Customer Base and the Advertising Sector

(preliminary draft. Please, do not circulate)

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Abstract

I develop a dynamic general equilibrium model of advertising where consumers are characterised by the subset of goods they are aware of, which evolves over time. Consumers are exposed to advertising during the time they are consuming media goods. Firms advertise for three motives: to persuade customers to spend more, to acquire new customers, and, given that consumers' attention is limited, to preclude consumers from learning about competitors. I use the model to study the evolution of the motives to advertise along the firm life cycle and their aggregate effects. I also compare the decentralised equilibrium with the planner's allocation, keeping the information constrains. Finally, I study whether we should tax or subsidize advertising, and the welfare gains from implementing an age-dependent tax.

1 Introduction

In Economics, there is an extensive literature exploring whether advertising is informative or persuasive, with supporting evidence for both (see Bagwell, 2007, for a review). The conclusion we are drawn to, then, is that advertising is a mixture of both, which calls for the need of a framework that accommodates both. If consumers face information frictions, then advertising can provide valuable information that mitigates these frictions, allowing consumers to enjoy more variety and promoting competition. Empirical evidence from Foster et al. (2016) suggests that the accumulation of customer base plays a key role in explaining the size difference between young and mature firms.[1] This lends support to the importance of these frictions and the idea that past customer base matters, indicating that it is a dynamic problem rather than a static one. On the persuasive side, advertising can enhance consumer preferences for a specific good. This can have both positive effects, by increasing the perceived utility of the good, and negative effects, as it increases product differentiation, which leads to more market power.
In addition, limited attention capacity of consumers implies that firms need to compete for the attention of consumers to make their way into their consumption sets. This introduces a novel effect of advertising: advertising by one firm diverts consumers' attention away from competitors. This effect is particularly relevant in settings like Google search or Amazon advertising, where firms compete to be placed in the top positions within a keyword. In Google, firms bid on specific keywords in a cost-per-click (CPC) auction, where they compete to have their ads displayed in top positions. Google doesn't charge firms just to be placed at the top, instead, the CPC is the amount the firm will be charged for each click their ad receives. Therefore, Google doesn't necessarily place higher a firm with a higher bid; it also takes into account the relevance of the ad, based on the so-called Quality Score. Figure 1 shows the heterogeneity in the average CPC in Google search ads across industries, which aligns with the modelling approach of this paper that firms compete for a limited ad space within their industry.[2]

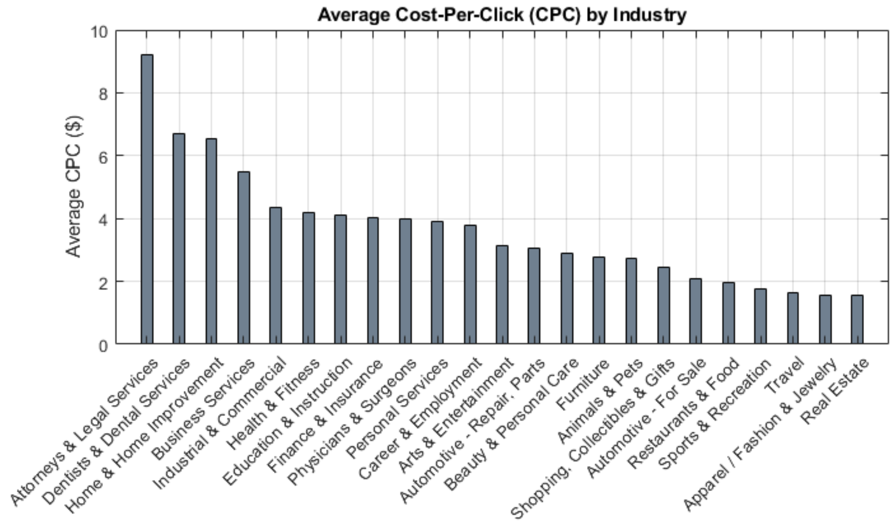
Since advertising has various effects and firms may benefit differently from each, then some firms may be motivated to advertise more due to one effect than another. Throughout the paper, I will use the terms (i) *informative motive* to refer to the incentives of a firm to advertise to inform consumers, (ii) *persuasive motive* to refer to the incentives of a firm to advertise to increase the spending by current customers, and (iii) *anticompetitive motive* to refer to the incentives of a firm to advertise to reduce consumers' attention to the competitors. I use the term 'anticompetitive', since, under this motive, the firm is doing advertising to avoid competition by precluding competitors to expand their customer base.

Advertising also plays a key role in the provision of media goods that consumers tend to enjoy at zero monetary price. The presence of media goods that get their revenue from

[1]In particular, Foster et al. (2016) take advantage of data on physical quantities in industries that are plausibly little subject to quality differentiation and find that the fact that older firms are bigger than younger firms cannot be explained by differences in productivity, and then they find support for the hypothesis that firms play an active role, not just a passive effect from aging. Einav et al. (2022), focusing on the retail sector, find that most of the variability in sales is accounted by the number of clients.

[2]The same applies if we look at CPC in Google shopping ads across industries, although these are considerably cheaper, rarely more than one dollar per click.

Figure 1: Average Coct-Per-Click (CPC) in Google search ads by industry



Notes. Adapted from Wordstream (2023). This figure displays the average CPC in Google ads by industry, calculated by dividing the overall cost of a campaign by the number of clicks it received. Each individual click has a different cost as it's determined by the Google Ads auction algorithm.

selling advertising space is pervasive, spanning traditional outlets like radio and TV as well as digital platforms such as YouTube, Instagram, Facebook, and Google. Although these media goods are barely reflected in GDP (see Greenwood et al., 2024), the time consumers spend on them suggests they have a significant impact on welfare. According to Statista, the average daily time spent on media in the United States in 2023 amounted to 751 minutes.

I present a model that accommodates for all these various aspects of advertising, in order to study the effects of advertising on welfare and industry dynamics. A key feature of the model is that, within an industry, the consumer's type is characterized by the set of goods the consumer knows, which I refer to as awareness sets. These awareness sets evolve stochastically over time, and advertising is the tool firms have to increase the probability their good enters the consumers' awareness sets. Advertising also has a static persuasive effect by increasing the demand shifter of the advertised good in preferences. The model also features a media good sector that provides the platform for firms to advertise and supplies media goods to consumers at a zero monetary price. Consumers are exposed to advertising during the time they spend on media. Key to the model is the idea that consumers' time and attention capacity are limited, which imply that the advertising space is limited. The fact that the advertising space is limited introduces the possibility that firms may want to advertise not to inform consumers, but rather to divert consumers' attention away from competitors (the anticompetitive motive).

I study how the three motives change along the firm life cycle. Intuitively, younger firms are the ones that have more potential customers to acquire, and so they tend to have a stronger informative motive. On the contrary, the anticompetitive motive, which is about retaining market power over the current customers by reducing the probability they learn about competitors, is stronger for older firms, as they have more to lose due to their larger customer base. The persuasive motive also tends to be stronger in older firms. Intuitively, if advertising persuades current customers to spend more, then the increase in revenues will be larger the larger the customer base.

I estimate the model by simulated method of moments to fit key empirical patterns regarding (i) the evolution of average firm growth by age, which is important to discipline the customer base building process in the model; (ii) the relationship between advertising expenditure and sales; and (iii) macroeconomic aggregates. The model does well in matching the targets. In addition, the model also features an inverted-U relationship between advertising and sales as documented in previous literature.

First, I use the estimated model to assess the aggregate effects of the anticompetitive and the persuasive motives. To do so, I compare the baseline economy with two counterfactual economies. In the first one, I shut down the anticompetitive motive from the firms' first order condition, and in the second one, in addition, I also shut down the persuasive motive. The persuasive motive turns out to be beneficial in the aggregate, although it raises markups and reduces entry. This positive effect arises from the effect the increased taste shifters have on preferences. The anticompetitive motive is detrimental to output, leading to higher markups and lower entry. However, although the anticompetitive motive doesn't seem to have any positive impact on the product market, the presence of firms spending on advertising for the anticompetitive motive is not necessarily bad. This is because, as it increases total advertising expenditure, it contributes to welfare through the provision of media goods, which may offset the negative effect on output, if consumers' value for these entertainment goods is big enough.

I also compare the decentralised equilibrium with the resulting from solving the social planner problem (keeping the consumers' information frictions). A novel feature of my model is that the social planner not only values media goods because they entertain consumers, but also because, through the advertising in media, consumers get information that allows them to improve consumption. The welfare gains that would obtain from implementing the planner's allocation are sizeable, even if we assume that the entertainment value of media goods is negligible. In this extreme case where the entertainment value is negligible, output would increase by almost 15%, whereas the quantity of media supplied would be lower than in the decentralized equilibrium. This seems to indicate that, for this case, there might be too much advertising. However, this conclusion is inaccurate as we also need to consider how the advertising space is allocated among firms. In other words, the 'overprovision' of media, through its effect on learning, may help mitigating the inefficiencies arising from the misallocation in the advertising space. This is confirmed in the uniform advertising tax exercise, which indicates that there is actually too little advertising (the optimal tax is a subsidy).

Finally, given the observation that the informative motive is decreasing with age, while the persuasive and the anticompetitive motives are increasing with age, and given that the motives have different aggregate implications, a reasonable question to ask is what would the welfare gains from allowing the advertising tax to be age-dependent be. Contrary to the prior, this exercise reveals that it would be optimal to set a higher tax for youger firms. The reason lies again on the effect of the taste shifters on preferences: although allocating more advertising space to young firms yields larger informative gains, these come at the cost of reduced utility from a lower taste shifter for older firms, which affects many consumers. In the calibrated model, this latter effect dominates.

Related literature. This paper relates to the literature that studies the implications of customer capital for firm, industry, and macroeconomic dynamics (e.g. Dinlersoz and Yorukoglu (2012), Gourio and Rudanko (2014), Molinari and Turino (2018), Argente et al. (2023), Einav et al. (2022), Ignaszak and Sedlacek (2023), Greenwood et al. (2024)). In these models, firms grow via increasing their idiosyncratic demand (customer capital). Together with Cavenaile et al. (2024b), we contribute to this literature by showing that it is not just the quantity of customers that matter, but also the degree of information the customers have about alternative goods. Relative to Cavenaile et al. (2024b), I allow for strategic advertising decisions as well as the interaction between firms of different size and age. In Cavenaile et al. (2024b), advertising also has the role of expanding product awareness, but the advertising choices are coordinated at the industry level and made once and for all at the industry inception, and firms are assumed to be symmetric. They focus on the improvements in targeting advertising, showing that, although better targeting technologies are a priori good, as they allow better matches between firms and consumers, they may lead to an increase in market power through larger market segmentation.

This paper is also related to Greenwood et al. (2024) that study the inefficiencies from advertising, focusing on the welfare implications of the rise of digital advertising. They present a static model of informative advertising whose equilibrium is not efficient for two reasons, which are also present in my model. The first source of inefficiency comes from firms not internalizing the social value of media goods in their advertising decisions. The second is that ads are shown to people that won't buy the good. Similarly, in my model, although consumers have heterogeneous gains from a particular ad, they all see the same ads. Here, my contribution is to include in the analysis the inefficiencies coming from markups typical of oligopoly frameworks, as well as identifying three novel sources of inefficiency. The first two are reminiscent of Schumpeterian growth models, namely: (i) lack of full appropriability, as the producers cannot extract the entire consumer surplus; and (ii) a business-stealing effect, as firms don't consider how the increase in their market share reduces the competitors' market shares. The third source of inefficiency arises from the anticompetitive motive particular to this model, as firms try to avoid suffering from the business-stealing effect. Note that the sources of inefficiency push to different directions, so it is not clear whether there is too much or too little advertising, and requires a quantitative answer. Finally, there is also inefficient entry, again due to lack of appropriability and the business-stealing effect.

For the persuasive aspect of advertising, I build on the literature that has taken the persuasive view, e.g. Cavenaile et al. (2024a), Rachel (2024), Molinari and Turino (2018). These papers model advertising as a static demand shifter. A novelty of the current paper is to combine the persuasive and informative views of advertising in a single framework. In particular, I relate to Cavenaile et al. (2024a), as they also have an oligopolistic model with endogenous market structure. In their paper, they study the interaction of R&D and advertising, and find that they are substitutes at the aggregate level, consistent with the empirical findings in Cavenaile and Roldan-Blanco (2021). They also study the question of whether advertising should be taxed or subsidized, and find a very high optimal tax. As in my model, in Rachel (2024) and Greenwood et al. (2024) advertising also finances the provision of media goods that improve utility. Similar to Rachel (2024), the model here features that as advertising expenditure increases, more media goods are produced, leading consumers to spend more time on media, which in turn makes advertising more effective (in his model, via a reduction of the price of brand equity).

Finally, the notion that limited attention capacity leads to consumers being aware of a subset of the available goods relates to a behavioural literature of rational inattention and bounded rationality (Masatlioglu et al. (2012), Manzini and Mariotti (2014), Matějka and McKay (2015)).[3] In particular, Manzini and Mariotti (2014) model choice as a two-stage process. In the first stage, some of the available alternatives are selected into a *consideration set*, with a probability that is linked to attention. In the second stage, the agent maximizes utility restricted to the consideration set.

2 Model

2.1 Environment

Market structure and the production sector. There is a continuum of mass 1 of industries indexed by i. In each industry, there is a generic good and a set $\mathcal{J}_{i,t}$ of firms, indexed by j, each one producing a single differentiated good with the production function $y_{j,i,t} = N_{j,i,t}$, where $N_{j,i,t}$ is the labor employed by firm j. Note that this implies that firms will be homogeneous regarding their production technology. The generic good is produced by many small firms with the production function $y_{0,i,t} = A_0 N_{0,i,t}$, where $N_{0,i,t}$ is the total labor employed by these small firms in industry i at period t.

Advertising and the media sector. There is a media sector populated by media firms that employ labor to produce media goods that are supplied to consumers at zero monetary price and they get revenue from selling advertising space to production firms. There is free entry. Each media firm produces a differentiated variety of media good of equal quality (so, consumers will allocate the time they spend on media equally among the different media goods). The media good is available during the whole unit of time of period t (the consumer decides when to watch it and for how long) but a fraction α is advertising, which is distributed randomly across time (so the consumer cannot avoid them and will watch ads for a fraction α of the time she spends on media).[4] Within this advertising space, each industry of the production sector has its own advertising space. This is a reasonable assumption for digital advertising, where each industry is associated to some keyword and firms compete for the attention in that keyword. Further, assume all sectors have the same advertising space, $\alpha_i = \alpha$. This is a reasonable assumption for search advertising: there is only one top position for a specific keyword, so higher demand only leads to higher price, as suggested in Figure 1.
The process whereby firms acquire advertising space follows a kind of auction, where media firms post a price per unit of ad space in industry i, $p_{a,i,t}$, which is the minimum bid accepted, and supply at most α_i units of ad space. Letting $e_{j,i,t}$ be the advertising expenditure of firm

[3] as well as in finance (e.g. Guido Menzio) and marketing literatures (check references).
[4] More generally, you could think of this α as some measure of attention.

j in industry i, then the final price per unit of ad space in industry i will be equal to $\max\left\{p_{a,i,t}, \frac{\sum\limits_{j\in\mathcal{J}_{i,t}} e_{j,i,t}}{\alpha_i}\right\}$. So, the advertising space acquired by firm j, $\alpha_{j,i,t}$, will be:

$$\alpha_{j,i,t} = \min\left\{\frac{e_{j,i,t}}{p_{a,i,t}}, e_{j,i,t}\frac{\alpha_i}{\sum\limits_{k\in\mathcal{J}_{i,t}} e_{k,i,t}}\right\} \tag{1}$$

The aggregate quality of media is given by $Q = AN_m^{\frac{1}{2}}$, where N_m is total labor employed in media. In order to rule out an equilibrium with no advertising expenditure and no media produced, I assume the government employs $\bar{N}_m$ units of labor in media, which is financed by a lump-sump tax to consumers.

Entry and Exit of firms. A firm is hit by a death shock with probability κ, and is independent of whether the other firms are hit by a death shock (so, the probability n firms exit is κ^n).[5] Regarding entry, there is a measure one of entrepreneurs that pay $h_{e,i,t}$ in units of the final good in order to create a new differentiated good in industry i with probability $z_{e,i,t} = \phi_s\left(\frac{h_{e,i,t}}{E_t}\right)^{\frac{1}{2}}$, where E_t is aggregate expenditure. Upon successfully creating a new good (and, for computational purposes, provided that the number of firms in the industry is below $\bar{J}$), a new firm enters the market and initially no consumer is aware of the new firm. Entry and exit occur simultaneously right at the start of $t+1$.

