Advertising Motives and Firm Life-Cycle Dynamics in a General Equilibrium Model

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Abstract

I develop a dynamic general equilibrium model of advertising where consumers are characterised by the subset of goods they are aware of, which evolves over time. Consumers are exposed to advertising while consuming media goods. Firms advertise for three motives: to persuade customers to spend more, to acquire new customers, and, given that consumers' attention is limited, to prevent consumers from learning about competitors. I study how these motives change over the firm life cycle and their aggregate effects, with the informative motive being stronger in younger firms and the persuasive and anticompetitive motives being stronger in mature firms. In the calibrated model, the informative motive is responsible for half of total advertising expenditure. I also compare the decentralised equilibrium with the planner's allocation, keeping the information constraints. A novel feature of the model is that the planner values media goods even when their entertainment value is negligible, as they serve as a vehicle for product awareness. Finally, I find that advertising should be subsidized, although the gains are small.

1 Introduction

There is an extensive literature exploring whether advertising is informative or persuasive, with supporting evidence for both.[1] The conclusion we are drawn to is that advertising is a mixture of both. If consumers face information frictions, advertising can provide valuable information that mitigates these frictions, allowing consumers to enjoy more variety and promoting competition. On the persuasive side, advertising can enhance consumer preferences for a specific good. This can have both positive effects, by increasing the utility from the good, and negative effects as it increases product differentiation, leading to more market power.

In addition, consumers' limited attention capacity implies that firms need to compete for the attention of consumers to make their way into their consumption sets. This introduces a novel effect of advertising: one firm's advertising diverts consumers' attention away from competitors. This effect seems particularly relevant in settings like Google search or Amazon advertising, where firms compete to be placed in the top positions within a keyword, as these receive most of the attention. On Google, firms bid on specific keywords in a cost-per-click (CPC) auction.[2] Figure 1 shows the heterogeneity in the average CPC in Google search ads across industries, aligning with the modelling approach of this paper, which assumes that firms compete for a limited ad space within their industry.[3]

Since advertising has these three effects and firms may benefit differently from each, some firms may be more motivated to advertise due to one effect than another. Throughout the paper, I use the terms (i) *informative motive*, (ii) *persuasive motive* and (iii) *anticompetitive motive* to refer to a firm's incentive to advertise to (i) inform consumers, (ii) increase the spending by current customers, and (iii) reduce consumers' attention to competitors. I use the term 'anticompetitive' because, under this motive, the firm advertises to avoid competition by hindering competitors' ability to expand their customer base.

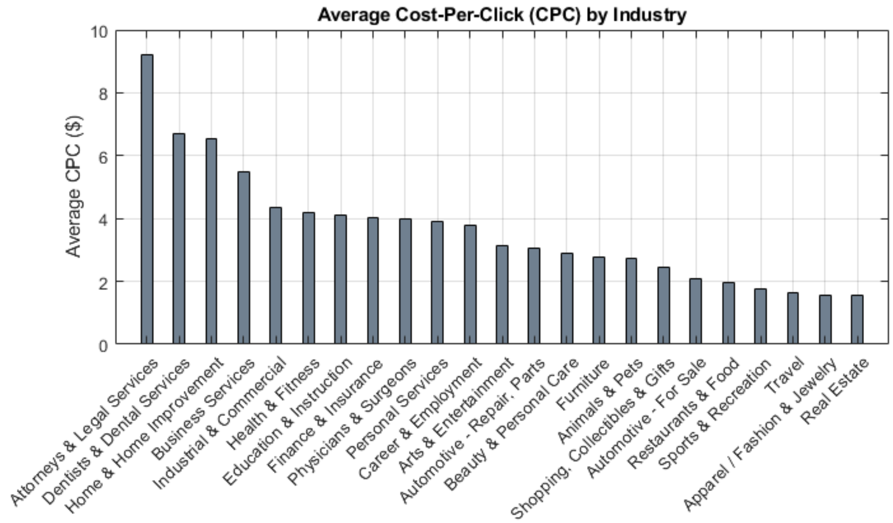
Advertising also plays a key role in the provision of media goods that consumers tend to enjoy at zero monetary price. The presence of media goods that get their revenue from selling advertising space is pervasive, spanning traditional outlets like radio and TV as well as digital platforms such as YouTube, Instagram, Facebook, and Google. Although these media goods are barely reflected in GDP (see Greenwood et al., 2024), the time consumers

[1] See Bagwell, 2007 for a review.

[2] Google doesn't charge firms just to be placed at the top, instead, the CPC is the amount the firm will be charged for each click their ad receives. Therefore, Google doesn't necessarily place the highest bidding firm at the top; it also considers the relevance of the ad.

[3] The same applies if we look at CPC in Google shopping ads across industries, although these are considerably cheaper, rarely more than one dollar per click.

Figure 1: Average Coct-Per-Click (CPC) in Google search ads by industry



Notes. Adapted from Wordstream (2023). This figure displays the average CPC in Google ads by industry, calculated by dividing the overall cost of a campaign by the number of clicks it received. Each individual click has a different cost as it's determined by the Google Ads auction algorithm.

spend on them suggests they have a significant impact on welfare.[4]

I present a model that accommodates these aspects of advertising to study the effects of advertising on welfare and industry dynamics. A key feature of the model is that, within an industry, the consumers are characterized by the set of goods they are aware of, which I refer to as awareness sets. These awareness sets evolve stochastically over time, and advertising is the tool firms use to increase the probability their goods enter consumers' awareness sets. Firms thus face a dynamic problem, as building a customer base takes time. This is motivated by the empirical evidence from Foster et al. (2016), which suggests that the accumulation of customer base plays a key role in explaining the size difference between young and mature firms.[5] Advertising also has a static persuasive effect by increasing the demand shifter of the advertised good. The model also includes a media sector that provides the platform for firms to advertise and supplies media goods to consumers at a zero monetary price. Consumers choose how much time to spend on media, during which they are exposed to advertising. A key idea in the model is that consumers' attention capacity

[4]According to Statista, the average daily time spent on media in the United States in 2023 amounted to 751 minutes.

[5]In particular, Foster et al. (2016) take advantage of data on physical quantities in industries that are plausibly little subject to quality differentiation and find that the fact that older firms are bigger than younger firms cannot be explained by differences in productivity, and then they find support for the hypothesis that firms play an active role, not just a passive effect from aging. Einav et al. (2022), focusing on the retail sector, find that most of the variability in sales is accounted by the number of clients.

is limited, which implies that the advertising space is limited. Given that the advertising space is limited, firms may want to advertise not to inform consumers, but rather to divert consumers' attention away from competitors (the anticompetitive motive).

I study how the three motives change along the firm life cycle. Intuitively, younger firms have more potential customers to acquire, and so they tend to have a stronger informative motive. In contrast, the anticompetitive motive, which is about retaining market power over the existing customers by reducing the probability they learn about competitors, is stronger for older firms, as they have more to lose due to their larger customer base. The persuasive motive also tends to be stronger in older firms. Intuitively, if advertising persuades existing customers to spend more, the revenue increase will be larger the bigger the customer base.

I estimate the model by simulated method of moments to fit key empirical patterns regarding (i) the evolution of average firm growth by age, which is important to discipline the customer base building process in the model; (ii) the relationship between advertising expenditure and sales; and (iii) macroeconomic aggregates. The model does well in matching the targets. In addition, the model also features an inverted-U relationship between advertising and sales as documented in previous literature.

First, I use the estimated model to assess the aggregate effects of the anticompetitive and persuasive motives. I do this by comparing the baseline economy with two counterfactual economies. In the first, I shut down the anticompetitive motive from the firms' first order condition. In the second, I also shut down the persuasive motive. The results show that while the persuasive motive increases markups and reduces entry, it has a net positive effect by increasing consumers' taste for the advertised good. The anticompetitive motive is detrimental to output, resulting from higher markups and lower entry. However, the anticompetitive motive does not necessarily have bad aggregate consequences. This is because, as it increases total advertising expenditure, it contributes to welfare through the provision of media goods. If consumers place enough value to the entertainment provided by these media goods, this may outweigh the negative impact on consumption. The counterfactual exercise shows that if firms neglected the anticompetitive motive, total advertising expenditure would decrease by 12%. A complementary decomposition exercise based on the firms' first order condition shows that 8.45% of the (marginal) incentives to advertise are attributable to the anticompetitive motive, 33.74% to the persuasive motive, and 57.81% to the informative motive.

I also compare the decentralized equilibrium with the one resulting from solving the social planner problem, while maintaining the consumers' information frictions. A novel feature of the model is that the social planner values media goods not only because they entertain consumers, but also because, through the advertising in media, consumers get information that allows them to improve consumption. In other words, media serves as a vehicle for product awareness. Unsurprisingly, as the entertainment value of media goods increases, the social planner reallocates more labor from the production sector to the media sector. After a certain point, consumption under the planner's allocation becomes lower than under the decentralized one. When the entertainment value of media is negligible, the optimal quantity of media is lower than in the decentralized equilibrium, suggesting excessive advertising expenditure. However, this conclusion is inaccurate as we must also consider how the advertising space is allocated among firms. In other words, the 'overprovision' of media, through its effect on learning, may help mitigating the inefficiencies arising from the misallocation in the advertising space. In this direction, the exercise looking at the optimal uniform tax on advertising reveals that advertising should be subsidized.[6]

Finally, given that the informative motive declines with firm age, while the persuasive and anticompetitive motives increase, and since these motives have different aggregate implications, a natural question to ask is what the welfare gains from allowing the advertising tax to be age-dependent would be. However, in the current version, the gains from such a policy are negligible.

The paper is organized as follows: In Section 2 I introduce the model and characterize the equilibrium. Section 3 estimates the model, studies the evolution of the motives, their contribution to total advertising expenditure, and their aggregate effect. Section 4 compares the informationally-constrained social optimal equilibrium to the decentralized one. Section 5 examines advertising taxation. Finally, section 6 concludes.

Related literature. This paper relates to the literature that studies the implications of customer capital for firm, industry, and macroeconomic dynamics (e.g. Dinlersoz and Yorukoglu (2012), Gourio and Rudanko (2014), Molinari and Turino (2018), Argente et al. (2023), Einav et al. (2022), Ignaszak and Sedlacek (2023), Greenwood et al. (2024)). In these models, firms grow via increasing their idiosyncratic demand (customer capital). Together with Cavenaile et al. (2024b), we contribute to this literature by showing that it is not just the quantity of customers that matters, but also the degree of information the customers have about alternative goods. Relative to Cavenaile et al. (2024b), I allow for strategic advertising decisions as well as the interaction between firms of different sizes and ages. In Cavenaile et al. (2024b), advertising also serves to expand product awareness, but the advertising choices are coordinated at the industry level, made once and for all at industry inception, and firms are assumed to be symmetric. They focus on the improvements in targeted advertising, showing that, although better targeting technologies are a priori beneficial, as they facilitate

[6]In this exercise, I compare the stationary equilibria resulting from the different tax levels, without considering the transition.

better matches between firms and consumers, they may lead to increased market power through greater market segmentation.

This paper is also related to Greenwood et al. (2024), which studies the inefficiencies from advertising, focusing on the welfare implications of the rise of digital advertising. They present a static model of informative advertising whose equilibrium is not efficient for two reasons, which are also present in my model. The first source of inefficiency arises from firms not internalizing the social value of media goods in their advertising decisions. The second is that ads are shown to individuals who will not purchase the goods. Similarly, in my model, although consumers have heterogeneous gains from a particular ad, they all see the same ads. Here, my contribution is to include in the analysis the inefficiencies arising from markups typical of oligopoly frameworks, as well as identifying three novel sources of inefficiency. The first two are reminiscent of Schumpeterian growth models, namely: (i) lack of full appropriability, as the producers cannot extract the entire consumer surplus; and (ii) a business-stealing effect, as firms don't consider how increases in their market share reduces the competitors' market shares. The third source of inefficiency arises from the anticompetitive motive particular to this model, as firms try to avoid suffering from the business-stealing effect. Note that the sources of inefficiency push in different directions, so it is not clear whether there is too much or too little advertising, and requires a quantitative answer. Finally, there is also inefficient entry, again due to lack of appropriability and the business-stealing effect.

For the persuasive aspect of advertising, I build on the literature that adopts the persuasive view, e.g. Cavenaile et al. (2024a), Rachel (2024), Molinari and Turino (2018). These papers model advertising as a static demand shifter. A novel contribution of the current paper is the combination of the persuasive and informative views of advertising within a single framework. In particular, I relate to Cavenaile et al. (2024a), as they also develop an oligopolistic model with endogenous market structure. In their paper, they study the interaction between R&D and advertising, finding that they are substitutes at the aggregate level, consistent with the empirical findings in Cavenaile and Roldan-Blanco (2021). They also examine the question of whether advertising should be taxed or subsidized, and, in contrast to the current paper, they find a very high optimal tax. As in my model, advertising in Rachel (2024) and Greenwood et al. (2024) also finances the provision of media goods that improve utility.

Finally, I use the concept of consideration sets introduced by Manzini and Mariotti (2014) that are widely used in other fields. Manzini and Mariotti (2014) model choice as a two-stage process. In the first stage, some of the available alternatives are selected into a *consideration set*, with a probability that is linked to attention. In the second stage, the agent maximizes utility restricted to the consideration set. In macroeconomics, this concept (using the term *awareness set*) was introduced by Cavenaile et al. (2024b). My contribution relative to them is to fully endogenize the evolution of awareness sets. The presence of awareness sets complicates the firm problem, as firms need to keep track of the distribution of consumers across these sets. In their model, they abstract from this by assuming the evolution of the awareness sets is determined at industry inception, so the only state variable is industry age.

2 Model

2.1 Environment

Market structure and the production sector. There is a continuum of mass 1 of industries indexed by i. In each industry, there is a generic good and a set $\mathcal{J}_{i,t}$ of firms, indexed by j, each one producing a single differentiated good with the production function $y_{j,i,t} = N_{j,i,t}$, where $N_{j,i,t}$ is the labor employed by firm j. Note that this implies that firms will be homogeneous regarding their production technology. The generic good is produced by many small firms with the production function $y_{0,i,t} = A_0 N_{0,i,t}$, where $N_{0,i,t}$ is the total labor employed by these small firms in industry i at period t.

Advertising and the media sector. There is a media sector populated by media firms that employ labor to produce media goods that are supplied to consumers at zero monetary price and they get revenue from selling advertising space to production firms. There is free entry. Each media firm produces a differentiated variety of media good of equal quality (so, consumers will allocate the time they spend on media equally among the different media goods). The media good is available during the whole unit of time of period t (the consumer decides when to watch it and for how long) but a fraction α is advertising, which is distributed randomly across time (so the consumer cannot avoid them and will watch ads for a fraction α of the time she spends on media).[7] Within this advertising space, each industry of the production sector has its own advertising space. This is a reasonable assumption for digital advertising, where each industry is associated to some keyword and firms compete for the attention in that keyword. Further, assume all sectors have the same advertising space, $\alpha_i = \alpha$. This is a reasonable assumption for search advertising: there is only one top position for a specific keyword, so higher demand only leads to higher price, as suggested in Figure 1.
The process whereby firms acquire advertising space follows a kind of auction, where media

[7] More generally, you could think of this α as some measure of attention.

firms post a price per unit of ad space in industry i, $p_{a,i,t}$, which is the minimum bid accepted, and supply at most α_i units of ad space. Letting $e_{j,i,t}$ be the advertising expenditure of firm j in industry i, then the final price per unit of ad space in industry i will be equal to $\max\left\{p_{a,i,t}, \frac{\sum_{j\in\mathcal{J}_{i,t}} e_{j,i,t}}{\alpha_i}\right\}$. So, the advertising space acquired by firm j, $\alpha_{j,i,t}$, will be:

$$\alpha_{j,i,t} = \min\left\{\frac{e_{j,i,t}}{p_{a,i,t}}, e_{j,i,t}\frac{\alpha_i}{\sum\limits_{k\in\mathcal{J}_{i,t}} e_{k,i,t}}\right\} \tag{1}$$

The aggregate quality of media is given by $Q = AN_m^{\frac{1}{2}}$, where N_m is total labor employed in media. In order to rule out an equilibrium with no advertising expenditure and no media produced, I assume the government employs $\bar{N}_m$ units of labor in media, which is financed by a lump-sump tax to consumers.

Entry and Exit of firms. A firm is hit by a death shock with probability κ, and is independent of whether the other firms are hit by a death shock (so, the probability n firms exit is κ^n).[8] Regarding entry, there is a measure one of entrepreneurs that employ $N_{e,i,t}$ units of labor in order to create a new differentiated good in industry i with probability $z_{e,i,t} = \phi_s N_{e,i,t}^{\frac{1}{2}}$, where E_t is aggregate expenditure. Upon successfully creating a new good (and, for computational purposes, provided that the number of firms in the industry is below $\bar{J}$), a new firm enters the market and initially no consumer is aware of the new firm. Entry and exit occur simultaneously right at the start of $t+1$.

Consumers. There is a unit mass of individuals indexed by ℓ who maximize lifetime utility, where the instantaneous utility is a function of her consumption (C_ℓ) and entertainment (L_ℓ) goods. Individuals die with an exogenous probability δ, they are replaced with an offspring who inherits the assets $a_{\ell,t}$, and individuals discount the offspring's utility with the same discount rate (see the next section of product learning for the effect of dying); thus, we can write utility as if they were infinitely lived:[9]

$$U_\ell = \sum_{t=0}^{\infty} \beta^t \left[\ln C_{\ell t} + L_{\ell t}\right] \tag{2}$$

[8] I plan to do an extension where this probability is decreasing in firm size. This is a more realistic assumption, and would imply a stronger anticompetitive motive to advertise, since by precluding the competitors from expanding they are also effectively increasing the probability they exit.

[9] Note that there is no uncertainty on $C_{\ell,t}$. There is uncertainty at the industry level due to the stochastic evolution of the awareness sets (see next section), but the law of large numbers over the continuum of industries removes the uncertainty at the aggregate level.

Each individual supplies inelastically N units of labor time and chooses how much time to allocate to media goods, $T_{\ell,t}$, in order to maximise her entertainment good $L_{\ell,t}$, which is defined as follows:

$$L_{\ell t} = \upsilon \left(Q_t T_{\ell,t} - \frac{T_{\ell,t}^2}{2} \right) \tag{3}$$

where Q_t is an output of the media sector production function. Anticipating that all individuals choose the same $T_{\ell,t}$, in what follows I drop the subindex ℓ from T_t and L_t. Individual ℓ gets her C_ℓ following a Cobb-Douglas aggregator of her consumption over the continuum of industries of mass 1

$$\ln C_{\ell,t} = \int_0^1 \ln C_{\ell,i,t} di \tag{4}$$

where the industry i consumption good of individual ℓ is a CES aggregator of her consumption on the generic good and each of the differentiated goods she is aware of:

$$C_{\ell,i,t} = \left(c_{\ell,0,i,t}^{\frac{\sigma-1}{\sigma}} + \sum_{j \in \mathcal{I}_{\ell,i,t}} \omega_{j,i,t} c_{\ell,j,i,t}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} , \quad \sigma > 1 \tag{5}$$

where $c_{\ell,j,i,t}$ is the quantity of good j consumed by ℓ at t; $\mathcal{I}_{\ell,i,t}$ will be referred to as the awareness set of individual ℓ in industry i at period t, as it is the subset of the differentiated goods $\mathcal{J}_{i,t}$ the individual is aware of at period t (in the next section I describe the evolution of this object); and $\omega_{j,i,t}$ is a demand shifter which depends on the exposure of individuals to the ad of good j. In particular:[10]

$$\omega_{j,i,t} = 1 + \nu_s (\alpha_{j,i,t} T_t)^{\nu_c}, \quad \nu_c \in (0,1) \tag{6}$$

$\omega_{j,i,t} = 1 + \nu_s (\alpha_{j,i,t} T_t)^{\nu_c}$, $\nu_c \in (0,1)$. Note that the more time consumers spend on media, the larger the effect of advertising on the demand shifter.

Individual ℓ's budget constraint writes:

$$w_t N_{\ell,t} + r_t a_{\ell t} = \int \sum_{j \in \mathcal{I}_{\ell,it}} c_{\ell,j,i,t} p_{j,i,t} di + a_{\ell,t+1} - a_{\ell,t} + \tau_t \tag{7}$$

where w_t is wage, $a_{\ell t}$ is the asset holdings of individual ℓ at period t, r_t is the return of each unit of asset in period t, and τ_t is the lump-sum tax the government uses to employ $\bar{N}_m$ units of labor in the media sector. All individuals start with the same level of assets a_0.

[10] In an extension (citation), I use the functional form $\omega_{\mathcal{I},j,i,t} = \frac{1+\nu_s(\alpha_{j,i,t}T_t)^{\nu_c}}{\frac{1}{\#\mathcal{I}} \sum_{j' \in \mathcal{I}} (1+\nu_s(\alpha_{j',i,t}T_t)^{\nu_c})}$, which is an adaptation of the functional form used in Cavenaile et al. (2024a) but using the awareness set of the individual.

Product learning and the evolution of the awareness sets. I assume that the probability an individual gets aware of a product thanks to advertising is an increasing and concave function of the exposure to the ad of that good. In particular, assume an individual will get aware of product j in industry i with probability[11]

$$\rho_{j,i,t} = \hat{\rho} + \psi_s(\alpha_{j,i,t} T_t)^{\psi_c}, \quad \psi_c \in (0,1) \tag{8}$$

Although I focus on advertising as an active way through which firms can increase their customer base, consumers can get to know a firm in other ways (word-of-mouth, seeing the product in a shop...), and these are captured by $\hat{\rho}$. The inclusion of T_t is to capture the idea that the more time consumers spend on media, the more they are exposed to ads, and so the more effective advertising is, just like in the demand shifter $\omega_{j,i,t}$.

The events of learning goods are assumed to be independent; that is, the probability of learning all the goods in $\mathcal{I} \subseteq \mathcal{J}_i$ is $\prod_{j \in \mathcal{I}} \rho_{j,i}$. And, given that the complements of independent events are also independent, then the probability of not learning any of the goods in $\mathcal{I} \subseteq \mathcal{J}_i$ is $\prod_{j \in \mathcal{I}} (1 - \rho_{j,i})$.[12]

In addition, when a consumer dies, they are replaced by a newborn individual who starts knowing only the generic good of each industry (i.e. $\mathcal{I}_{\ell,i} = \emptyset$ for all i). This is completely equivalent to say that individuals forget all the differentiated goods they know with an exogenous probability δ. This assumption is not crucial for the results, and its only implication is that, even if a firm lived forever, there would always be some consumers that are not aware of it.

Then, we have all the information to find the probability of moving between any pair of awareness sets. Letting $\Theta_{(\mathcal{I} \to \mathcal{I}')}$ be the probability of moving from $\mathcal{I}$ to $\mathcal{I}'$, since the awareness set of an individual can only expand or move to containing only the generic good of the fringe, then if $\mathcal{I}'$ doesn't contain $\mathcal{I}$, then the transition is only possible (i.e. $\Theta_{(\mathcal{I} \to \mathcal{I}')} > 0$) if $\mathcal{I}' = \emptyset$, which happens with the probability of dying δ. On the contrary, if $\mathcal{I}'$ contains $\mathcal{I}$, then the probability this transition takes place is the probability an individual doesn't die, $(1-\delta)$, times the probability of learning all the goods that are in $\mathcal{I}'$ but not in $\mathcal{I}$, $\prod_{j \in \mathcal{I}' \setminus \mathcal{I}} \rho_{j,i}$, times

[11] In a version of the model, I have congestion, $\alpha_i^{-\zeta}$, $\zeta \in [0, \psi_c]$, which allows to play with the intensity of the anticompetitive motive.

[12] Given two independent events A, B, we have $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = P(A)P(B) + P(A|B^c)(1 - P(B))$, and so $P(A|B^c) = P(A)$; so: $P(A^c|B^c) = 1 - P(A|B^c) = 1 - P(A) = P(A^c)$.

the probability of not learning any of the goods that are not in $\mathcal{I}'$, $\prod_{j \notin \mathcal{I}'} (1 - \rho_{j,i})$. Formally:

$$\Theta_{(\mathcal{I} \to \mathcal{I}')} = \begin{cases} 0, & \text{if } \mathcal{I} \nsubseteq \mathcal{I}' \neq \emptyset \\ \delta, & \text{if } \mathcal{I} \nsubseteq \mathcal{I}' = \emptyset \\ (1-\delta) \prod_{j \in \mathcal{I}' \setminus \mathcal{I}} \rho_{j,i} \cdot \prod_{j \notin \mathcal{I}'} (1 - \rho_{j,i}), & \text{if } \mathcal{I} \subseteq \mathcal{I}' \neq \emptyset \\ (1-\delta) \prod_{j \in \mathcal{J}_{i,t}} (1 - \rho_{j,i}) + \delta, & \text{if } \mathcal{I} = \mathcal{I}' = \emptyset \end{cases} \tag{9}$$

2.2 Equilibrium

In this section I characterize the stationary equilibrium, that is such that the time spent in media T_t and the relative wage $\hat{w}_t = \frac{w_t}{E_t}$ are constant.

2.2.1 Consumption.

On the one hand, logarithmic preferences on $C_{\ell,t}$, together with $a_{\ell,0} = a_0$, imply that all consumers choose the same expenditure at all t: $E_{\ell,t} = E_t$. On the other hand, CES preferences over the varieties within an industry implies that consumer's spending in an industry is independent of her industry price index: $E_{\ell,i,t} = E_t$. Therefore, the awareness sets $\mathcal{I}_{\ell,i,t}$ only affect the allocation of the expenditure within each industry. That is, in order to characterize consumer ℓ's consumption choices in industry i, we only need to know her awareness set in i, $\mathcal{I}_{\ell,i,t}$. In other words, within industry i, there are as many types of consumers as subsets $\mathcal{I} \subseteq \mathcal{J}_{i,t}$. So, the set of consumer types in industry i is identified by the power set of $\mathcal{J}_{i,t}$, $\mathcal{P}(\mathcal{J}_{i,t})$, and, within an industry, I'll use subindex $\mathcal{I}$ to denote the choice of an individual with awareness set $\mathcal{I}$.
The optimal choices satisfy: [13] [14]

$$c_{\mathcal{I},j,i,t} = E_t P_{\mathcal{I},i,t}^{\sigma-1} p_{j,i,t}^{-\sigma} \omega_{j,i,t}^{\sigma}, \quad j \in \mathcal{I} \tag{10}$$

$$\frac{E_{t+1}}{E_t} = \beta(1 + r_{t+1}) \tag{11}$$

$$T_t = Q_t \tag{12}$$

[13]Together with the No-Ponzi condition $\lim_{\tau \to \infty} \frac{a_{t+\tau}}{\prod_{s=0}^{\tau}(1+r_{t+s})} = 0$.

