,s} y_{i,s} - \Omega_v \sqrt{\sum_{i \in I} \sum_{s \in S_i} y_{i,s}^2 (\sigma_{i,s}^v)^2}$$

得证。

Proof 2:

和证明1基本一致，细微差异如下。

问题2的矩阵形式：

$$min_{\tilde{c}} \quad y_{1 \times P}^T \cdot \tilde{c}_{P \times 1}$$

$$s.t. \quad (\tilde{c} - \bar{c})_{1 \times P}^T \cdot \Sigma_{P \times P}^{-1} \cdot (\tilde{c} - \bar{c})_{P \times 1} \leq \Omega^2$$

由KKT条件：

$$f(\tilde{c}, \mu) = y^T \cdot \tilde{c} + \mu \cdot [\Omega^2 - (\tilde{c} - \bar{c})^T \cdot \Sigma^{-1} \cdot (\tilde{c} - \bar{c})]$$

$$\frac{\partial f(\tilde{c}, \mu)}{\partial \tilde{c}} = y - 2\mu \Sigma^{-1} (\tilde{c} - \bar{c}) = 0 \quad (1)$$

$$\mu \cdot [(\tilde{c} - \bar{c})^T \cdot \Sigma^{-1} \cdot (\tilde{c} - \bar{c}) - \Omega^2] = 0 \quad (2)$$

最终求得：

$$\tilde{c} = \bar{c} + \frac{\Omega}{\sqrt{y^T \Sigma y}} \cdot \Sigma \cdot y \quad (*)$$