Consumers. There is a unit mass of individuals indexed by ℓ who maximize lifetime utility, where the instantaneous utility is a function of her consumption (C_ℓ) and entertainment (L_ℓ) goods. Individuals die with an exogenous probability δ, they are replaced with an offspring who inherits the assets $a_{\ell,t}$, and individuals discount the offspring's utility with the same discount rate (see the next section of product learning for the effect of dying); thus, we can write utility as if they were infinitely lived:[6]

$$U_\ell = \sum_{t=0}^{\infty}\beta^t\ [\mathbb{E}\ln C_{\ell t} + L_{\ell t}]$$

Each individual supplies inelastically N units of labor time and chooses how much of the remaining time $1-N$ to allocate to media goods, $T_{\ell,t}$, in order to maximise her entertainment good $L_{\ell,t}$, which is defined as follows:

$$L_{\ell t} = \upsilon\left(Q_t T_{\ell,t} - \frac{T_{\ell,t}^2}{2}\right) \tag{2}$$

[5] I plan to do an extension where this probability is decreasing in firm size. This is a more realistic assumption, and would imply a stronger anticompetitive motive to advertise, since by precluding the competitors from expanding they are also effectively increasing the probability they exit.

[6] The expectation is due to the stochastic evolution of the awareness sets at the individual level, as explained in the next section.

where Q_t is an output of the media sector production function. Anticipating that all individuals choose the same $T_{\ell,t}$, in what follows I drop the subindex ℓ from T_t and L_t. Individual ℓ gets her C_ℓ following a Cobb-Douglas aggregator of her consumption over the continuum of industries of mass 1

$$\ln C_{\ell,t} = \int_0^1 \ln C_{\ell,i,t} di \tag{3}$$

where the industry i consumption good of individual ℓ is a CES aggregator of her consumption on the generic good and each of the differentiated goods she is aware of:

$$C_{\ell,i,t} = \left(c_{\ell,0,i,t}^{\frac{\sigma-1}{\sigma}} + \sum_{j \in \mathcal{I}_{\ell,i,t}} \omega_{j,i,t} c_{\ell,j,i,t}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \quad \sigma > 1 \tag{4}$$

where $c_{\ell,j,i,t}$ is the quantity of good j consumed by ℓ at t; $\mathcal{I}_{\ell,i,t}$ will be referred to as the awareness set of individual ℓ in industry i at period t, as it is the subset of the differentiated goods $\mathcal{J}_{i,t}$ the individual is aware of at period t (in the next section I describe the evolution of this object); and $\omega_{j,i,t}$ is a demand shifter which depends on the exposure of individuals to the ad of good j. In particular, $\omega_{j,i,t} = 1 + \nu_s(\alpha_{j,i,t}T_t)^{\nu_c}$, $\nu_c \in (0,1)$. Note that the more time consumers spend on media, the larger the effect of advertising on the demand shifter.
Individual ℓ's budget constraint writes:

$$w_t N_{\ell,t} + r_t a_{\ell t} = \int \sum_{j \in \mathcal{I}_{\ell,it}} c_{\ell,j,i,t} p_{j,i,t} di + a_{\ell,t+1} - a_{\ell,t} + \tau_t \tag{5}$$

where w_t is wage, $a_{\ell t}$ is the asset holdings of individual ℓ at period t, r_t is the return of each unit of asset in period t, and τ_t is the lump-sum tax the government uses to employ $\bar{N}_m$ units of labor in the media sector. All individuals start with the same level of assets a_0.

Product learning and the evolution of the awareness sets. I assume that the probability an individual gets aware of a product thanks to advertising is an increasing and concave function of the exposure to the ad of that good. In particular, assume an individual will get aware of product j in industry i with probability[7]

$$\rho_{j,i,t} = \hat{\rho} + \psi_s(\alpha_{j,i,t}T_t)^{\psi_c}, \quad \psi_c \in (0,1) \tag{6}$$

Although I focus on advertising as an active way through which firms can increase their customer base, consumers can get to know a firm in other ways (word-of-mouth, seeing the product in a shop...), and these are captured by $\hat{\rho}$. The inclusion of T_t is to capture the idea that the more time consumers spend on media, the more they are exposed to ads, and so the more effective advertising is, just like in the demand shifter $\omega_{j,i,t}$.
The events of learning goods are assumed to be independent; that is, the probability of

[7] In a version of the model, I have congestion, $\alpha_i^{-\zeta}$, $\zeta \in [0, \psi_c]$, which allows to play with the intensity of the anticompetitive motive.

learning all the goods in $\mathcal{I} \subseteq \mathcal{J}_i$ is $\prod_{j\in\mathcal{I}} \rho_{j,i}$. And, given that the complements of independent events are also independent, then the probability of not learning any of the goods in $\mathcal{I} \subseteq \mathcal{J}_i$ is $\prod_{j\in\mathcal{I}}(1-\rho_{j,i})$.[8]

In addition, when a consumer dies, they are replaced by a newborn individual who starts knowing only the generic good of each industry (i.e. $\mathcal{I}_{\ell,i} = \emptyset$ for all i). This is completely equivalent to say that individuals forget all the differentiated goods they know with an exogenous probability δ. This assumption is not crucial for the results, and its only implication is that, even if a firm lived forever, there would always be some consumers that are not aware of it.

Then, we have all the information to find the probability of moving between any pair of awareness sets. Letting $\Theta_{(\mathcal{I}\to\mathcal{I}')}$ be the probability of moving from $\mathcal{I}$ to $\mathcal{I}'$, since the awareness set of an individual can only expand or move to containing only the generic good of the fringe, then if $\mathcal{I}'$ doesn't contain $\mathcal{I}$, then the transition is only possible (i.e. $\Theta_{(\mathcal{I}\to\mathcal{I}')} > 0$) if $\mathcal{I}' = \emptyset$, which happens with the probability of dying δ. On the contrary, if $\mathcal{I}'$ contains $\mathcal{I}$, then the probability this transition takes place is the probability an individual doesn't die, $(1-\delta)$, times the probability of learning all the goods that are in $\mathcal{I}'$ but not in $\mathcal{I}$, $\prod_{j\in\mathcal{I}'\setminus\mathcal{I}} \rho_{j,i}$, times the probability of not learning any of the goods that are not in $\mathcal{I}'$, $\prod_{j\notin\mathcal{I}'}(1-\rho_{j,i})$. Formally:

$$\Theta_{(\mathcal{I}\to\mathcal{I}')} = \begin{cases} 0, & \text{if } \mathcal{I} \nsubseteq \mathcal{I}' \neq \emptyset \\ \delta, & \text{if } \mathcal{I} \nsubseteq \mathcal{I}' = \emptyset \\ (1-\delta) \prod\limits_{j\in\mathcal{I}'\setminus\mathcal{I}} \rho_{j,i} \cdot \prod\limits_{j\notin\mathcal{I}'} (1-\rho_{j,i}), & \text{if } \mathcal{I} \subseteq \mathcal{I}' \neq \emptyset \\ (1-\delta) \prod\limits_{j\in\mathcal{J}_{i,t}} (1-\rho_{j,i}) + \delta, & \text{if } \mathcal{I} = \mathcal{I}' = \emptyset \end{cases} \tag{7}$$

2.2 Equilibrium

In this section I characterize the stationary equilibrium, that is such that the time spent in media T_t and the relative wage $\hat{w}_t = \frac{w_t}{E_t}$ are constant.

2.2.1 Consumption.

On the one hand, logarithmic preferences on $C_{\ell,t}$, together with $a_{\ell,0} = a_0$, imply that all consumers choose the same expenditure at all t: $E_{\ell,t} = E_t$. On the other hand, CES preferences over the varieties within an industry implies that consumer's spending in an industry is independent of her industry price index: $E_{\ell,i,t} = E_t$. Therefore, the awareness sets $\mathcal{I}_{\ell,i,t}$ only affect the allocation of the expenditure within each industry. That is, in order to characterize consumer ℓ's consumption choices in industry i, we only need to know her awareness set in i, $\mathcal{I}_{\ell,i,t}$. In other words, within industry i, there are as many types of consumers as subsets $\mathcal{I} \subseteq \mathcal{J}_{i,t}$. So, the set of consumer types in industry i is identified by the

[8] Given two independent events A, B, we have $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = P(A)P(B) + P(A|B^c)(1-P(B))$, and so $P(A|B^c) = P(A)$; so: $P(A^c|B^c) = 1 - P(A|B^c) = 1 - P(A) = P(A^c)$.

power set of $\mathcal{J}_{i,t}$, $\mathcal{P}(\mathcal{J}_{i,t})$, and, within an industry, I'll use subindex $\mathcal{I}$ to denote the choice of an individual with awareness set $\mathcal{I}$.
The optimal choices satisfy: [9]

$$c_{\mathcal{I},j,i,t} = E_t P_{\mathcal{I},i,t}^{\sigma-1} p_{j,i,t}^{-\sigma} \omega_{j,i,t}^{\sigma}, \quad j \in \mathcal{I} \tag{8}$$

$$\frac{E_{t+1}}{E_t} = \beta(1 + r_{t+1}) \tag{9}$$

$$T_t = \min\{Q_t, 1 - N\} \tag{10}$$

where $P_{\mathcal{I},i,t} = \left(p_{0,i,t}^{1-\sigma} + \sum_{j \in \mathcal{I}} \omega_{j,i,t}^{\sigma} p_{j,i,t}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$. Note that consumers consume a positive amount of all the goods they are aware of. And from $E_t = P_{\mathcal{I},i,t} C_{\mathcal{I},i,t}$ we see one of the channels through which (the informative) advertising will increase welfare: advertising will increase the amount of products the consumer is aware of, which reduces the price $P_{\mathcal{I},i,t}$ of her industry composite good.[10] This is a standard love for variety effect.

2.2.2 The industry state and its evolution

Given that firms have the same production technology, all the heterogeneity comes from the consumer side. The relevant state of the industry is characterised by the triple $(\mathcal{J}_{i,t}, \mathcal{P}(\mathcal{J}_{i,t}), \vec{M}_{i,t})$, where $\vec{M}_{i,t} = (M_{i,t}(\mathcal{I}))_{\mathcal{I} \in \mathcal{P}(\mathcal{J}_{i,t})}$ is the mass of consumers in each awareness set (that is, how consumers are distributed over the awareness sets). Note that since there is a mapping from $\mathcal{J}_{i,t}$ to $\mathcal{P}(\mathcal{J}_{i,t})$, I may write the state simply as $(\mathcal{J}_{i,t}, \vec{M}_{i,t})$. There are two processes that shape the evolution of the industry state.
On the one hand, the industry state changes as consumers' awareness sets evolve due to learning and death, which, by law of large numbers, is a deterministic process at the industry level.[11] Calling Θ_t the transition matrix, where the element in row r and column s indicates the probability of going from subset $\mathcal{I}_r$ to $\mathcal{I}_s$ at time t (i.e. $\Theta_{(\mathcal{I}_r \to \mathcal{I}_s)}$), and calling $\vec{M}_{i,t}$ the $2^{\#\mathcal{J}_{i,t}}$-dimensional row vector (where $\#\mathcal{J}_{i,t}$ is the cardinal of $\mathcal{J}_{i,t}$) containing the masses of consumers in each awareness set at time t; then, by the law of large numbers, the distribution of consumers in $t+1$ in the absence of entry and exit of goods, which I denote by $\vec{\hat{M}}_{i,t+1}$, would be:

$$\vec{\hat{M}}_{i,t+1} = \vec{M}_{i,t} \Theta_t \tag{11}$$

On the other hand, the industry state changes stochastically due to entry and exit of firms. If the realisation of exit and entry changes the set of firms in industry i from $\mathcal{J}$ to $\mathcal{J}'$, then the next period industry state is obtained using the application $(\mathcal{J}, \vec{\hat{M}}, \mathcal{J}') \longmapsto (\mathcal{J}', \vec{M}')$

[9] Together with the No-Ponzi condition $\lim_{\tau \to \infty} \frac{a_{t+\tau}}{\prod_{s=0}^{\tau}(1+r_{t+s})} = 0$.

[10] Formally, since $\sigma > 1$ and $p_{k,i,t}, \omega_{k,i,t} > 0$, we have $P_{\mathcal{I} \cup k,i,t} = \left(p_{0,i,t}^{1-\sigma} + \sum_{j \in \mathcal{I}} \omega_{j,i,t}^{\sigma} p_{j,i,t}^{1-\sigma} + \omega_{k,i,t}^{\sigma} p_{k,i,t}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} < \left(p_{0,i,t}^{1-\sigma} + \sum_{j \in \mathcal{I}} \omega_{j,i,t}^{\sigma} p_{j,i,t}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} = P_{\mathcal{I},i,t}$.

[11] In case there were mixed strategies (although this is not the case in the equilibrium studied) this process would be stochastic.

defined as follows.

$$\text{For } \mathcal{I}' \in \mathcal{P}(\mathcal{J}'), \qquad M'(\mathcal{I}') = \begin{cases} \sum\limits_{\{\mathcal{I}\in\mathcal{P}(\mathcal{J}):\mathcal{I}\cap\mathcal{J}'=\mathcal{I}'\}} \hat{M}(\mathcal{I}) & \text{, if } \mathcal{I}' \subseteq \mathcal{J} \\ 0 & \text{, if } \mathcal{I}' \nsubseteq \mathcal{J} \end{cases} \tag{12}$$

where the first case says that two consumers become identical in industry i if all the firms in which they differed exit, whereas the second case says that there are no consumers who are aware of a newborn firm. The last piece of infornation needed to compute expected values is the probabilities that the set of differentiated goods moves from $\mathcal{J}$ to $\mathcal{J}' \subseteq \mathcal{J} \cup \{e\}$, where e denotes an entrant. These probabilities are given by:

$$\text{For } \mathcal{J}' \in \mathcal{P}(\mathcal{J} \cup e)), \qquad Prob\{\mathcal{J} \to \mathcal{J}'\} = \begin{cases} (1-z_{e,i,t}) \prod\limits_{j\in\mathcal{J}\cap\mathcal{J}'} (1-\kappa) \prod\limits_{j\in\mathcal{J}\setminus\mathcal{J}'} \kappa & \text{, if } e \notin \mathcal{J}' \\ z_{e,i,t} \prod\limits_{j\in\mathcal{J}\cap\mathcal{J}'} (1-\kappa) \prod\limits_{j\in\mathcal{J}\setminus\mathcal{J}'} \kappa & \text{, if } e \in \mathcal{J}' \end{cases} \tag{13}$$

where $z_{e,i,t}$ is the probability of an entrant, $\prod\limits_{j\in\mathcal{J}\cap\mathcal{J}'} (1-\kappa)$ is the probability that all the firms in $\mathcal{J} \cap \mathcal{J}'$ survive, and $\prod\limits_{j\in\mathcal{J}\setminus\mathcal{J}'} \kappa$ is the probability that all the firms in $\mathcal{J} \setminus \mathcal{J}'$ exit.

In the Appendix 7.3 I show that assuming that individuals don't die (i.e. $\delta = 0$) allows a simpler sufficient industry state given by the vector of customer bases. That is, instead of requiring the mass of consumers in each awareness set, we would only need to know the mass of consumers aware of each good.

2.2.3 Production firms problem

On the one hand, given the large number of small firms producing a homogeneous product, the price of the generic good is equal to its marginal cost, $p_{0,i,t} = \frac{w_t}{A_0}$.
On the other hand, the differentiated firms compete in prices a la Bertrand and in advertising expenditures for the attention of consumers. Both decisions are made simultaneously.