[14]Note that $N + Q_t$ can exceed 1; this is consistent with the way media time is measured in the data where multitasking is counted separately, see Appendix 7.1.

where $P_{\mathcal{I},i,t} = \left(p_{0,i,t}^{1-\sigma} + \sum_{j\in\mathcal{I}} \omega_{j,i,t}^{\sigma} p_{j,i,t}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$.[15] Note that consumers consume a positive amount of all the goods they are aware of. And from $E_t = P_{\mathcal{I},i,t}C_{\mathcal{I},i,t}$ we see one of the channels through which (the informative) advertising will increase welfare: advertising will increase the amount of products the consumer is aware of, which reduces the price $P_{\mathcal{I},i,t}$ of her industry composite good.[16] This is a standard love for variety effect.

2.2.2 The industry state and its evolution

Given that firms have the same production technology, all the heterogeneity comes from the consumer side. The relevant state of the industry is characterised by the triple $(\mathcal{J}_{i,t}, \mathcal{P}(\mathcal{J}_{i,t}), \vec{M}_{i,t})$, where $\vec{M}_{i,t} = (M_{i,t}(\mathcal{I}))_{\mathcal{I}\in\mathcal{P}(\mathcal{J}_{i,t})}$ is the mass of consumers in each awareness set (that is, how consumers are distributed over the awareness sets). Note that since there is a mapping from $\mathcal{J}_{i,t}$ to $\mathcal{P}(\mathcal{J}_{i,t})$, I may write the state simply as $(\mathcal{J}_{i,t}, \vec{M}_{i,t})$. There are two processes that shape the evolution of the industry state.

On the one hand, the industry state changes as consumers' awareness sets evolve due to learning and death, which, by law of large numbers, is a deterministic process at the industry level.[17] Calling Θ_t the transition matrix, where the element in row r and column s indicates the probability of going from subset $\mathcal{I}_r$ to $\mathcal{I}_s$ at time t (i.e. $\Theta_{(\mathcal{I}_r\to\mathcal{I}_s)}$), and calling $\vec{M}_{i,t}$ the $2^{\#\mathcal{J}_{i,t}}$-dimensional row vector (where $\#\mathcal{J}_{i,t}$ is the cardinal of $\mathcal{J}_{i,t}$) containing the masses of consumers in each awareness set at time t; then, by the law of large numbers, the distribution of consumers in $t+1$ in the absence of entry and exit of goods, which I denote by $\vec{\hat{M}}_{i,t+1}$, would be:

$$\vec{\hat{M}}_{i,t+1} = \vec{M}_{i,t}\Theta_t \tag{13}$$

On the other hand, the industry state changes stochastically due to entry and exit of firms. If the realisation of exit and entry changes the set of firms in industry i from $\mathcal{J}$ to $\mathcal{J}'$, then the next period industry state is obtained using the application $(\mathcal{J}, \vec{\hat{M}}, \mathcal{J}') \longmapsto (\mathcal{J}', \vec{M}')$ defined as follows.

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[15] Note that consumers may not only have different industry price index, $P_{\ell,i,t}$, but also a different aggregate price index $P_{\ell,t} = exp\left(\int_0^1 \ln P_{\ell,i,t} di\right)$. In particular, as explained in section 2.2.8, individuals with the same age have the same aggregate price index, and the numeraire of the economy is the geometric mean of $P_{\ell,t}$.

[16] Formally, since $\sigma > 1$ and $p_{k,i,t}, \omega_{k,i,t} > 0$, we have $P_{\mathcal{I}\cup k,i,t} = \left(p_{0,i,t}^{1-\sigma} + \sum_{j\in\mathcal{I}} \omega_{j,i,t}^{\sigma} p_{j,i,t}^{1-\sigma} + \omega_{k,i,t}^{\sigma} p_{k,i,t}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} < \left(p_{0,i,t}^{1-\sigma} + \sum_{j\in\mathcal{I}} \omega_{j,i,t}^{\sigma} p_{j,i,t}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} = P_{\mathcal{I},i,t}$.

[17] In case there were mixed strategies (although this is not the case in the equilibrium studied) this process would be stochastic.

$$\text{For } \mathcal{I}' \in \mathcal{P}(\mathcal{J}'), \qquad M'(\mathcal{I}') = \begin{cases} \sum\limits_{\{\mathcal{I} \in \mathcal{P}(\mathcal{J}) : \mathcal{I} \cap \mathcal{J}' = \mathcal{I}'\}} \hat{M}(\mathcal{I}) & , \text{if } \mathcal{I}' \subseteq \mathcal{J} \\ 0 & , \text{if } \mathcal{I}' \nsubseteq \mathcal{J} \end{cases} \tag{14}$$

where the first case says that two consumers become identical in industry i if all the firms in which they differed exit, whereas the second case says that there are no consumers who are aware of a newborn firm. The last piece of infornation needed to compute expected values is the probabilities that the set of differentiated goods moves from $\mathcal{J}$ to $\mathcal{J}' \subseteq \mathcal{J} \cup \{e\}$, where e denotes an entrant. These probabilities are given by:

$$\text{For } \mathcal{J}' \in \mathcal{P}(\mathcal{J} \cup e)), \qquad Prob\{\mathcal{J} \to \mathcal{J}'\} = \begin{cases} (1 - z_{e,i,t}) \prod\limits_{j \in \mathcal{J} \cap \mathcal{J}'} (1-\kappa) \prod\limits_{j \in \mathcal{J} \setminus \mathcal{J}'} \kappa & , \text{if } e \notin \mathcal{J}' \\ z_{e,i,t} \prod\limits_{j \in \mathcal{J} \cap \mathcal{J}'} (1-\kappa) \prod\limits_{j \in \mathcal{J} \setminus \mathcal{J}'} \kappa & , \text{if } e \in \mathcal{J}' \end{cases} \tag{15}$$

where $z_{e,i,t}$ is the probability of an entrant, $\prod\limits_{j \in \mathcal{J} \cap \mathcal{J}'} (1-\kappa)$ is the probability that all the firms in $\mathcal{J} \cap \mathcal{J}'$ survive, and $\prod\limits_{j \in \mathcal{J} \setminus \mathcal{J}'} \kappa$ is the probability that all the firms in $\mathcal{J} \setminus \mathcal{J}'$ exit.

In the Appendix 7.3 I show that assuming that individuals don't die (i.e. $\delta = 0$) allows a simpler sufficient industry state given by the vector of customer bases. That is, instead of requiring the mass of consumers in each awareness set, we would only need to know the mass of consumers aware of each good.

2.2.3 Production firms problem

On the one hand, given the large number of small firms producing a homogeneous product, the price of the generic good is equal to its marginal cost, $p_{0,i,t} = \frac{w_t}{A_0}$.

On the other hand, the differentiated firms compete in prices a la Bertrand and in advertising expenditures for the attention of consumers. Both decisions are made simultaneously.

Profits. Using the production function $y_{j,i,t} = N_{j,i,t}$, we can express profits decomposed as

$$\pi_{j,i,t} = \underbrace{M_{j,i,t}}_{\substack{\text{Customer} \\ \text{Base}}} \cdot \underbrace{(1 - \mathcal{M}^{-1}_{j,i,t}) \underbrace{\sum_{\mathcal{I} \in \mathcal{P}_j(\mathcal{J}_{i,t})} \frac{M_{i,t}(\mathcal{I})}{M_{j,i,t}} s_{\mathcal{I},j,i,t} E_t}_{\substack{\text{Average spending} \\ \text{by customers}}}}_{\text{Average rents from customers}} \tag{16}$$

where $\mathcal{M}_{j,i,t} = \frac{p_{j,i,t}}{w_t}$ is the markup of firm j, $s_{\mathcal{I},j,i,t} = \frac{p_{j,i,t}c_{\mathcal{I},j,i,t}}{E_t}$ is type $\mathcal{I}$ individual's share of expenditure in good j, $\mathcal{P}_j(\mathcal{J}_{i,t}) = \{\mathcal{I} \in \mathcal{P}(\mathcal{J}_{i,t}) \mid j \in \mathcal{I}\}$ is the family of awareness sets containing good j, and $M_{j,i,t} = \sum_{\mathcal{I}\in\mathcal{P}_j(\mathcal{J}_{i,t})} M_{i,t}(\mathcal{I})$ is the customer base of firm j.

This expression offers a first intuition of the motives driving firms to advertise. First, they want to advertise to increase their customer base. I refer to this as the informative motive. Second, as shown in the Appendix 7.4, all else equal, firms prefer to have customers that know as fewer competitors as possible. Intuitively, the fewer alternative goods they know, the more they will spend in j (i.e. higher $s_{\mathcal{I},j,i,t}$) and the lower their demand elasticity (so, the firm is able to extract more rents by rising the markup). So, given that by increasing the advertising space they occupy, firms reduce the attention of consumers to the competitors' goods and so the probability they will add them to their awareness sets; then, firms may have the incentive to do advertising for the mere purpose of reducing the mass of customers who learn about competitors. I refer to this as the anticompetitive motive, as under this motive the firm is doing advertising to avoid competition by precluding competitors to expand their customer base. Finally, given that the demand shifter $\omega_{j,i,t}$ increases with the the advertising space, firms want to do advertising to persuade current consumers to buy more. This is the persuasive motive.

Price setting. Given that, as standard, I focus on Markov-Perfect equilibrium (so, decisions just depend on the current state, not on history), and given that the price has no direct effect on the evolution of the industry state and that advertising and price choices are made simultaneously, then the price setting problem is static. The optimal markup $\mathcal{M}_{j,i,t}$ is such that profits (16) are maximised, given its own demand-shifter $\omega_{j,i,t}$, the markups and demand-shifters of the competitors $\{\mathcal{M}_{k,i,t}, \omega_{k,i,t}\}_{k\in\mathcal{J}_{i,t},k\neq j}$, and the distribution of consumers over the awareness sets $\vec{M}_{i,t} = (M_{i,t}(\mathcal{I}))_{\mathcal{I}\in\mathcal{P}(\mathcal{J}_{i,t})}$, and taking into account that individuals' spending shares are given by

$$s_{\mathcal{I},j,i,t} = \left[(A_0\mathcal{M}_{j,i,t})^{\sigma-1}\omega_{j,i,t}^{-\sigma} + \sum_{k\in\mathcal{I}} \left(\frac{\omega_{k,i,t}}{\omega_{j,i,t}}\right)^{\sigma} \left(\frac{\mathcal{M}_{j,i,t}}{\mathcal{M}_{k,i,t}}\right)^{\sigma-1} \right]^{-1} \tag{17}$$

The equilibrium markups are given by:

$$\mathcal{M}_{j,i,t} = \frac{\frac{\sigma}{\sigma-1} - \bar{s}_{j,i,t}}{1 - \bar{s}_{j,i,t}} \tag{18}$$

Where $\bar{s}_{j,i,t} = \sum_{\mathcal{I}\in\mathcal{P}_j(\mathcal{J}_{i,t})} \frac{M_{i,t}(\mathcal{I})p_{j,i,t}c_{\mathcal{I},j,i,t}}{p_{j,i,t}y_{j,i,t}} s_{\mathcal{I},j,i,t}$ is a sales-weighted average of firm j customers' share of expenditure in industry i allocated to good j.

Note that in a standard oligopoly model with Bertrand competition, the optimal markup is given be the expression in (18) but with the market share $s_{j,i,t}$ instead of $\bar{s}_{j,i,t}$. So, while the optimal markup in a standard oligopoly model with Bertrand is increasing with size, this is not necessarily the case here. Here, the markup depends on the composition of the customers, not in the size: a smaller firm can have a higher markup if a larger fraction of its customers spend a larger share of expenditure on it. However, the model will still predict that, within an industry, larger firms have higher markups. The intuition is as follows: a firm that entered earlier had more time to accumulate customers (so older firms will be larger); but also, since as time passes consumers get aware of more goods and advertising is undirected, then a firm that enters later will get consumers that, on average, know more goods (and we have seen that customers with more alternative goods spend a smaller share). So, within an industry, larger firms will have customers that on average spend a larger share of expenditure, and thus they set higher markups.

Advertising choice. Each firm chooses dynamically its advertising expenditure $e_{j,i,t}$, taking into account (i) the advertising expenditure choices of its competitors $\{e_{k,i,t}\}_{k\in\mathcal{J}_{i,t},k\neq j}$; (ii) markups $\{\mathcal{M}_{k,i,t}\}_{k\in\mathcal{J}_{i,t}}$; (iii) the time consumers spend on media, T_t; (iv) the law of motion of the industry state; and (v) that the actual advertising space purchased by each firm is given by (1). In practice, given that in equilibrium $p_{a,i,t}$ is such that total advertising expenditure in industry i exactly purchases α_i units of ad space, then, in all industries with more than one differentiated firms, $\alpha_{j,i,t}$ will be given by $\alpha_{j,i,t} = e_{j,i,t}\frac{\alpha_i}{\sum\limits_{k\in\mathcal{J}_{i,t}} e_{k,i,t}}$, and so there will be an anticompetitive motive to advertise: by increasing $e_{j,i,t}$, firm j will achieve to increase the actual price for advertising space and thus reduce the advertising space of competitors, which will reduce the probability consumers learn about competitors. In industry states with only one differentiated firm, there is trivially no anticompetitive motive because there is no competitor and so the unique firm has no incentive to spend more than $p_{a,i,t}\alpha_i$, and so in such industry states $\alpha_{j,i,t}$ will be given by $\alpha_{j,i,t} = \frac{e_{j,i,t}}{p_{a,i,t}}$.
Given that I focus on Markov-Perfect equilibrium, then the firm problem can be expressed in recursive form, with the value of the firm being a function of the state. Given that profits are linear on E_t, by guess and verify, the value of the firm is also linear on E_t. Therefore, defining $V_j(\mathcal{J}_{i,t}, \vec{M}_{i,t}) = \frac{V_{j,i,t}}{E_t}$, $\hat{e}_{j,i,t} = \frac{e_{j,i,t}}{E_t}$, $\hat{p}_{a,i,t} = \frac{p_{a,i,t}}{E_t}$ and $\pi_j(\omega_{j,i,t}, \mathcal{J}_{i,t}, \vec{M}_{i,t}) = \frac{\pi_{j,i,t}}{E_t}$ and using the Euler equation and that in the stationary equilibrium it will be $T_t = T$, we can write the dynamic firm problem recursively as

$$V_j\Big(\mathcal{J}, \vec{M}\Big) = \max_{\hat{e}_j}\Big\{\pi_j\Big(\omega_j, \mathcal{J}, \vec{M}\Big) - \hat{e}_j + \beta\mathbb{E}V_j\Big(\mathcal{J}', \vec{M}'\Big)\Big\}$$

$$\text{s.t. } \{\hat{e}_k\}_{k\in\mathcal{J}\setminus\{j\}},\ \{\mathcal{M}_k\}_{k\in\mathcal{J}},\ T,\ (8),\ (9),\ (13),\ (31),\ (32),\ (1)$$

We can decompose the FOC into the three motives to advertise: the informative motive (increase ρ_j), the anticompetitive motive (decrease $\rho_{j'}$, $j' \neq j$), and the persuasive motive (increase $s_{\mathcal{I},j,i}$ for $\mathcal{I} \in \mathcal{P}_j$):

$$1 = \frac{\partial \pi_{j,i}}{\partial e_j} + \frac{\partial V_j}{\partial \rho_j}\frac{\partial \rho_j}{\partial \alpha_j}\frac{\partial \alpha_j}{\partial e_j} + \sum_{j' \neq j}\left(-\frac{\partial V_j}{\partial \rho_{j'}}\right)\frac{\partial \rho_{j'}}{\partial \alpha_{j'}}\left(-\frac{\partial \alpha_{j'}}{\partial e_j}\right) \tag{19}$$

In section 3.2 I show how the intensity of the three motives evolve with firm age and size. The intuition is clear. Smaller or younger firms (that is, those firms that are unknown for most consumers) are the ones that have more potential customers to acquire. In the extreme case, a firm that was known by all consumers wouldn't have any incentive to advertise to inform consumers. On the contrary, the anticompetitive motive, which is about retaining market power over the current customers by reducing the probability they learn about competitors, is stronger the bigger the customer base. A firm that is unknown for everybody also has some incentives to preclude consumers from learning about other firms (as it is internalising that these consumers may eventually become customers and so the firm wants consumers to know as fewer goods as possible), but intuitively, the incentives to avoid a consumer learning about a competitor are higher if the consumer is a current customer rather than a potential one. Finally, the persuasive motive also tends to be bigger in older/bigger firms. Intuitively, if advertising persuades current customers to spend more, then the increase in revenues will be larger if there are more customers.

2.2.4 Entrepreneurs problem.

The entrepreneurs in an industry $(\mathcal{J}, \vec{M})$ choose N_e to maximise their expected value:

$$v^e(\mathcal{J}, \vec{M}) = \max_{N_e}\left\{-N_e\hat{w} + \beta z_e \mathbb{E}_e V_e\Big(\mathcal{J}' \cup \{e\}, \vec{M}'\Big)\right\}, \text{ s.t. } z_e = \phi_s N_e^{\frac{1}{2}}$$

where $\beta\mathbb{E}_e V_e\Big(\mathcal{J}' \cup \{e\}, \vec{M}'\Big)$ is the expected value of being a new firm conditional on successfully creating a new differentiated good (so, the expectation comes from the uncertainty on which of the $\mathcal{J}$ incumbents will survive). Then, the equilibrium labor employed in entry in an industry $(\mathcal{J}, \vec{M})$ will be:

$$N_{e,(\mathcal{J},\vec{M})} = \left(\beta\frac{\phi_s}{2\hat{w}}\mathbb{E}_e V_e\Big(\mathcal{J}' \cup \{e\}, \vec{M}'\Big)\right)^2 \tag{20}$$

2.2.5 Stationary distribution.

In the Appendix 7.8 I prove that, for any aggregates $\hat{w}$ and T given, with their associated solutions of the firms and entrepreneurs problems $\{\alpha_{j,(\mathcal{J},\vec{M})}, N_{e,(\mathcal{J},\vec{M})}\}$, the probability that an industry is at a given state $(\mathcal{J}, \vec{M})$ converges to an ergodic distribution (existence), which is independent of the initial state (uniqueness), and satisfies that the set of different states realised, call it Ω, is at most countably infinite.[18]

By Law of large numbers, this implies that the economy converges to a stationary distribution associated to the aggregates $\hat{w}, T$. Let $\mu_{(\mathcal{J},\vec{M})}$ be the mass of industries in state $(\mathcal{J},\vec{M}) \in \Omega$ in this stationary distribution. If $\hat{w}, T$ are consistent with this stationary distribution, then we are in the stationary equilibrium.

Computationally, the stationary distribution is a complicated object, and the method used to obtain it, described in the Appendix 7.9, is a computational contribution of this paper.

2.2.6 Media sector problem.

Given the symmetry of media firms, consumers allocate their media time T equally among the media firms, and production firms allocate their advertising expenditure equally among the media firms. Therefore, all media firms have the same profits, and so each media firm has positive profits if and only if the overall profits in the media sector are positive. Then, since there is free entry into the media sector, profits in the media sector must be zero in equilibrium; so, the equilibrium Q_t satisfies:

$$\int_0^1 \sum_{j\in\mathcal{J}_{i,t}} \hat{e}_{j,i,t} E_t di + w_t \bar{N}_m - w_t \left(\frac{Q_t}{A}\right)^2 = 0 \tag{21}$$

where recall that $\bar{N}_m$ is the labor in media employed by the public sector. In the stationary equilibrium, $Q_t = Q$ is constant.

2.2.7 Labor market clearing.

The labor market must clear, that is, the amount of labor supplied has to be equal to the labor demanded by the production firms, media firms and entrepreneurs. Without any loss of generality (just a change in the units we measure labor), I can normalize labor supply N to 1.

$$1 = N = \int_0^1 \left(\sum_{j\in\{0\}\cup\mathcal{J}_{i,t}} N_{j,i,t} + N_{e,i,t} \right) di + N_{m,t} \tag{22}$$

where $N_{j,i,t} = \frac{s_{j,i,t}}{\mathcal{M}_{j,i,t}} \hat{w}^{-1}$, and $N_{e,i,t}$ and $N_{m,t}$ are given by (20) and (21), respectively. This pins down the equilibrium relative wage $\hat{w}_t$, and verifies that it is constant in the stationary equilibrium.

[18] Note that I haven't proved whether the solution of the firms and entrepreneurs problems is unique. One could prove unicity by imposing restrictions on how the players make their decisions.

2.2.8 Aggregate output and representative consumer conditional on age.

Define the aggregate consumption good as the geometric mean of the individuals' aggregate consumption goods; that is $\ln C_t = \int_0^1 \ln C_{\ell,t} d\ell$. Using the definitions of $C_{\ell,t}$ and $C_{\ell,i,t}$, together with $c_{\ell,j,i,t} = \frac{s_{\ell,j,i,t}}{\mathcal{M}_{j,i,t}} \hat{w}^{-1}$ and $c_{\ell,0,i,t} = s_{\ell,0,i,t} A_0 \hat{w}^{-1}$, and interchanging the integrals over ℓ and i, we obtain the level of the consumption good:

$$\ln C = -\ln \hat{w} + \sum_{(\mathcal{J},\vec{M})\in\Omega} \mu(\mathcal{J},\vec{M}) \sum_{\mathcal{I}\in\mathcal{P}(\mathcal{J})} M(\mathcal{I}) \frac{\sigma}{\sigma-1} \ln \left((s_{\mathcal{I},0,(\mathcal{J},\vec{M})} A_0)^{\frac{\sigma-1}{\sigma}} + \sum_{j\in\mathcal{I}} \omega_{j,(\mathcal{J},\vec{M})} \left(\frac{s_{\mathcal{I},j,(\mathcal{J},\vec{M})}}{\mathcal{M}_{j,(\mathcal{J},\vec{M})}} \right)^{\frac{\sigma-1}{\sigma}} \right) \tag{23}$$

The aggregate price index of the economy is P_t such that $P_t C_t = E_t$, and it is the numeraire (i.e. $P_t = 1$). GDP in the economy is given by $Y = E + \sum_{(\mathcal{J},\vec{M})\in\Omega} \mu_{(\mathcal{J},\vec{M})} N_{e,(\mathcal{J},\vec{M})} \hat{w} E$.

Finally, note that applying a law of large numbers to the continuum of industries, two consumers with the same age will have the same level of aggregate consumption good. That is, although they will differ on their awareness sets for particular industries, at the aggregate level they will have the same distribution of awareness sets. This points to a potentially interesting extension where firms can target consumers based on the observable age.

3 Quantitative and empirical analysis

3.1 Calibration

In this section, I describe the calibration of the model, and the details of the data sources and how the moments are computed are provided in the Appendix 7.1.

One of the main components of the model is firms' customer base accumulation, which has a strong relationship with firm size, both in the model and in the data, as pointed by the empirical literature cited in the introduction. Therefore, it is important that the model reproduces the evolution of the average firm sales growth by age, in order to calibrate this customer base building process. In particular, I target the average firm sales growth at age 1, and the slope of the fitted line that has no regression error for age 1 (I plan to do another calibration where, instead, I minimise the sum of squared errors between the data and the model for each age). Also, as shown in section 3.2, the intensity of the different motives to advertise varies with firm size, so the coefficient from a regression of advertising expenditure and sales is a good candidate to discipline the model.

To compute these three moments I use Compustat data. Given that firms typically enter Compustat a few years after their foundation (and certainly not with zero customers as it is assumed in the model for new firms), for the computation of the model-implied moments of these three targets, I assume that firms in the model are unobserved until they are at least two years old.

I estimate the model for the US at an annual frequency and set the consumer discount rate to $\beta = 0.98$. I also set (i) $\delta = 0.01$ corresponding to the mortality rate of 1% in the data, (ii) the concavity parameter for the persuasive advertising $\nu_c = 0.2972$ is taken from (the inverse) Cavenaile et al. (2024a), (iii) given that public sector spending on media represents roughly 0.008% of US GDP, dividing this by the (capital-adjusted) labor share, I set $\bar{N}_m = \frac{8 \cdot 10^{-5}}{0.8359}$, and (iv) I set $\kappa = 0.1151$ corresponding to the entry rate in the data. Acknowledging the difficulty to find good proxies for the utility value of media goods, I leave the weight of the entertainment good on the utility function, υ, uncalibrated and all the exercises involving welfare are made for a range of values of υ. This leaves 8 moments to estimate: the elasticity of substitution parameter, σ; the relative productivity of the small firms producing the generic goods, A_0; the scale parameter for the persuasive effect of advertising, ν_s; the scale and convexity parameters for the informative effect of advertising, (ψ_s, ψ_c); the exogenous learning probability, $\hat{\rho}$; the scale parameter regulating the creation of new products, ϕ; and the aggregate productivity of the media sector, A. Apart from A, which can be derived directly from 21 using the target values for aggregate advertising expenditure and labor shares and the fraction of time in media, the rest of the 7 parameters are estimated jointly through a Simulated Method of Moments estimation procedure. Apart from the three moments described above concerning the average firm growth by age and the relationship between advertising and sales, at the aggregate level, I target the sales-weighted average markup, the aggregate advertising expenditure as a percentage of GDP, the firm entry rate, the fraction of time spent in media, and the labor share. Given that there is no physical capital in the model, for comparability, I take the labor share as the share of labor income among labor income and profits, following Cavenaile et al. (2024a). Table 1 summarises the results of the calibration. Panel A reports the parameter values, while Panel B reports both the model-implied moments and the empirical ones. The model does well in matching the moments. In addition to the targeted moments, the calibrated model also features an inverted-U relationship between advertising expenditure and relative sales as documented in Cavenaile et al. (2024a).

Note that Compustat is not the ideal dataset to discipline the growth pattern of firms in the model for the following reasons. First, age don't automatically enter Compustat when they are born, and they may enter at different stages of the life cycle. Second, contrary to the model, firms may grow through expanding to new geographical markets or new product lines. Figure 2 plots the average firm sales growth rate both in the model and in the data. Note that, in the model, if a firm had a constant $\rho_{j,i}$ (this is the case of a firm that

Table 1: Parameter values and targeted moments

A. Parameters

	Parameter	Description	Calibration	Value
Preferences	β	Discount rate	External	0.98
	σ	CES consumption	Internal	7.0438
	υ	weight of leisure	Uncalibrated	-
Persuasive	ν_s	Scale parameter	Internal	0.2548
	ν_c	Convexity parameter	External	0.2972
Learning	ψ_s	Scale parameter	Internal	0.1238
	ψ_c	Convexity parameter	Internal	0.1988
	$\hat{\rho}$	Exogenous learning	Internal	0.0058
	δ	Mortality rate	External	0.01
Media sector	A	productivity media firms	Internal	3.2087
	$\bar{N}_m$	public sector media	External	$9.5705 \cdot 10^{-5}$
Generic good	A_0	Productivity	Internal	0.4063
Entry/Exit	κ	Exit rate	External	0.1151
	ϕ	Entry scale	Internal	0.4917

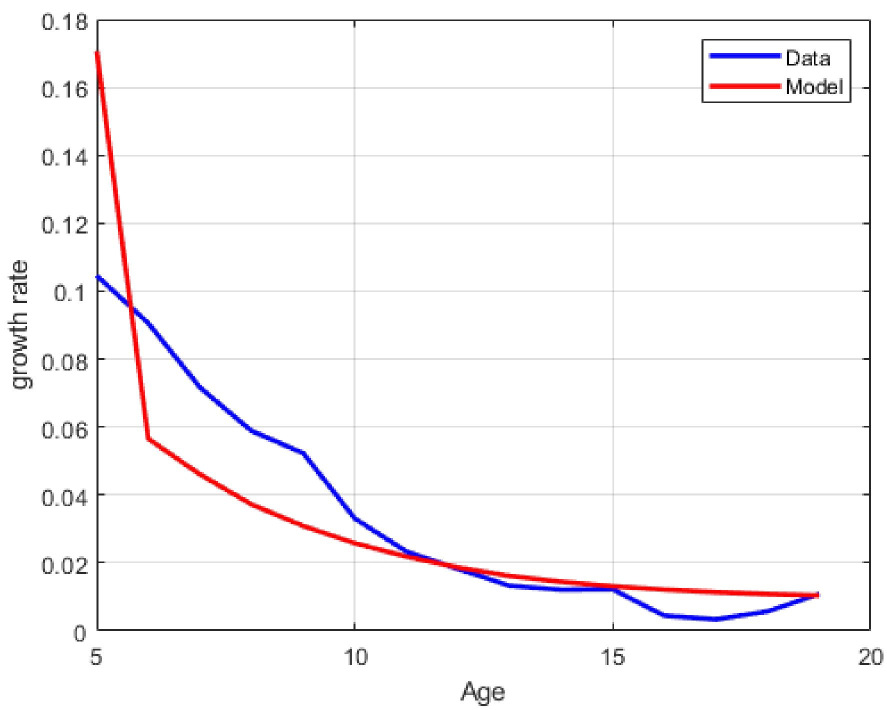
B. Moments

Moment	Data value	Model value
Sales-weighted average markup	1.3498	1.3560
Labor share (capital-adjusted)	0.8359	0.8375
Advertising/GDP	2.2%	2.1931%
Fraction of time in media	0.552	0.552
Entry rate	0.1151	0.1154
Average firm growth at age 1	16.5889%	12.3002
Slope firm growth by age	-0.8244	-0.8470
Coefficient advertising vs sales	0.6710	0.6710

Notes. Panel A reports the parameter values. Panel B reports the simulated and empirical moments. Details on data sources and how this moments are computed can be found in the Appendix 7.1

has always been the single differentiated firm of the industry, which I could show as a case study), then growth would be monotonically decreasing, pushed by a mechanical force: given that the population is constant, as the firm's customer base expands, the growth rate slows because (i) a given increase in customers has a smaller relative impact, and (ii) there are fewer non-customers left. Things get noisier when there are other competitors, and there is entry and exit. Given that advertising tends to increase with size (as it is targeted), and size is positively related with age; then, older firms will tend to occupy a larger share of the ad space, which increases growth through two channels: increasing the probability non-customers learn about the good, and encouraging current customers to spend more due to the persuasive aspect of advertising. This explains why in the model we observe that average firm growth is first decreasing from 95% at age 1 (unreported for readability of the plot) to 12.3% at age 2 (as the mechanical force dominates because the firm is initially very small), then increasing (due to the positive relationship of advertising and size), and finally decreasing (as eventually the mechanical force dominates).

Figure 2: Average firm growth by age

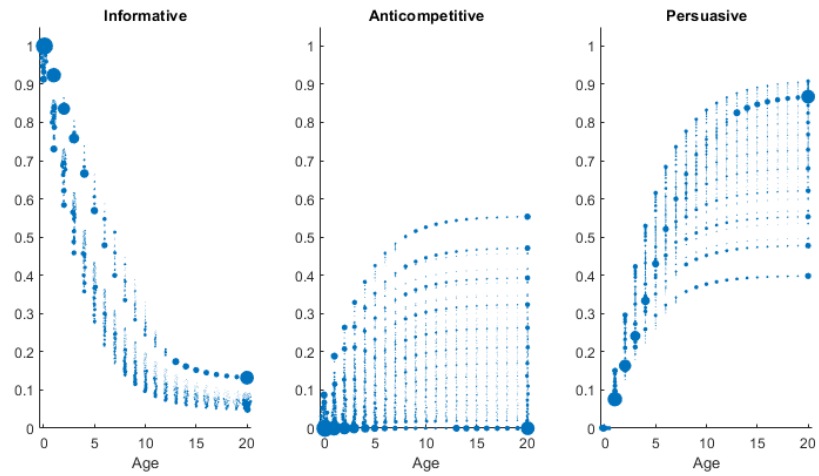


Notes. This figure displays the average firm sales growth by age both in the data (blue) and in the model (red). Given that firms typically enter Compustat a few years after their foundation, for comparability I assume age 1 in Compustat corresponds to age 2 in the model.

3.2 Relationship of the motives with firm age and size

The decomposition of the FOC of advertising expenditure in 19 allows us to see the share of the firm's marginal value of advertising coming from each of the three motives. Using this observation, Figure 3 displays the share of the marginal value of advertising attributable to

Figure 3: Marginal intensity of the advertising motives by age



Notes. This figure displays the values from the terms of the FOC corresponding to the informative motive (left panel), the anticompetitive motive (middle panel) and the persuasive motive (right panel), for all the firms in the stationary equilibrium, where the relative size of each dot indicates the share of this firm type in the stationary distribution.

each of the three motives for all the firms in the stationary equilibrium. Three observations can be drawn. First, that there is significant heterogeneity, which indicates that age is far from being a sufficient statistic. This shows that industry dynamics play a key role (i.e. competition matters). Second, despite the variability, it can be observed that the informative motive is negatively associated with age, while the anticompetitive and the persuasive motives are positively associated with it. Finally, another aspect that can be observed from the figure is that most of the action in terms of changes in the share of each motive seems to be concentrated in the first five years. This is connected to the concern raised in the calibration section regarding the targets for the evolution of the average firm growth by age, which implies that firms in the model grow too much in the early stages (with the alternative targeting strategy, firms should display a growing pattern more aligned with the data).

Because of the positive link between age and firm size (either sales or customer base), we obtain qualitatively similar plots when firm size is used instead of age. This numerical result is further supported by the following analytical result:

Proposition 1 *If distribution $\vec{M}_2$ is obtained from $\vec{M}_1$ by adding $\{j\}$ to the awareness sets of some consumers (that is, formally, if $\vec{M}_1$, $\vec{M}_2$ satisfy $M_2(\mathcal{I}\cup\{j\})-M_1(\mathcal{I}\cup\{j\}) = M_1(\mathcal{I})-M_2(\mathcal{I}) \geq 0$ for every $\mathcal{I} \in \mathcal{P}_{-j} = \{\mathcal{I} \in \mathcal{P} | j \notin \mathcal{I}\}$), then (for now, the proof is keeping the advertising choices fixed):*

1. *Firm j's informative motive is smaller in $\vec{M}_2$. That is, the informative motive is stronger in smaller firms.*

2. *Firm j's anticompetitive and persuasive motives are bigger in $\vec{M}_2$. That is, both the anticompetitive and the persuasive motives are stronger in bigger firms.*

Proof. See the Appendix 7.5 ■

3.3 Counterfactuals shutting the persuasive and/or the anticompetitive motives

How does each of the three motives affect the aggregates? Is the anticompetitive motive necessarily bad? This section addresses these questions. To do so, I compare the baseline economy with two counterfactual ones. The first one is an economy where firms neglect the anticompetitive motive; that is, they don't internalise that by increasing their advertising expenditure they are effectively reducing the amount of consumers who learn about competitors. To be precise, this is done by shutting down the anticompetitive component from the firm's first order condition. In the second counterfactual, in addition, firms also neglect the persuasive motive; that is, they don't internalise that advertising increases current customers' spending.

Note that this exercise does not indicate how much of the advertising in the baseline is anticompetitive, for example. For this, one should compare the advertising share in the baseline against the share that would result if firms made the advertising choices as in the first counterfactual, while keeping the stationary distribution of the baseline (I plan to do this decomposition exercise). Instead, the current exercise illustrates what the economy would look like if firms neglected some of the motives to advertise; that is, this negligence changes firms' decisions, but this in turn also has general equilibrium consequences.

Table 2 reports some relevant statistics for both counterfactuals and the benchmark. The second line shows output assuming that the persuasive advertising is deceiving (that is, consumers make their purchasing decisions based on $\omega_{j,i,t}$, but then they derive utility as if $\omega_{j,i,t} = 1$).

On the one hand, shutting down the anticompetitive motive leads to a higher final output. As the intuition would suggest, without the anticompetitive motive, smaller firms face less competition for advertising space, and so they can grow faster, increasing competition and, thus, lowering markups. In addition, this improvement in growing prospects increases the incentives of entrepreneurs to create new products, pushing the entry rate up. On the negative side, removing one incentive to advertise decreases the demand for advertising, which

Table 2: Comparison of counterfactuals with firms neglecting the anticompetitive and/or the persuasive motives

	Benchmark	No Anticompetitive	Only Informative
C	0.7260	0.7268	0.7226
C no taste shifter	0.6898	0.6916	0.6935
Q	0.5150	0.4860	0.3536
Adv/GDP	2.1535	1.9137	1.0054
w	0.8392	0.8376	0.8345
Avg Number of Firms	1.4148	1.4233	1.4406
Sales-Weighted Average Markup	1.3281	1.3260	1.3135
Coefficient advertising vs market share	0.6501	-2.5961	-15.0744

Notes. In the 'No Anticompetitive' counterfactual, firms make their decisions neglecting the anticompetitive motive, and in the 'Only Informative' counterfactual, firms neglect both the anticompetitive and the persuasive motives.

reduces aggregate advertising expenditure and thus lowers the quality of media. Therefore, although the anticompetitive motive has negative effects for final output, it is not necessarily the case that welfare would be higher in a counterfactual economy without it, due to its contribution on the provision of media goods.

On the other hand, the counterfactual where, in addition, the persuasive motive is shut down, indicates that the persuasive motive overall has a positive effect on output. The second line shows that this positive effect arises from its impact on the taste shifters in preferences. Similar to the anticompetitive motive, shutting down the persuasive motive allows smaller firms to get a larger share of the advertising space, which pushes the entry rate up and markups down. Additionally, because firms are losing out on profits by not considering that advertising would lead to higher customer spending and higher market power, this further reduces markups but diminishes the increase in entry incentives.

Finally, the last row of the table provides further validation to the previous section result that both the anticompetitive and the persuasive motives are positively associated with size. As expected, as we shut down each of these two motives, since this reduces the incentives to advertise relatively more for bigger firms, the coefficient of the regression of advertising expenses on market share decreases, and becomes negative in the counterfactual economy with only the informative motive.

4 Social planner problem

The planner aims to maximize the aggregate utility (all individuals are weighted the same). The planner has full control on the production, media, and entrepreneurial decisions but cannot affect consumers' behaviour; that is, the learning process and the consumption and media time choices are as in the decentralised equilibrium. Formally, the planner solves:

$$\max_{\{N_{M,t},N_{j,i,t},h_{e,i,t},p_{j,i,t},\alpha_{j,i,t}\}} U = \sum_{t=0}^{\infty} \beta^t \int_0^1 [\ln C_{\ell t} + L_{\ell t}]\, d\ell$$

s.t. $C_{\ell,t}$ from (4), $C_{\ell,i,t}$ from (5), $c_{\ell,j,i,t}$ from (10), and $L_{\ell,t}$ from 3, with $T_t = Q_t$ (Consumer choices)

$$y_{j,i,t} = N_{j,i,t}, \quad y_{0,i,t} = A_0 N_{0,i,t}, \quad Q_t = A N_{m,t}^{\frac{1}{2}} \quad \text{(Production functions)}$$

$$1 = N_{m,t} + \int_0^1 \left(\sum_{j\in\{0\}\cup\mathcal{J}_{i,t}} N_{j,i,t} + N_{e,i,t} \right) di \quad , \quad w_t = E_t \quad \text{(Resource constraints)}$$

$$\sum_{j\in\mathcal{J}_{i,t}} \alpha_{j,i,t} = \alpha_i, (8), (9), (13), (31), (32) \quad \text{(Learning process)}$$

$$z_{e,i,t} = \phi N_{e,i,t}^{\frac{1}{2}}, (31), (32) \quad \text{(Entry and exit)}$$

I leave the details of the solution in the Appendix 7.7. The planner sets prices equal to marginal cost times a markup (or a tax) that allows the planner to pay for the labor to produce the media goods and for entry. That is, $p_{j,i,t} = \tau \frac{w_t}{A_j}$, with $\tau = \frac{E_t}{w_t N_t^P}$, where N_t^P is the labor used in the production sector.

For the dynamic problem of advertising and media, as in the baseline model, I focus on the stationary Markov-Perfect equilibrium. The social planner has to decide on (i) how to allocate the ad space among the differentiated firms of each industry, $\alpha_{j,i,t}$, (ii) how much labor to allocate to the media sector, $N_{m,t}$, and (iii) how much final good to allocate to creating new products in each sector.

First, let's see the social planner choice of $\alpha_{j,i,t}$. The allocation of the ad space has to be such that the marginal social gain of increasing the ad space given to each firm is the same, since otherwise we could improve the allocation. Formally, letting $\ln C_{i,t} = \int_0^1 \ln C_{\ell,i,t} d\ell$ be the total consumption good of industry i, and $U_X = \sum_{t=0}^{\infty} \beta^t \mathbb{E} \ln C_{i,t}$ be the expected life-time utility derived from an industry whose current state is X; it must be $\frac{\partial \ln C_{X,t}}{\partial \alpha_{j,X}} + \beta \frac{\partial \mathbb{E} U_{X'}}{\partial \rho_{j,i}} \frac{\partial \rho_{j,i}}{\partial \alpha_{j,i}} = \hat{h}_X$ for some $\hat{h}_X$ and all $j \in \mathcal{J}_X$, together with $\sum_{j\in\mathcal{J}_X} \alpha_{j,X} = \alpha_X$. Note that the anticompetitive motive plays no role in the social planner's allocation of $\alpha_{j,X,t}$, as the planner directly chooses the ad space occupied by each firm. So, in deciding whether to give more ad space to one firm over another, the planner only considers the utility gains from informing more consumers and from enhancing customers' taste for that good.

Second, let's see the social planner choice of N_m. The planner takes into account that by employing more labor in media it will increase the aggregate quality Q of media, which has two effects: (i) it increases the level of entertainment L; and, by increasing the time spent in media, (ii) it increases the consumption good by increasing the probability of learning goods. The optimal N_m is given by

$$N_m = \frac{N^P}{2}\left(\upsilon Q^2 + \sum_{X\in\Omega}\mu(X)\hat{h}_X\alpha_X\right) \tag{24}$$

Note that, unlike the papers with a media sector cited in the related literature section, here the planner would value the provision of media goods even if their entertainment value was negligible (i.e. even if $\upsilon = 0$), due to their role as a vehicle for spreading product awareness. Finally, the labor employed in entry in each industry satisfies:

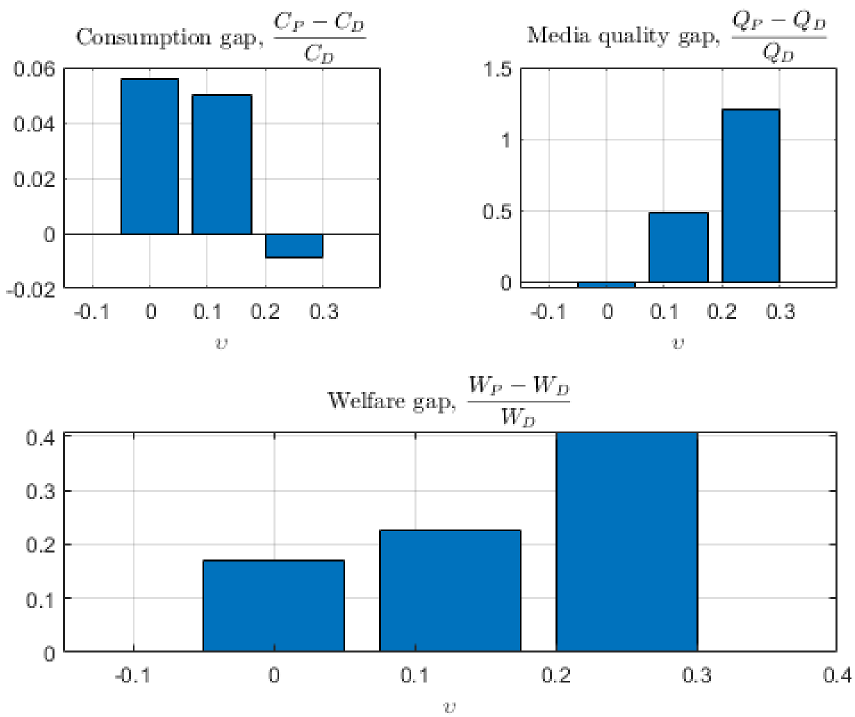
$$N_{e,X} = \left(\frac{\phi N^P}{2}\beta\left(\mathbb{E}_e U_{X'} - \mathbb{E}_{-e} U_{X'}\right)\right)^2 \tag{25}$$

where $\mathbb{E}_e U_{X'}$ (resp. $\mathbb{E}_{-e} U_{X'}$) is the expected industry-utility conditional on successfully creating (resp. not creating) a new differentiated good (so the expectation comes from the probabilities the incumbents exit). The relative wage is $\hat{w} = 1$ as consumers spend all the income they receive, which is w. The labor market clearing, using 46 and 24 pins down N^P:

$$1 = N^P + N_e + N_m \tag{26}$$

Figure 4 compares the planner economy with the decentralized one for different values of the relative utility weight of the entertainment good, υ, ranging from 0 to 0.5. Interestingly, when $\upsilon = 0$ (that is, when spending time on media doesn't provide any direct utility gain to consumers), we see that final output would be almost 15% higher under the planner's allocation, whereas media quality Q would be lower. The statement that the planner would choose a lower Q seems to indicate that when $\upsilon = 0$ there is too much advertising in the decentralised economy. However, this conclusion is inaccurate (as shown later in the taxation exercise in section 5) because we also need to consider how the advertising space is allocated among firms. In other words, the inefficiencies from the misallocation in the advertising space may be mitigated with the 'overprovision' of media, through its effect on learning.
As the importance of media goods in utility increases (i.e. as υ increases), the planner puts more weight on producing media goods, at the expense of final output, which eventually is lower in the planner's allocation than in the decentralised one. Welfare gains from implementing the planner's allocation are sizeable, and increasing with υ, reflecting the inefficiency arising from the fact that firms don't take into account consumers' utility gains from the media goods financed by advertising when making their advertising decisions. (to be done, a decomposition of the welfare gains)

Figure 4: Welfare comparison of the planner and decentralised allocations



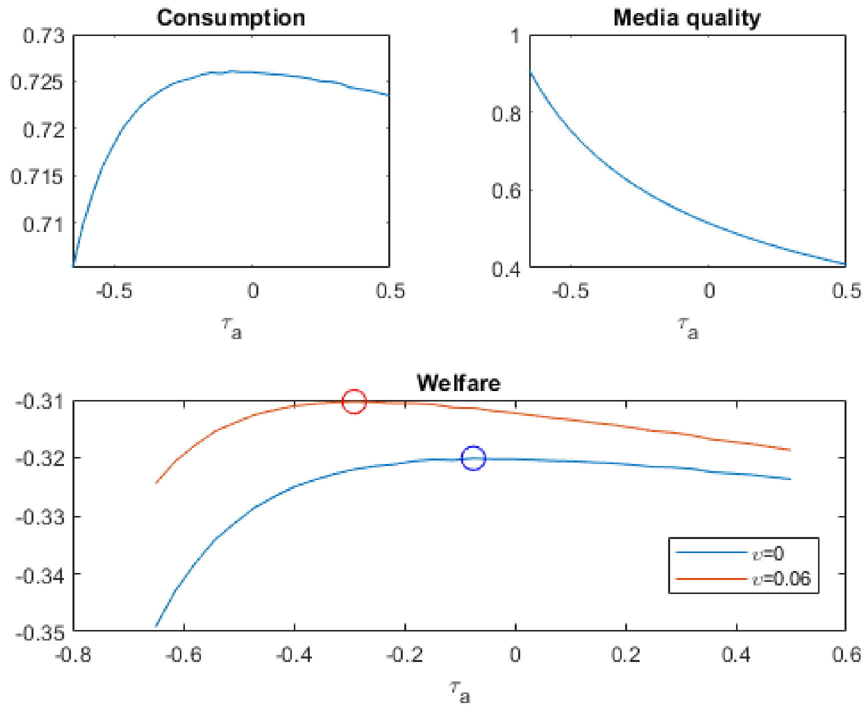
Notes. This figure displays the difference in final output (upper-left panel), in media time (upper-right panel), and welfare (bottom panel) between the planner's equilibrium and the decentralized one, relative to the decentralized one.

5 Taxing advertising

5.1 Uniform tax

In this section, I explore whether there is too much advertising, in which case we should tax it, or if, on the contrary, there is too little advertising and we should subsidize it. Here, in addition to $e_{j,i,t}$, firms pay $\tau_a e_{j,i,t}$ as taxes to the government, which are distributed as transfers to consumers. Figure 5 depicts the effect of this tax on final output, the quality of media, and welfare for different values of υ. As expected, the higher the entertainment value of media, the more valuable advertising becomes, leading to a lower optimal tax (i.e. higher subsidy). More interestingly, recall that we have seen that the decentralised equilibrium supplies more media than the planner's one for the case of $\upsilon = 0$, which seems to point to an overprovision of media. Actually, it turns out that there is too little advertising and the optimal tax is a subsidy. This confirms the hypothesis that this 'overprovision' of media, via increasing the time spent on media and thus the effectiveness of advertising, mitigates the inefficiencies from the misallocation of the advertising space.

Figure 5: Welfare under uniform tax on advertising



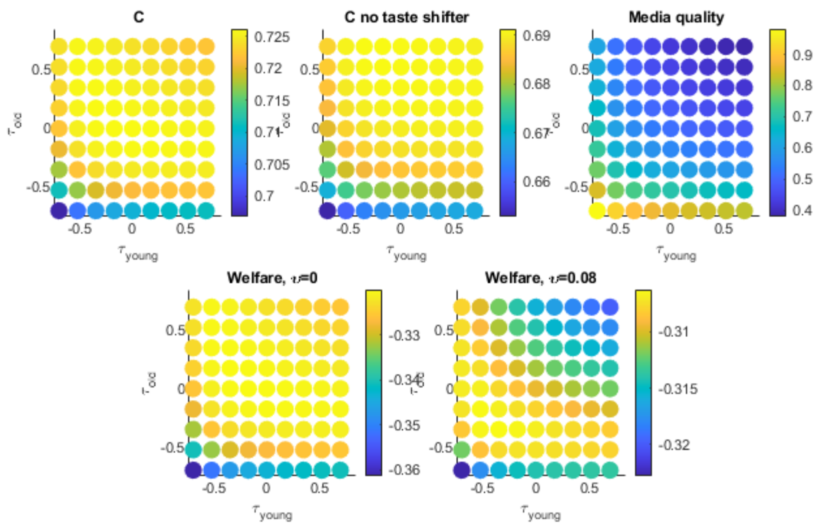
5.2 Age-dependent tax

Observation to make: note that a larger share of firms are older than the threshold, and so τ_O affects more firms.

The observation in section 3.2 that the informative motive is decreasing with age whereas the perusasive and anticompetitive motives are increasing on age, together with the fact that they have different welfare implications, leads us to think that we may have a significant welfare improvement by considering an age-dependent tax rather than applying the same tax for all firms. Assume now firms pay τ_Y if their age is smaller than the age cutoff $\bar{a}$, and τ_O if their age is greater or equal than $\bar{a}$. In particular, for this exercise I have set $\bar{a} = 3$; meaning that firms get a different tax treatment in their first three periods of life than afterwards. Note that this differential policy treatment makes the vector of ages of the firms to be an additional state. Note that firms that are at least $\bar{a}$ years old are indistinguishable by age (if all firms are older, then the firm problem is identical to the baseline with a uniform tax). However, for $a_j < \bar{a}$, we need to keep track of the particular age a_j (how close you are to $\bar{a}$ makes a difference). So, if $(a_1, \ldots, a_J)$ is the vector of ages (from older to younger), then the relevant ages state is $\vec{a} = (\hat{a}_1, \ldots, \hat{a}_J)$, where $\hat{a}_j = \min(a_j, \bar{a})$.