Profits. Using the production function $y_{j,i,t} = N_{j,i,t}$, we can express profits decomposed as

$$\pi_{j,i,t} = \underbrace{M_{j,i,t}}_{\substack{\text{Customer}\\\text{Base}}} \cdot \underbrace{(1-\mathcal{M}_{j,i,t}^{-1}) \underbrace{\sum_{\mathcal{I}\in\mathcal{P}_j(\mathcal{J}_{i,t})} \frac{M_{i,t}(\mathcal{I})}{M_{j,i,t}} s_{\mathcal{I},j,i,t} E_t}_{\substack{\text{Average spending}\\\text{by customers}}}}_{\text{Average rents from customers}} \tag{14}$$

where $\mathcal{M}_{j,i,t} = \frac{p_{j,i,t}}{w_t}$ is the markup of firm j, $s_{\mathcal{I},j,i,t} = \frac{p_{j,i,t} c_{\mathcal{I},j,i,t}}{E_t}$ is type $\mathcal{I}$ individual's share of expenditure in good j, $\mathcal{P}_j(\mathcal{J}_{i,t}) = \{\mathcal{I} \in \mathcal{P}(\mathcal{J}_{i,t}) \mid j \in \mathcal{I}\}$ is the family of awareness sets containing good j, and $M_{j,i,t} = \sum\limits_{\mathcal{I}\in\mathcal{P}_j(\mathcal{J}_{i,t})} M_{i,t}(\mathcal{I})$ is the customer base of firm j.
This expression offers a first intuition of the motives driving firms to advertise. First, they want to advertise to increase their customer base. I refer to this as the informative motive. Second, as shown in the Appendix 7.4, all else equal, firms prefer to have customers that

know as fewer competitors as possible. Intuitively, the fewer alternative goods they know, the more they will spend in j (i.e. higher $s_{\mathcal{I},j,i,t}$) and the lower their demand elasticity (so, the firm is able to extract more rents by rising the markup). So, given that by increasing the advertising space they occupy, firms reduce the attention of consumers to the competitors' goods and so the probability they will add them to their awareness sets; then, firms may have the incentive to do advertising for the mere purpose of reducing the mass of customers who learn about competitors. I refer to this as the anticompetitive motive, as under this motive the firm is doing advertising to avoid competition by precluding competitors to expand their customer base. Finally, given that the demand shifter $\omega_{j,i,t}$ increases with the the advertising space, firms want to do advertising to persuade current consumers to buy more. This is the persuasive motive.

Price setting. Given that, as standard, I focus on Markov-Perfect equilibrium (so, decisions just depend on the current state, not on history), and given that the price has no direct effect on the evolution of the industry state and that advertising and price choices are made simultaneously, then the price setting problem is static. The optimal markup $\mathcal{M}_{j,i,t}$ is such that profits (14) are maximised, given its own demand-shifter $\omega_{j,i,t}$, the markups and demand-shifters of the competitors $\{\mathcal{M}_{k,i,t}, \omega_{k,i,t}\}_{k \in \mathcal{J}_{i,t}, k \neq j}$, and the distribution of consumers over the awareness sets $\vec{M}_{i,t} = (M_{i,t}(\mathcal{I}))_{\mathcal{I} \in \mathcal{P}(\mathcal{J}_{i,t})}$, and taking into account that individuals' spending shares are given by

$$s_{\mathcal{I},j,i,t} = \left[(A_0 \mathcal{M}_{j,i,t})^{\sigma-1} \omega_{j,i,t}^{-\sigma} + \sum_{k \in \mathcal{I}} \left(\frac{\omega_{k,i,t}}{\omega_{j,i,t}} \right)^{\sigma} \left(\frac{\mathcal{M}_{j,i,t}}{\mathcal{M}_{k,i,t}} \right)^{\sigma-1} \right]^{-1} \tag{15}$$

The equilibrium markups are given by:

$$\mathcal{M}_{j,i,t} = \frac{\frac{\sigma}{\sigma-1} - \bar{s}_{j,i,t}}{1 - \bar{s}_{j,i,t}} \tag{16}$$

Where $\bar{s}_{j,i,t} = \sum_{\mathcal{I} \in \mathcal{P}_j(\mathcal{J}_{i,t})} \frac{M_{i,t}(\mathcal{I}) p_{j,i,t} c_{\mathcal{I},j,i,t}}{p_{j,i,t} y_{j,i,t}} s_{\mathcal{I},j,i,t}$ is a sales-weighted average of firm j customers' share of expenditure in industry i allocated to good j.

Note that in a standard oligopoly model with Bertrand competition, the optimal markup is given be the expression in (16) but with the market share $s_{j,i,t}$ instead of $\bar{s}_{j,i,t}$. So, while the optimal markup in a standard oligopoly model with Bertrand is increasing with size, this is not necessarily the case here. Here, the markup depends on the composition of the customers, not in the size: a smaller firm can have a higher markup if a larger fraction of its customers spend a larger share of expenditure on it. However, the model will still predict that, within an industry, larger firms have higher markups. The intuition is as follows: a firm that entered earlier had more time to accumulate customers (so older firms will be larger); but also, since as time passes consumers get aware of more goods and advertising is undirected, then a firm that enters later will get consumers that, on average, know more goods (and we have seen that customers with more alternative goods spend a smaller share). So, within an industry, larger firms will have customers that on average spend a larger share of expenditure, and thus they set higher markups.

Advertising choice. Each firm chooses dynamically its advertising expenditure $e_{j,i,t}$, taking into account (i) the advertising expenditure choices of its competitors $\{e_{k,i,t}\}_{k\in\mathcal{J}_{i,t},k\neq j}$; (ii) markups $\{\mathcal{M}_{k,i,t}\}_{k\in\mathcal{J}_{i,t}}$; (iii) the time consumers spend on media, T_t; (iv) the law of motion of the industry state; and (v) that the actual advertising space purchased by each firm is given by (1). In practice, given that in equilibrium $p_{a,i,t}$ is such that total advertising expenditure in industry i exactly purchases α_i units of ad space, then, in all industries with more than one differentiated firms, $\alpha_{j,i,t}$ will be given by $\alpha_{j,i,t} = e_{j,i,t}\frac{\alpha_i}{\sum\limits_{k\in\mathcal{J}_{i,t}} e_{k,i,t}}$, and so there will be an anticompetitive motive to advertise: by increasing $e_{j,i,t}$, firm j will achieve to increase the actual price for advertising space and thus reduce the advertising space of competitors, which will reduce the probability consumers learn about competitors. In industry states with only one differentiated firm, there is trivially no anticompetitive motive because there is no competitor and so the unique firm has no incentive to spend more than $p_{a,i,t}\alpha_i$, and so in such industry states $\alpha_{j,i,t}$ will be given by $\alpha_{j,i,t} = \frac{e_{j,i,t}}{p_{a,i,t}}$.

Given that I focus on Markov-Perfect equilibrium, then the firm problem can be expressed in recursive form, with the value of the firm being a function of the state. Given that profits are linear on E_t, by guess and verify, the value of the firm is also linear on E_t. Therefore, defining $V_j(\mathcal{J}_{i,t}, \vec{M}_{i,t}) = \frac{V_{j,i,t}}{E_t}$, $\hat{e}_{j,i,t} = \frac{e_{j,i,t}}{E_t}$, $\hat{p}_{a,i,t} = \frac{p_{a,i,t}}{E_t}$ and $\pi_j(\omega_{j,i,t}, \mathcal{J}_{i,t}, \vec{M}_{i,t}) = \frac{\pi_{j,i,t}}{E_t}$ and using the Euler equation and that in the stationary equilibrium it will be $T_t = T$, we can write the dynamic firm problem recursively as

$$V_j\Big(\mathcal{J}, \vec{M}\Big) = \max_{\hat{e}_j} \Big\{\pi_j\Big(\omega_j, \mathcal{J}, \vec{M}\Big) - \hat{e}_j + \beta\mathbb{E}V_j\Big(\mathcal{J}', \vec{M}'\Big)\Big\}$$
$$\text{s.t. } \{\hat{e}_k\}_{k\in\mathcal{J}\setminus\{j\}},\ \{\mathcal{M}_k\}_{k\in\mathcal{J}},\ T,\ (6),\ (7),\ (11),\ (12),\ (13),\ (1)$$

We can decompose the FOC into the three motives to advertise: the informative motive (increase ρ_j), the anticompetitive motive (decrease $\rho_{j'}$, $j' \neq j$), and the persuasive motive (increase $s_{\mathcal{I},j,i}$ for $\mathcal{I} \in \mathcal{P}_j$):

$$1 = \frac{\partial \pi_{j,i}}{\partial e_j} + \frac{\partial V_j}{\partial \rho_j}\frac{\partial \rho_j}{\partial \alpha_j}\frac{\partial \alpha_j}{\partial e_j} + \sum_{j'\neq j}\left(-\frac{\partial V_j}{\partial \rho_{j'}}\right)\frac{\partial \rho_{j'}}{\partial \alpha_{j'}}\left(-\frac{\partial \alpha_{j'}}{\partial e_j}\right) \tag{17}$$

In section 3.2 I show how the intensity of the three motives evolve with firm age and size. The intuition is clear. Smaller or younger firms (that is, those firms that are unknown for most consumers) are the ones that have more potential customers to acquire. In the extreme case, a firm that was known by all consumers wouldn't have any incentive to advertise to inform consumers. On the contrary, the anticompetitive motive, which is about retaining market power over the current customers by reducing the probability they learn about competitors, is stronger the bigger the customer base. A firm that is unknown for everybody also has some incentives to preclude consumers from learning about other firms (as it is internalising that these consumers may eventually become customers and so the firm wants consumers to know as fewer goods as possible), but intuitively, the incentives to avoid a consumer learning about a competitor are higher if the consumer is a current customer rather than a potential one. Finally, the persuasive motive also tends to be bigger in older/bigger firms. Intuitively, if advertising persuades current customers to spend more, then the increase in revenues will be larger if there are more customers.

2.2.4 Entrepreneurs problem.

The entrepreneurs in an industry $(\mathcal{J}, \vec{M})$ choose $\hat{h}_e = \frac{h_e}{E}$ to maximise their expected profits:

$$v^e(\mathcal{J}, \vec{M}) = \max_{h_e} \left\{ -\hat{h}_e + \beta z_e \mathbb{E}_e V_e \Big(\mathcal{J}' \cup \{e\}, \vec{M}' \Big) \right\}, \text{ s.t. } z_e = \phi_s \hat{h}_e^{\frac{1}{2}}$$

where $\beta \mathbb{E}_e V_e \Big(\mathcal{J}' \cup \{e\}, \vec{M}' \Big)$ is the expected value of being a new firm conditional on successfully creating a new differentiated good (so, the expectation comes from the uncertainty on which of the $\mathcal{J}$ incumbents will survive). Then, the equilibrium spending in entry in an industry $(\mathcal{J}, \vec{M})$ will be:

$$\hat{h}_{e,(\mathcal{J},\vec{M})} = \left(\frac{\phi_s}{2} \beta \mathbb{E}_e V_e \Big(\mathcal{J}' \cup \{e\}, \vec{M}' \Big) \right)^2 \tag{18}$$

2.2.5 Stationary distribution.

In the Appendix 7.8 I prove that, for any aggregates $\hat{w}$ and T given, with their associated solutions of the firms and entrepreneurs problems $\{\alpha_{j,(\mathcal{J},\vec{M})}, \hat{h}_{e,(\mathcal{J},\vec{M})}\}$, the probability that an industry is at a given state $X = (\mathcal{J}, \vec{M})$ converges to an ergodic distribution (existence), which is independent of the initial state (uniqueness), and satisfies that the set of different states realised, call it Ω, is at most countably infinite.[12]

By Law of large numbers, this implies that the economy converges to a stationary distribution associated to the aggregates $\hat{w}, T$. Let $\mu(X)$ be the mass of industries in state $X \in \Omega$ in this stationary distribution. If $\hat{w}, T$ are consistent with this stationary distribution, then we are in the stationary equilibrium.
Computationally, the stationary distribution is a complicated object, and the method used to obtain it, described in the Appendix 7.9, is a computational contribution of this paper.

2.2.6 Media sector problem.

Given the symmetry of media firms, consumers allocate their media time T equally among the media firms, and production firms allocate their advertising expenditure equally among the media firms. Therefore, all media firms have the same profits, and so each media firm has positive profits if and only if the overall profits in the media sector are positive. Then, since there is free entry into the media sector, profits in the media sector must be zero in equilibrium; so, the equilibrium Q_t satisfies:

$$\int_0^1 \sum_{j \in \mathcal{J}_{i,t}} \hat{e}_{j,i,t} E_t di + w_t \bar{N}_m - w_t \left(\frac{Q_t}{A} \right)^2 = 0 \tag{19}$$

where recall that $\bar{N}_m$ is the labor in media employed by the public sector. In the stationary equilibrium, $Q_t = Q$ is constant.

[12]Note that I haven't proved whether the solution of the firms and entrepreneurs problems is unique. One could prove unicity by imposing restrictions on how the players make their decisions.

2.2.7 Labor market clearing.

The labor market must clear, that is, the amount of labor supplied has to be equal to the labor demanded by the production firms, media firms and entrepreneurs. Without any loss of generality (just a change in the units we measure labor), I can normalize labor supply N to 1.

$$1 = N = \int_0^1 \sum_{j \in \{0\} \cup \mathcal{J}_{i,t}} N_{j,i,t} di + N_{m,t}$$

where $N_{j,i,t} = \frac{s_{j,i,t}}{\mathcal{M}_{j,i,t}} \frac{E_t}{w_t}$, and $N_{m,t}$ is given by (19). This pins down the equilibrium relative wage $\hat{w}_t$, and verifies that it is constant in the stationary equilibrium.

2.2.8 Aggregate output and representative consumer conditional on age.

Define the aggregate output as the geometric mean of the individuals' aggregate consumption goods; that is $\ln Y_t = \int_0^1 \ln C_{\ell,t} d\ell$. Using the definitions of $C_{\ell,t}$ and $C_{\ell,i,t}$, together with $c_{\ell,j,i,t} = \frac{s_{\ell,j,i,t}}{\mathcal{M}_{j,i,t}} \hat{w}^{-1}$ and $c_{\ell,0,i,t} = s_{\ell,0,i,t} A_0 \hat{w}^{-1}$, and interchanging the integrals over ℓ and i, we have:

$$\ln Y = -\ln \hat{w} + \sum_{(\mathcal{J},\vec{M}) \in \Omega} \mu(\mathcal{J},\vec{M}) \sum_{\mathcal{I} \in \mathcal{P}(\mathcal{J})} M(\mathcal{I}) \frac{\sigma}{\sigma - 1} \ln \left((s_{\mathcal{I},0,(\mathcal{J},\vec{M})} A_0)^{\frac{\sigma-1}{\sigma}} + \sum_{j \in \mathcal{I}} \omega_{j,(\mathcal{J},\vec{M})} \left(\frac{s_{\mathcal{I},j,(\mathcal{J},\vec{M})}}{\mathcal{M}_{j,(\mathcal{J},\vec{M})}} \right)^{\frac{\sigma-1}{\sigma}} \right) \tag{20}$$

The aggregate price index of the economy is P_t such that $P_t Y_t = \sum_{\mathcal{I} \in \mathcal{P}} M_t(\mathcal{I}) P_{\mathcal{I},t} C_{\mathcal{I},t} = E_t$, and it is the numeraire (i.e. $P_t = 1$).

Finally, note that applying a law of large numbers to the continuum of industries, two consumers with the same age will have the same level of aggregate consumption good. That is, although they will differ on their awareness sets for particular industries, at the aggregate level they will have the same distribution of awareness sets. This points to a potentially interesting extension where firms can target consumers based on the observable age.

3 Quantitative and empirical analysis

3.1 Calibration

In this section, I describe the calibration of the model, and the details of the data sources and how the moments are computed are provided in the Appendix 7.1.

One of the main components of the model is firms' customer base accumulation, which has a strong relationship with firm size, both in the model and in the data, as pointed by the empirical literature cited in the introduction. Therefore, it is important that the model reproduces the evolution of the average firm sales growth by age, in order to calibrate this customer base building process. In particular, I target the average firm sales growth at age 1, and the slope of the fitted line that has no regression error for age 1 (I plan to do another calibration where, instead, I minimise the sum of squared errors between the data and the

model for each age). Also, as shown in section 3.2, the intensity of the different motives to advertise varies with firm size, so the coefficient from a regression of advertising expenditure and sales is a good candidate to discipline the model.
To compute these three moments I use Compustat data. Given that firms typically enter Compustat a few years after their foundation (and certainly not with zero customers as it is assumed in the model for new firms), for the computation of the model-implied moments of these three targets, I assume that firms in the model are unobserved until they are at least two years old.