Figure 6 illustrates the effect of this age-dependent advertising tax on (i) output, (ii) output if we assumed that the persuasive effect of advertising is deceiving, (iii) media quality, and (iv) welfare for two values of υ.

Figure 6: Welfare under age-dependent tax on advertising



Why is it preferable to tax advertising more for young firms than for old ones? The reason relies on the taste-shifter effect of advertising. Allocating more advertising space to young firms yields larger informative gains, but these come at the cost of reduced utility from a lower taste shifter for older firms, which affects many consumers due to their large customer base. In the calibrated model, the latter effect dominates. To support this explanation, I show in the second plot what would be the final output if we shut down this taste shifter effect (that is, assuming the persuasive effect of advertising is deceiving as in section 2). As expected, with this assumption, output would be improved by setting a higher tax for older firms (if it helps the visualisation, I could plot instead the difference between the output for $(\tau_Y = a, \tau_O = b)$ and the output for $(\tau_Y = b, \tau_O = a)$ for $a < b$, and we would see it is positive).

6 Concluding remarks

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7 Appendix

7.1 Calibration Appendix: Data sources and Computation of moments

1. **Sales-weighted average markup, labor share, entry rate, and aggregate advertising expenditure as a percentage of GDP.** Taken from Cavenaile et al. (2024a). Following Cavenaile et al. (2024a), given that there is no physical capital in the model, I target the labor share among labor income and profits. Given that $\frac{wL}{wL+\pi+rK} = \frac{wL}{wL+\pi}\frac{wL+\pi}{wL+\pi+rK} = \frac{wL}{wL+\pi}\left(1 - \frac{rK}{wL+\pi+rK}\right)$; then, the target used is obtained from dividing the labor share by one minus the capital share. In the model, given that labor supply is normalised to 1, then labor share equals w.
   The entry rate in the model is the average number of new firms (i.e. the average probability of creating a new product).

2. **Fraction of time in media.** According to Statista, people in the US spend on average 751 minutes per day in media, which corresponds to the 0.521528 of time. Note that in this measure of media time multitasking is counted separately; that is: it counts the time spend in media while also doing other activities (e.g. commuting to work, breaks at work, listening a podcast while cooking or running), and duplicated media time when using multiple forms of media simultaneously (e.g. watching the TV while using a phone will count double).

3. **Coefficient of a regression of advertising expenditure on relative sales.** This and the growth by age moments are computed using Compustat data for the time period 1976-2018. Both in the model and in the data, I take the logarithm of advertising expenditure and then I standardise it by subtracting their means and dividing by their standard deviation for comparability. In the data, I regress the standardised logarithm of advertising expenses on relative sales of the firm in its SIC4 industry, controlling for the same set of controls used in Cavenaile et al., namely: profitability, leverage, market-to-book ratio, log R&D stock, firm age, the coefficient of variation of the firm's stock price, the number of firms in the industry, and a full set of year and SIC4 industry fixed effects. In the model, I regress the standardised logarithm of advertising expenses, $p_{a,i,t}e_{j,i,t}$, on market shares, $s_{j,i,t}$, with industry fixed effects. Table 3 shows the results of the empirical regression:

4. **Average firm growth at age 1 and the slope of the fitted line.** In Compustat, I define age as the number of years since the first appearance of the firm in Compustat.

Table 3: Advertising and relative sales in the data

	log advertising expenses
Relative sales	0.671 (0.0448)***
R^2	0.6056
N	40,007

Notes. Robust asymptotic standard errors (in parenthesis) are clustered at the firm level. The sample period is from 1976 to 2018. The regression controls for profitability, leverage, market-to-book ratio, log R&D stock, firm age, the coefficient of variation of the firm's stock price, the number of firms in the industry, and a full set of year and SIC4 industry fixed effects.

Firms in the data may experience big jumps on sales through expansion to new markets or via mergers and acquisitions, and I am interested in the average evolution of firm growth in the absence of such disruptive events; therefore, I drop all the observations of a firm posterior to a big change in their sales. In particular, once a firm attains a growth rate bigger than 100% in absolute value, this observation and the posterior ones of this firm are dropped. Then, I take the average firm sales growth grouping all the observations with the same age. Given the average firm sales growth by age, $\bar{g}_a$, I define the fitted line $\hat{g}_a = \bar{g}_1 + \beta_g a$, where $\bar{g}_1$ is the actual average growth for firms aged 1 in the data (resp. in the model) for the regression using actual data (resp. model-implied data), and β_g is the coefficient that minimises the sum of square errors $\sum_a (\bar{g}_a - \hat{g}_a)^2$.

5. **Calibration of the public sector financed media $\bar{N}_m$.** According to the US Government Accountability Office, the federal government spent \$14.9 billion over the last 10 fiscal years (2014-2023). Then, I use that federal governments spent roughly \$1.49 billion per year. In addition, federal appropriations for CPB (Corporation for Public Broadcasting) amounted to \$477 million in fiscal year 2023. So, the estimate I use for public sector spending on media is (\$1.49 + \$0.477) billion, which I divide for the US GDP in 2023, \$27360 billion. This gives 0.008% of GDP, which divided by $w = 0.8359$ gives the $\bar{N}_m = 9.5705 \cdot 10^5$.

7.2 Preferences

$$\max_{\{\{c_{\ell,j,i,t}\},a_{\ell,t+1},N_{\ell,t},T_{F,\ell,t},T_{\ell,i,t}\}} U_\ell = \sum_{t=0}^{\infty} \beta^t \left[\frac{C_{\ell t}^{1-\theta}-1}{1-\theta} + L_{\ell t}\right]$$

$$\text{s.t. } C_{\ell,t} = \left(\int_0^1 C_{\ell,i,t}^{\frac{\chi-1}{\chi}} di\right)^{\frac{\chi}{\chi-1}}, \quad L_{\ell t} = \upsilon\left(Q_t T_{\ell,i,t} - \frac{T_{\ell,t}^2}{2}\right)$$

$$C_{\ell,i,t} = \left(c_{\ell,0,i,t}^{\frac{\sigma-1}{\sigma}} + \sum_{j\in\mathcal{I}_{\ell,i,t}} \omega_{\ell,j,i,t} c_{\ell,j,i,t}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

$$w_t N + r_t a_{\ell t} = \int_0^1 \sum_{j\in\mathcal{I}_{\ell,i,t}} c_{\ell,j,i,t} p_{j,i,t} di + a_{\ell,t+1} - a_{\ell,t}$$

(for the case $\theta = 1$, $\lim_{\theta\to 1} \frac{c^{1-\theta}}{1-\theta} = \lim_{\theta\to 1} \frac{c^{1-\theta}-1}{1-\theta} + \lim_{\theta\to 1} \frac{1}{1-\theta} = \ln c + \lim_{\theta\to 1} \frac{1}{1-\theta}$)
We can already plug $C_{\ell,t}$ into the objective function.
The FOCs read:

$$[c_{\ell jt}]: \quad \frac{\partial U_{\ell,t}}{\partial C_{\ell,t}} \frac{\partial C_{\ell,t}}{\partial C_{\ell,i,t}} \frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = \mu_{\ell,t} p_{j,i,t}$$

where $\frac{\partial U_{\ell,t}}{\partial C_{\ell,t}} = \beta^t C_{\ell,t}^{-\theta}$, $\frac{\partial C_{\ell,t}}{\partial C_{\ell,i,t}} = C_{\ell,t}^{\frac{1}{\chi}} C_{\ell,i,t}^{-\frac{1}{\chi}}$, and $\frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = C_{\ell,i,t}^{\frac{1}{\sigma}} c_{\ell,j,i,t}^{-\frac{1}{\sigma}} \omega_{\ell,j,i,t}$.

We can break down the FOC into three conditions, by defining $P_{\ell,i,t}$ as $P_{\ell,i,t} C_{\ell,i,t} = \sum_{j\in\mathcal{I}_{\ell,i,t}} c_{\ell,j,i,t} p_{j,i,t}$, and $P_{\ell,t}$ as $P_{\ell,t} C_{\ell,t} = \int_0^1 C_{\ell,i,t} P_{i,t} di$:

1. $\frac{\partial U_{\ell,t}}{\partial C_{\ell,t}} \frac{\partial C_{\ell,t}}{\partial c_{\ell,j,i,t}} = \mu_{\ell,t} \frac{\partial E_{\ell,t}}{\partial C_{\ell,t}} \frac{\partial C_{\ell,t}}{\partial c_{\ell,j,i,t}} \implies \left[\frac{\partial U_{\ell,t}}{\partial C_{\ell,t}} - \mu_{\ell,t} \frac{\partial E_{\ell,t}}{\partial C_{\ell,t}}\right] \frac{\partial C_{\ell,t}}{\partial c_{\ell,j,i,t}} = 0 \implies \beta^t C_{\ell,t}^{-\theta} = \mu_{\ell,t} P_{\ell,t}$

2. $\frac{\partial U_{\ell,t}}{\partial C_{\ell,t}} \frac{\partial C_{\ell,t}}{\partial C_{\ell,i,t}} \frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = \mu_{\ell,t} \frac{\partial E_{\ell,t}}{\partial C_{\ell,i,t}} \frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} \implies \left[\frac{\partial U_{\ell,t}}{\partial C_{\ell,t}} \frac{\partial C_{\ell,t}}{\partial C_{\ell,i,t}} - \mu_{\ell,t} \frac{\partial E_{\ell,t}}{\partial C_{\ell,i,t}}\right] \frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = 0$, where using from the previous condition that $\mu_{\ell,t} = \beta^t C_{\ell,t}^{-\theta} P_{\ell,t}^{-1}$, we get: $C_{\ell,t}^{\frac{1}{\chi}} C_{\ell,i,t}^{-\frac{1}{\chi}} = \frac{P_{\ell,i,t}}{P_{\ell,t}} \implies C_{\ell,i,t} = C_{\ell,t} \left(\frac{P_{\ell,t}}{P_{\ell,i,t}}\right)^{\chi}$, and plugging it into the definition of $C_{\ell,t}$, we get $P_{\ell,t} = \left(\int_0^1 P_{\ell,i,t}^{1-\chi} di\right)^{\frac{1}{1-\chi}}$. Note that if Cobb-Douglas (i.e. $\chi = 1$), then $E_{\ell,t} = P_{\ell,t} C_{\ell,t} = P_{\ell,i,t} C_{\ell,i,t}$.

3. $\frac{\partial U_{\ell,t}}{\partial C_{\ell,t}} \frac{\partial C_{\ell,t}}{\partial C_{\ell,i,t}} \frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = \mu_{\ell,t} \frac{\partial E_{\ell,t}}{\partial c_{\ell,j,i,t}} = \mu_{\ell,t} p_{j,i,t}$, where using from the previous conditions that $\mu_{\ell,t} = \beta^t C_{\ell,t}^{-\theta} P_{\ell,t}^{-1}$ and $C_{\ell,t}^{\frac{1}{\chi}} C_{\ell,i,t}^{-\frac{1}{\chi}} = \frac{P_{\ell,i,t}}{P_{\ell,t}}$, we get: $\frac{P_{\ell,i,t}}{P_{\ell,t}} C_{\ell,i,t}^{\frac{1}{\sigma}} c_{\ell,j,i,t}^{-\frac{1}{\sigma}} \omega_{\ell,j,i,t} = \frac{p_{j,i,t}}{P_{\ell,t}} \implies c_{\ell,j,i,t} = C_{\ell,i,t} \left(\frac{\omega_{\ell,j,i,t} P_{\ell,i,t}}{p_{j,i,t}}\right)^{\sigma}$, and plugging it into the definition of $C_{\ell,i,t}$, we get: $P_{\ell,i,t} = \left(p_{0,i,t}^{1-\sigma} + \sum_{j\in\mathcal{I}_{\ell,i,t}} \omega_{\ell,j,i,t}^{\sigma} p_{j,i,t}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$

The FOC for assets is:

$$[a_{\ell,t+1}]: \quad \mu_{\ell t} = (1 + r_{t+1}) \mu_{\ell,t+1}$$

From the first one, using that $\mu_{\ell,t} = \beta^t C_{\ell,t}^{-\theta} P_{\ell,t}^{-1}$, we get the Euler equation: $\beta^t C_{\ell,t}^{-\theta} P_{\ell,t}^{-1} = (1 + r_{t+1}) \beta^{t+1} C_{\ell,t+1}^{-\theta} P_{\ell,t+1}^{-1}$, which assuming $\theta = 1$ (i.e. logarithmic preferences on $C_{\ell,t}$), then the expenditure choice is independent of the price indices (so, the awareness set just affects the intratemporal allocation of expenditure).

So, assuming $\chi = \theta = 1$, we have:

$$c_{\ell,j,i,t} = E_{\ell,t} P_{\ell,i,t}^{\sigma-1} p_{j,i,t}^{-\sigma} \omega_{\ell,j,i,t}^{\sigma}$$
$$\frac{E_{\ell,t+1}}{E_{\ell,t}} = \beta(1 + r_{t+1})$$

where E_ℓ is the expenditure of individual ℓ. So, the growth of expenditure is symmetric for all individuals (and the level is also identical if all individuals start with the same level of assets).

Since the individual is characterised by the awareness set, from now on I use the subindex $\mathcal{I}$, instead of ℓ. The share of expenditure of each consumer on each good they know is: $s_{\mathcal{I},j} = \frac{p_j c_{\mathcal{I},j}}{E} = P_{\mathcal{I}}^{\sigma-1} p_j^{1-\sigma} \omega_j^{\sigma} = p_j^{1-\sigma} \omega_j^{\sigma} \left[p_{0,t}^{1-\sigma} + \sum_{k\in\mathcal{I}_\ell} \omega_{\ell,k}^{\sigma} p_k^{1-\sigma}\right]^{-1} = \left[\left(\frac{p_{0,t}}{p_j}\right)^{1-\sigma} \left(\frac{1}{\omega_j}\right)^{\sigma} + \sum_{k\in\mathcal{I}_\ell} \left(\frac{p_{k,t}}{p_j}\right)^{1-\sigma} \left(\frac{\omega_k}{\omega_j}\right)^{\sigma}\right]^{-1}$
So, using the definition of markup $\mathcal{M}_j = \frac{p_j A_j}{w}$:

$$s_{\mathcal{I},j} = \left[\left(\frac{1}{\mathcal{M}_j A_0}\right)^{1-\sigma} \left(\frac{1}{\omega_j}\right)^{\sigma} + \sum_{k\in\mathcal{I}_\ell} \left(\frac{\mathcal{M}_k A_j}{\mathcal{M}_j A_k}\right)^{1-\sigma} \left(\frac{\omega_k}{\omega_j}\right)^{\sigma}\right]^{-1}$$

Next, the choice of media time is straightforward: $\frac{\partial L_{\ell,t}}{\partial T_{\ell,t}} = \upsilon\left(Q_t - T_{\ell,t}\right)$, so $T_t = Q_t$. And so, optimal leisure as a function of Q is: $L_t^* = \upsilon \frac{Q_t^2}{2}$.

7.3 Proof that $\delta = 0$ allows a simpler state

Setting $\delta = 0$ allows a simpler sufficient state:
If individuals don't die (or forget), then, given that learning is independent for each good, we have that a sufficient information that allows us to identify the industry state is the mass of consumers that are aware of good j for each $j \in \mathcal{J}_{i,t}$. Intuitively, the reason why $\delta > 0$ doesn't allow this simplification is that the fact that consumers die with positive probability breaks the independence of the events of being aware of a particular good. That is: given that the older the consumer the more likely she is aware of the good, then, knowing that the consumer is aware of a good allows us to get a better guess of the consumer's age.

Proposition 2 *If $\delta = 0$ and $M_{j,it}$ is the mass of consumers aware of good j, then the distribution of consumers over the awareness sets is given by* $M_{i,t}(\mathcal{I}) = \prod_{j\in\mathcal{I}} M_{j,it} \prod_{j\notin\mathcal{I}} (1 - M_{j,it})$

Proof. By induction on t. Set $t = 0$ as the first period a differentiated firm enters. Then, it is trivially satisfied for $t = 0$. We'll see that if it is true for $t - 1$ that $M_{i,t-1}(\mathcal{I}) = \prod_{j\in\mathcal{I}} M_{j,it-1} \prod_{j\notin\mathcal{I}} (1 - M_{j,it-1})$, then it is true for t that $M_{i,t}(\mathcal{I}) = \prod_{j\in\mathcal{I}} M_{j,it} \prod_{j\notin\mathcal{I}} (1 - M_{j,it})$:

By the law of motion of consumers:

$$M_{i,t}(\mathcal{I}) = (1-\delta)\sum_{\mathcal{I}'\subseteq\mathcal{I}} M_{i,t-1}(\mathcal{I}') \prod_{k\in\mathcal{I}\backslash\mathcal{I}'} \rho_{k,i,t-1} \prod_{k\notin\mathcal{I}}(1-\rho_{k,i,t-1})$$

Using the induction hypothesis:

$$M_{i,t}(\mathcal{I}) = (1-\delta)\sum_{\mathcal{I}'\subseteq\mathcal{I}} \prod_{j\in\mathcal{I}'} M_{j,it-1} \prod_{j\notin\mathcal{I}'}(1-M_{j,it-1}) \prod_{k\in\mathcal{I}\backslash\mathcal{I}'} \rho_{k,i,t-1} \prod_{k\notin\mathcal{I}}(1-\rho_{k,i,t-1})$$

Note that $\prod_{j\notin\mathcal{I}'}(1-M_{j,it-1}) = \prod_{j\in\mathcal{I}\backslash\mathcal{I}'}(1-M_{j,it-1})\prod_{j\notin\mathcal{I}}(1-M_{j,it-1})$; so, we can write it as:

$$M_{i,t}(\mathcal{I}) = (1-\delta)\sum_{\mathcal{I}'\subseteq\mathcal{I}} \prod_{j\in\mathcal{I}'} M_{j,it-1} \prod_{k\in\mathcal{I}\backslash\mathcal{I}'} [(1-M_{k,it-1})\rho_{k,i,t-1}] \prod_{k\notin\mathcal{I}}[(1-M_{k,it-1})(1-\rho_{k,i,t-1})]$$

And using that it holds $\prod_{j\in\mathcal{J}}(a_j+b_j) = \sum_{\mathcal{I}\subseteq\mathcal{J}}\prod_{j\in\mathcal{I}} a_j \prod_{j\notin\mathcal{I}} b_j$; then, we have:

$$M_{i,t}(\mathcal{I}) = (1-\delta)\prod_{j\in\mathcal{I}}[M_{j,it-1}+(1-M_{j,it-1})\rho_{j,i,t-1}] \prod_{k\notin\mathcal{I}}[(1-M_{k,it-1})(1-\rho_{k,i,t-1})]$$

On the other hand, the law of motion of $M_{k,it}$ is:
$M_{k,it} = (1-\delta)M_{k,it-1} + (1-\delta)(1-M_{k,it-1})\rho_{k,i,t-1} \implies (1-M_{k,it-1})(1-\rho_{k,i,t-1}) = 1-\frac{M_{k,it}}{1-\delta}$.
So, if $\delta=0$, we have, as wanted:

$$M_{i,t}(\mathcal{I}) = \prod_{j\in\mathcal{I}} M_{j,it}\prod_{j\notin\mathcal{I}}(1-M_{j,it})$$

■

7.4 Production Firms

7.4.1 Derivatives of profits and expenditure shares

1. $\pi_{j,i} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I}) s_{\mathcal{I},j,i}$
   (a) $\frac{\partial\pi_{j,i}}{\partial e_{k,i}} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial s_{\mathcal{I},j,i}}{\partial e_{k,i}}$, $k\in\mathcal{J}_i$
   (b) $\frac{\partial\pi_{j,i}}{\partial\mathcal{M}_{j,i}} = \frac{s_{j,i}}{\mathcal{M}_{j,i}^2} + (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial s_{\mathcal{I},j,i}}{\partial\mathcal{M}_{j,i}}$
   (c) $\frac{\partial\pi_{j,i}}{\partial\mathcal{M}_{k,i}} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial s_{\mathcal{I},j,i}}{\partial\mathcal{M}_{k,i}}$, $k\neq j$
2. $s_{\mathcal{I},j,i} = \frac{p_{j,i}c_{\mathcal{I},j,i}}{E} = \frac{p_{j,i}^{1-\sigma}\omega_{j,i}^{\sigma}}{P_{\mathcal{I},i}^{1-\sigma}} = \frac{p_{j,i}^{1-\sigma}\omega_{j,i}^{\sigma}}{p_{0,i}^{1-\sigma}+\sum_{k\in\mathcal{I}}\omega_{k,i}^{\sigma}p_{k,i}^{1-\sigma}} = \frac{\mathcal{M}_{j,i}^{1-\sigma}\omega_{j,i}^{\sigma}}{(\frac{\mathcal{M}_{0,i}}{A_0})^{1-\sigma}+\sum_{k\in\mathcal{I}}\omega_{k,i}^{\sigma}\mathcal{M}_{k,i}^{1-\sigma}}$, and $s_{\mathcal{I},j,i}=0$ if

$j \notin \mathcal{I}$. So:[19]

(a) $\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{j,i}} = s_{\mathcal{I},j,i}(1 - s_{\mathcal{I},j,i})\frac{\sigma}{\omega_{j,i}}$

(b) $\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{k,i}} = -s_{\mathcal{I},j,i}s_{\mathcal{I},k,i}\frac{\sigma}{\omega_{k,i}}$

(c) $\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}} = -s_{\mathcal{I},j,i}(1 - s_{\mathcal{I},j,i})\frac{\sigma-1}{\mathcal{M}_{j,i}} = -\frac{\sigma}{\sigma-1}\frac{\mathcal{M}_{j,i}}{\omega_{j,i}}\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{j,i}}$

(d) $\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{k,i}} = s_{\mathcal{I},j,i}s_{\mathcal{I},k,i}\frac{\sigma-1}{\mathcal{M}_{k,i}} = -\frac{\sigma}{\sigma-1}\frac{\mathcal{M}_{k,i}}{\omega_{k,i}}\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{k,i}}$

3. $\omega_{j,i} = 1 + \nu_s(\alpha_{j,i}T)^{\nu_c}$, $\alpha_{j,i} = \frac{e_{j,i}}{p_{a,i}}$, where $p_{a,i} = \frac{\sum_k e_{k,i}}{\alpha_i}$ if limited ad space is binding, otherwise $p_{a,i} = \bar{p}_{a,i}$.

   (a) $\frac{\partial \omega_{j,i}}{\partial e_{j,i}} = \nu_c\nu_s T^{\nu_c}\alpha_{j,i}^{\nu_c-1}\frac{\partial \alpha_{j,i}}{\partial e_{j,i}}$

   (b) $\frac{\partial \omega_{k,i}}{\partial e_{j,i}} = \nu_c\nu_s T^{\nu_c}\alpha_{k,i}^{\nu_c-1}\frac{\partial \alpha_{k,i}}{\partial e_{j,i}}$

   (c) $\frac{\partial \alpha_{k,i}}{\partial e_{j,i}} = -\frac{\alpha_{k,i}}{\sum_k e_{k,i}}$ if the limited ad space is binding, otherwise $\frac{\partial \alpha_{k,i}}{\partial e_{j,i}} = 0$.

   (d) $\frac{\partial \alpha_{j,i}}{\partial e_{j,i}} = \frac{\alpha_i}{\sum_k e_{k,i}} - \frac{\alpha_{j,i}}{\sum_k e_{k,i}} = \sum_{j' \neq j}\frac{\alpha_{j',i}}{\sum_k e_{k,i}}$ if the limited ad space is binding, otherwise $\frac{\partial \alpha_{j,i}}{\partial e_{j,i}} = \frac{1}{p_{a,i}}$.

So: $\frac{\partial \pi_{j,i}}{\partial e_{j,i}} = (1 - \mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\sum_{k\in\mathcal{I}}\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{k,i}}\frac{\partial \omega_{k,i}}{\partial e_{j,i}}$. And using the above expressions we have:

- for $k \neq j$: $\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{k,i}}\frac{\partial \omega_{k,i}}{\partial e_{j,i}} = -s_{\mathcal{I},j,i}s_{\mathcal{I},k,i}\frac{\sigma}{\omega_{k,i}}\nu_c\nu_s T^{\nu_c}\alpha_{k,i}^{\nu_c-1}\frac{\partial \alpha_{k,i}}{\partial e_{j,i}}$.
- $\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{j,i}}\frac{\partial \omega_{j,i}}{\partial e_{j,i}} = s_{\mathcal{I},j,i}(1 - s_{\mathcal{I},j,i})\frac{\sigma}{\omega_{j,i}}\nu_c\nu_s T^{\nu_c}\alpha_{j,i}^{\nu_c-1}\frac{\partial \alpha_{j,i}}{\partial e_{j,i}}$.

So: $\frac{\partial \pi_{j,i}}{\partial e_{j,i}} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\sigma\nu_s\nu_c T^{\nu_c}s_{\mathcal{I},j,i}\left[\sum_{k\in\mathcal{I}\setminus\{j\}}\frac{s_{\mathcal{I},k,i}\alpha_{k,i}^{\nu_c-1}}{\omega_{k,i}}\left(-\frac{\partial \alpha_{k,i}}{\partial e_{j,i}}\right) + \frac{(1-s_{\mathcal{I},j,i})\alpha_{j,i}^{\nu_c-1}}{\omega_{j,i}}\frac{\partial \alpha_{j,i}}{\partial e_{j,i}}\right]$

And substituting $\frac{\partial \alpha_{k,i}}{\partial e_{j,i}}$:

1. With binding limited ad space (i.e. $J > 1$):
   $\frac{\partial \pi_{j,i}}{\partial e_{j,i}} = (1 - \mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\sigma\nu_s\nu_c}{\sum_k e_{k,i}}T^{\nu_c}s_{\mathcal{I},j,i}\left[\sum_{k\in\mathcal{I}\setminus\{j\}}\frac{s_{\mathcal{I},k,i}\alpha_{k,i}^{\nu_c}}{\omega_{k,i}} + \frac{(1-s_{\mathcal{I},j,i})\alpha_{j,i}^{\nu_c-1}}{\omega_{j,i}}(\alpha_i - \alpha_{j,i})\right]$
   Equivalently:
   $\frac{\partial \pi_{j,i}}{\partial e_{j,i}} = (1 - \mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\sigma\nu_s\nu_c}{\sum_k e_{k,i}}T^{\nu_c}s_{\mathcal{I},j,i}\left[\sum_{k\in\mathcal{I}}\frac{s_{\mathcal{I},k,i}\alpha_{k,i}^{\nu_c}}{\omega_{k,i}} + \frac{\alpha_{j,i}^{\nu_c-1}}{\omega_{j,i}}(\alpha_i(1 - s_{\mathcal{I},j,i}) - \alpha_{j,i})\right]$
   Or equivalently:
   $\frac{\partial \pi_{j,i}}{\partial e_{j,i}} = (1 - \mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\sigma\nu_c}{\sum_k e_{k,i}}s_{\mathcal{I},j,i}\left[\sum_{k\in\mathcal{I}}\frac{s_{\mathcal{I},k,i}\hat{\omega}_{k,i}}{\omega_{k,i}} + \frac{\hat{\omega}_{j,i}}{\omega_{j,i}}\left(\alpha_i\frac{1-s_{\mathcal{I},j,i}}{\alpha_{j,i}} - 1\right)\right]$, where $\hat{\omega}_{j,i} = \nu_s(T\alpha_{j,i})^{\nu_c}$

2. With non-binding ad space (i.e. $J = 1$):
   $\frac{\partial \pi_{j,i}}{\partial e_{j,i}} = (1 - \mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\sigma\nu_s\nu_c T^{\nu_c}s_{\mathcal{I},j,i}\frac{(1-s_{\mathcal{I},j,i})\alpha_{j,i}^{\nu_c-1}}{\omega_{j,i}p_{a,i}}$

[19] Also, in terms of relative consumption: $s_{\mathcal{I},j,i} = \frac{p_{j,i}c_{\mathcal{I},j,i}}{E} = \frac{c_{\mathcal{I},j,i}^{\frac{\sigma-1}{\sigma}}\omega_j P_{\mathcal{I},j,i}^{\frac{\sigma-1}{\sigma}}}{E_{\mathcal{I},j,i}^{\frac{\sigma-1}{\sigma}}} = \frac{c_{\mathcal{I},j,i}^{\frac{\sigma-1}{\sigma}}\omega_j}{Y_{\mathcal{I},j,i}^{\frac{\sigma-1}{\sigma}}}$

7.4.2 Extension: Persuasive effect as in Cavenaile et al.

Here, $\omega_{\mathcal{I},j,i,t} = \frac{1+\nu_s(T_t e_{j,i,t})^{\nu_c}}{\frac{1}{\#\mathcal{I}}\sum\limits_{j'\in\mathcal{I}}(1+\nu_s(T_t e_{j',i,t})^{\nu_c})}$. With this, now the derivatives with respect to $e_{j,i,t}$ write:

$$\frac{\partial\omega_{\mathcal{I},j,i}}{\partial e_{j,i}} = \omega_{\mathcal{I},j,i}\frac{\nu_s(T_t e_{j,i,t})^{\nu_c}}{1+\nu_s(T_t e_{j,i,t})^{\nu_c}}\frac{\nu_c}{e_{j,i}} - \omega_{\mathcal{I},j,i}\frac{1+\nu_s(T_t e_{j,i,t})^{\nu_c}}{\sum\limits_{j'\in\mathcal{I}}(1+\nu_s(T_t e_{j',i,t})^{\nu_c})}\frac{\nu_s(T_t e_{j,i,t})^{\nu_c}}{1+\nu_s(T_t e_{j,i,t})^{\nu_c}}\frac{\nu_c}{e_{j,i}} \quad (27)$$

$$= \omega_{\mathcal{I},j,i}\frac{\nu_s(T_t e_{j,i,t})^{\nu_c}}{1+\nu_s(T_t e_{j,i,t})^{\nu_c}}\frac{\nu_c}{e_{j,i}}\frac{\sum\limits_{j'\neq j}(1+\nu_s(T_t e_{j',i,t})^{\nu_c})}{\sum\limits_{j'\in\mathcal{I}}(1+\nu_s(T_t e_{j',i,t})^{\nu_c})} \quad (28)$$

$$\frac{\partial\omega_{\mathcal{I},k,i}}{\partial e_{j,i}} = -\omega_{\mathcal{I},k,i}\frac{1+\nu_s(T_t e_{j,i,t})^{\nu_c}}{\sum\limits_{j'\in\mathcal{I}}(1+\nu_s(T_t e_{j',i,t})^{\nu_c})}\frac{\nu_s(T_t e_{j,i,t})^{\nu_c}}{1+\nu_s(T_t e_{j,i,t})^{\nu_c}}\frac{\nu_c}{e_{j,i}} \quad (29)$$

And recall that $\frac{\partial\pi_{j,i}}{\partial e_{j,i}} = (1-\mathcal{M}_{j,i}^{-1})\sum\limits_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial s_{\mathcal{I},j,i}}{\partial e_{j,i}}$, $\frac{\partial s_{\mathcal{I},j,i}}{\partial\omega_{\mathcal{I},j,i}} = s_{\mathcal{I},j,i}(1-s_{\mathcal{I},j,i})\frac{\sigma}{\omega_{\mathcal{I},j,i}}$, and $\frac{\partial s_{\mathcal{I},j,i}}{\partial\omega_{\mathcal{I},k,i}} = -s_{\mathcal{I},j,i}s_{\mathcal{I},k,i}\frac{\sigma}{\omega_{\mathcal{I},k,i}}$; so:

$$\frac{\partial\pi_{j,i}}{\partial e_{j,i}} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\sum_{k\in\mathcal{I}}\frac{\partial s_{\mathcal{I},j,i}}{\partial\omega_{\mathcal{I},k,i}}\frac{\partial\omega_{\mathcal{I},k,i}}{\partial e_{j,i}}$$

And using the previous expressions we have:

- $\frac{\partial s_{\mathcal{I},j,i}}{\partial\omega_{\mathcal{I},j,i}}\frac{\partial\omega_{\mathcal{I},j,i}}{\partial e_{j,i}} = \sigma s_{\mathcal{I},j,i}(1-s_{\mathcal{I},j,i})\frac{\nu_s(T_t e_{j,i,t})^{\nu_c}}{1+\nu_s(T_t e_{j,i,t})^{\nu_c}}\frac{\nu_c}{e_{j,i}}\frac{\sum\limits_{j'\neq j}(1+\nu_s(T_t e_{j',i,t})^{\nu_c})}{\sum\limits_{j'\in\mathcal{I}}(1+\nu_s(T_t e_{j',i,t})^{\nu_c})}$
- $\sum\limits_{k\neq j}\frac{\partial s_{\mathcal{I},j,i}}{\partial\omega_{\mathcal{I},k,i}}\frac{\partial\omega_{\mathcal{I},k,i}}{\partial e_{j,i}} = \sum\limits_{k\neq j}\sigma s_{\mathcal{I},j,i}s_{\mathcal{I},k,i}\frac{1+\nu_s(T_t e_{j,i,t})^{\nu_c}}{\sum\limits_{j'\in\mathcal{I}}(1+\nu_s(T_t e_{j',i,t})^{\nu_c})}\frac{\nu_s(T_t e_{j,i,t})^{\nu_c}}{1+\nu_s(T_t e_{j,i,t})^{\nu_c}}\frac{\nu_c}{e_{j,i,t}}$
  $= \sigma s_{\mathcal{I},j,i}(1-s_{\mathcal{I},j,i}-s_{\mathcal{I},0,i})\frac{\nu_s(T_t e_{j,i,t})^{\nu_c}}{1+\nu_s(T_t e_{j,i,t})^{\nu_c}}\frac{\nu_c}{e_{j,i,t}}\frac{1+\nu_s(T_t e_{j,i,t})^{\nu_c}}{\sum\limits_{j'\in\mathcal{I}}(1+\nu_s(T_t e_{j',i,t})^{\nu_c})}$

So, adding them, we get that:

$$\frac{\partial\pi_{j,i,t}}{\partial e_{j,i,t}} = (1-\mathcal{M}_{j,i,t}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i,t}} M_{i,t}(\mathcal{I})\sigma s_{\mathcal{I},j,i}\frac{\nu_s(T_t e_{j,i,t})^{\nu_c}}{1+\nu_s(T_t e_{j,i,t})^{\nu_c}}\frac{\nu_c}{e_{j,i}}\left(1-s_{\mathcal{I},j,i,t}-s_{\mathcal{I},0,i,t}\frac{\omega_{\mathcal{I},j,i,t}}{\#\mathcal{I}}\right) \quad (30)$$

7.4.3 The effect of learning about another good on $s_{\mathcal{I},j,i}$ and the demand elasticity

Proposition 3 *If $j\in\mathcal{I}\subset\mathcal{I}'$, then:*

1. $s_{\mathcal{I},j,i} > s_{\mathcal{I}',j,i}$
2. $|\epsilon_{\mathcal{I},j,i}| < |\epsilon_{\mathcal{I}',j,i}|$, *where* $\epsilon_{\mathcal{I},j,i} = \frac{p_{j,i}}{c_{\mathcal{I},j,i}}\frac{\partial c_{\mathcal{I},j,i}}{\partial p_{j,i}}$

Proof. If $j\in\mathcal{I}\subset\mathcal{I}'$, then, since $\omega_{k,i},\mathcal{M}_{k,i} > 0$ for all firm k and $\sigma > 1$:

$$s_{\mathcal{I},j,i} = \left[(A_0 \mathcal{M}_{j,i})^{\sigma-1} \omega_{j,i}^{-\sigma} + \sum_{k\in\mathcal{I}} \left(\frac{\omega_{k,i}}{\omega_{j,i}}\right)^{\sigma} \left(\frac{\mathcal{M}_{j,i}}{\mathcal{M}_{k,i}}\right)^{\sigma-1} \right]^{-1}$$

$$> \left[(A_0 \mathcal{M}_{j,i})^{\sigma-1} \omega_{j,i}^{-\sigma} + \sum_{k\in\mathcal{I}} \left(\frac{\omega_{k,i}}{\omega_{j,i}}\right)^{\sigma} \left(\frac{\mathcal{M}_{j,i}}{\mathcal{M}_{k,i}}\right)^{\sigma-1} + \sum_{k\in\mathcal{I}'\setminus\mathcal{I}} \left(\frac{\omega_{k,i}}{\omega_{j,i}}\right)^{\sigma} \left(\frac{\mathcal{M}_{j,i}}{\mathcal{M}_{k,i}}\right)^{\sigma-1} \right]^{-1} = s_{\mathcal{I}',j,i}$$

For 2, define $\epsilon_{\mathcal{I},j,i} = \frac{p_{j,i}}{c_{\mathcal{I},j,i}} \frac{\partial c_{\mathcal{I},j,i}}{\partial p_{j,i}}$, and note that $\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}} = \frac{\partial s_{\mathcal{I},j,i}}{\partial p_{j,i}} \frac{\partial p_{j,i}}{\partial \mathcal{M}_{j,i}} = \frac{1}{E}\left[c_{\mathcal{I},j,i} + p_{j,i}\frac{\partial c_{\mathcal{I},j,i}}{\partial p_{j,i}}\right] w = \frac{s_{\mathcal{I},j,i}}{\mathcal{M}_{j,i}}(1+\epsilon_{\mathcal{I},j,i})$. And using the expression for $\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}}$ in 7.4.1, we have:

$$-\frac{s_{\mathcal{I},j,i}}{\mathcal{M}_{j,i}}(\sigma-1)(1-s_{\mathcal{I},j,i}) = \frac{s_{\mathcal{I},j,i}}{\mathcal{M}_{j,i}}(1+\epsilon_{\mathcal{I},j,i}) \implies \epsilon_{\mathcal{I},j,i} = -\sigma + s_{\mathcal{I},j,i}(\sigma-1)$$

So, using 1, we have $j \in \mathcal{I} \subset \mathcal{I}'$ $0 > \epsilon_{\mathcal{I},j,i} > \epsilon_{\mathcal{I}',j,i}$ ■

7.4.4 Derivation of optimal markup:

$0 = \frac{\partial \pi_{j,i}}{\partial p_{j,i}} = y_{j,i} + \left(p_{j,i} - \frac{\partial w N_{j,i}}{\partial N_{j,i}}\frac{\partial N_{j,i}}{\partial y_{j,i}}\right) \sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I}) \frac{\partial c_{\mathcal{I},j,i}}{\partial p_{j,i}}$

On the one hand: $\frac{\partial c_{\mathcal{I},j,i}}{\partial p_{j,i}} = c_{\mathcal{I},j,i}\left[-\sigma p_{j,i}^{-1} + (\sigma-1)P_{\mathcal{I},i}^{-1}\frac{\partial P_{\mathcal{I},i}}{\partial p_{j,i}}\right] = \frac{c_{\mathcal{I},j,i}}{p_{j,i}}\left[-\sigma + (\sigma-1)\frac{p_{j,i}}{P_{\mathcal{I},i}}\frac{\partial P_{\mathcal{I},i}}{\partial p_{j,i}}\right]$.
And $\frac{\partial P_{\mathcal{I},i}}{\partial p_{j,i}} = P_{\mathcal{I},i}^{\sigma} p_{j,i}^{-\sigma} \omega_{j,i}^{\sigma}$. So, we have:

$0 = y_{j,i} + (p_{j,i} - w)\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I}) \frac{c_{\mathcal{I},j,i}}{p_{j,i}}\left[-\sigma + (\sigma-1)\left(\frac{P_{\mathcal{I},i}}{p_{j,i}}\right)^{\sigma-1}\omega_{j,i}^{\sigma}\right]$

Since $s_{\mathcal{I},j,i} = \left(\frac{P_{\mathcal{I},i}}{p_{j,i}}\right)^{\sigma-1}\omega_{j,i}^{\sigma}$ and multiplying by $\frac{p_{j,i}}{E}$:
$0 = s_{j,i} + \left(1 - \mathcal{M}_{j,i}^{-1}\right)\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I}) s_{\mathcal{I},j,i}\left[-\sigma + (\sigma-1)s_{\mathcal{I},j,i}\right]$
Equivalently, we can write it:
$0 = 1 + \left(1 - \frac{1}{\mathcal{M}_{j,i}}\right)\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} \frac{M_i(\mathcal{I})c_{\mathcal{I},j,i}}{y_{j,i}}\left[-\sigma + (\sigma-1)s_{\mathcal{I},j,i}\right]$
And using that $\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} \frac{M_i(\mathcal{I})c_{\mathcal{I},j,i}}{y_{j,i}} = 1$ and defining $\bar{s}_{j,i} = \sum_{\mathcal{I}\in\mathcal{P}_{j,i}} \frac{M_i(\mathcal{I})c_{\mathcal{I},j,i}}{y_{j,i}} s_{\mathcal{I},j,i}$:
$1 = \left(1 - \frac{1}{\mathcal{M}_{j,i}}\right)\left[\sigma - (\sigma-1)\bar{s}_{j,i}\right] \implies \left[\sigma - (\sigma-1)\bar{s}_{j,i}\right]^{-1} = 1 - \frac{1}{\mathcal{M}_{j,i}}$
Rearranging:
$\frac{1}{\mathcal{M}_{j,i}} = \frac{\sigma-1-(\sigma-1)\bar{s}_{j,i}}{\sigma-(\sigma-1)\bar{s}_{j,i}}$

7.5 Derivation of the expression for the FOC:

Derivative of $\rho_{j,i}$ with respect to $e_{k,i}$. $\frac{\partial \rho_k}{\partial e_j} = \psi_c \psi_s T^{\psi_c} \alpha_k^{\psi_c-1}\frac{\partial \alpha_k}{\partial e_j}$

- If $J = 1$ (ad space not binding), then $\frac{\partial \alpha_k}{\partial e_j} = 0$, so $\frac{\partial \rho_k}{\partial e_j} = 0$; and $\frac{\partial \alpha_j}{\partial e_j} = \frac{1}{p_a}$, so: $\frac{\partial \rho_j}{\partial e_j} = \psi_c \psi_s T^{\psi_c}\alpha_j^{\psi_c-1}\frac{1}{p_a}$
- If $J > 1$ (ad space binding), then $\frac{\partial \alpha_k}{\partial e_j} = -\frac{a_k}{\sum_s e_s}$, so $\frac{\partial \rho_k}{\partial e_j} = -\psi_c \psi_s T^{\psi_c}\alpha_k^{\psi_c-1}\frac{a_k}{\sum_s e_s}$; and $\frac{\partial \alpha_j}{\partial e_j} = \frac{\alpha}{\sum_s e_s} - \frac{\alpha_j}{\sum_s e_s} = \frac{\sum_{k\neq j}\alpha_k}{\sum_s e_s}$, so: $\frac{\partial \rho_j}{\partial e_j} = \psi_c \psi_s T^{\psi_c}\alpha_j^{\psi_c-1}\frac{\sum_{k\neq j}\alpha_k}{\sum_s e_s}$.
  Note that since $\alpha_j = \alpha\frac{e_j}{\sum_k e_k}$, we can rewrite them as: $\frac{\partial \rho_k}{\partial e_j} = -\psi_c \psi_s T^{\psi_c}\frac{\alpha^{\psi_c}}{(\sum_s e_s)^{\psi_c+1}} e_k^{\psi_c-1} e_k$ and $\frac{\partial \rho_j}{\partial e_j} = \psi_c \psi_s T^{\psi_c}\frac{\alpha^{\psi_c}}{(\sum_s e_s)^{\psi_c+1}} e_j^{\psi_c-1}\sum_{k\neq j} e_k$

Derivative of a function of next period vector of masses.

Lemma 1 *If $f : \vec{\hat{M}}' \to \mathbb{R}$, then we have:*

1. $\frac{\partial f}{\partial \rho_j} = \sum_{\mathcal{I}\in\mathcal{P}_{-j}} M(\mathcal{I}) \sum_{\mathcal{I}'\in\mathcal{P}_{-j},\mathcal{I}'\supseteq\mathcal{I}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right]$

2. *For the anticompettive motive, it will be useful:*
$$\frac{\partial f}{\partial \rho_j} = \sum_{\mathcal{I}\in\mathcal{P}_{-k,-j}} M(\mathcal{I}) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}\left[\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right] - \left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k,j\})} - \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k\})}\right]\right]$$
$$+ \sum_{\mathcal{I}\in\mathcal{P}_{-k,-j}} (M(\mathcal{I}) + M(\mathcal{I}\cup\{k\})) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\cup\{k\}\to\mathcal{I}'\cup\{k\})}{1-\rho_j}\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k,j\})} - \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k\})}\right]$$

Proof. First, recall that $\hat{M}'(\mathcal{I}') = \sum_{\mathcal{I}\in\mathcal{P}(\mathcal{J})} M(\mathcal{I})\Theta(\mathcal{I}\to\mathcal{I}')$.
Next, the derivatives of $\Theta(\mathcal{I}\to\mathcal{I}')$ wrt ρ_j are:

1. If $\mathcal{I} \nsubseteq \mathcal{I}'$: $\frac{\partial\Theta(\mathcal{I}\to\mathcal{I}')}{\partial\rho_j} = 0$

2. If $\mathcal{I} \subseteq \mathcal{I}'$:

   (a) If $j \in \mathcal{I}$: $\frac{\partial\Theta(\mathcal{I}\to\mathcal{I}')}{\partial\rho_j} = 0$

   (b) If $j \in \mathcal{I}'\setminus\mathcal{I}$: $\frac{\partial\Theta(\mathcal{I}\to\mathcal{I}')}{\partial\rho_j} = (1-\delta)\prod_{k\in\mathcal{I}'\setminus(\mathcal{I}\cup\{j\})}\rho_k \prod_{k\notin\mathcal{I}'}(1-\rho_k) = \frac{\Theta(\mathcal{I}\to\mathcal{I}'\setminus\{j\})}{1-\rho_j}$

   (c) If $j \notin \mathcal{I}'$: $\frac{\partial\Theta(\mathcal{I}\to\mathcal{I}')}{\partial\rho_j} = -(1-\delta)\prod_{k\in\mathcal{I}'\setminus\mathcal{I}}\rho_k \prod_{k\notin(\mathcal{I}'\cup\{j\})}(1-\rho_k) = -\frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}$

Using this, the derivative of $\hat{M}'(\mathcal{I}')$ wrt ρ_j is:

1. If $j \in \mathcal{I}'$: $\frac{\partial\hat{M}'(\mathcal{I}')}{\partial\rho_j} = \sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subset\mathcal{I}'} M(\mathcal{I})\frac{\partial\Theta(\mathcal{I}\to\mathcal{I}')}{\partial\rho_j} = \sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subset\mathcal{I}'} M(\mathcal{I})\frac{\Theta(\mathcal{I}\to\mathcal{I}'\setminus\{j\})}{1-\rho_j}$

2. If $j \notin \mathcal{I}'$: $\frac{\partial\hat{M}'(\mathcal{I}')}{\partial\rho_j} = \sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subseteq\mathcal{I}'} M(\mathcal{I})\frac{\partial\Theta(\mathcal{I}\to\mathcal{I}')}{\partial\rho_j} = -\sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subseteq\mathcal{I}'} M(\mathcal{I})\frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}$

And the derivative of a generic function $f : \vec{\hat{M}}' \to \mathbb{R}$ wrt ρ_j is:

$$\frac{\partial f}{\partial\rho_j} = \sum_{\mathcal{I}'\in\mathcal{P}} \frac{\partial f}{\partial M'(\mathcal{I}')}\frac{\partial M'(\mathcal{I}')}{\partial\rho_j} = \sum_{\mathcal{I}'\in\mathcal{P}_j} \frac{\partial f}{\partial M'(\mathcal{I}')}\frac{\partial M'(\mathcal{I}')}{\partial\rho_j} + \sum_{\mathcal{I}'\in\mathcal{P}_{-j}} \frac{\partial f}{\partial M'(\mathcal{I}')}\frac{\partial M'(\mathcal{I}')}{\partial\rho_j}$$
$$= \sum_{\mathcal{I}'\in\mathcal{P}_j} \frac{\partial f}{\partial M'(\mathcal{I}')} \sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subset\mathcal{I}'} M(\mathcal{I})\frac{\Theta(\mathcal{I}\to\mathcal{I}'\setminus\{j\})}{1-\rho_j} - \sum_{\mathcal{I}'\in\mathcal{P}_{-j}} \frac{\partial f}{\partial M'(\mathcal{I}')} \sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subseteq\mathcal{I}'} M(\mathcal{I})\frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}$$

Now we are going to merge the two summations using that $\mathcal{P}_j = \{\mathcal{I}\cup\{j\}|\mathcal{I}\in\mathcal{P}_{-j}\}$ [**Proof:** from any set $\mathcal{I}$ that doesn't contain j we can build one by adding j to $\mathcal{I}$ (that is, $\{\mathcal{I}\cup\{j\}|\mathcal{I}\in\mathcal{P}_{-j}\} \subseteq \mathcal{P}_j$), and that from any $\mathcal{I}'$ that contains j we can build another one that doesn't contain j by removing j from $\mathcal{I}'$ (that is, $\mathcal{P}_j = \{(\mathcal{I}'\setminus\{j\})\cup\{j\}|\mathcal{I}'\in\mathcal{P}_j\} \subseteq \{\mathcal{I}\cup\{j\}|\mathcal{I}\in\mathcal{P}_{-j}\}$].

Using this in the previous expression, we get:

$$\frac{\partial f}{\partial \rho_j} = \sum_{\mathcal{I}'\in\mathcal{P}_{-j}} \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} \sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subseteq\mathcal{I}'} M(\mathcal{I}) \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j} - \sum_{\mathcal{I}'\in\mathcal{P}_{-j}} \frac{\partial f}{\partial M'(\mathcal{I}')} \sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subseteq\mathcal{I}'} M(\mathcal{I}) \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}$$
$$= \sum_{\mathcal{I}'\in\mathcal{P}_{-j}} \sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subseteq\mathcal{I}'} M(\mathcal{I}) \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j} \left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right]$$
$$= \sum_{\mathcal{I}\in\mathcal{P}_{-j}} M(\mathcal{I}) \sum_{\mathcal{I}'\in\mathcal{P}_{-j},\mathcal{I}'\supseteq\mathcal{I}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j} \left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right]$$

where for the last equality, I have used that $\{(\mathcal{I},\mathcal{I}')|\mathcal{I}\in\mathcal{P},\mathcal{I}'\subseteq\mathcal{I}\} = \{(\mathcal{I},\mathcal{I}')|\mathcal{I}'\in\mathcal{P},\mathcal{I}\supseteq\mathcal{I}'\}$.