I estimate the model for the US at an annual frequency and set the consumer discount rate to $\beta = 0.98$. I also set (i) $\delta = 0.01$ corresponding to the mortality rate of 1% in the data, (ii) the concavity parameter for the persuasive advertising $\nu_c = 0.2972$ is taken from (the inverse) Cavenaile et al. (2024a), (iii) given that public sector spending on media represents roughly 0.008% of US GDP, dividing this by the (capital-adjusted) labor share, I set $\bar{N}_m = \frac{8 \cdot 10^{-5}}{0.8359}$, and (iv) I set $\kappa = 0.1151$ corresponding to the entry rate in the data. Acknowledging the difficulty to find good proxies for the utility value of media goods, I leave the weight of the entertainment good on the utility function, υ, uncalibrated and all the exercises involving welfare are made for a range of values of υ. This leaves 8 moments to estimate: the elasticity of substitution parameter, σ; the relative productivity of the small firms producing the generic goods, A_0; the scale parameter for the persuasive effect of advertising, ν_s; the scale and convexity parameters for the informative effect of advertising, (ψ_s, ψ_c); the exogenous learning probability, $\hat{\rho}$; the scale parameter regulating the creation of new products, ϕ; and the aggregate productivity of the media sector, A. Apart from A, which can be derived directly from 19 using the target values for aggregate advertising expenditure and labor shares and the fraction of time in media, the rest of the 7 parameters are estimated jointly through a Simulated Method of Moments estimation procedure. Apart from the three moments described above concerning the average firm growth by age and the relationship between advertising and sales, at the aggregate level, I target the sales-weighted average markup, the aggregate advertising expenditure as a percentage of GDP, the firm entry rate, the fraction of time spent in media, and the labor share. Given that there is no physical capital in the model, for comparability, I take the labor share as the share of labor income among labor income and profits, following Cavenaile et al. (2024a). Table 1 summarises the results of the calibration. Panel A reports the parameter values, while Panel B reports both the model-implied moments and the empirical ones. The model does well in matching the moments. In addition to the targeted moments, the calibrated model also features an inverted-U relationship between advertising expenditure and relative sales as documented in Cavenaile et al. (2024a).

Note that Compustat is not the ideal dataset to discipline the growth pattern of firms in the model for the following reasons. First, age don't automatically enter Compustat when they are born, and they may enter at different stages of the life cycle. Second, contrary to the model, firms may grow through expanding to new geographical markets or new product lines. Figure 2 plots the average firm sales growth rate both in the model and in the data. Note that, in the model, if a firm had a constant $\rho_{j,i}$ (this is the case of a firm that has always been the single differentiated firm of the industry, which I could show as a case study), then growth would be monotonically decreasing, pushed by a mechanical force: given

Table 1: Parameter values and targeted moments

A. Parameters

	Parameter	Description	Calibration	Value
Preferences	β	Discount rate	Externally calibrated	0.98
	σ	CES consumption	Internally calibrated	6.8875
	υ	weight of leisure	Uncalibrated	-
Persuasive	ν_s	Scale parameter	Internally calibrated	0.2063
	ν_c	Convexity parameter	Externally calibrated	0.2972
Learning	ψ_s	Scale parameter	Internally calibrated	0.2204
	ψ_c	Convexity parameter	Internally calibrated	0.2650
	$\hat{\rho}$	Exogenous learning	Internally calibrated	0.0843
	δ	Mortality rate	Externally calibrated	0.01
Media sector	A	productivity media firms	Internally calibrated	2.8757
	$\bar{N}_m$	public sector media	Externally calibrated	0.0105
Generic good	A_0	Productivity	Internally calibrated	0.4688
Entry/Exit	κ	Exit rate	Externally calibrated	0.1151
	ϕ	Entry scale	Internally calibrated	0.4448

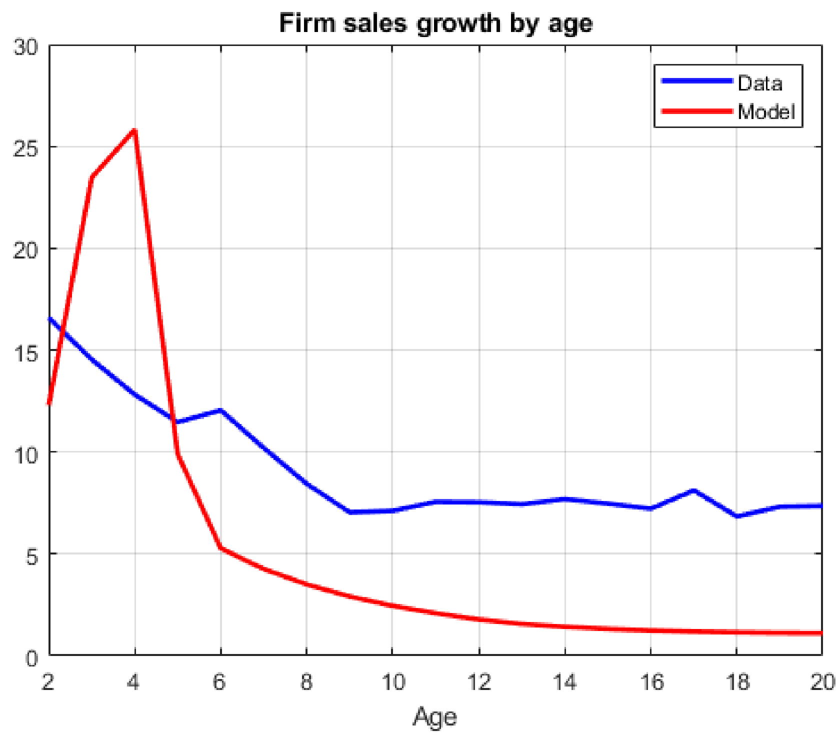
B. Moments

Moment	Data value	Model value
Sales-weighted average markup	1.3498	1.3560
Labor share (capital-adjusted)	0.8359	0.8375
Advertising/GDP	2.2%	2.1931%
Fraction of time in media	0.552	0.552
Entry rate	0.1151	0.1154
Average firm growth at age 1	16.5889%	12.3002
Slope firm growth by age	-0.8244	-0.8470
Coefficient advertising vs sales	0.6710	0.6710

Notes. Panel A reports the parameter values. Panel B reports the simulated and empirical moments. Details on data sources and how this moments are computed can be found in the Appendix 7.1

that the population is constant, as the firm's customer base expands, the growth rate slows because (i) a given increase in customers has a smaller relative impact, and (ii) there are fewer non-customers left. Things get noisier when there are other competitors, and there is entry and exit. Given that advertising tends to increase with size (as it is targeted), and size is positively related with age; then, older firms will tend to occupy a larger share of the ad space, which increases growth through two channels: increasing the probability non-customers learn about the good, and encouraging current customers to spend more due to the persuasive aspect of advertising. This explains why in the model we observe that average firm growth is first decreasing from 95% at age 1 (unreported for readability of the plot) to 12.3% at age 2 (as the mechanical force dominates because the firm is initially very small), then increasing (due to the positive relationship of advertising and size), and finally decreasing (as eventually the mechanical force dominates).

Figure 2: Average firm growth by age

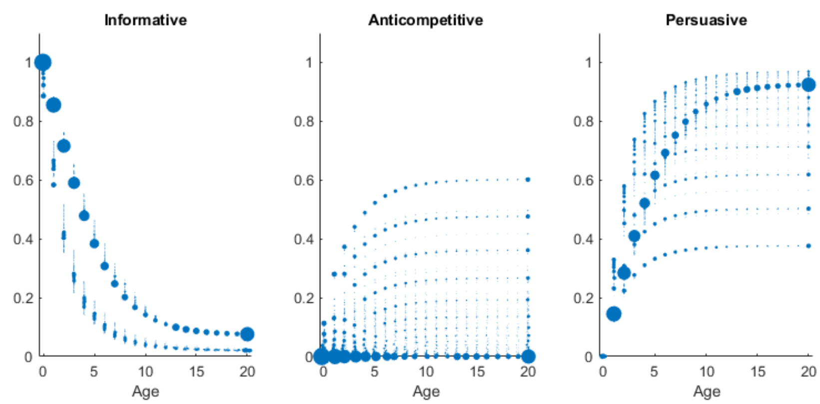


Notes. This figure displays the average firm sales growth by age both in the data (blue) and in the model (red). Given that firms typically enter Compustat a few years after their foundation, for comparability I assume age 1 in Compustat corresponds to age 2 in the model.

3.2 Relationship of the motives with firm age and size

The decomposition of the FOC of advertising expenditure in 17 allows us to see the share of the firm's marginal value of advertising coming from each of the three motives. Using this observation, Figure 3 displays the share of the marginal value of advertising attributable to each of the three motives for all the firms in the stationary equilibrium. Three observations can be drawn. First, that there is significant heterogeneity, which indicates that age is far from being a sufficient statistic. This shows that industry dynamics play a key role (i.e. competition matters). Second, despite the variability, it can be observed that the informative motive is negatively associated with age, while the anticompetitive and the persuasive motives are positively associated with it. Finally, another aspect that can be observed from the figure is that most of the action in terms of changes in the share of each motive seems to be concentrated in the first five years. This is connected to the concern

Figure 3: Marginal intensity of the advertising motives by age



Notes. This figure displays the values from the terms of the FOC corresponding to the informative motive (left panel), the anticompetitive motive (middle panel) and the persuasive motive (right panel), for all the firms in the stationary equilibrium, where the relative size of each dot indicates the share of this firm type in the stationary distribution.

raised in the calibration section regarding the targets for the evolution of the average firm growth by age, which implies that firms in the model grow too much in the early stages (with the alternative targeting strategy, firms should display a growing pattern more aligned with the data).

Because of the positive link between age and firm size (either sales or customer base), we obtain qualitatively similar plots when firm size is used instead of age. This numerical result is further supported by the following analytical result:

Proposition 1 *If distribution $\vec{M}_2$ is obtained from $\vec{M}_1$ by adding $\{j\}$ to the awareness sets of some consumers (that is, formally, if $\vec{M}_1$, $\vec{M}_2$ satisfy $M_2(\mathcal{I}\cup\{j\})-M_1(\mathcal{I}\cup\{j\}) = M_1(\mathcal{I})-M_2(\mathcal{I}) \geq 0$ for every $\mathcal{I} \in \mathcal{P}_{-j} = \{\mathcal{I} \in \mathcal{P} | j \notin \mathcal{I}\}$), then (for now, the proof is keeping the advertising choices fixed):*

1. *Firm j's informative motive is smaller in $\vec{M}_2$. That is, the informative motive is stronger in smaller firms.*

2. *Firm j's anticompetitive and persuasive motives are bigger in $\vec{M}_2$. That is, both the anticompetitive and the persuasive motives are stronger in bigger firms.*

Proof. See the Appendix. ■

3.3 Counterfactuals shutting the persuasive and/or the anticompetitive motives

How does each of the three motives affect the aggregates? Is the anticompetitive motive necessarily bad? This section addresses these questions. To do so, I compare the baseline economy with two counterfactual ones. The first one is an economy where firms neglect the anticompetitive motive; that is, they don't internalise that by increasing their advertising expenditure they are effectively reducing the amount of consumers who learn about competitors. To be precise, this is done by shutting down the anticompetitive component from the firm's first order condition. In the second counterfactual, in addition, firms also neglect the persuasive motive; that is, they don't internalise that advertising increases current customers' spending.

Note that this exercise does not indicate how much of the advertising in the baseline is anticompetitive, for example. For this, one should compare the advertising share in the baseline against the share that would result if firms made the advertising choices as in the first counterfactual, while keeping the stationary distribution of the baseline (I plan to do this decomposition exercise). Instead, the current exercise illustrates what the economy would look like if firms neglected some of the motives to advertise; that is, this negligence changes firms' decisions, but this in turn also has general equilibrium consequences.

Table 2 reports some relevant statistics for both counterfactuals and the benchmark. The second line shows output assuming that the persuasive advertising is deceiving (that is, consumers make their purchasing decisions based on $\omega_{j,i,t}$, but then they derive utility as if $\omega_{j,i,t} = 1$).

Table 2: Comparison of counterfactuals with firms neglecting the anticompetitive and/or the persuasive motives

	Benchmark	No Anticompetitive	Only Informative
Y	0.6950	0.6960	0.6947
Y no taste shifter	0.6440	0.6457	0.6533
Q	0.5546	0.5342	0.3429
Adv/GDP	2.2	2.0393	0.8306
w	0.8283	0.8274	0.8226
Entry rate	11.54	11.61	11.82
Sales-Weighted Average Markup	1.356	1.3546	1.335
Coefficient advertising vs market share	0.671	0.2281	-1.8217

Notes. In the 'No Anticompetitive' counterfactual, firms make their decisions neglecting the anticompetitive motive, and in the 'Only Informative' counterfactual, firms neglect both the anticompetitive and the persuasive motives.

On the one hand, shutting down the anticompetitive motive leads to a higher final output. As the intuition would suggest, without the anticompetitive motive, smaller firms face less competition for advertising space, and so they can grow faster, increasing competition and, thus, lowering markups. In addition, this improvement in growing prospects increases the incentives of entrepreneurs to create new products, pushing the entry rate up. On the negative side, removing one incentive to advertise decreases the demand for advertising, which reduces aggregate advertising expenditure and thus lowers the quality of media. Therefore, although the anticompetitive motive has negative effects for final output, it is not necessarily the case that welfare would be higher in a counterfactual economy without it, due to its contribution on the provision of media goods.

On the other hand, the counterfactual where, in addition, the persuasive motive is shut down, indicates that the persuasive motive overall has a positive effect on output. The second line shows that this positive effect arises from its impact on the taste shifters in preferences. Similar to the anticompetitive motive, shutting down the persuasive motive allows smaller firms to get a larger share of the advertising space, which pushes the entry rate up and markups down. Additionally, because firms are losing out on profits by not considering that advertising would lead to higher customer spending and higher market power, this further reduces markups but diminishes the increase in entry incentives.
Finally, the last row of the table provides further validation to the previous section result that both the anticompetitive and the persuasive motives are positively associated with size. As expected, as we shut down each of these two motives, since this reduces the incentives to advertise relatively more for bigger firms, the coefficient of the regression of advertising expenses on market share decreases, and becomes negative in the counterfactual economy with only the informative motive.

4 Social planner problem

The planner aims to maximize the aggregate utility (all individuals are weighted the same). The planner has full control on the production, media, and entrepreneurial decisions but cannot affect consumers' behaviour; that is, the learning process and the consumption and media time choices are as in the decentralised equilibrium. Formally, the planner solves:

$$\max_{\{N_{M,t},N_{j,i,t},h_{e,i,t},p_{j,i,t},\alpha_{j,i,t}\}} U=\sum_{t=0}^{\infty}\beta^t\int_0^1[\ln C_{\ell t}+L_{\ell t}]\,d\ell$$

$$\text{s.t. } C_{\ell,t} \text{ from (3)}, C_{\ell,i,t} \text{ from (4)}, c_{\ell,j,i,t} \text{ from (8), and } L_{\ell,t} \text{ from 2, with } T_t=Q_t \qquad \text{(Consumer choices)}$$

$$y_{j,i,t}=N_{j,i,t},\quad y_{0,i,t}=A_0N_{0,i,t},\quad Q_t=AN_{m,t}^{\varphi}\qquad\text{(Production functions)}$$

$$1=N_{m,t}+\int_0^1[N_{0,i,t}+\sum_{j\in\mathcal{J}_{i,t}}N_{j,i,t}]di\quad,\quad w_t+\int_0^1 h_{e,i,t}di=E_t\qquad\text{(Resource constraints)}$$

$$\sum_{j\in\mathcal{J}_{i,t}}\alpha_{j,i,t}=\alpha_i,(6),(7),(11),(12),(13)\qquad\text{(Learning process)}$$

$$z_{e,i,t}=\phi\left(\frac{h_{e,i,t}}{E_t}\right)^{\frac{1}{2}},(12),(13)\qquad\text{(Entry and exit)}$$

I leave the details of the solution in the Appendix 7.7. The planner sets prices equal to marginal cost times a markup (or a tax) that allows the planner to pay for the labor to produce the media goods and the entry costs. That is, $p_{j,i,t}=\tau\frac{w_t}{A_j}$, with $\tau=\frac{E_t}{w_tN_t^P}$, where N_t^P is the labor used in the production sector.
For the dynamic problem of advertising and media, as in the baseline model, I focus on the stationary Markov-Perfect equilibrium. The social planner has to decide on (i) how to allocate the ad space among the differentiated firms of each industry, $\alpha_{j,i,t}$, (ii) how much labor to allocate to the media sector, $N_{m,t}$, and (iii) how much final good to allocate to creating new products in each sector.
First, let's see the social planner choice of $\alpha_{j,i,t}$. The allocation of the ad space has to be such that the marginal social gain of increasing the ad space given to each firm is the same, since otherwise we could improve the allocation. Formally, letting $\ln C_{i,t}=\int_0^1\ln C_{\ell,i,t}d\ell$ be the total consumption good of industry i, and $U_X=\sum_{t=0}^{\infty}\beta^t\mathbb{E}\ln C_{i,t}$ be the expected life-time utility

derived from an industry whose current state is X; it must be $\frac{\partial \ln C_{X,t}}{\partial \alpha_{j,X}} + \beta \frac{\partial \mathbb{E} U_{X'}}{\partial \rho_{j,i}} \frac{\partial \rho_{j,i}}{\partial \alpha_{j,i}} = \hat{h}_X$ for some $\hat{h}_X$ and all $j \in \mathcal{J}_X$, together with $\sum_{j \in \mathcal{J}_X} \alpha_{j,X} = \alpha_X$. Note that the anticompetitive motive plays no role in the social planner's allocation of $\alpha_{j,X,t}$, as the planner directly chooses the ad space occupied by each firm. So, in deciding whether to give more ad space to one firm over another, the planner only considers the utility gains from informing more consumers and from enhancing customers' taste for that good.