This proves the first expression of the lemma. For the second:
First, note that above we have shown that $\{\mathcal{I}\cup\{j\}|\mathcal{I}\in\mathcal{P}_{-j}\} = \mathcal{P}_j$, which implies that $\mathcal{P} = \mathcal{P}_{-j}\cup\{\mathcal{I}\cup\{j\}|\mathcal{I}\in\mathcal{P}_{-j}\}$. Analogously, defining $\mathcal{P}_{-k,-j} = \{\mathcal{I}\in\mathcal{P}|j,k\notin\mathcal{I}\}$, we have: $\mathcal{P}_{-j} = \mathcal{P}_{-k,-j}\cup\{\mathcal{I}\cup\{k\}|\mathcal{I}\in\mathcal{P}_{-k,-j}\}$, so the previous expression becomes

$$\frac{\partial f}{\partial \rho_j} = \sum_{\mathcal{I}\in\mathcal{P}_{-k,-j}} M(\mathcal{I}) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j} \left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right]$$
$$+ M(\mathcal{I}\cup\{k\}) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-j}\\ \mathcal{I}'\supseteq\mathcal{I}\cup\{k\}}} \frac{\Theta(\mathcal{I}\cup\{k\}\to\mathcal{I}')}{1-\rho_j} \left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right]$$

Now, for the first line, I use the following equivalence: for each $\mathcal{I}\in\mathcal{P}_{-k,-j}$ we have $\{\mathcal{I}'\in\mathcal{P}_{-j}|\mathcal{I}'\supseteq\mathcal{I}\} = \{\mathcal{I}'\in\mathcal{P}_{-k,-j}|\mathcal{I}'\supseteq\mathcal{I}\}\cup\{\mathcal{I}'\cup\{k\}|\mathcal{I}'\in\mathcal{P}_{-k,-j},\mathcal{I}'\supseteq\mathcal{I}\}$. And for the second line, I use the equivalence: for $\mathcal{I}\cup\{k\}$ we have $\{\mathcal{I}'\in\mathcal{P}_{-j}|\mathcal{I}'\supseteq\mathcal{I}\cup\{k\}\} = \{\mathcal{I}'\cup\{k\}|\mathcal{I}'\in\mathcal{P}_{-k,-j},\mathcal{I}'\supseteq\mathcal{I}\}$.

$$\frac{\partial f}{\partial \rho_j} = \sum_{\mathcal{I}\in\mathcal{P}_{-k,-j}} M(\mathcal{I}) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \left[\frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j} \left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right] + \frac{\Theta(\mathcal{I}\to\mathcal{I}'\cup\{k\})}{1-\rho_j} \left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k,j\})} - \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k\})}\right]\right]$$
$$+ M(\mathcal{I}\cup\{k\}) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\cup\{k\}\to\mathcal{I}'\cup\{k\})}{1-\rho_j} \left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k,j\})} - \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k\})}\right]$$

Finally, I use that for $\mathcal{I},\mathcal{I}'\in\mathcal{P}_{-k,-j}$ with $\mathcal{I}'\supseteq\mathcal{I}$, we have $\Theta(\mathcal{I}\cup\{k\}\to\mathcal{I}'\cup\{k\}) = \prod_{h\in\mathcal{I}'\setminus\mathcal{I}}\rho_h \prod_{h\notin\mathcal{I}'\cup\{k\}}(1-\rho_h)\cdot(\rho_k+1-\rho_k) = \Theta(\mathcal{I}\to\mathcal{I}'\cup\{k\}) + \Theta(\mathcal{I}\to\mathcal{I}')$. So, I substitute in the first line $\Theta(\mathcal{I}\to\mathcal{I}'\cup\{k\}) = \Theta(\mathcal{I}\cup\{k\}\to\mathcal{I}'\cup\{k\}) - \Theta(\mathcal{I}\to\mathcal{I}')$.

$$\frac{\partial f}{\partial \rho_j} = \sum_{\mathcal{I}\in\mathcal{P}_{-k,-j}} M(\mathcal{I}) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j} \left[\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right] - \left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k,j\})} - \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k\})}\right]\right]$$
$$+ (M(\mathcal{I}) + M(\mathcal{I}\cup\{k\})) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\cup\{k\}\to\mathcal{I}'\cup\{k\})}{1-\rho_j} \left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k,j\})} - \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k\})}\right]$$

■

With uncertainty:

$$\text{For } \mathcal{I}' \in \mathcal{P}(\mathcal{J}'), \qquad M'(\mathcal{I}') = \begin{cases} \sum_{\{\mathcal{I}\in\mathcal{P}(\mathcal{J}):\mathcal{I}\cap\mathcal{J}'=\mathcal{I}'\}} \hat{M}(\mathcal{I}) & \text{, if } \mathcal{I}' \subseteq \mathcal{J} \\ 0 & \text{, if } \mathcal{I}' \nsubseteq \mathcal{J} \end{cases} \tag{31}$$

where the first case says that two consumers become identical in industry i if all the firms in which they differed exit, whereas the second case says that there are no consumers who are aware of a newborn firm. The last piece of infornation needed to compute expected values is the probabilities that the set of differentiated goods moves from $\mathcal{J}$ to $\mathcal{J}' \subseteq \mathcal{J} \cup \{e\}$, where e denotes an entrant. These probabilities are given by:

$$\text{For } \mathcal{J}' \in \mathcal{P}(\mathcal{J} \cup e)), \qquad Prob\{\mathcal{J} \to \mathcal{J}'\} = \begin{cases} (1 - z_{e,i,t}) \prod_{j\in\mathcal{J}\cap\mathcal{J}'} (1-\kappa) \prod_{j\in\mathcal{J}\setminus\mathcal{J}'} \kappa & \text{, if } e \notin \mathcal{J}' \\ z_{e,i,t} \prod_{j\in\mathcal{J}\cap\mathcal{J}'} (1-\kappa) \prod_{j\in\mathcal{J}\setminus\mathcal{J}'} \kappa & \text{, if } e \in \mathcal{J}' \end{cases} \tag{32}$$

In the model, there is uncertainty on $\mathcal{J}'$, so I am more interested in finding the derivative of the expected value of a function $g : \vec{M}' \to \mathbb{R}$ rather than the derivative of a function $f : \vec{\hat{M}}' \to \mathbb{R}$ (note that f is defined on $\vec{\hat{M}}'$, that is, the next period distribution if there weren't entry and exit, whereas g is defined on the actual next period distribution after the uncertainty has been resolved). Recall that for each $\mathcal{J}' \subseteq \mathcal{J} \cup \{e\}$, the probability of this transition is given by $Prob\{\mathcal{J} \to \mathcal{J}'\}$ defined in 32 and the mapping between $\vec{\hat{M}}'$ and $\vec{M}'$ is given by $F_{\mathcal{J},\mathcal{J}'} : \vec{\hat{M}}' \to \vec{M}'$ defined in 31:

$$F_{\mathcal{J},\mathcal{J}'}(\vec{\hat{M}}') = \begin{cases} M'(\mathcal{I}) = \sum_{\{\mathcal{I}\in\mathcal{P}(\mathcal{J}):\mathcal{I}\cap\mathcal{J}'=\mathcal{I}'\}} \hat{M}(\mathcal{I}) & \text{, for } \mathcal{I}' \subseteq \mathcal{J} \\ M'(\mathcal{I}) = 0 & \text{, for } \mathcal{I}' \nsubseteq \mathcal{J} \end{cases}$$

Going in the reverse order, each $\mathcal{I} \in \mathcal{P}(\mathcal{J})$ is associated to $\mathcal{I}' = \mathcal{I}\cap\mathcal{J}' \in \mathcal{P}(\mathcal{J}')$; therefore $\frac{\partial g\left(F_{\mathcal{J},\mathcal{J}'}(\vec{\hat{M}}')\right)}{\partial \hat{M}'(\mathcal{I})} = \frac{\partial g(\vec{M}')}{\partial \hat{M}'(\mathcal{I})} = \frac{\partial g(\vec{M}')}{\partial M'(\mathcal{I}\cap\mathcal{J}')} \frac{\partial M'(\mathcal{I}\cap\mathcal{J}')}{\partial \hat{M}'(\mathcal{I})} = \frac{\partial g(\vec{M}')}{\partial M'(\mathcal{I}\cap\mathcal{J}')}$.
Then, we can apply this to the result of the case without entry and exit and we have:

$$\frac{\partial g\left(F_{\mathcal{J},\mathcal{J}'}(\vec{\hat{M}}')\right)}{\partial \rho_j} = \sum_{\mathcal{I}\in\mathcal{P}_{-j}} M(\mathcal{I}) \sum_{\mathcal{I}'\in\mathcal{P}_{-j},\mathcal{I}'\supseteq\mathcal{I}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j} \left[\frac{\partial g}{\partial M'((\mathcal{I}'\cup\{j\})\cap\mathcal{J}')} - \frac{\partial g}{\partial M'(\mathcal{I}'\cap\mathcal{J}')}\right]$$

Note that if $j \notin \mathcal{J}'$, then this derivative is 0 since in this case $\frac{\partial g}{\partial M'((\mathcal{I}'\cup\{j\})\cap\mathcal{J}')} = \frac{\partial g}{\partial M'(\mathcal{I}'\cap\mathcal{J}')}$.
With this, the expected value is defined as:

$$\mathbb{E}g\left(\vec{M}\right) = \sum_{\mathcal{J}'\subseteq\mathcal{J}\cup\{e\}} Prob\{\mathcal{J}\to\mathcal{J}'\} g\left(F_{\mathcal{J},\mathcal{J}'}(\vec{\hat{M}}')\right)$$

Proof of the Proposition 1: For the informative motive it is straightforward from applying 1 of Lemma 1, together with the note on the Uncertainty case:

$$\frac{\partial \mathbb{E}V_j(\mathcal{J}',\vec{M}')}{\partial \rho_j}\frac{\partial \rho_j}{\partial e_j} = \left(\sum_{\mathcal{J}'\subseteq\mathcal{J}\cup\{e\}} Prob\{\mathcal{J}\to\mathcal{J}'\} \sum_{\mathcal{I}\in\mathcal{P}_{-j}} M(\mathcal{I}) \sum_{\mathcal{I}'\in\mathcal{P}_{-j},\mathcal{I}'\supseteq\mathcal{I}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j} \left[\frac{\partial V_j(\mathcal{J}',\vec{M}')}{\partial M'((\mathcal{I}'\cup\{j\})\cap\mathcal{J}')} - \frac{\partial V_j(\mathcal{J}',\vec{M}')}{\partial M'(\mathcal{I}'\cap\mathcal{J}')}\right]\right)\frac{\partial \rho_j}{\partial e_j} > 0$$

where the positive comes from the fact that $\frac{\partial V_j(\mathcal{J}',\vec{M}')}{\partial M'((\mathcal{I}'\cup\{j\})\cap\mathcal{J}')} \geq \frac{\partial V_j(\mathcal{J}',\vec{M}')}{\partial M'(\mathcal{I}'\cap\mathcal{J}')}$, since firm j's value increases more if we add a consumer that besides $\mathcal{I}' \cap \mathcal{J}'$ she is also aware of j (there is equality if j has exited in the scenario with $\mathcal{J}'$). For the result that the informative motive decreases if we add $\{j\}$ to some consumers that weren't aware, note that the above expression is a summation over the awareness sets that don't contain j, and the change described implies a reduction of the masses in these sets.

For the anticompetitive motive, I use 2 of 1, which together with the note on the Uncertainty case, implies:

$$\sum_{k\neq j}\left(-\frac{\partial \mathbb{E}V_j(\mathcal{J}',\vec{M}')}{\partial \rho_{j'}}\right)\left(-\frac{\partial \rho_k}{\partial e_j}\right) =$$

$$\sum_{k\neq j}\sum_{\mathcal{J}'\subseteq\mathcal{J}\cup\{e\}} Prob\{\mathcal{J}\to\mathcal{J}'\}\left(\sum_{\mathcal{I}\in\mathcal{P}_{-k,-j}} M(\mathcal{I}) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_k} \underbrace{\left[\left[\frac{\partial V_j(\mathcal{J}',\vec{M}')}{\partial M'(\mathcal{I}')} - \frac{\partial V_j(\mathcal{J}',\vec{M}')}{\partial M'(\mathcal{I}'\cup\{k\})}\right] - \left[\frac{\partial V_j(\mathcal{J}',\vec{M}')}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial V_j(\mathcal{J}',\vec{M}')}{\partial M'(\mathcal{I}'\cup\{j,k\})}\right]\right]}_{<0}\right.$$

$$\left. + \sum_{\mathcal{I}\in\mathcal{P}_{-k,-j}} (M(\mathcal{I}) + M(\mathcal{I}\cup\{j\})) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\cup\{j\}\to\mathcal{I}'\cup\{j\})}{1-\rho_k} \underbrace{\left[\frac{\partial V_j(\mathcal{J}',\vec{M}')}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial V_j(\mathcal{J}',\vec{M}')}{\partial M'(\mathcal{I}'\cup\{j,k\})}\right]}_{>0}\right)\left(-\frac{\partial \rho_k}{\partial e_j}\right) > 0$$

where the negative sign of the first underbrace is due to $\left(\frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j\})} - \frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j\}\cup\{j'\})}\right) > \left(\frac{\partial V_j}{\partial M'(\mathcal{I})} - \frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j'\})}\right)$ (i.e. the firm value clearly is more affected if it is a customer who learns about another good (since she will reduce the spending in j) rather than if it is a non-customer who learns about another good). And the positive sign in the second underbrace is because the value of a firm decreases if a customer learns about another good. So, we see that the anticompetitive motive increases more if $M(\mathcal{I}\cup\{j\})$ increases rather than $M(\mathcal{I})$ (direct from taking the derivatives with respect to $M(\mathcal{I}\cup\{j\})$ and $M(\mathcal{I})$ in the previous expression. Finally, that the anticompetitive motive is positive follows from observing that $\Theta(\mathcal{I}\to\mathcal{I}') = (1-\rho_j)\Theta(\mathcal{I}\cup\{j\}\to\mathcal{I}'\cup\{j\}) < \Theta(\mathcal{I}\cup\{j\}\to\mathcal{I}'\cup\{j\})$, and then the negative part of the first line is offset by the second line.

The result for the persuasive motive is straightforward from observing that 30 is a summation over the awareness sets that contain j.

7.6 Extension: Different definition of the demand shifter. Persuasive effect no harm to the taste of other varieties

Here, $\omega_{j,i,t} = 1 + \nu_s(T_t\alpha_{j,i,t})^{\nu_c}$. With this, now the derivatives with respect to $e_{j,i,t}$ write:

7.7 Social planner problem

$$\max_{\{N_{M,t},N_{j,i,t},h_{e,i,t},p_{j,i,t},\alpha_{j,i,t}\}} U = \sum_{t=0}^{\infty} \beta^t \int_0^1 [\ln C_{\ell t} + L_{\ell t}] \, d\ell$$

s.t. $C_{\ell,t}$ from (4), $C_{\ell,i,t}$ from (5), $c_{\ell,j,i,t}$ from (10), and $L_{\ell,t}$ from 3, with $T_t = Q_t$

$$y_{j,i,t} = N_{j,i,t}, \quad y_{0,i,t} = A_0 N_{0,i,t}, \quad Q_t = AN_{m,t}^{\varphi} \quad \text{(Production functions)}$$

$$1 = N_{m,t} + \int_0^1 \left(\sum_{j\in\{0\}\cup\mathcal{J}_{i,t}} N_{j,i,t} + N_{e,i,t} \right) di \quad , \quad w_t = E_t \quad \text{(Resource constraints)}$$

$$\sum_{j\in\mathcal{J}_{i,t}} \alpha_{j,i,t} = \alpha_i, (8), (9), (13), (31), (32) \quad \text{(Learning process)}$$

$$z_{e,i,t} = \phi N_{e,i,t}^{\frac{1}{2}}, (31), (32) \quad \text{(Entry and exit)}$$

Plugging $C_{\ell,t}$ and $L_{\ell,t}$ with $T_t = Q_t$ into the objective function and interchanging the integrals over ℓ and i:

$$\max_{\{N_{M,t},N_{e,i,t},N_{j,i,t},p_{j,i,t},\alpha_{j,i,t}\}} U = \int_0^1 \int_0^1 \sum_{t=0}^{\infty} \beta^t \ln C_{\ell,i,t} d\ell di + \sum_{t=0}^{\infty} \beta^t \upsilon \frac{Q_t^2}{2}$$

s.t. $C_{\ell,i,t}$ from (5), $c_{\ell,j,i,t}$ from (10)

$$y_{j,i,t} = N_{j,i,t}, \quad y_{0,i,t} = A_0 N_{0,i,t}, \quad Q_t = AN_{m,t}^{\varphi} \quad \text{(Production functions)}$$

$$1 = N_{m,t} + \int_0^1 \left(\sum_{j\in\{0\}\cup\mathcal{J}_{i,t}} N_{j,i,t} + N_{e,i,t} \right) di \quad , \quad w_t = E_t \quad \text{(Resource constraints)}$$

$$\sum_{j\in\mathcal{J}_{i,t}} \alpha_{j,i,t} = \alpha_i, (8), (9), (13), (31), (32) \quad \text{(Learning process)}$$

$$z_{e,i,t} = \phi N_{e,i,t}^{\frac{1}{2}}, (31), (32) \quad \text{(Entry and exit)}$$

The planner decides how much to produce for each individual and, accordingly, sets the prices that induce the consumers to consume these quantities. Let N_t^P be the labor used to produce all the goods in the production sector. Then, The FOC for $c_{\mathcal{I},j,i,t}$ writes:

$$[c_{\mathcal{I},j,i,t}] : \quad \beta^t \frac{1}{C_t} \frac{\partial C_t}{\partial C_{i,t}} \frac{\partial C_{i,t}}{\partial C_{\mathcal{I},i,t}} \frac{\partial C_{\mathcal{I},i,t}}{\partial c_{\mathcal{I},j,i,t}} = \beta^t \lambda \frac{\partial N_t^P}{\partial C_t} \frac{\partial C_t}{\partial C_{i,t}} \frac{\partial C_{i,t}}{\partial C_{\mathcal{I},i,t}} \frac{\partial C_{\mathcal{I},i,t}}{\partial c_{\mathcal{I},j,i,t}} \tag{33}$$

1. Dividing both sides $\frac{\partial C_t}{\partial C_{i,t}} \frac{\partial C_{i,t}}{\partial C_{\mathcal{I},i,t}} \frac{\partial C_{\mathcal{I},i,t}}{\partial c_{\mathcal{I},j,i,t}} > 0$, and defining $\hat{P}_t = \frac{w_t N_t^P}{C_t}$, we get:

$$\frac{1}{C_t} = \lambda \frac{\hat{P}_t}{w_t} \implies \lambda = \frac{w_t}{\hat{P}_t C_t} \tag{34}$$

2. $\ln C_t = \int_0^1 \ln C_{i,t} di$. Dividing both sides of 33 by $\frac{\partial C_{i,t}}{\partial C_{\mathcal{I},i,t}} \frac{\partial C_{\mathcal{I},i,t}}{\partial c_{\mathcal{I},j,i,t}} > 0$, letting $N_{i,t}$ be the the labor used in sector i, $\frac{\partial N_t^P}{\partial C_t} \frac{\partial C_t}{\partial C_{i,t}} = \frac{\partial N_t^P}{\partial C_{i,t}}$ and defining $\hat{P}_{i,t} = \frac{w_t N_{i,t}}{C_{i,t}}$, we get:

$$\frac{1}{C_{i,t}} = \lambda \frac{\hat{P}_{i,t}}{w_t} \implies \lambda = \frac{w_t}{\hat{P}_{i,t} C_{i,t}} \implies C_{i,t} = C_t \frac{\hat{P}_t}{\hat{P}_{i,t}} \tag{35}$$

where for the last expression I have used 34. Plugging $C_{i,t}$ into the definition of C_t, we get:

$$\ln \hat{P}_t = \int_0^1 \ln \hat{P}_{i,t} di \tag{36}$$

3. $\ln C_{i,t} = \int_0^1 \ln C_{\ell,i,t} d\ell$. Dividing both sides of 33 by $\frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} > 0$, letting $N_{\ell,i,t}$ be the the labor used in sector i by ℓ, $\frac{\partial N_{i,t}}{\partial C_{i,t}} \frac{\partial C_{i,t}}{\partial C_{\ell,i,t}} = \frac{\partial N_{i,t}}{\partial C_{\ell,i,t}}$ and defining $\hat{P}_{\ell,i,t} = \frac{w_t N_{\ell,i,t}}{C_{\ell,i,t}}$, we get:

$$\frac{1}{C_{\ell,i,t}} = \lambda \frac{\hat{P}_{\ell,i,t}}{w_t} \implies \lambda = \frac{w_t}{\hat{P}_{\ell,i,t} C_{\ell,i,t}} \implies C_{\ell,i,t} = C_{i,t} \frac{\hat{P}_{i,t}}{\hat{P}_{\ell,i,t}} \tag{37}$$

where for the last expression I have used 35. Plugging $C_{\ell,i,t}$ into the definition of $C_{i,t}$, we get:

$$\ln \hat{P}_{i,t} = \int_0^1 \ln \hat{P}_{\ell,i,t} di \tag{38}$$

4. $C_{\ell,i,t}$ given by 5. Letting $N_{\ell,j,i,t}$ be the the labor used in good j in sector i by ℓ, $\frac{\partial N_{\ell,i,t}}{\partial C_{\ell,i,t}} \frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = \frac{\partial N_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = \frac{1}{A_j}$, we get:

$$\frac{1}{C_{\ell,i,t}} \left(\frac{C_{\ell,i,t}}{c_{\ell,j,i,t}} \right)^{\frac{1}{\sigma}} \omega_{j,i,t} = \lambda \frac{1}{A_j} \implies \left(\frac{C_{\ell,i,t}}{c_{\ell,j,i,t}} \right)^{\frac{1}{\sigma}} \omega_{j,i,t} = \frac{w_t}{\hat{P}_{\ell,i,t} A_j} \implies c_{\ell,j,i,t} = C_{\ell,i,t} \hat{P}^{\sigma}_{\mathcal{I},i,t} \left(\omega_{j,i,t} \frac{A_j}{w_t} \right)^{\sigma} \tag{39}$$

where I have used λ from 37. Plugging $c_{\ell,j,i,t}$ into the definition of $C_{\ell,i,t}$, we get:

$$\hat{P}_{\mathcal{I},i,t} = \left(\left(\frac{A_0}{w_t} \right)^{\sigma-1} + \sum_{j \in \mathcal{I}} \omega^{\sigma}_{j,i,t} \left(\frac{1}{w_t} \right)^{\sigma-1} \right)^{\frac{1}{1-\sigma}} \tag{40}$$

Since we have $\hat{P}_{\ell,i,t} C_{\ell,i,t} = \hat{P}_{i,t} C_{i,t} = \hat{P}_t C_t = w_t N^P_t$ and λ from 34, then we have:

$$c_{\mathcal{I},j,i,t} = w_t N^P_t \hat{P}^{\sigma-1}_{\mathcal{I},i,t} \left(\omega_{j,i,t} \frac{A_j}{w_t} \right)^{\sigma}, \quad N^P_t = \frac{1}{\lambda} \tag{41}$$

Comparing this with the consumer choices:

$$c_{\mathcal{I},j,i,t} = E_t P^{\sigma-1}_{\mathcal{I},i,t} p^{-\sigma}_{j,i,t} \omega^{\sigma}_{j,i,t}, \quad P_{\mathcal{I},i,t} = \left(p^{1-\sigma}_{0,i,t} + \sum_{j \in \mathcal{I}} \omega^{\sigma}_{j,i,t} p^{1-\sigma}_{j,i,t} \right)^{\frac{1}{1-\sigma}}$$

It is straightforward to check that the planner can induce the consumer to consume the quantities in 41 by setting prices equal to the marginal cost times a markup (or a tax) equal to the ratio of expenditure to the production costs; i.e. $p_{j,i,t} = \frac{w_t}{A_j} \tau_t$, where $\tau_t = \frac{E_t}{w_t N^P_t}$.