Second, let's see the social planner choice of N_m. The planner takes into account that by employing more labor in media it will increase the aggregate quality Q of media, which has two effects: (i) it increases the level of entertainment L; and, by increasing the time spent in media, (ii) it increases the consumption good by increasing the probability of learning goods. The optimal N_m is given by

$$N_m = \frac{N_t^P}{2} \left(\upsilon Q^2 + \sum_{X \in \Omega} \mu(X) \hat{h}_X \alpha_X \right) \tag{21}$$

Note that, unlike the papers with a media sector cited in the related literature section, here the planner would value the provision of media goods even if their entertainment value was negligible (i.e. even if $\upsilon = 0$), due to their role as a vehicle for spreading product awareness. The labor market clearing condition imposes that $N_{m,t} + N_t^P = 1$.

Finally, the spending in entry in an industry whose current state is X is given by (where η is the Lagrange multiplier of the planner's budget constraint):

$$\frac{h_{e,X,t}}{E_t} = \left(\frac{\phi}{2} \frac{E_t}{w_t} \beta \left(\mathbb{E}_e U_{X'} - \mathbb{E}_{-e} U_{X'} \right) \right)^2 \tag{22}$$

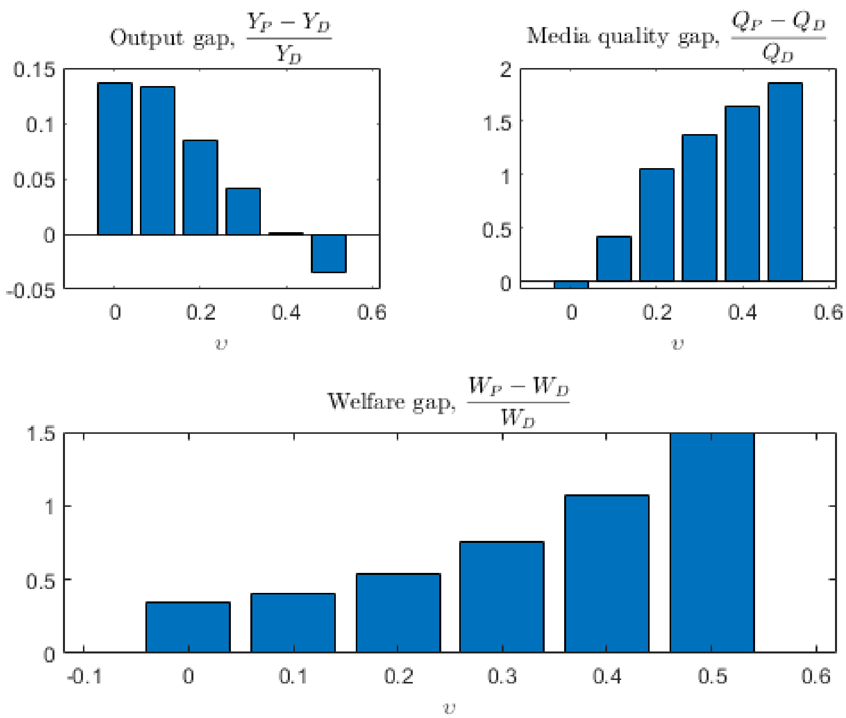
where $\mathbb{E}_e U_{X'}$ (resp. $\mathbb{E}_{-e} U_{X'}$) is the expected industry-utility conditional on successfully creating (resp. not creating) a new differentiated good (so, the expectation comes from the probabilities the incumbents exit). The relative wage is pinned down by using 22 into the budget constraint

$$\frac{w_t}{E_t} + \int_0^1 \frac{h_{e,i,t}}{E_t} di = 1$$

Figure 4 compares the planner economy with the decentralized one for different values of the relative utility weight of the entertainment good, υ, ranging from 0 to 0.5. Interestingly, when $\upsilon = 0$ (that is, when spending time on media doesn't provide any direct utility gain to consumers), we see that final output would be almost 15% higher under the planner's allocation, whereas media quality Q would be lower. The statement that the planner would choose a lower Q seems to indicate that when $\upsilon = 0$ there is too much advertising in the decentralised economy. However, this conclusion is inaccurate (as shown later in the taxation exercise in section 5) because we also need to consider how the advertising space is allocated among firms. In other words, the inefficiencies from the misallocation in the advertising space may be mitigated with the 'overprovision' of media, through its effect on learning.

As the importance of media goods in utility increases (i.e. as υ increases), the planner puts more weight on producing media goods, at the expense of final output, which eventually is lower in the planner's allocation than in the decentralised one. Welfare gains from implementing the planner's allocation are sizeable, and increasing with υ, reflecting the inefficiency arising from the fact that firms don't take into account consumers' utility gains from the media goods financed by advertising when making their advertising decisions. (to be done, a decomposition of the welfare gains)

Figure 4: Welfare comparison of the planner and decentralised allocations



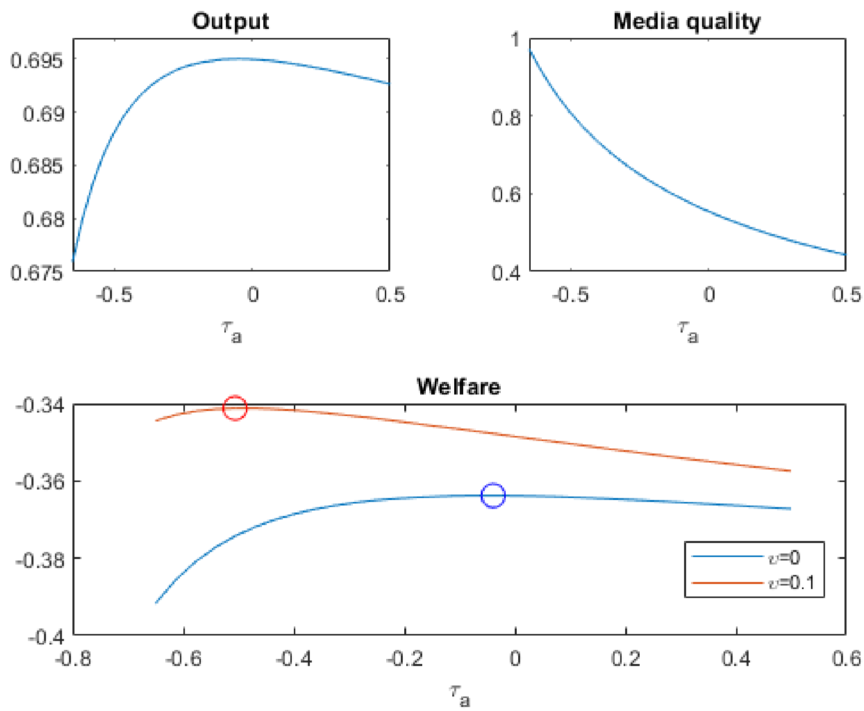
Notes. This figure displays the difference in final output (upper-left panel), in media time (upper-right panel), and welfare (bottom panel) between the planner's equilibrium and the decentralized one, relative to the decentralized one.

5 Taxing advertising

5.1 Uniform tax

In this section, I explore whether there is too much advertising, in which case we should tax it, or if, on the contrary, there is too little advertising and we should subsidize it. Here, in addition to $e_{j,i,t}$, firms pay $\tau_a e_{j,i,t}$ as taxes to the government, which are distributed as transfers to consumers. Figure 5 depicts the effect of this tax on final output, the quality of media, and welfare for different values of υ. As expected, the higher the entertainment value of media, the more valuable advertising becomes, leading to a lower optimal tax (i.e. higher subsidy). More interestingly, recall that we have seen that the decentralised equilibrium supplies more media than the planner's one for the case of $\upsilon = 0$, which seems to point to an overprovision of media. Actually, it turns out that there is too little advertising and the optimal tax is a subsidy. This confirms the hypothesis that this 'overprovision' of media, via increasing the time spent on media and thus the effectiveness of advertising, mitigates the inefficiencies from the misallocation of the advertising space.

Figure 5: Welfare under uniform tax on advertising



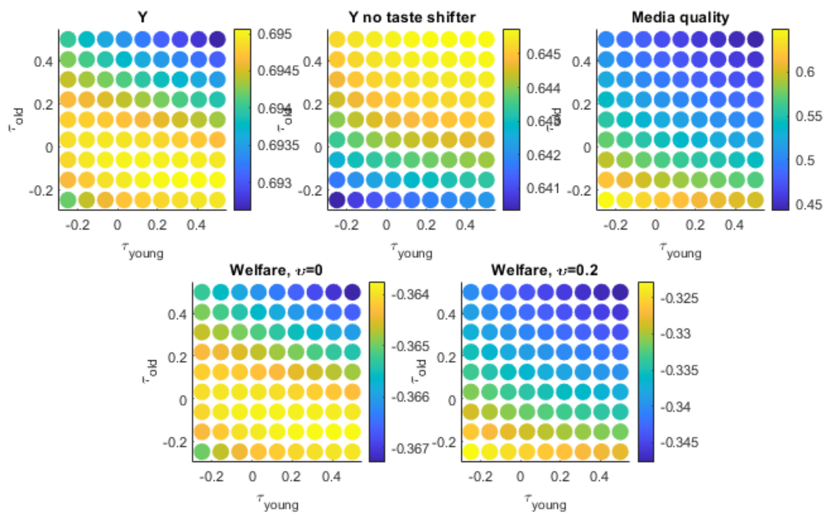
5.2 Age-dependent tax

Observation to make: note that a larger share of firms are older than the threshold, and so τ_O affects more firms.

The observation in section 3.2 that the informative motive is decreasing with age whereas the perusasive and anticompetitive motives are increasing on age, together with the fact that they have different welfare implications, leads us to think that we may have a significant welfare improvement by considering an age-dependent tax rather than applying the same tax for all firms. Assume now firms pay τ_Y if their age is smaller than the age cutoff $\bar{a}$, and τ_O if their age is greater or equal than $\bar{a}$. In particular, for this exercise I have set $\bar{a} = 3$; meaning that firms get a different tax treatment in their first three periods of life than afterwards. Note that this differential policy treatment makes the vector of ages of the firms to be an additional state. Note that firms that are at least $\bar{a}$ years old are indistinguishable by age (if all firms are older, then the firm problem is identical to the baseline with a uniform tax). However, for $a_j < \bar{a}$, we need to keep track of the particular age a_j (how close you are to $\bar{a}$ makes a difference). So, if $(a_1, \ldots, a_J)$ is the vector of ages (from older to younger), then the relevant ages state is $\vec{a} = (\hat{a}_1, \ldots, \hat{a}_J)$, where $\hat{a}_j = \min(a_j, \bar{a})$.
Figure 6 illustrates the effect of this age-dependent advertising tax on (i) output, (ii) output

if we assumed that the persuasive effect of advertising is deceiving, (iii) media quality, and (iv) welfare for two values of υ.

Figure 6: Welfare under age-dependent tax on advertising



Why is it preferable to tax advertising more for young firms than for old ones? The reason relies on the taste-shifter effect of advertising. Allocating more advertising space to young firms yields larger informative gains, but these come at the cost of reduced utility from a lower taste shifter for older firms, which affects many consumers due to their large customer base. In the calibrated model, the latter effect dominates. To support this explanation, I show in the second plot what would be the final output if we shut down this taste shifter effect (that is, assuming the persuasive effect of advertising is deceiving as in section 2). As expected, with this assumption, output would be improved by setting a higher tax for older firms (if it helps the visualisation, I could plot instead the difference between the output for $(\tau_Y = a, \tau_O = b)$ and the output for $(\tau_Y = b, \tau_O = a)$ for $a < b$, and we would see it is positive).

6 Concluding remarks

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7 Appendix

7.1 Calibration Appendix: Data sources and Computation of moments

1. **Sales-weighted average markup, labor share, entry rate, and aggregate advertising expenditure as a percentage of GDP.** Taken from Cavenaile et al. (2024a). Following Cavenaile et al. (2024a), given that there is no physical capital in the model, I target the labor share among labor income and profits. Given that $\frac{wL}{wL+\pi+rK} = \frac{wL}{wL+\pi}\frac{wL+\pi}{wL+\pi+rK} = \frac{wL}{wL+\pi}\left(1 - \frac{rK}{wL+\pi+rK}\right)$; then, the target used is obtained from dividing the labor share by one minus the capital share. In the model, given that labor supply is normalised to 1, then labor share equals w.
   The entry rate in the model is the average number of new firms (i.e. the average probability of creating a new product).

2. **Fraction of time in media.** According to Statista, people in the US spend on average 751 minutes per day in media, which corresponds to the 0.521528 of time.

3. **Coefficient of a regression of advertising expenditure on relative sales.** This and the growth by age moments are computed using Compustat data for the time period 1976-2018. Both in the model and in the data, I take the logarithm of advertising expenditure and then I standardise it by subtracting their means and dividing by their standard deviation for comparability. In the data, I regress the standardised logarithm of advertising expenses on relative sales of the firm in its SIC4 industry, controlling for the same set of controls used in Cavenaile et al., namely: profitability, leverage, market-to-book ratio, log R&D stock, firm age, the coefficient of variation of the firm's stock price, the number of firms in the industry, and a full set of year and SIC4 industry fixed effects. In the model, I regress the standardised logarithm of advertising expenses, $p_{a,i,t}e_{j,i,t}$, on market shares, $s_{j,i,t}$, with industry fixed effects. Table 3 shows the results of the empirical regression:

Table 3: Advertising and relative sales in the data

	log advertising expenses
Relative sales	0.671 (0.0448)***
R^2	0.6056
N	40,007

Notes. Robust asymptotic standard errors (in parenthesis) are clustered at the firm level. The sample period is from 1976 to 2018. The regression controls for profitability, leverage, market-to-book ratio, log R&D stock, firm age, the coefficient of variation of the firm's stock price, the number of firms in the industry, and a full set of year and SIC4 industry fixed effects.

4. **Average firm growth at age 1 and the slope of the fitted line.** In Compustat, I define age as the number of years since the first appearance of the firm in Compustat. Firms in the data may experience big jumps on sales through expansion to new markets or via mergers and acquisitions, and I am interested in the average evolution of firm growth in the absence of such disruptive events; therefore, I drop all the observations of a firm posterior to a big change in their sales. In particular, once a firm attains a growth rate bigger than 100% in absolute value, this observation and the posterior ones of this firm are dropped. Then, I take the average firm sales growth grouping all the observations with the same age. Given the average firm sales growth by age, $\bar{g}_a$, I define the fitted line $\hat{g}_a = \bar{g}_1 + \beta_g a$, where $\bar{g}_1$ is the actual average growth for firms aged 1 in the data (resp. in the model) for the regression using actual data (resp. model-implied data), and β_g is the coefficient that minimises the sum of square errors $\sum\limits_a (\bar{g}_a - \hat{g}_a)^2$.

5. **Calibration of the public sector financed media $\bar{N}_m$.** According to the US Government Accountability Office, the federal government spent \$14.9 billion over the last 10 fiscal years (2014-2023). Then, I use that federal governments spent roughly \$1.49 billion per year. In addition, federal appropriations for CPB (Corporation for Public Broadcasting) amounted to \$477 million in fiscal year 2023. So, the estimate I use for public sector spending on media is (\$1.49 + \$0.477) billion, which I divide for the US GDP in 2023, \$27360 billion.

7.2 Preferences

$$\max_{\{\{c_{\ell,j,i,t}\},a_{\ell,t+1},N_{\ell,t},T_{F,\ell,t},T_{\ell,i,t}\}} U_\ell = \sum_{t=0}^{\infty} \beta^t \left[\frac{C_{\ell t}^{1-\theta} - 1}{1-\theta} + L_{\ell t} \right]$$

$$\text{s.t. } C_{\ell,t} = \left(\int_0^1 C_{\ell,i,t}^{\frac{\chi-1}{\chi}} di \right)^{\frac{\chi}{\chi-1}}, \quad L_{\ell t} = \upsilon \left(Q_t T_{\ell,i,t} - \frac{T_{\ell,t}^2}{2} \right)$$

$$C_{\ell,i,t} = \left(c_{\ell,0,i,t}^{\frac{\sigma-1}{\sigma}} + \sum_{j \in \mathcal{I}_{\ell,i,t}} \omega_{\ell,j,i,t} c_{\ell,j,i,t}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

$$w_t N + r_t a_{\ell t} = \int_0^1 \sum_{j \in \mathcal{I}_{\ell,i,t}} c_{\ell,j,i,t} p_{j,i,t} di + a_{\ell,t+1} - a_{\ell,t}$$

(for the case $\theta = 1$, $\lim_{\theta \to 1} \frac{c^{1-\theta}}{1-\theta} = \lim_{\theta \to 1} \frac{c^{1-\theta}-1}{1-\theta} + \lim_{\theta \to 1} \frac{1}{1-\theta} = \ln c + \lim_{\theta \to 1} \frac{1}{1-\theta}$)
We can already plug $C_{\ell,t}$ into the objective function.
The FOCs read:

$$[c_{\ell jt}]: \quad \frac{\partial U_{\ell,t}}{\partial C_{\ell,t}} \frac{\partial C_{\ell,t}}{\partial C_{\ell,i,t}} \frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = \mu_{\ell,t} p_{j,i,t}$$

where $\frac{\partial U_{\ell,t}}{\partial C_{\ell,t}} = \beta^t C_{\ell,t}^{-\theta}$, $\frac{\partial C_{\ell,t}}{\partial C_{\ell,i,t}} = C_{\ell,t}^{\frac{1}{\chi}} C_{\ell,i,t}^{-\frac{1}{\chi}}$, and $\frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = C_{\ell,i,t}^{\frac{1}{\sigma}} c_{\ell,j,i,t}^{-\frac{1}{\sigma}} \omega_{\ell,j,i,t}$.
We can break down the FOC into three conditions, by defining $P_{\ell,i,t}$ as $P_{\ell,i,t} C_{\ell,i,t} = \sum_{j \in \mathcal{I}_{\ell,i,t}} c_{\ell,j,i,t} p_{j,i,t}$, and $P_{\ell,t}$ as $P_{\ell,t} C_{\ell,t} = \int_0^1 C_{\ell,i,t} P_{i,t} di$:

1. $\frac{\partial U_{\ell,t}}{\partial C_{\ell,t}}\frac{\partial C_{\ell,t}}{\partial c_{\ell,j,i,t}} = \mu_{\ell,t}\frac{\partial E_{\ell,t}}{\partial C_{\ell,t}}\frac{\partial C_{\ell,t}}{\partial c_{\ell,j,i,t}} \implies \left[\frac{\partial U_{\ell,t}}{\partial C_{\ell,t}} - \mu_{\ell,t}\frac{\partial E_{\ell,t}}{\partial C_{\ell,t}}\right]\frac{\partial C_{\ell,t}}{\partial c_{\ell,j,i,t}} = 0 \implies \beta^t C_{\ell,t}^{-\theta} = \mu_{\ell,t}P_{\ell,t}$

2. $\frac{\partial U_{\ell,t}}{\partial C_{\ell,t}}\frac{\partial C_{\ell,t}}{\partial C_{\ell,i,t}}\frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = \mu_{\ell,t}\frac{\partial E_{\ell,t}}{\partial C_{\ell,i,t}}\frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} \implies \left[\frac{\partial U_{\ell,t}}{\partial C_{\ell,t}}\frac{\partial C_{\ell,t}}{\partial C_{\ell,i,t}} - \mu_{\ell,t}\frac{\partial E_{\ell,t}}{\partial C_{\ell,i,t}}\right]\frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = 0$, where using from the previous condition that $\mu_{\ell,t} = \beta^t C_{\ell,t}^{-\theta}P_{\ell,t}^{-1}$, we get: $C_{\ell,t}^{\frac{1}{\chi}}C_{\ell,i,t}^{-\frac{1}{\chi}} = \frac{P_{\ell,i,t}}{P_{\ell,t}} \implies C_{\ell,i,t} = C_{\ell,t}\left(\frac{P_{\ell,t}}{P_{\ell,i,t}}\right)^{\chi}$, and plugging it into the definition of $C_{\ell,t}$, we get $P_{\ell,t} = \left(\int_0^1 P_{\ell,i,t}^{1-\chi}di\right)^{\frac{1}{1-\chi}}$. Note that if Cobb-Douglas (i.e. $\chi = 1$), then $E_{\ell,t} = P_{\ell,t}C_{\ell,t} = P_{\ell,i,t}C_{\ell,i,t}$.

3. $\frac{\partial U_{\ell,t}}{\partial C_{\ell,t}}\frac{\partial C_{\ell,t}}{\partial C_{\ell,i,t}}\frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = \mu_{\ell,t}\frac{\partial E_{\ell,t}}{\partial c_{\ell,j,i,t}} = \mu_{\ell,t}p_{j,i,t}$, where using from the previous conditions that $\mu_{\ell,t} = \beta^t C_{\ell,t}^{-\theta}P_{\ell,t}^{-1}$ and $C_{\ell,t}^{\frac{1}{\chi}}C_{\ell,i,t}^{-\frac{1}{\chi}} = \frac{P_{\ell,i,t}}{P_{\ell,t}}$, we get: $\frac{P_{\ell,i,t}}{P_{\ell,t}}C_{\ell,i,t}^{\frac{1}{\sigma}}c_{\ell,j,i,t}^{-\frac{1}{\sigma}}\omega_{\ell,j,i,t} = \frac{p_{j,i,t}}{P_{\ell,t}} \implies c_{\ell,j,i,t} = C_{\ell,i,t}\left(\frac{\omega_{\ell,j,i,t}P_{\ell,i,t}}{p_{j,i,t}}\right)^{\sigma}$, and plugging it into the definition of $C_{\ell,i,t}$, we get: $P_{\ell,i,t} = \left(p_{0,i,t}^{1-\sigma} + \sum_{j\in\mathcal{I}_{\ell,i,t}}\omega_{\ell,j,i,t}^{\sigma}p_{j,i,t}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$

The FOC for assets is:

$$[a_{\ell,t+1}]: \qquad \mu_{\ell t} = (1 + r_{t+1})\mu_{\ell,t+1}$$

From the first one, using that $\mu_{\ell,t} = \beta^t C_{\ell,t}^{-\theta}P_{\ell,t}^{-1}$, we get the Euler equation: $\beta^t C_{\ell,t}^{-\theta}P_{\ell,t}^{-1} = (1 + r_{t+1})\beta^{t+1}C_{\ell,t+1}^{-\theta}P_{\ell,t+1}^{-1}$, which assuming $\theta = 1$ (i.e. logarithmic preferences on $C_{\ell,t}$), then the expenditure choice is independent of the price indices (so, the awareness set just affects the intratemporal allocation of expenditure).

So, assuming $\chi = \theta = 1$, we have:

$$c_{\ell,j,i,t} = E_{\ell,t}P_{\ell,i,t}^{\sigma-1}p_{j,i,t}^{-\sigma}\omega_{\ell,j,i,t}^{\sigma}$$
$$\frac{E_{\ell,t+1}}{E_{\ell,t}} = \beta(1 + r_{t+1})$$

where E_ℓ is the expenditure of individual ℓ. So, the growth of expenditure is symmetric for all individuals (and the level is also identical if all individuals start with the same level of assets).

Since the individual is characterised by the awareness set, from now on I use the subindex $\mathcal{I}$, instead of ℓ. The share of expenditure of each consumer on each good they know is: $s_{\mathcal{I},j} = \frac{p_j c_{\mathcal{I},j}}{E} = P_{\mathcal{I}}^{\sigma-1}p_j^{1-\sigma}\omega_j^{\sigma} = p_j^{1-\sigma}\omega_j^{\sigma}\left[p_{0,t}^{1-\sigma} + \sum_{k\in\mathcal{I}_\ell}\omega_{\ell,k}^{\sigma}p_k^{1-\sigma}\right]^{-1} = \left[\left(\frac{p_{0,t}}{p_j}\right)^{1-\sigma}\left(\frac{1}{\omega_j}\right)^{\sigma} + \sum_{k\in\mathcal{I}_\ell}\left(\frac{p_{k,t}}{p_j}\right)^{1-\sigma}\left(\frac{\omega_k}{\omega_j}\right)^{\sigma}\right]^{-1}$

So, using the definition of markup $\mathcal{M}_j = \frac{p_j A_j}{w}$:

$$s_{\mathcal{I},j} = \left[\left(\frac{1}{\mathcal{M}_j A_0}\right)^{1-\sigma}\left(\frac{1}{\omega_j}\right)^{\sigma} + \sum_{k\in\mathcal{I}_\ell}\left(\frac{\mathcal{M}_k A_j}{\mathcal{M}_j A_k}\right)^{1-\sigma}\left(\frac{\omega_k}{\omega_j}\right)^{\sigma}\right]^{-1}$$

Next, the choice of media time is straightforward: $\frac{\partial L_{\ell,t}}{\partial T_{\ell,t}} = \upsilon\left(Q_t - T_{\ell,t}\right)$, so $T_t = Q_t$. And so, optimal leisure as a function of Q is: $L_t^* = \upsilon\frac{Q_t^2}{2}$.

7.3 Proof that $\delta = 0$ allows a simpler state

Setting $\delta = 0$ allows a simpler sufficient state:

If individuals don't die (or forget), then, given that learning is independent for each good, we have that a sufficient information that allows us to identify the industry state is the mass of consumers that are aware of good j for each $j \in \mathcal{J}_{i,t}$. Intuitively, the reason why $\delta > 0$ doesn't allow this simplification is that the fact that consumers die with positive probability breaks the independence of the events of being aware of a particular good. That is: given that the older the consumer the more likely she is aware of the good, then, knowing that the consumer is aware of a good allows us to get a better guess of the consumer's age.

Proposition 2 *If $\delta = 0$ and $M_{j,it}$ is the mass of consumers aware of good j, then the distribution of consumers over the awareness sets is given by* $M_{i,t}(\mathcal{I}) = \prod_{j\in\mathcal{I}} M_{j,it} \prod_{j\notin\mathcal{I}}(1-M_{j,it})$

Proof. By induction on t. Set $t = 0$ as the first period a differentiated firm enters. Then, it is trivially satisfied for $t = 0$. We'll see that if it is true for $t-1$ that $M_{i,t-1}(\mathcal{I}) = \prod_{j\in\mathcal{I}} M_{j,it-1} \prod_{j\notin\mathcal{I}}(1-M_{j,it-1})$, then it is true for t that $M_{i,t}(\mathcal{I}) = \prod_{j\in\mathcal{I}} M_{j,it} \prod_{j\notin\mathcal{I}}(1-M_{j,it})$:
By the law of motion of consumers:

$$M_{i,t}(\mathcal{I}) = (1-\delta)\sum_{\mathcal{I}'\subseteq\mathcal{I}} M_{i,t-1}(\mathcal{I}') \prod_{k\in\mathcal{I}\setminus\mathcal{I}'} \rho_{k,i,t-1} \prod_{k\notin\mathcal{I}}(1-\rho_{k,i,t-1})$$

Using the induction hypothesis:

$$M_{i,t}(\mathcal{I}) = (1-\delta)\sum_{\mathcal{I}'\subseteq\mathcal{I}} \prod_{j\in\mathcal{I}'} M_{j,it-1} \prod_{j\notin\mathcal{I}'}(1-M_{j,it-1}) \prod_{k\in\mathcal{I}\setminus\mathcal{I}'} \rho_{k,i,t-1} \prod_{k\notin\mathcal{I}}(1-\rho_{k,i,t-1})$$

Note that $\prod_{j\notin\mathcal{I}'}(1-M_{j,it-1}) = \prod_{j\in\mathcal{I}\setminus\mathcal{I}'}(1-M_{j,it-1}) \prod_{j\notin\mathcal{I}}(1-M_{j,it-1})$; so, we can write it as:

$$M_{i,t}(\mathcal{I}) = (1-\delta)\sum_{\mathcal{I}'\subseteq\mathcal{I}} \prod_{j\in\mathcal{I}'} M_{j,it-1} \prod_{k\in\mathcal{I}\setminus\mathcal{I}'} [(1-M_{k,it-1})\rho_{k,i,t-1}] \prod_{k\notin\mathcal{I}}[(1-M_{k,it-1})(1-\rho_{k,i,t-1})]$$

And using that it holds $\prod_{j\in\mathcal{J}}(a_j+b_j) = \sum_{\mathcal{I}\subseteq\mathcal{J}} \prod_{j\in\mathcal{I}} a_j \prod_{j\notin\mathcal{I}} b_j$; then, we have:

$$M_{i,t}(\mathcal{I}) = (1-\delta)\prod_{j\in\mathcal{I}} [M_{j,it-1} + (1-M_{j,it-1})\rho_{j,i,t-1}] \prod_{k\notin\mathcal{I}}[(1-M_{k,it-1})(1-\rho_{k,i,t-1})]$$

On the other hand, the law of motion of $M_{k,it}$ is:
$M_{k,it} = (1-\delta)M_{k,it-1} + (1-\delta)(1-M_{k,it-1})\rho_{k,i,t-1} \implies (1-M_{k,it-1})(1-\rho_{k,i,t-1}) = 1 - \frac{M_{k,it}}{1-\delta}$.
So, if $\delta = 0$, we have, as wanted:

$$M_{i,t}(\mathcal{I}) = \prod_{j\in\mathcal{I}} M_{j,it} \prod_{j\notin\mathcal{I}}(1-M_{j,it})$$

■

7.4 Production Firms

7.4.1 Derivatives of profits and expenditure shares

1. $\pi_{j,i} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})s_{\mathcal{I},j,i}$

   (a) $\frac{\partial \pi_{j,i}}{\partial e_{k,i}} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial s_{\mathcal{I},j,i}}{\partial e_{k,i}}$, $k\in\mathcal{J}_i$

   (b) $\frac{\partial \pi_{j,i}}{\partial \mathcal{M}_{j,i}} = \frac{s_{j,i}}{\mathcal{M}_{j,i}^2} + (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}}$

   (c) $\frac{\partial \pi_{j,i}}{\partial \mathcal{M}_{k,i}} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{k,i}}$, $k\neq j$

   (d) $\frac{\partial^2 \pi_{j,i}}{\partial \mathcal{M}_{j,i}^2} = -2\frac{s_{j,i}}{\mathcal{M}_{j,i}^3} + 2\frac{1}{\mathcal{M}_{j,i}^2}\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}} + (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial^2 s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}^2}$
   Using that the FOC says that $-\frac{s_j}{\mathcal{M}_j^2} = (1-\mathcal{M}_j^{-1})\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}}$ and the expression of $\frac{\partial^2 s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}^2}$ (in the next point), we have:
   $\frac{\partial^2 \pi_{j,i}}{\partial \mathcal{M}_{j,i}^2} = 2\frac{1}{\mathcal{M}_{j,i}}(1-\mathcal{M}_j^{-1})\frac{\partial s_{j,i}}{\partial \mathcal{M}_{j,i}} + 2\frac{1}{\mathcal{M}_{j,i}^2}\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}} + (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial^2 s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}^2}$
   $\frac{\partial^2 \pi_{j,i}}{\partial \mathcal{M}_{j,i}^2} = \sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}}\frac{1}{\mathcal{M}_{j,i}}\left[2\left((1-\mathcal{M}_{j,i}^{-1}) + \frac{1}{\mathcal{M}_{j,i}}\right) - (1-\mathcal{M}_{j,i}^{-1})\left[1+(\sigma-1)(1-2s_{\mathcal{I},j,i})\right]\right]$
   $\frac{\partial^2 \pi_{j,i}}{\partial \mathcal{M}_{j,i}^2} = \sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}}\frac{1}{\mathcal{M}_{j,i}}\left[2 - (1-\mathcal{M}_{j,i}^{-1})\left[1+(\sigma-1)(1-2s_{\mathcal{I},j,i})\right]\right]$
   And substituting $\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}}$, we have:
   $\frac{\partial^2 \pi_{j,i}}{\partial \mathcal{M}_{j,i}^2} = -\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{s_{\mathcal{I},j,i}}{\mathcal{M}_{j,i}}\frac{(1-s_{\mathcal{I},j,i})}{\mathcal{M}_{j,i}}(\sigma-1)\left[2 - (1-\mathcal{M}_{j,i}^{-1})\left[1+(\sigma-1)(1-2s_{\mathcal{I},j,i})\right]\right]$,
   and we can rearrange the term in the brackets:
   $\frac{\partial^2 \pi_{j,i}}{\partial \mathcal{M}_{j,i}^2} = -\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{s_{\mathcal{I},j,i}}{\mathcal{M}_{j,i}}\frac{(1-s_{\mathcal{I},j,i})}{\mathcal{M}_{j,i}}(\sigma-1)\left[1+\frac{1}{\mathcal{M}_{j,i}} - (1-\mathcal{M}_{j,i}^{-1})(\sigma-1)(1-2s_{\mathcal{I},j,i})\right]$