The particular level of τ affects the level of consumption, but not the share of expenditure allocated to each good, since, as seen in the following expression, $s_{\mathcal{I},j,i,t}$ is independent of τ (that is, τ doesn't distort how N_t^P is allocated among the production goods):

$$s_{\mathcal{I},j,i,t} = \frac{p_{j,i,t}c_{\mathcal{I},j,i,t}}{E_{\mathcal{I},i,t}} = \frac{\tau \frac{w_t}{A_j} c_{\mathcal{I},j,i,t}}{\tau w_t N_{\mathcal{I},i,t}} = \omega_{j,i,t}^{\sigma} \left(\frac{w_t}{A_j \hat{P}_{\mathcal{I},i,t}} \right)^{1-\sigma} \implies s_{\mathcal{I},j,i} = \frac{\omega_{j,i}^{\sigma}}{A_0^{\sigma-1} + \sum_{k \in \mathcal{I}} \omega_{k,i}^{\sigma}} \tag{42}$$

Using that $c_{\mathcal{I},j,i} = A_j N_{\mathcal{I},j,i} = A_j \frac{N_{\mathcal{I},j,i}}{N_{\mathcal{I},i}} N_{\mathcal{I},i} = A_j s_{\mathcal{I},j,i} \frac{E_t}{w_t} \frac{1}{\tau}$; then, we can write $C_{\mathcal{I},i}$ as:

$$C_{\mathcal{I},i,t} = \frac{1}{\tau} \frac{E_t}{w_t} \left((A_0 s_{\mathcal{I},0,i,t})^{\frac{\sigma-1}{\sigma}} + \sum_{k \in \mathcal{I}} \omega_{k,i,t} (s_{\mathcal{I},k,i,t})^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{43}$$

and combining this with $\frac{E_t}{\tau} = w_t N_t^P = C_{\mathcal{I},i,t} \hat{P}_{\mathcal{I},i,t}$, we get (where for the second equality I use 42):

$$\left((A_0 s_{\mathcal{I},0,i,t})^{\frac{\sigma-1}{\sigma}} + \sum_{k \in \mathcal{I}} \omega_{k,i,t} (s_{\mathcal{I},k,i,t})^{\frac{\sigma-1}{\sigma}} \right) = \left(\frac{w_t}{\hat{P}_{\mathcal{I},i,t}} \right)^{\frac{\sigma-1}{\sigma}} = \omega_{j,i,t} s_{\mathcal{I},j,i,t}^{-\frac{1}{\sigma}} \tag{44}$$

Next, I move to the advertising part of the planner problem. We will use the following derivatives:

$$\frac{\partial C_{\mathcal{I},i,t}}{\partial \omega_{j,i}} = \frac{\sigma}{\sigma-1} C_{\mathcal{I},i,t}^{\frac{1}{\sigma}} \left[s_{\mathcal{I},j,i,t}^{\frac{\sigma-1}{\sigma}} + \frac{\sigma-1}{\sigma} \left(\sum_{k \in \mathcal{I}} \omega_{k,i,t} s_{\mathcal{I},k,i,t}^{\frac{-1}{\sigma}} \frac{\partial s_{\mathcal{I},k,i,t}}{\partial \omega_{j,i}} + A_0^{\frac{\sigma-1}{\sigma}} s_{\mathcal{I},0,i,t}^{\frac{-1}{\sigma}} \frac{\partial s_{\mathcal{I},0,i,t}}{\partial \omega_{j,i}} \right) \right]$$

$$\frac{\partial s_{\mathcal{I},j,i,t}}{\partial \omega_{j,i}} = s_{\mathcal{I},j,i,t}(1 - s_{\mathcal{I},j,i,t}) \frac{\sigma}{\omega_{j,i,t}}, \qquad \frac{\partial s_{\mathcal{I},k,i,t}}{\partial \omega_{j,i}} = -s_{\mathcal{I},j,i,t} s_{\mathcal{I},k,i,t} \frac{\sigma}{\omega_{j,i,t}}$$

$$\frac{\partial \omega_{j,i}}{\partial \alpha_{j,i,t}} = \nu_c \nu_s T^{\nu_c} \alpha_{j,i,t}^{\nu_c - 1} = \frac{T}{\alpha_{j,i,t}} \frac{\partial \omega_{j,i}}{\partial T}$$

The term in parenthesis of the first line can be rewritten as (in the second expression, I use 44):

$$\left(\omega_{j,i,t} - \sum_{k \in \mathcal{I}} \omega_{k,i,t} s_{\mathcal{I},k,i,t}^{\frac{\sigma-1}{\sigma}} - A_0^{\frac{\sigma-1}{\sigma}} s_{\mathcal{I},0,i,t}^{\frac{\sigma-1}{\sigma}} \right) s_{\mathcal{I},j,i,t} \frac{\sigma}{\omega_{j,i,t}} = \left(\omega_{j,i,t} - \omega_{j,i,t} s_{\mathcal{I},j,i,t}^{-\frac{1}{\sigma}} \right) s_{\mathcal{I},j,i,t} \frac{\sigma}{\omega_{j,i,t}} = \left(1 - s_{\mathcal{I},j,i,t}^{-\frac{1}{\sigma}} \right) s_{\mathcal{I},j,i,t} \sigma < 0$$

so, the term in the parenthesis is negative. And we have:

$$\frac{\partial \ln C_{\mathcal{I},i,t}}{\partial \omega_{j,i}} = \frac{\sigma}{\sigma-1} C_{\mathcal{I},i,t}^{\frac{1-\sigma}{\sigma}} \left[s_{\mathcal{I},j,i,t}^{\frac{\sigma-1}{\sigma}} + (\sigma-1) \left(1 - s_{\mathcal{I},j,i,t}^{-\frac{1}{\sigma}} \right) s_{\mathcal{I},j,i,t} \right] = \frac{\sigma}{\sigma-1} \left(\frac{s_{\mathcal{I},j,i,t}}{C_{\mathcal{I},i,t}} \right)^{\frac{\sigma-1}{\sigma}} \left[1 + (\sigma-1) \left(s_{\mathcal{I},j,i,t}^{\frac{1}{\sigma}} - 1 \right) \right]$$

So:

$$\frac{\partial \ln C_{\mathcal{I},i,t}}{\partial \alpha_{j,i}} = \left(\frac{s_{\mathcal{I},j,i,t}}{C_{\mathcal{I},i,t}} \right)^{\frac{\sigma-1}{\sigma}} \left[\frac{\sigma}{\sigma-1} + \sigma \left(s_{\mathcal{I},j,i,t}^{\frac{1}{\sigma}} - 1 \right) \right] \nu_c \nu_s T^{\nu_c} \alpha_{j,i,t}^{\nu_c - 1}$$

For the dynamic problem of advertising/media, it is useful to define $U_X = \int_0^1 \sum_{t=0}^{\infty} \beta^t \ln C_{\ell,i,t} d\ell$ as the expected life-time industry-consumption utility of an industry with the current industry state being X.

The social planner has to decide on (i) how much labor to allocate to the media sector, $N_{m,t}$, and (ii) how to allocate the ad space among the differentiated firms of each industry, $\alpha_{j,i,t}$.
First, let's see the social planner choice of $\alpha_{j,i,t}$. The allocation of the ad space has to be such that the marginal social gain of increasing the ad space given to each firm is the same, since otherwise we could improve the allocation. Formally, it must be $\beta\frac{\partial \mathbb{E}U_{X'}}{\partial \rho_{j,i}}\frac{\partial \rho_{j,i}}{\partial \alpha_{j,i}}+\frac{\partial \ln C_{X,t}}{\partial \alpha_{j,X}}=\hat{h}_X$ for some $\hat{h}_X$ and all $j\in\mathcal{J}_X$, together with $\sum\limits_{j\in\mathcal{J}_X}\alpha_{j,X}=\alpha_X$.
Second, let's see the social planner choice of N_m.

$$\left[\frac{\partial L}{\partial Q}+\sum_{X\in\Omega}\mu_t(X)\sum_{j\in\mathcal{J}_X}\left[\beta\frac{\partial \mathbb{E}U_{X'}}{\partial \rho_{j,X}}\frac{\partial \rho_{j,X}}{\partial T}+\frac{\partial \ln C_{X,t}}{\partial \omega_{j,X}}\frac{\partial \omega_{j,X}}{\partial T}\right]\frac{\partial T}{\partial Q}\right]\frac{\partial Q}{\partial N_m}=\lambda$$

where $\frac{\partial L}{\partial \bar{Q}}=\upsilon Q$, $\frac{\partial T}{\partial Q}=1$ (if $Q<1$, otherwise it is 0). Also, using that $\frac{\partial \rho_{j,X}}{\partial T}=\frac{\alpha_{j,X}}{T}\frac{\partial \rho_{j,X}}{\partial \alpha_{j,X}}$, $\frac{\partial \omega_{j,X}}{\partial T}=\frac{\alpha_{j,X}}{T}\frac{\partial \omega_{j,X}}{\partial \alpha_{j,X}}$, and $\frac{\partial Q}{\partial N_m}=\varphi\frac{Q}{N_m}$

$$\left[\upsilon Q+\sum_{X\in\Omega}\mu_t(X)\sum_{j\in\mathcal{J}_X}\left[\beta\frac{\partial \mathbb{E}U_{X'}}{\partial \rho_{j,X}}\frac{\partial \rho_{j,X}}{\partial \alpha_{j,X}}+\frac{\partial \ln C_{X,t}}{\partial \alpha_{j,X}}\right]\frac{\alpha_{j,X}}{T}\right]\varphi\frac{Q}{N_m}=\lambda$$

and using that $\frac{\partial \mathbb{E}U_{X'}}{\partial \rho_{j,X}}\frac{\partial \rho_{j,X}}{\partial \alpha_{j,X}}+\frac{\partial \ln C_{X,t}}{\partial \alpha_{j,X}}=\hat{h}_X$ for some value $\hat{h}_X$ and all j, that $\sum_j \alpha_{j,X}=\alpha_X$, and $Q=T$, then the condition for N_m writes:

$$\upsilon Q^2+\sum_{X\in\Omega}\mu(X)\hat{h}_X\alpha_X=\frac{\lambda}{\varphi}N_m \tag{45}$$

Finally, the labor employed in entry in each industry satisfies:

$$\lambda=\frac{\phi}{2}N_{e,X}^{-\frac{1}{2}}\beta\left(\mathbb{E}_e U_{X'}-\mathbb{E}_{-e}U_{X'}\right) \tag{46}$$

where $\mathbb{E}_e U_{X'}$ (resp. $\mathbb{E}_{-e}U_{X'}$) is the expected industry-utility conditional on successfully creating (resp. not creating) a new differentiated good (so the expectation comes from the probabilities the incumbents exit).
Using 41, 46 and 45, the labor market clearing condition writes:

$$1=N^P+N_e+N_m \implies \lambda=1+\upsilon Q^2\varphi+\sum_{X\in\Omega}\mu(X)\left(\varphi\hat{h}_X\alpha_X+\left(\frac{\phi}{2}\beta\left(\mathbb{E}_e U_{X'}-\mathbb{E}_{-e}U_{X'}\right)\right)^2\lambda^{-1}\right) \tag{47}$$

Note that this clearly implies $\lambda>1$. Finally, the budget constraint implies the relative wage is 1, $\hat{w}=\frac{w}{E}=1$. Therefore the planner's markup (or tax) is $\tau=\frac{1}{N^P\hat{w}}=\frac{1}{N^P}=\lambda>1$.

7.8 Proof of convergence to an ergodic distribution and uniqueness

Uniqueness:

Let τ be the first period that we arrive at state $\mathcal{J}=\emptyset$, and $P_{t,0}(X)$ be he probability that

we are at X after t periods starting from $\mathcal{J} = \emptyset$; then the probability we are at state X starting from a given state is:
$P_t\{X\} = \sum_{k=1}^{t} P\{\tau = k\}P_{t-k,0}\{X\} + P\{\tau > t\}P_t\{X|\tau > K\}$
As $t \to \infty$, $P\{\tau > t\} \to 0$ since every period there is a positive probability that all differentiated firms die and we arrive at $\mathcal{J} = \emptyset$. Therefore, this tells us that if $P_{t,0}\{X\}$ converges (which later I prove that this is the case), then, the only stationary distribution we can have is $P_0(X) = \lim_{t\to\infty} P_{t,0}(X)$.

The set of possible states is at most countably infinite

This is a consequence of two things: (i) from a given state you can directly move to a finite number of states; (ii) with probability 1 any industry will pass through the state $\mathcal{J} = \emptyset$ at some point in time. Just as in the proof of Uniqueness, (ii) is telling us that the only stationary distribution we can have (if any, since I haven't proved this yet) is the one we would converge to starting from the state $\mathcal{J} = \emptyset$, which (i) tells us that at most will have a countably infinite number of different states.

Convergence (Existence)

Suppose there are $n \in \mathbb{N} \cup \{\infty\}$ possible states and the probability of moving from state j to state i is $a_{i,j}$, then the transition matrix is

$$Q = \begin{pmatrix} 1 - \sum_{j=2}^{n} a_{j,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & 1 - \sum_{j\neq 2} a_{j,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & 1 - \sum_{j=1}^{n} a_{n,n} \end{pmatrix}$$

Let $m_t = (m_{1,t}, \ldots, m_{n,t})$ be the vector of masses in each state, and call $M_t := m_{t+1} - m_t = (Q - \mathbb{I}_n)m_t$; so $M_{i,t} = \sum_{k\neq i} m_{k,t}a_{i,k} - m_{i,t}\sum_{k\neq i} a_{k,i}$.

Lemma 2 $\sum_{k=1}^{n} M_{k,t} = 0$

Proof. Given that $M_{i,t} = \sum_{k\neq i} m_{k,t}a_{i,k} - m_{i,t}\sum_{k\neq i} a_{k,i}$; then
$\sum_{i=1}^{n} M_{i,t} = \sum_{i=1}^{n}\left[\sum_{k\neq i} m_{k,t}a_{i,k} - m_{i,t}\sum_{k\neq i} a_{k,i}\right] = \sum_{i=1}^{n}\sum_{k\neq i} m_{k,t}a_{i,k} - \sum_{i=1}^{n}\sum_{k\neq i} m_{i,t}a_{k,i}$
$= \sum_{i=1}^{n}\sum_{k\neq i} m_{k,t}a_{i,k} - \sum_{k=1}^{n}\sum_{i\neq k} m_{k,t}a_{i,k}$.
So, we just need to see that $\{k \neq i | i, k \in \{1, \ldots, n\}\} = \{i \neq k | i, k \in \{1, \ldots, n\}\}$, which is clearly satisfied by symmetry of the $\neq$-relationship.

■

And the following lemma expresses M_{t+q} for $q \in \mathbb{N}$ in terms of M_t:

Lemma 3 *For any $q \in \mathbb{N}$, $M_{t+q} = Q^q M_t$, with $M_{i,t+q} = \left(1 - \sum_{k \neq i} a_{k,i}^{(q)}\right) M_{i,t} + \sum_{k \neq i} a_{i,k}^{(q)} M_{k,t}$, where $a_{i,k}^{(q)}$ is the probability of moving from k to i in q periods.*

Proof. By definition, $M_{t+q} = (Q - \mathbb{I}_n)m_{t+q} = (Q - \mathbb{I}_n)Q^q m_t = (Q^{q+1} - Q^q)m_t = Q^q(Q - \mathbb{I}_n)m_t = Q^q M_t$.

■

My main goal here is to study the convergence of M_t towards the null vector; so, we want to establish some result that compares M_t to M_{t+q} for some $q \in \mathbb{N}$. Since M_t is an n-dimensional object, it is important to specify under which metric. To see the importance of this, let's see a counterexample that shows that not necessarily each component of M_t has to monotonically decrease in absolute value:

Lemma 4 *It is not necessarily true that $|m_{t+1}(k) - m_t(k)| \geq |m_{t+2}(k) - m_{t+1}(k)|$ for all k.*

Proof. Suppose that m_t is only non-zero in position i, where $m_{i,t} = 1$. Then

$$M_t = (Q - \mathbb{I}_n)m_t = \begin{pmatrix} a_{1,i} \\ \vdots \\ -\sum_{k \neq i} a_{k,i} \\ \vdots \\ a_{n,i} \end{pmatrix}, \quad m_{t+2} - m_{t+1} = \begin{pmatrix} B_1 \\ \vdots \\ B_i \\ \vdots \\ B_n \end{pmatrix}$$

where $B_j = a_{j,1}\left(1 - \sum_{k \neq i} a_{k,i} - \sum_{k \neq j} a_{k,j}\right) + \sum_{k \notin \{i,j\}} a_{j,k}a_{k,1}$ for $j \neq i$ and $B_i = -\left(\sum_{k \neq i} a_{k,i}\right)\left(1 - \sum_{k \neq i} a_{k,i}\right) + \sum_{k \neq i} a_{i,k}a_{k,i}$.

Then, we can find a counterexample by just supposing $a_{1,i} = 0$ and that there exists $k \notin \{1, i\}$ such that $a_{1,k}a_{k,i} > 0$. Then, we have: $|m_{1,t+1} - m_{1,t}| = a_{1,i} = 0 < a_{1,k}a_{k,i} \leq \sum_{k \notin \{1,i\}} a_{1,k}a_{k,i} = |B_1| = |m_{1,t+2} - m_{1,t+1}|$

■

The norm that will prove useful is $||M_t|| := \max\limits_{\mathcal{A} \subset \{1,\ldots,n\}} \{|\sum_{k \in \mathcal{A}} M_{k,t}|\}$. Define $\mathcal{B}^+ := \{i \in \{1, \ldots, n\} | M_{i,t} > 0\}$ and $\mathcal{B}^- := \{i \in \{1, \ldots, n\} | M_{i,t} < 0\}$

Proposition 4 *It is satisfied that $\max\limits_{\mathcal{A} \subset \{1,\ldots,n\}} \{|\sum_{k \in \mathcal{A}} M_{k,t+1}|\} \leq \max\limits_{\mathcal{A} \subset \{1,\ldots,n\}} \{|\sum_{k \in \mathcal{A}} M_{k,t}|\} = \sum\limits_{k \in \mathcal{B}^+} M_{k,t}$ Further, if $i \in \mathcal{B}^+$, $j \in \mathcal{B}^-$ and there exists $\ell \in \{1, \ldots, n\}$ such that $a_{\ell,i}^{(q)}$, $a_{\ell,j}^{(q)} > 0$ (ℓ can be equal to i or j, which means that this state has period smaller or equal than q), then $\max\limits_{\mathcal{A} \subset \{1,\ldots,n\}} \{|\sum_{k \in \mathcal{A}} M_{k,t+q}|\} < \max\limits_{\mathcal{A} \subset \{1,\ldots,n\}} \{|\sum_{k \in \mathcal{A}} M_{k,t}|\} = \sum_{k \in \mathcal{B}^+} M_{k,t}$*

Proof. First, $\max\limits_{\mathcal{A} \subset \{1,\ldots,n\}} \{|\sum_{k \in \mathcal{A}} M_{k,t}|\} = \sum_{k \in \mathcal{B}^+} M_{k,t}$ is because $\max\limits_{\mathcal{A} \subset \{1,\ldots,n\}} \{|\sum_{k \in \mathcal{A}} [m_{t+1}(k) - m_t(k)]|\} = \max\{\sum_{k \in \mathcal{B}^+} M_{k,t}, -\sum_{k \notin \mathcal{B}^+} M_{k,t}\}$ and the fact that both terms have the same value by Lemma 2.

Now, from Lemma 3, for any given $q \in \mathbb{N}$ and $\mathcal{A} \subset \{1, \ldots, n\}$, we have:

$$\sum_{k \in \mathcal{A}} M_{k,t+q} = \sum_{k \in \mathcal{A}} \left[(1 - \sum_{j \neq k} a_{j,k}^{(q)})M_{k,t} + \sum_{j \neq k} a_{k,j}^{(q)} M_{j,t}\right]$$

And I group the terms with same $M_{i,t}$. Let's focus first on the positive terms (i.e. $M_{i,t}$ for $i \in \mathcal{B}^+$):

- If $i \in \mathcal{B}^+ \cap \mathcal{A}$, then: (i) for $k = i$ we have the term $(1 - \sum\limits_{j \neq i} a_{j,i}^{(q)}) M_{i,t}$; (ii) for each $k \in \mathcal{A} \setminus \{i\}$, we have the term $a_{k,i}^{(q)} M_{i,t}$.
- If $i \in \mathcal{B}^+ \setminus \mathcal{A}$: for each $k \in \mathcal{A}$, we have the term $a_{k,i}^{(q)} M_{i,t}$.

Then, the positive terms can be written as:

$$\sum_{i \in \mathcal{B}^+ \cap \mathcal{A}} M_{i,t} \left(1 - \sum_{j \neq i} a_{j,i}^{(q)} + \sum_{k \in \mathcal{A} \setminus \{i\}} a_{k,i}^{(q)}\right) + \sum_{i \in \mathcal{B}^+ \setminus \mathcal{A}} M_{i,t} \left(\sum_{k \in \mathcal{A}} a_{k,i}^{(q)}\right) = \sum_{i \in \mathcal{B}^+ \cap \mathcal{A}} M_{i,t} \left(1 - \sum_{k \notin \mathcal{A}} a_{k,i}^{(q)}\right) + \sum_{i \in \mathcal{B}^+ \setminus \mathcal{A}} M_{i,t} \left(\sum_{k \in \mathcal{A}} a_{k,i}^{(q)}\right)$$

And using that for any i $\sum_{k \neq i} a_{k,i}^{(q)} \in [0, 1]$, we have

$$\sum_{i \in \mathcal{B}^+ \cap \mathcal{A}} M_{i,t} \left(1 - \sum_{k \notin \mathcal{A}} a_{k,i}^{(q)}\right) + \sum_{i \in \mathcal{B}^+ \setminus \mathcal{A}} M_{i,t} \left(\sum_{k \in \mathcal{A}} a_{k,i}^{(q)}\right) \leq \sum_{i \in \mathcal{B}^+} M_{i,t} \tag{48}$$

Analogously, for the negative terms (i.e $M_{i,t}$ for $i \in \mathcal{B}^-$):

- If $i \in \mathcal{A} \cap \mathcal{B}^-$, then: (i) for $k = i$ we have the term $(1 - \sum\limits_{j \neq i} a_{j,i}^{(q)}) M_{i,t}$; (ii) for each $k \in \mathcal{A} \setminus \{i\}$, we have the term $a_{k,i}^{(q)} M_{i,t}$
- If $i \in \mathcal{B}^- \setminus \mathcal{A}$: for each $k \in \mathcal{A}$, we have the term $a_{k,i}^{(q)} M_{i,t}$

Then, the negative terms can be written as:

$$\sum_{i \in \mathcal{B}^- \cap \mathcal{A}} M_{i,t} \left(1 - \sum_{j \neq i} a_{j,i}^{(q)} + \sum_{k \in \mathcal{A} \setminus \{i\}} a_{k,i}^{(q)}\right) + \sum_{i \in \mathcal{B}^- \setminus \mathcal{A}} M_{i,t} \left(\sum_{k \in \mathcal{A}} a_{k,i}^{(q)}\right) = \sum_{i \in \mathcal{B}^- \cap \mathcal{A}} M_{i,t} \left(1 - \sum_{k \notin \mathcal{A}} a_{k,i}^{(q)}\right) + \sum_{i \in \mathcal{B}^- \setminus \mathcal{A}} M_{i,t} \left(\sum_{k \in \mathcal{A}} a_{k,i}^{(q)}\right)$$

So, again using that for any i $\sum_{k \neq i} a_{k,i}^{(q)} \in [0, 1]$, we have

$$\sum_{i \in \mathcal{B}^- \cap \mathcal{A}} (-M_{i,t}) \left(1 - \sum_{k \notin \mathcal{A}} a_{k,i}^{(q)}\right) + \sum_{i \in \mathcal{B}^- \setminus \mathcal{A}} (-M_{i,t}) \left(\sum_{k \in \mathcal{A}} a_{k,i}^{(q)}\right) \leq - \sum_{i \in \mathcal{B}^-} M_{i,t} \tag{49}$$

The first part of the proposition follows directly from the fact that the previous two inequalities for $q = 1$ imply that for any $\mathcal{A} \subset \{1, \ldots, n\}$, we have $|\sum_{k \in \mathcal{A}} M_{k,t+1}| \leq \sum_{k \in \mathcal{B}^+} M_{k,t}$, and so the inequality is also true for the maximum.

For the second part of the proposition, suppose the condition holds and I will show by contradiction that we cannot find any $\mathcal{A}$ such that $|\sum_{k \in \mathcal{A}} M_{k,t+q}| \leq \sum_{k \in \mathcal{B}^+} M_{k,t}$ holds with equality, and so the inequality has to be strict.

In order for the equality to hold, it must be one of the following two cases:

- Case A: The positive terms are equal to the upper bound, and the negative terms are zero. For this to be the case, we need: (i) for $i \in \mathcal{B}^+ \cap \mathcal{A}$, $\sum_{k \notin \mathcal{A}} a_{k,i}^{(q)} = 0$; (ii) for $i \in \mathcal{B}^+ \setminus \mathcal{A}$, $\sum_{k \in \mathcal{A}} a_{k,i}^{(q)} = 1$; (iii) for $i \in \mathcal{B}^- \cap \mathcal{A}$, $\sum_{k \notin \mathcal{A}} a_{k,i}^{(q)} = 1$; and (iv) for $\mathcal{B}^- \setminus \mathcal{A}$, it must be $\sum_{k \in \mathcal{A}} a_{k,i}^{(q)} = 0$

  If $i \in \mathcal{A}$, then condition (i) implies that also $\ell \in \mathcal{A}$, since otherwise we would have the contradiction $0 = \sum_{k \notin \mathcal{A}} a_{k,i}^{(q)} \geq a_{\ell,i}^{(q)} > 0$. If $i \notin \mathcal{A}$, then condition (ii) again implies that $\ell \in \mathcal{A}$, since otherwise we would have the contradiction $1 = \sum_{k \in \mathcal{A}} a_{k,i}^{(q)} \leq 1 - a_{\ell,i}^{(q)} < 0$. Therefore, ℓ must be in $\mathcal{A}$ in order for the positive terms to reach the upper bound.

  Next, if $j \in \mathcal{A}$, then condition (iii) implies that $\ell \notin \mathcal{A}$, since otherwise we would have the contradiction $1 = \sum_{k \notin \mathcal{A}} a_{k,j}^{(q)} \leq 1 - a_{\ell,j}^{(q)} < 1$. So, the only possibility is that $j \notin \mathcal{A}$, but then condition (iv) contradicts that $\ell \in \mathcal{A}$, since then we would have the contradiction $0 = \sum_{k \in \mathcal{A}} a_{k,j}^{(q)} \geq a_{\ell,j}^{(q)} > 0$. Therefore, Case A is not possible.

- Case B: The positive terms are equal to zero, and the negative terms are equal to the lower bound. For this to be the case, we need: (i) for $i \in \mathcal{B}^+ \cap \mathcal{A}$, $\sum_{k \notin \mathcal{A}} a_{k,i}^{(q)} = 1$; (ii) for $i \in \mathcal{B}^+ \setminus \mathcal{A}$, $\sum_{k \in \mathcal{A}} a_{k,i}^{(q)} = 0$; (iii) for $i \in \mathcal{B}^- \cap \mathcal{A}$, $\sum_{k \notin \mathcal{A}} a_{k,i}^{(q)} = 0$; and (iv) for $\mathcal{B}^- \setminus \mathcal{A}$, it must be $\sum_{k \in \mathcal{A}} a_{k,i}^{(q)} = 1$
  Analogously as in the previous case, we get to the conclusion that this case is not possible.
  If $i \in \mathcal{A}$, then (i) implies $\ell \notin \mathcal{A}$, since otherwise $1 = \sum_{k \notin \mathcal{A}} a_{k,i}^{(q)} \leq 1 - a_{\ell,i}^{(q)} < 1$. If $i \notin \mathcal{A}$, then (ii) also implies that $\ell \notin \mathcal{A}$, since otherwise $0 = \sum_{k \in \mathcal{A}} a_{k,i}^{(q)} \geq a_{\ell,i}^{(q)} > 0$. So, it must be $\ell \notin \mathcal{A}$.
  If $j \in \mathcal{A}$, (iii) implies the contradiction $0 = \sum_{k \notin \mathcal{A}} a_{k,j}^{(q)} \geq a_{\ell,j} > 0$. But if $j \notin \mathcal{A}$, (iv) also implies the contradiction $1 = \sum_{k \in \mathcal{A}} a_{k,j}^{(q)} \leq 1 - a_{\ell,j} < 1$. So, we conclude that this case is not possible.