   (e) $\frac{\partial^2 \pi_{j,i}}{\partial \mathcal{M}_{k,i}\partial \mathcal{M}_{j,i}} = \frac{1}{\mathcal{M}_{j,i}^2}\frac{\partial s_{j,i}}{\partial \mathcal{M}_{k,i}} + (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\partial^2 s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{k,i}\partial \mathcal{M}_{j,i}}$, and substituting $\frac{\partial^2 s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{k,i}\partial \mathcal{M}_{j,i}}$, we have:
   $\frac{\partial^2 \pi_{j,i}}{\partial \mathcal{M}_{k,i}\partial \mathcal{M}_{j,i}} = \sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{1}{\mathcal{M}_{j,i}}\frac{\partial s_{j,i}}{\partial \mathcal{M}_{k,i}}\left[\frac{1}{\mathcal{M}_{j,i}} - (1-\mathcal{M}_{j,i}^{-1})(\sigma-1)(1-2_{\mathcal{I},j,i})\right]$, and substituting $\frac{\partial s_{j,i}}{\partial \mathcal{M}_{k,i}}$, we have:
   $\frac{\partial^2 \pi_{j,i}}{\partial \mathcal{M}_{k,i}\partial \mathcal{M}_{j,i}} = \sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{s_{\mathcal{I},j,i}}{\mathcal{M}_{j,i}}\frac{s_{\mathcal{I},k,i}}{\mathcal{M}_{k,i}}(\sigma-1)\left[\frac{1}{\mathcal{M}_{j,i}} - (1-\mathcal{M}_{j,i}^{-1})(\sigma-1)(1-2_{\mathcal{I},j,i})\right]$

2. $s_{\mathcal{I},j,i} = \frac{p_{j,i}c_{\mathcal{I},j,i}}{E} = \frac{p_{j,i}^{1-\sigma}\omega_{j,i}^{\sigma}}{P_{\mathcal{I},i}^{1-\sigma}} = \frac{p_{j,i}^{1-\sigma}\omega_{j,i}^{\sigma}}{p_{0,i}^{1-\sigma}+\sum_{k\in\mathcal{I}}\omega_{k,i}^{\sigma}p_{k,i}^{1-\sigma}} = \frac{\mathcal{M}_{j,i}^{1-\sigma}\omega_{j,i}^{\sigma}}{(\frac{\mathcal{M}_{0,i}}{A_0})^{1-\sigma}+\sum\limits_{k\in\mathcal{I}}\omega_{k,i}^{\sigma}\mathcal{M}_{k,i}^{1-\sigma}}$, and $s_{\mathcal{I},j,i}=0$ if $j\notin\mathcal{I}$. So:[13]

   (a) $\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{j,i}} = s_{\mathcal{I},j,i}(1-s_{\mathcal{I},j,i})\frac{\sigma}{\omega_{j,i}}$

   (b) $\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{k,i}} = -s_{\mathcal{I},j,i}s_{\mathcal{I},k,i}\frac{\sigma}{\omega_{k,i}}$

   (c) $\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}} = -s_{\mathcal{I},j,i}(1-s_{\mathcal{I},j,i})\frac{\sigma-1}{\mathcal{M}_{j,i}} = -\frac{\sigma}{\sigma-1}\frac{\mathcal{M}_{j,i}}{\omega_{j,i}}\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{j,i}}$

   (d) $\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{k,i}} = s_{\mathcal{I},j,i}s_{\mathcal{I},k,i}\frac{\sigma-1}{\mathcal{M}_{k,i}} = -\frac{\sigma}{\sigma-1}\frac{\mathcal{M}_{k,i}}{\omega_{k,i}}\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{k,i}}$

   (e) $\frac{\partial^2 s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}^2} = -\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}}\frac{1}{\mathcal{M}_{j,i}}\left[1+(\sigma-1)(1-2s_{\mathcal{I},j,i})\right]$

[13] Also, in terms of relative consumption: $s_{\mathcal{I},j,i} = \frac{p_{j,i}c_{\mathcal{I},j,i}}{E} = \frac{c_{\mathcal{I},j,i}^{\frac{\sigma-1}{\sigma}}\omega_j P_{\mathcal{I},j,i}^{\frac{\sigma-1}{\sigma}}}{E_{\mathcal{I},j,i}^{\frac{\sigma-1}{\sigma}}} = \frac{c_{\mathcal{I},j,i}^{\frac{\sigma-1}{\sigma}}\omega_j}{Y_{\mathcal{I},j,i}^{\frac{\sigma-1}{\sigma}}}$

(f) $\frac{\partial^2 s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{k,i}\partial \mathcal{M}_{j,i}} = -\frac{1}{\mathcal{M}_{j,i}}\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{k,i}}(\sigma-1)(1-2s_{\mathcal{I},j,i})$

3. $\omega_{j,i} = 1+\nu_s(\alpha_{j,i}T)^{\nu_c}$, $\alpha_{j,i} = \frac{e_{j,i}}{p_{a,i}}$, where $p_{a,i} = \frac{\sum_k e_{k,i}}{\alpha_i}$ if limited ad space is binding, otherwise $p_{a,i} = \bar{p}_{a,i}$.

   (a) $\frac{\partial \omega_{j,i}}{\partial e_{j,i}} = \nu_c\nu_s T^{\nu_c}\alpha_{j,i}^{\nu_c-1}\frac{\partial \alpha_{j,i}}{\partial e_{j,i}}$

   (b) $\frac{\partial \omega_{k,i}}{\partial e_{j,i}} = \nu_c\nu_s T^{\nu_c}\alpha_{k,i}^{\nu_c-1}\frac{\partial \alpha_{k,i}}{\partial e_{j,i}}$

   (c) $\frac{\partial \alpha_{k,i}}{\partial e_{j,i}} = -\frac{\alpha_{k,i}}{\sum_k e_{k,i}}$ if the limited ad space is binding, otherwise $\frac{\partial \alpha_{k,i}}{\partial e_{j,i}} = 0$.

   (d) $\frac{\partial \alpha_{j,i}}{\partial e_{j,i}} = \frac{\alpha_i}{\sum_k e_{k,i}} - \frac{\alpha_{j,i}}{\sum_k e_{k,i}}$ if the limited ad space is binding, otherwise $\frac{\partial \alpha_{j,i}}{\partial e_{j,i}} = \frac{1}{p_{a,i}}$.

So: $\frac{\partial \pi_{j,i}}{\partial e_{j,i}} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\sum_{k\in\mathcal{I}}\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{k,i}}\frac{\partial \omega_{k,i}}{\partial e_{j,i}}$. And using the above expressions we have:

- for $k \neq j$: $\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{k,i}}\frac{\partial \omega_{k,i}}{\partial e_{j,i}} = -s_{\mathcal{I},j,i}s_{\mathcal{I},k,i}\frac{\sigma}{\omega_{k,i}}\nu_c\nu_s T^{\nu_c}\alpha_{k,i}^{\nu_c-1}\frac{\partial \alpha_{k,i}}{\partial e_{j,i}}$.
- $\frac{\partial s_{\mathcal{I},j,i}}{\partial \omega_{j,i}}\frac{\partial \omega_{j,i}}{\partial e_{j,i}} = s_{\mathcal{I},j,i}(1-s_{\mathcal{I},j,i})\frac{\sigma}{\omega_{j,i}}\nu_c\nu_s T^{\nu_c}\alpha_{j,i}^{\nu_c-1}\frac{\partial \alpha_{j,i}}{\partial e_{j,i}}$.

So: $\frac{\partial \pi_{j,i}}{\partial e_{j,i}} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\sigma\nu_s\nu_c T^{\nu_c}s_{\mathcal{I},j,i}\left[\sum_{k\in\mathcal{I}\setminus\{j\}}\frac{s_{\mathcal{I},k,i}\alpha_{k,i}^{\nu_c-1}}{\omega_{k,i}}\left(-\frac{\partial \alpha_{k,i}}{\partial e_{j,i}}\right)+\frac{(1-s_{\mathcal{I},j,i})\alpha_{j,i}^{\nu_c-1}}{\omega_{j,i}}\frac{\partial \alpha_{j,i}}{\partial e_{j,i}}\right]$

And substituting $\frac{\partial \alpha_{k,i}}{\partial e_{j,i}}$:

1. With binding limited ad space (i.e. $J>1$):
   $\frac{\partial \pi_{j,i}}{\partial e_{j,i}} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\sigma\nu_s\nu_c}{\sum_k e_{k,i}}T^{\nu_c}s_{\mathcal{I},j,i}\left[\sum_{k\in\mathcal{I}\setminus\{j\}}\frac{s_{\mathcal{I},k,i}\alpha_{k,i}^{\nu_c}}{\omega_{k,i}}+\frac{(1-s_{\mathcal{I},j,i})\alpha_{j,i}^{\nu_c-1}}{\omega_{j,i}}(\alpha_i-\alpha_{j,i})\right]$
   Equivalently:
   $\frac{\partial \pi_{j,i}}{\partial e_{j,i}} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\sigma\nu_s\nu_c}{\sum_k e_{k,i}}T^{\nu_c}s_{\mathcal{I},j,i}\left[\sum_{k\in\mathcal{I}}\frac{s_{\mathcal{I},k,i}\alpha_{k,i}^{\nu_c}}{\omega_{k,i}}+\frac{\alpha_{j,i}^{\nu_c-1}}{\omega_{j,i}}(\alpha_i(1-s_{\mathcal{I},j,i})-\alpha_{j,i})\right]$
   Or equivalently:
   $\frac{\partial \pi_{j,i}}{\partial e_{j,i}} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\frac{\sigma\nu_c}{\sum_k e_{k,i}}s_{\mathcal{I},j,i}\left[\sum_{k\in\mathcal{I}}\frac{s_{\mathcal{I},k,i}\omega_{k,i}}{\omega_{k,i}}+\frac{\omega_{j,i}}{\omega_{j,i}}\left(\alpha_i\frac{1-s_{\mathcal{I},j,i}}{\alpha_{j,i}}-1\right)\right]$

2. With non-binding ad space (i.e. $J=1$):
   $\frac{\partial \pi_{j,i}}{\partial e_{j,i}} = (1-\mathcal{M}_{j,i}^{-1})\sum_{\mathcal{I}\in\mathcal{P}_{j,i}} M_i(\mathcal{I})\sigma\nu_s\nu_c T^{\nu_c}s_{\mathcal{I},j,i}\frac{(1-s_{\mathcal{I},j,i})\alpha_{j,i}^{\nu_c-1}}{\omega_{j,i}p_{a,i}}$

7.4.2 The effect of learning about another good on $s_{\mathcal{I},j,i}$ and the demand elasticity

Proposition 3 *If $j\in\mathcal{I}\subset\mathcal{I}'$, then:*

1. $s_{\mathcal{I},j,i} > s_{\mathcal{I}',j,i}$
2. $|\epsilon_{\mathcal{I},j,i}| < |\epsilon_{\mathcal{I}',j,i}|$, *where* $\epsilon_{\mathcal{I},j,i} = \frac{p_{j,i}}{c_{\mathcal{I},j,i}}\frac{\partial c_{\mathcal{I},j,i}}{\partial p_{j,i}}$

Proof. If $j \in \mathcal{I} \subset \mathcal{I}'$, then, since $\omega_{k,i}, \mathcal{M}_{k,i} > 0$ for all firm k and $\sigma > 1$:

$$s_{\mathcal{I},j,i} = \left[(A_0 \mathcal{M}_{j,i})^{\sigma-1} \omega_{j,i}^{-\sigma} + \sum_{k \in \mathcal{I}} \left(\frac{\omega_{k,i}}{\omega_{j,i}} \right)^{\sigma} \left(\frac{\mathcal{M}_{j,i}}{\mathcal{M}_{k,i}} \right)^{\sigma-1} \right]^{-1}$$
$$> \left[(A_0 \mathcal{M}_{j,i})^{\sigma-1} \omega_{j,i}^{-\sigma} + \sum_{k \in \mathcal{I}} \left(\frac{\omega_{k,i}}{\omega_{j,i}} \right)^{\sigma} \left(\frac{\mathcal{M}_{j,i}}{\mathcal{M}_{k,i}} \right)^{\sigma-1} + \sum_{k \in \mathcal{I}' \setminus \mathcal{I}} \left(\frac{\omega_{k,i}}{\omega_{j,i}} \right)^{\sigma} \left(\frac{\mathcal{M}_{j,i}}{\mathcal{M}_{k,i}} \right)^{\sigma-1} \right]^{-1} = s_{\mathcal{I}',j,i}$$

For 2, define $\epsilon_{\mathcal{I},j,i} = \frac{p_{j,i}}{c_{\mathcal{I},j,i}} \frac{\partial c_{\mathcal{I},j,i}}{\partial p_{j,i}}$, and note that $\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}} = \frac{\partial s_{\mathcal{I},j,i}}{\partial p_{j,i}} \frac{\partial p_{j,i}}{\partial \mathcal{M}_{j,i}} = \frac{1}{E} \left[c_{\mathcal{I},j,i} + p_{j,i} \frac{\partial c_{\mathcal{I},j,i}}{\partial p_{j,i}} \right] w = \frac{s_{\mathcal{I},j,i}}{\mathcal{M}_{j,i}} (1 + \epsilon_{\mathcal{I},j,i})$. And using the expression for $\frac{\partial s_{\mathcal{I},j,i}}{\partial \mathcal{M}_{j,i}}$ in 7.4.1, we have:

$$-\frac{s_{\mathcal{I},j,i}}{\mathcal{M}_{j,i}} (\sigma - 1)(1 - s_{\mathcal{I},j,i}) = \frac{s_{\mathcal{I},j,i}}{\mathcal{M}_{j,i}} (1 + \epsilon_{\mathcal{I},j,i}) \implies \epsilon_{\mathcal{I},j,i} = -\sigma + s_{\mathcal{I},j,i}(\sigma - 1)$$

So, using 1, we have $j \in \mathcal{I} \subset \mathcal{I}'$ $0 > \epsilon_{\mathcal{I},j,i} > \epsilon_{\mathcal{I}',j,i}$ ■

7.4.3 Derivation of optimal markup:

$0 = \frac{\partial \pi_{j,i}}{\partial p_{j,i}} = y_{j,i} + \left(p_{j,i} - \frac{\partial w N_{j,i}}{\partial N_{j,i}} \frac{\partial N_{j,i}}{\partial y_{j,i}} \right) \sum_{\mathcal{I} \in \mathcal{P}_{j,i}} M_i(\mathcal{I}) \frac{\partial c_{\mathcal{I},j,i}}{\partial p_{j,i}}$

On the one hand: $\frac{\partial c_{\mathcal{I},j,i}}{\partial p_{j,i}} = c_{\mathcal{I},j,i} \left[-\sigma p_{j,i}^{-1} + (\sigma - 1) P_{\mathcal{I},i}^{-1} \frac{\partial P_{\mathcal{I},i}}{\partial p_{j,i}} \right] = \frac{c_{\mathcal{I},j,i}}{p_{j,i}} \left[-\sigma + (\sigma - 1) \frac{p_{j,i}}{P_{\mathcal{I},i}} \frac{\partial P_{\mathcal{I},i}}{\partial p_{j,i}} \right]$.