■

This proposition tells us that a sufficient condition to guarantee convergence to an ergodic distribution is that whenever we are not in a stationary distribution, we can find states that have changed in opposite directions in the previous iteration (period) such that there exists some state which can be reached from each of the two states with positive probability in the same number of periods (in other words, if two points start from state i and j respectively, there is positive probability they will meet at some future period).
The next definitions and proposition show a sufficient condition for this condition to hold:

Definition 1 *A Markov chain is* ***irreducible*** *if for any pair of states i, j, there exists $q \in \mathbb{N}$ such that $a_{j,i}^{(q)} > 0$. (that is, it is possible to get to any state from any other state)*

Definition 2 *Let the* ***longitude of the shortest path between two states*** *i, j be $d_{i,j} = \min\{q \in \mathbb{N} | a_{i,j}^{(q)} > 0\}$ (that is, the smallest number of periods required to go from one state to the other).*

Proposition 5 *In an irreducible Markov chain that contains at least one state i with $d_{i,i} = 1$, as long as we are not in the stationary distribution, it is always possible to find states $j \in \mathcal{B}^+$ and $k \in \mathcal{B}^-$, and a state ℓ such that $a_{\ell,j}^{(q)}, a_{\ell,k}^{(q)} > 0$ for some $q \in \mathbb{N}$ (and so, in such Markov chain we can guarantee convergence to an ergodic distribution).*

Proof. If we are not in a stationary distribution then there are j with $M_{j,t} \neq 0$, and by Lemma 2 there must be $j \in \mathcal{B}^+$ and $k \in \mathcal{B}^-$. .Let i be the state such that $d_{i,i} = 1$. Then, it is sufficient to see that we can find $q \in \mathbb{N}$ such that $a_{i,j}^{(q)}, a_{i,k}^{(q)} > 0$, which is straightforward. We can check that $q := \max(d_{i,j}, d_{i,k})$ satisfies this (intuitively, the first to arrive from one of the two states then stays with positive probability in i, and at some point the one that started from the other state will also arrive to i). Without loss of generality, assume $\max(d_{i,j}, d_{i,k}) = d_{i,k}$. $a_{i,k}^{d_{i,k}} > 0$ by definition of $d_{i,k}$. But also $a_{i,j}^{d_{i,k}} \geq a_{i,j}^{d_{i,j}} a_{i,i}^{(d_{i,k} - d_{i,j})} \geq a_{i,j}^{d_{i,j}} \left(a_{i,i}^{(1)}\right)^{d_{i,k} - d_{i,j}} > 0$
■

So, in the Uniqueness section I proved that the only possible stationary distribution is the one we would obtain if the initial sate is $\mathcal{J} = \emptyset$ (if this converges). Now, the previous proposition tells us that $P_{t,0}(X)$ converges, since the Markov chain obtained is irreducible (if a state is possible, it means that there was positive probability of arriving to it starting from $\mathcal{J} = \emptyset$; and, from any state, there is probability 1 of eventually going back to $\mathcal{J} = \emptyset$) and the state $\mathcal{J} = \emptyset$ satisfies that the longitude of its shortest path connecting it to itself is 1 (with positive probability there will be no entrant and we stay at $\mathcal{J} = \emptyset$).

7.9 Summary of the method to solve the model

1. First, for each possible number of firms J:
   - Define the different awareness sets $\mathcal{P}_J$. There are 2^J awareness sets (think on how many different ways we can assign $\{0, 1\}$ to J variables).
   - Define the N_J grid nodes we will use, $\vec{M}_n$, $n = 1, \ldots, N_J$. Each node is a vector of the mass of consumers in each awareness set. As we have seen, the solutions of the model are functions of the form $f(\mathcal{J}, \vec{M})$ on a continuous m-dimensional space, with $m = 2^J - 1$ ($\vec{M}$ is $(m+1)$-dimensional, but since the masses have to add up to 1, one is redundant). To deal with this exponentially increasing state space and alleviate the curse of dimensionality, I introduce a piecewise multivariate Newton interpolation method described in detail in section 7.10. Using this method, increasing the number of grid points leads to a better approximation, as in standard univariate methods using a grid and linear interpolation, with the advantage that the higher degree of the interpolating polynomial allows to reduce the number of necessary grid points for a given fit.[20]

     Also, note that $\mathcal{J}$ has information of the identity of the firm. Therefore, some nodes are just a reordering of firms, so I use this to avoid solving again nodes that are just a reordering of a node that has already been solved.
2. Define initial guesses for the aggregate states w and T, as well as initialize the policy functions for advertising expenditure and entry; that is, assign a value for the grid nodes $\{\{\{e_{j,n}\}_{j=1}^{J}\}_{n=1}^{N_J}\}_{J=1}^{\bar{J}}$ and entry $\{\{N_{e,n}\}_{n=1}^{N_J}\}_{J=1}^{\bar{J}}$.
3. Given the aggregate states:
   (a) Solve the firm problem:
      i. Given the guess of the policy functions for advertising expenditures and entry:

[20] Using piecewise interpolation is important because increasing the degree of an interpolating polynomial doesn't necessary lead to a better approximation (Runge's phenomenon).

- Solve the static price-setting problem. Note that this has to be updated in every iteration of the firm problem because the advertising choices affect the demand shifters $\omega_{j,i,t}$. This gives us profits and markups at each node: $\{\{\mathcal{M}_j(J,\vec{M}_n),\pi_j(J,\vec{M}_n)\}_{n=1}^{N_J}\}_{J=1}^{\bar{J}}$.
- Solve for the value function implied by the policy functions of advertising and entry and the profit function. Note that this implies solving a linear system on $\{\{V(J,\vec{M}_n)\}_{n=1}^{N_J}\}_{J=1}^{\bar{J}}$.

ii. Given the functions for markups and firm value found in the previous points, $\{\{\mathcal{M}_j(J,\vec{M}_n),V_j(J,\vec{M}_n)\}_{n=1}^{N_J}\}_{J=1}^{\bar{J}}$, compute the best responses $\{\{\{e'_{j,n},N'_{e,n}\}_{j=1}^{J}\}_{n=1}^{N_J}\}_{J=1}^{\bar{J}}$ (i.e. the optimal choice keeping the competitors' choices fixed). If the difference between these best responses and the previous guess is small enough, we are done (in this case we have found a Nash equilibrium); otherwise, update the guesses and go back to (i).

(b) Solve for the unique stationary distribution given the solution of the firm problem (in particular, we need the policy functions for the advertising space $\{\{\{\alpha_{j,n}\}_{j=1}^{J}\}_{n=1}^{N_J}\}_{J=1}^{\bar{J}}$ and entry $\{\{N_{e,n}\}_{n=1}^{N_J}\}_{J=1}^{\bar{J}}$ and entry in an industry with $J=0$: $N_{e,0}$). For the details of the method, see section 7.9.1

4. Given the firm policy functions and the stationary distribution, compute the implied aggregates w and T, using 22, 21, together with $T=Q$. If the difference between the guesses and the implied values of w and T are close enough, we are done; otherwise, update the new guesses for w and T and go back to 3.

7.9.1 Method used to find the stationary distribution

The method has two parts.

1. In section 7.8, I show that the set of industry states observed in the stationary distribution is at most countably infinite; and so the stationary distribution is a discrete probability function defined on a potentially infinite set of points, and so, computationally, the set of states needs to be bounded some way. In the following I describe the approach used in the baseline to bound the set of states. As a robustness, I compare the stationary distribution obtained from this approach to the one obtained by bounding the space by a grid (that is, restricting $\vec{M}$ to take only values from a grid). The baseline approach tends to be much faster.

   (a) Given that in section 7.8 I show that the unique stationary distribution is the one we would obtain if the initial sate is $\mathcal{J}=\emptyset$, then:

- I initialize the *List* of states with this state. For each state in the *List*, I store (1) the number of firms, (2) the vector of masses corresponding to this state, (3) the vector of ages, and (4) the probability *Prob*, which I now describe. *Prob* is the probability of going from state $\mathcal{J} = \emptyset$ to the particular state X in the shortest path from $\mathcal{J} = \emptyset$ to X. That is, for this initial state $\mathcal{J} = \emptyset$, we have $Prob = 1$.
- To facilitate the process of looking up whether we have already encountered a state before (i.e. whether a state is already in *List*, I order the states in a library *LibraryStates* lexicographically based on (i) the number of active firms, (ii) the vector of ages, and (iii) the vector of masses. Initially, $LibraryStates = 1$.
- I also initialize $iter = 0$ and the list of states I will take as starting point in the following iteration, $NewStates_{iter}$. Initially, $NewStates_1 = 1$.
- Finally, we need the transition matrix with the probabilities of going from each state to the others. However, since this matrix is very sparse (the states are just directly connected to few others) and storing the whole matrix with all the zeros would be highly costly for memory storage (and solving the system would also be very slow), I only store the non-zero elements of the transition matrix in a *Library*, where each book contains three pieces of information (the books are ordered lexicographically based on the same order of these three pieces of information): the row in the transition matrix (i.e., state of origin), the column in the transition matrix (i.e., state of destination), and the value in this position of the matrix resulting from subtracting the transition matrix from the identity matrix. I initialize it as $Library = [1, 1, 1]$ (the third 1 is the 1 from the identity matrix).

(b) Then, as long as $NewStates_{iter+1}$ is not empty, increase iter by 1 and do the following for each state $s \in NewStates_{iter+1}$:

i. Calculate the next period vector of masses if there weren't entry/exit, and the probability of an entrant. Then, for each of the possible cases of entry/exit, letting q be the probability of the particular event of entry/exit, I do the following:

ii. Look up whether this state is already in *List*, using the order in *LibraryStates*. Here is where I bound the problem.

- If the probability *Prob* is above a threshold, then I check for an exact match (that is, they match in the three elements: J, the vector of ages, and the vector of masses (note that, although the vector of ages is not a state in the baseline firm problem, it is useful to distinguish it for the quantitative exercises).

- If the probability *Prob* is below the threshold (that is, it is a rare state), then I just check for J and the vector of ages. If there is no state in *List* matching J and the vector of ages, then we will treat this state as a new state; otherwise, I will treat it as if it were identical to the first state in *List* with the same J and ages. The intuition is that, although the vector of ages is not a sufficient statistic (because history matters), it serves as a good first approximation. The other boundary I set is on the firm age; in particular, I don't distinguish ages above a threshold (which I set to 20 years old). The intuition is that for firms older than 20 years old very few consumers remain unaware of the firm, so the error from not distinguishing older firms is negligible.

iii. If the outcome from the previous point is that it is not a new state, then we index it by s' equal to the index of the state we have matched it to and go to (iv); else, if it is a new state, then we index it by s' equal to the current size of *List* plus one and do the following:

- Add the one of the identity matrix to *Library*; that is: add $[s', s', 1]$.
- Add the four pieces of information relative to this state in *List*. *Prob* will be equal to the *Prob* of state s time q.
- Add s' to $NewStates_{iter+1}$.

iv. Add $[s, s', -q]$ to the *Library*. If there is already an element at position $[s, s']$, then just add $-q$.

2. In the second part, we need to solve for the stationary distribution. The matrix found in the previous step is singular (note that the sum of all the elements in row s is $1 - \sum_{s'} p_{s,s'} = 1 - 1 = 0$, where $p_{s,s'}$ is the probability of moving from state s to s'). So, we need to add a new condition to have a compatible and determinate system: it is the condition that the solution must add up to 1; so, I add to *Library* $[s, 0, 1]$, for all the states s. Then, we also need the vector of independent coefficients, which again is very sparse (there is only one non-zero value), so again I store it in a library called $LibraryB = [0, 1]$.

7.10 Multivariate Newton Interpolation

First, as a recap of the univariate Newton interpolation, given $n+1$ different points (nodes) defined as a pair $(x_i, f(x_i))$ with $x_i \neq x_j$ for any $j \neq i$, then the unique n-degree polynomial that passes through these $n+1$ points expressed in the Newton basis polynomials (which are defined as $w_j(x) = \prod_{k=0}^{j-1}(x - x_k)$, $j = 1, 2, \ldots, n$ and $w_0(x) = 1$) is $P_n(x) = \sum_{j=0}^{n} a_j w_j(x)$, where the coefficients a_i are the solutions of the system (note that $w_j(x_i) = 0$ when $i < j$):

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & (x_1 - x_0) & 0 & \ldots & 0 \\ 1 & (x_2 - x_0) & \prod_{k=0}^{1}(x_2 - x_k) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_n - x_0) & \prod_{k=0}^{1}(x_n - x_k) & \ldots & \prod_{k=0}^{n-1}(x_n - x_k) \end{pmatrix}_{(n+1)\times(n+1)} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}$$

This system can be solved reducing the system iteratively. To this purpose, note that a_0 is already solved: $a_0 = f(x_0)$ so we can forget about row 1 and the rest writes:

$$\begin{pmatrix} f(x_0) \\ f(x_0) \\ \vdots \\ f(x_0) \end{pmatrix} + \begin{pmatrix} (x_1 - x_0) & 0 & \ldots & 0 \\ (x_2 - x_0) & \prod_{k=0}^{1}(x_2 - x_k) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (x_n - x_0) & \prod_{k=0}^{1}(x_n - x_k) & \ldots & \prod_{k=0}^{n}(x_n - x_k) \end{pmatrix}_{n\times n} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}$$

Next, note that each element of row i in the $n \times n$ matrix (the Vandermonde matrix) contains $(x_i - x_0)$, so we can divide both sides of row i by $(x_i - x_0)$ and calling $f[x_i, x_0] = \frac{f(x_i) - f(x_0)}{x_i - x_0}$ (divided difference) we obtain an analogous system as the initial one but with one dimension less:

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & (x_2 - x_1) & 0 & \ldots & 0 \\ 1 & (x_3 - x_1) & \prod_{k=1}^{2}(x_3 - x_k) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_n - x_1) & \prod_{k=1}^{2}(x_n - x_k) & \ldots & \prod_{k=1}^{n-1}(x_n - x_k) \end{pmatrix}_{n\times n} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f[x_1, x_0] \\ f[x_2, x_0] \\ f[x_3, x_0] \\ \vdots \\ f[x_n, x_0] \end{pmatrix}$$

Now, $a_1 = f[x_1, x_0]$ is already solved; so, we repeat the procedure: we pass subtracting the $1 \cdot a_i$ of each row to the right hand side and we divide by the common factor of the left side $(x_i - x_1)$, we call $f[x_i, x_1, x_0] = \frac{f[x_i, x_0] - f[x_1, x_0]}{x_i - x_1}$, and we obtain:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & (x_3-x_2) & 0 & \dots & 0 \\ 1 & (x_4-x_2) & \prod_{k=2}^{3}(x_4-x_k) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_n-x_2) & \prod_{k=2}^{3}(x_n-x_k) & \dots & \prod_{k=2}^{n-1}(x_n-x_k) \end{pmatrix}_{(n-1)\times(n-1)} \begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f[x_2,x_1,x_0] \\ f[x_3,x_1,x_0] \\ f[x_4,x_1,x_0] \\ \vdots \\ f[x_n,x_1,x_0] \end{pmatrix}$$

Iterating, in the r-th iteration we will get:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & (x_r-x_{r-1}) & 0 & \dots & 0 \\ 1 & (x_{r+1}-x_{r-1}) & \prod_{k=r-1}^{r}(x_{r+1}-x_k) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_n-x_{r-1}) & \prod_{k=r-1}^{r}(x_n-x_k) & \dots & \prod_{k=r-1}^{n-1}(x_n-x_k) \end{pmatrix}_{(n-r+2)\times(n-r+2)} \begin{pmatrix} a_{r-1} \\ a_r \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f[x_{r-1},x_{r-2}\dots,x_0] \\ f[x_r,x_{r-2}\dots,x_0] \\ f[x_{r+1},x_{r-2}\dots,x_0] \\ \vdots \\ f[x_n,x_{r-2}\dots,x_0] \end{pmatrix}$$

Summarizing, the coefficients of the newton interpolation polynomial are given by the divided differences $a_i = f[x_i,x_{i-1},\dots,x_0] = \frac{f[x_i,x_{i-2},\dots,x_0]-f[x_{i-1},x_{i-2},\dots,x_0]}{x_i-x_{i-1}}$.

We can extend this to the multivariate case as follows. Suppose we want to interpolate a function $f:\mathbb{R}^m \to \mathbb{R}$ by a polynomial of m variables and degree n.

Definition 3 ***(Generating points):*** *For each dimension $i=1,\dots,m$, we define $n+1$ points $x_{i,k}$, $k=0,\dots,n$. $\{\{x_{i,k}\}_{k=0}^{n}\}_{i=1}^{m}$ are called the generating points.*

Definition 4 (Multiindices): *Let $\vec{\alpha}=(\alpha_1,\dots,\alpha_m)\in\Lambda_{m,n}:=\{\vec{\alpha}\in\{0,\dots,n\}^m | \sum_{i=1}^{m}\alpha_i \leq n\}$, and $\vec{x}_{\vec{\alpha}}=(x_{1,\alpha_1},\dots,x_{m,\alpha_m})$.*

The cardinal of $\Lambda_{m,n}$ (i.e. the number of different multiindices) is given by $N(m,n)=\binom{n+m}{n}$ (to see this, you can think of $1^{\alpha_0}x_1^{\alpha_1}\cdots x_m^{\alpha_m}$ with $\sum_{i=0}^{m}\alpha_i=n$, which we can transcribe as $\underbrace{1\dots1}_{\alpha_0}\#\underbrace{x_1\dots x_1}_{\alpha_1}\#\dots\#\underbrace{x_m\dots x_m}_{\alpha_m}$; so the problem of finding the number of different multiindices is equivalent to finding the number of different ways we can choose m boxes from $n+m$ boxes (i.e. the position of the m hashtags), which is $\binom{n+m}{m}$).

Definition 5 (Newton polynomial): $w_{\vec{\alpha}}(\vec{x})=\prod_{i=1}^{m}\prod_{k=0}^{\alpha_i-1}(x_i-x_{i,k})$.

Definition 6 *The m-dimensional* ***Newton interpolating polynomial*** *of degree n of the function f is $p_{m,n}(\vec{x})=\sum_{\vec{\alpha}\in\Lambda_{m,n}} a_{\vec{\alpha}}w_{\vec{\alpha}}(\vec{x})$, satisfying $f(\vec{x}_{\vec{\alpha}})=p_{m,n}(\vec{x}_{\vec{\alpha}})$, for all $\vec{\alpha}\in\Lambda_{m,n}$.*

Lemma 5 *Note that given $\vec{\beta},\vec{\alpha}\in\Lambda_{m,n}$, if $\beta_i-1\geq\alpha_i$, then $w_{\vec{\beta}}(\vec{x}_{\vec{\alpha}})$ contains the term $(x_{i,\alpha_i}-x_{i\alpha_i})=0$.*

Corollary 1 *Then,* $f(\vec{x}_{\vec{\alpha}}) = p_{m,n}(\vec{x}_{\vec{\alpha}}) = \sum_{k_m=-1}^{\alpha_m-1} \cdots \sum_{k_1=-1}^{\alpha_1-1} \prod_{s_m=0}^{k_m} (x_{m,\alpha_m} - x_{m,s_m}) \cdots \prod_{s_m=0}^{k_m} (x_{1,\alpha_1} - x_{1,s_1}) a_{(k_1+1,\ldots,k_m+1)}$

To allow generality, I define:

Definition 7 *Given* $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$*, define:*
(i) $\vec{\alpha}^{(i,k)} = (\alpha_1, \ldots, \alpha_{i-1}, k, \alpha_{i+1}, \ldots, \alpha_m)$ *(i.e.* $\vec{\alpha}^{(i,k)}$ *equals* $\vec{\alpha}$ *except* k *in position* i*)*
(ii) $\vec{\alpha}^{(k)} = (\alpha_{k+1}, \ldots, \alpha_m)$.
(iii) $\vec{\beta}^{(i,k)} = (\vec{\alpha}^{(i,k-1)}, \ldots, \vec{\alpha}^{(i,0)}, \ldots, \vec{\alpha}^{(m,\alpha_m-1)}, \ldots, \vec{\alpha}^{(m,0)})$*, and let* $\vec{\beta}^{(i,0)} = \vec{\beta}^{(i+1,\alpha_{i+1}-1)}$ *and* $\vec{\beta}^{(m,0)} = \emptyset$.

Definition 8 (Divided differences):
$f[\vec{\alpha}, \vec{\beta}^{(i,b)}] = \sum_{k_i=b-1}^{\alpha_i-1} \cdots \sum_{k_1=-1}^{\alpha_1-1} \prod_{s_i=b}^{k_i} (x_{m,\alpha_m} - x_{m,s_m}) \cdots \prod_{s_1=0}^{k_1} (x_{1,\alpha_1} - x_{1,s_1}) a_{(k_1+1,\ldots,k_i+1,\vec{\alpha}^{(i)})}$ *if* $\alpha_i > b$*; and* $f[\vec{\alpha}, \vec{\beta}^{(i,b)}] = f[\vec{\alpha}, \vec{\beta}^{(i-1,0)}]$ *otherwise.*

Note that by Corollary 1, and since $\vec{\alpha}^{(m)} = \emptyset$*, then* $f[\vec{\alpha}, \vec{\beta}^{(m,0)}] = f(\vec{x}_{\vec{\alpha}})$*. The algorithm to find the coefficients* $a_{\vec{\alpha}}$ *is defined as follows:*

1. *Start setting* $i = m$ *and* $b = 0$.

2. *If there is some* $\vec{\alpha} \in \Lambda_{m,n}$ *such that* $\alpha_i > b$*, then:*

   (a) *For all the* $\vec{\alpha} \in \Lambda_{m,n}$ *such that* $\alpha_i > b$*: Noting that* $f[\vec{\alpha}^{(i,b)}, \vec{\beta}^{(i,b)}]$ *contains all the terms of* $f[\vec{\alpha}, \vec{\beta}^{(i,b)}]$ *with* $k_i = b - 1$*, and so the remaining terms will all contain* $(x_{i,\alpha_i} - x_{i,b})$*; then:*

$$
\begin{aligned}
f[\vec{\alpha}, \vec{\beta}^{(i,b+1)}] &= \frac{f[\vec{\alpha}, \vec{\beta}^{(i,b)}] - f[\vec{\alpha}^{(i,b)}, \vec{\beta}^{(i,b)}]}{x_{i,\alpha_i} - x_{i,b}} \\
&= \sum_{k_i=b}^{\alpha_i-1} \cdots \sum_{k_1=-1}^{\alpha_1-1} \prod_{s_i=b+1}^{k_i} (x_{m,\alpha_m} - x_{m,s_m}) \cdots \prod_{s_1=0}^{k_1} (x_{1,\alpha_1} - x_{1,s_1}) a_{(k_1+1,\ldots,k_i+1,\vec{\alpha}^{(i)})}
\end{aligned}
$$

   (b) *For all the* $\vec{\alpha} \in \Lambda_{m,n}$ *such that* $\alpha_i \le b$*, then* $f[\vec{\alpha}, \vec{\beta}^{(i,b+1)}] = f[\vec{\alpha}, \vec{\beta}^{(i,b)}]$ *(satisfies the definition since* $\alpha_i \le b < b + 1$*, so* $f[\vec{\alpha}, \vec{\beta}^{(i,b+1)}] = f[\vec{\alpha}, \vec{\beta}^{(i,b)}] = f[\vec{\alpha}, \vec{\beta}^{(i-1,0)}]$

   Set $b = b + 1$*, and go back to step 2.*

3. *If* $\alpha_i \le b$ *for all* $\vec{\alpha} \in \Lambda_{m,n}$ *(which is satisfied if and only if* $b \le n$*), then make* $f[\vec{\alpha}, \vec{\beta}^{(i-1,0)}] = f[\vec{\alpha}, \vec{\beta}^{(i,b)}]$*, and set* $i = i - 1$ *and* $b = 0$*. If* $i = 0$*, we are done; otherwise, go back to step 2.*

All is left to do is to show that the $f[\vec{\alpha}, \vec{\beta}^{(0,0)}] = a_{\vec{\alpha}}$ for all $\vec{\alpha} \in \Lambda_{m,n}$. Given that the divided difference of $\vec{\alpha}$ just changes when we apply (2a) to it, then it is sufficient to see that in the last time that we select $\vec{\alpha}$ for (2a) it is $f[\vec{\alpha}, \vec{\beta}^{(i,b+1)}] = a_{\vec{\alpha}}$; since then it will be $f[\vec{\alpha}, \vec{\beta}^{(0,0)}] = f[\vec{\alpha}, \vec{\beta}^{(i,b+1)}] = a_{\vec{\alpha}}$.

Proof. *If we have used $a_{\vec{\alpha}}$ in (2a), it means that $\alpha_i > b$, which implies that exactly one of the following is true:*

1. *$\alpha_i > b+1$, in which case $a_{\vec{\alpha}}$ would also be selected in the next iteration, contradicting it was the last time it was selected;*

2. *$\alpha_i = b+1$, in which case $a_{\vec{\alpha}}$ it is the last iteration for variable i that $a_{\vec{\alpha}}$ is selected. In this case there are two possibilities:*

   - *$\alpha_k > 0$ for some $k < i$, in which case in iteration $(k, 0)$ $\vec{\alpha}$ would be selected, contradicting the hypothesis.*
   - *$\alpha_k = 0$ for all $k < i$, in which case we have:*

$$\begin{aligned} f[\vec{\alpha}, \vec{\beta}^{(i,b+1)}] &= \sum_{k_i=\alpha_i-1}^{\alpha_i-1} \cdots \sum_{k_1=-1}^{\alpha_1-1} \prod_{s_i=\alpha_i}^{k_i} (x_{m,\alpha_m} - x_{m,s_m}) \cdots \prod_{s_1=0}^{k_1} (x_{1,\alpha_1} - x_{1,s_1}) a_{(k_1+1,\ldots,k_i+1,\vec{\alpha}^{(i)})} \\ &= \prod_{s_i=\alpha_i}^{\alpha_i-1} (x_{m,\alpha_m} - x_{m,s_m}) \cdots \prod_{s_1=0}^{-1} (x_{1,\alpha_1} - x_{1,s_1}) a_{(0,\ldots,0,\alpha_i,\vec{\alpha}^{(i)})} \\ &= a_{(0,\ldots,0,\alpha_i,\vec{\alpha}^{(i)})} = a_{\vec{\alpha}} \end{aligned}$$

■