And $\frac{\partial P_{\mathcal{I},i}}{\partial p_{j,i}} = P_{\mathcal{I},i}^{\sigma} p_{j,i}^{-\sigma} \omega_{j,i}^{\sigma}$. So, we have:

$0 = y_{j,i} + (p_{j,i} - w) \sum_{\mathcal{I} \in \mathcal{P}_{j,i}} M_i(\mathcal{I}) \frac{c_{\mathcal{I},j,i}}{p_{j,i}} \left[-\sigma + (\sigma - 1) \left(\frac{P_{\mathcal{I},i}}{p_{j,i}} \right)^{\sigma-1} \omega_{j,i}^{\sigma} \right]$

Since $s_{\mathcal{I},j,i} = \left(\frac{P_{\mathcal{I},i}}{p_{j,i}} \right)^{\sigma-1} \omega_{j,i}^{\sigma}$ and multiplying by $\frac{p_{j,i}}{E}$:

$0 = s_{j,i} + \left(1 - \mathcal{M}_{j,i}^{-1} \right) \sum_{\mathcal{I} \in \mathcal{P}_{j,i}} M_i(\mathcal{I}) s_{\mathcal{I},j,i} \left[-\sigma + (\sigma - 1) s_{\mathcal{I},j,i} \right]$

Equivalently, we can write it:

$0 = 1 + \left(1 - \frac{1}{\mathcal{M}_{j,i}} \right) \sum_{\mathcal{I} \in \mathcal{P}_{j,i}} \frac{M_i(\mathcal{I}) c_{\mathcal{I},j,i}}{y_{j,i}} \left[-\sigma + (\sigma - 1) s_{\mathcal{I},j,i} \right]$

And using that $\sum_{\mathcal{I} \in \mathcal{P}_{j,i}} \frac{M_i(\mathcal{I}) c_{\mathcal{I},j,i}}{y_{j,i}} = 1$ and defining $\bar{s}_{j,i} = \sum_{\mathcal{I} \in \mathcal{P}_{j,i}} \frac{M_i(\mathcal{I}) c_{\mathcal{I},j,i}}{y_{j,i}} s_{\mathcal{I},j,i}$:

$1 = \left(1 - \frac{1}{\mathcal{M}_{j,i}} \right) \left[\sigma - (\sigma - 1) \bar{s}_{j,i} \right] \implies \left[\sigma - (\sigma - 1) \bar{s}_{j,i} \right]^{-1} = 1 - \frac{1}{\mathcal{M}_{j,i}}$

Rearranging:

$\frac{1}{\mathcal{M}_{j,i}} = \frac{\sigma - 1 - (\sigma - 1) \bar{s}_{j,i}}{\sigma - (\sigma - 1) \bar{s}_{j,i}}$

7.5 Derivation of the expression for the FOC:

Derivative of $\rho_{j,i}$ with respect to $e_{k,i}$. $\frac{\partial \rho_k}{\partial e_j} = \psi_c \psi_s T^{\psi_c} \alpha_k^{\psi_c - 1} \frac{\partial \alpha_k}{\partial e_j}$

- If $J = 1$ (ad space not binding), then $\frac{\partial \alpha_k}{\partial e_j} = 0$, so $\frac{\partial \rho_k}{\partial e_j} = 0$; and $\frac{\partial \alpha_j}{\partial e_j} = \frac{1}{p_a}$, so: $\frac{\partial \rho_j}{\partial e_j} = \psi_c \psi_s T^{\psi_c} \alpha_j^{\psi_c - 1} \frac{1}{p_a}$

- If $J > 1$ (ad space binding), then $\frac{\partial \alpha_k}{\partial e_j} = -\frac{a_k}{\sum_s e_s}$, so $\frac{\partial \rho_k}{\partial e_j} = -\psi_c \psi_s T^{\psi_c} \alpha_k^{\psi_c - 1} \frac{a_k}{\sum_s e_s}$; and $\frac{\partial \alpha_j}{\partial e_j} = \frac{\alpha}{\sum_s e_s} - \frac{\alpha_j}{\sum_s e_s} = \frac{\sum_{k \neq j} \alpha_k}{\sum_s e_s}$, so: $\frac{\partial \rho_j}{\partial e_j} = \psi_c \psi_s T^{\psi_c} \alpha_j^{\psi_c - 1} \frac{\sum_{k \neq j} \alpha_k}{\sum_s e_s}$.

Note that since $\alpha_j = \alpha \frac{e_j}{\sum_k e_k}$, we can rewrite them as: $\frac{\partial \rho_k}{\partial e_j} = -\psi_c \psi_s T^{\psi_c} \frac{\alpha^{\psi_c}}{(\sum_s e_s)^{\psi_c+1}} e_k^{\psi_c - 1} e_k$ and $\frac{\partial \rho_j}{\partial e_j} = \psi_c \psi_s T^{\psi_c} \frac{\alpha^{\psi_c}}{(\sum_s e_s)^{\psi_c+1}} e_j^{\psi_c-1} \sum_{k \neq j} e_k$

$$p_a = \sum_{j'=1}^{J} \left(\sum_{\mathcal{I} \in \mathcal{P}} \frac{\partial V_j}{\partial M'(\mathcal{I})} \left(\sum_{\mathcal{I}' \in \mathcal{P}} M(\mathcal{I}') \frac{\partial \Theta(\mathcal{I}' \to \mathcal{I})}{\partial \rho_{j'}} \right) \right) \frac{\partial \rho_{j'}}{\partial \alpha_j}$$

First, I rewrite $\sum_{\mathcal{I}' \in \mathcal{P}} M(\mathcal{I}') \frac{\partial \Theta(\mathcal{I}' \to \mathcal{I})}{\partial \rho_j}$ using:

1. For $\mathcal{I} = \emptyset$, if:
   - $\mathcal{I}' \neq \emptyset$ then $\frac{\partial \Theta(\mathcal{I}' \to \mathcal{I})}{\partial \rho_{j'}} = 0$, since $\Theta(\mathcal{I}' \to \emptyset) = \delta$.
   - $\mathcal{I}' = \emptyset$ then $\frac{\partial \Theta(\mathcal{I}' \to \mathcal{I})}{\partial \rho_{j'}} =$
2. For the rest, if $\mathcal{I}' \not\subseteq \mathcal{I} \implies \Theta(\mathcal{I}' \to \mathcal{I}) = 0$. Using these two statements, we rewrite the FOC as:
$$p_a = \sum_{j'=1}^{J} \left(\sum_{\mathcal{I} \in \mathcal{P}} \frac{\partial V_j}{\partial M'(\mathcal{I})} \left(\sum_{\mathcal{I}' \subseteq \mathcal{I}} M(\mathcal{I}') \frac{\partial \Theta(\mathcal{I}' \to \mathcal{I})}{\partial \rho_{j'}} \right) \right) \frac{\partial \rho_{j'}}{\partial \alpha_j}$$
3. It is straightforward that $\{(\mathcal{I}, \mathcal{I}') | \mathcal{I} \in \mathcal{P}, \mathcal{I}' \subseteq \mathcal{I}\} = \{(\mathcal{I}, \mathcal{I}') | \mathcal{I}' \in \mathcal{P}, \mathcal{I} \supseteq \mathcal{I}'\}$ (if $(\mathcal{I}, \mathcal{I}')$ is in the first set, it clearly is in the second and viceversa). So:
$$p_a = \sum_{j'=1}^{J} \left(\sum_{\mathcal{I}' \in \mathcal{P}} M(\mathcal{I}') \left(\sum_{\mathcal{I} \supseteq \mathcal{I}'} \frac{\partial \Theta(\mathcal{I}' \to \mathcal{I})}{\partial \rho_{j'}} \frac{\partial V_j}{\partial M'(\mathcal{I})} \right) \right) \frac{\partial \rho_{j'}}{\partial \alpha_j}$$
4. Now, if $j' \in \mathcal{I}' \subseteq \mathcal{I}$, then $\frac{\partial \Theta(\mathcal{I}' \to \mathcal{I})}{\partial \rho_{j'}} = 0$. Letting $\mathcal{P}_{-j} = \{\mathcal{I} \in \mathcal{P} | j \notin \mathcal{I}\}$, then:
$$p_a = \sum_{j'=1}^{J} \left(\sum_{\mathcal{I}' \in \mathcal{P}_{-j'}} M(\mathcal{I}') \left(\sum_{\mathcal{I} \supseteq \mathcal{I}'} \frac{\partial \Theta(\mathcal{I}' \to \mathcal{I})}{\partial \rho_{j'}} \frac{\partial V_j}{\partial M'(\mathcal{I})} \right) \right) \frac{\partial \rho_{j'}}{\partial \alpha_j}$$
5. Next, noting that $\frac{\partial \Theta(\mathcal{I}' \to \mathcal{I})}{\partial \rho_{j'}} = \frac{\Theta(\mathcal{I}' \to \mathcal{I})}{\rho_{j'}}$ if $j \in \mathcal{I} \setminus \mathcal{I}'$ and $\frac{\partial \Theta(\mathcal{I}' \to \mathcal{I})}{\partial \rho_{j'}} = -\frac{\Theta(\mathcal{I}' \to \mathcal{I})}{1 - \rho_{j'}}$ if $j \notin \mathcal{I}$
$$p_a = \sum_{j'=1}^{J} \left(\sum_{\mathcal{I}' \in \mathcal{P}_{-j'}} M(\mathcal{I}') \left(\sum_{\substack{\mathcal{I} \supseteq \mathcal{I}' \\ j' \in \mathcal{I}}} \frac{\Theta(\mathcal{I}' \to \mathcal{I})}{\rho_{j'}} \frac{\partial V_j}{\partial M'(\mathcal{I})} - \sum_{\substack{\mathcal{I} \supseteq \mathcal{I}' \\ j' \notin \mathcal{I}}} \frac{\Theta(\mathcal{I}' \to \mathcal{I})}{1 - \rho_{j'}} \frac{\partial V_j}{\partial M'(\mathcal{I})} \right) \right) \frac{\partial \rho_{j'}}{\partial \alpha_j}$$
6. Next, note that $\{\mathcal{I} \cup \{j'\} | \mathcal{I} \supseteq \mathcal{I}', j' \notin \mathcal{I}\} = \{\mathcal{I} | \mathcal{I} \supseteq \mathcal{I}', j' \in \mathcal{I}\}$, so:
$$p_a = \sum_{j'=1}^{J} \left(\sum_{\mathcal{I}' \in \mathcal{P}_{-j'}} M(\mathcal{I}') \sum_{\substack{\mathcal{I} \supseteq \mathcal{I}' \\ j' \notin \mathcal{I}}} \left(\frac{\Theta(\mathcal{I}' \to \mathcal{I} \cup \{j'\})}{\rho_{j'}} \frac{\partial V_j}{\partial M'(\mathcal{I} \cup \{j'\})} - \frac{\Theta(\mathcal{I}' \to \mathcal{I})}{1 - \rho_{j'}} \frac{\partial V_j}{\partial M'(\mathcal{I})} \right) \right) \frac{\partial \rho_{j'}}{\partial \alpha_j}$$

7. Next, note that $\frac{\Theta(\mathcal{I}'\to\mathcal{I}\cup\{j'\})}{\rho_{j'}} = \prod_{j\in\mathcal{I}\setminus\mathcal{I}'}\rho_j \prod_{j\notin\mathcal{I}}(1-\rho_j) = \frac{\Theta(\mathcal{I}'\to\mathcal{I})}{1-\rho_{j'}}$; so:

$$p_a = \sum_{j'=1}^{J}\left(\sum_{\mathcal{I}'\in\mathcal{P}_{-j'}} M(\mathcal{I}') \sum_{\substack{\mathcal{I}\supseteq\mathcal{I}'\\ j'\notin\mathcal{I}}} \frac{\Theta(\mathcal{I}'\to\mathcal{I})}{1-\rho_{j'}}\left(\frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j'\})} - \frac{\partial V_j}{\partial M'(\mathcal{I})}\right)\right)\frac{\partial \rho_{j'}}{\partial \alpha_j}$$

8. For the informative motive we are done. For the anticompetitive motive, first,

9. Almost finally, calling $\mathcal{P}_{-j,-j'} = \{\mathcal{I}\in\mathcal{P}_{-j'}|j\notin\mathcal{I}\}$, and analogously, $\mathcal{P}_{j,-j'} = \{\mathcal{I}\in\mathcal{P}_{-j'}|j\in\mathcal{I}\}$. So: $\mathcal{P}_{-j'} = \mathcal{P}_{j,-j'}\cup\mathcal{P}_{-j,-j'}$, and in addition we have $\mathcal{P}_{j,-j'} = \{\mathcal{I}\cup\{j\}|\mathcal{I}\in\mathcal{P}_{-j,-j'}\}$ (that is, for each $\mathcal{I}\in\mathcal{P}_{-j,-j'}$, we have $\mathcal{I}\subset\mathcal{I}\cup\{j\}\in\mathcal{P}_{j,-j'}$).

$$p_a = \sum_{j'=1}^{J}\left(\sum_{\mathcal{I}'\in\mathcal{P}_{-j,-j'}}\left[M(\mathcal{I}')\sum_{\substack{\mathcal{I}\supseteq\mathcal{I}'\\ j'\notin\mathcal{I}}}\frac{\Theta(\mathcal{I}'\to\mathcal{I})}{1-\rho_{j'}}\left(\frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j'\})} - \frac{\partial V_j}{\partial M'(\mathcal{I})}\right) + \right.\right.$$
$$\left.\left. + M(\mathcal{I}'\cup\{j\})\sum_{\substack{\mathcal{I}\supseteq\mathcal{I}'\cup\{j\}\\ j'\notin\mathcal{I}}}\frac{\Theta(\mathcal{I}'\cup\{j\}\to\mathcal{I})}{1-\rho_{j'}}\left(\frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j'\})} - \frac{\partial V_j}{\partial M'(\mathcal{I})}\right)\right]\right)\frac{\partial \rho_{j'}}{\partial \alpha_j}$$

10. For each $\mathcal{I}'\in\mathcal{P}_{-j,-j'}$ we have $\{\mathcal{I}\in\mathcal{P}_{-j'}|\mathcal{I}\supseteq\mathcal{I}'\} = \{\mathcal{I}\in\mathcal{P}_{-j,-j'}|\mathcal{I}\supseteq\mathcal{I}'\}\cup\{\mathcal{I}\cup\{j\}|\mathcal{I}\in\mathcal{P}_{-j,-j'},\mathcal{I}\supseteq\mathcal{I}'\}$. And for $\mathcal{I}'\cup\{j\}$ we have $\{\mathcal{I}\in\mathcal{P}_{-j'}|\mathcal{I}\supseteq\mathcal{I}'\cup\{j\}\} = \{\mathcal{I}\cup\{j\}|\mathcal{I}\in\mathcal{P}_{-j,-j'},\mathcal{I}\supseteq\mathcal{I}'\}$.

11. For $\mathcal{I},\mathcal{I}'\in\mathcal{P}_{-j,-j'}$ with $\mathcal{I}\supseteq\mathcal{I}'$, we have $\Theta(\mathcal{I}'\cup\{j\}\to\mathcal{I}\cup\{j\}) = \prod_{k\in\mathcal{I}\setminus\mathcal{I}'}\rho_k\prod_{k\notin\mathcal{I}\cup\{j\}}(1-\rho_k) = \rho_j\prod_{k\in\mathcal{I}\setminus\mathcal{I}'}\rho_k\prod_{k\notin\mathcal{I}\cup\{j\}}(1-\rho_k) + \prod_{k\in\mathcal{I}\setminus\mathcal{I}'}\rho_k\prod_{k\notin\mathcal{I}\cup\{j\}}(1-\rho_k)(1-\rho_j) = \Theta(\mathcal{I}'\to\mathcal{I}\cup\{j\}) + \Theta(\mathcal{I}'\to\mathcal{I})$. So, substitute $\Theta(\mathcal{I}'\to\mathcal{I}\cup\{j\}) = \Theta(\mathcal{I}'\cup\{j\}\to\mathcal{I}\cup\{j\}) - \Theta(\mathcal{I}'\to\mathcal{I})$

$$p_a = \left(\sum_{\mathcal{I}'\in\mathcal{P}_{-j}} M(\mathcal{I}')\sum_{\substack{\mathcal{I}\supseteq\mathcal{I}'\\ j\notin\mathcal{I}}}\frac{\Theta(\mathcal{I}'\to\mathcal{I})}{1-\rho_j}\left(\frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j\})} - \frac{\partial V_j}{\partial M'(\mathcal{I})}\right)\right)\frac{\partial\rho_j}{\partial\alpha_j}$$
$$+\sum_{j'\neq j}\left(\sum_{\mathcal{I}'\in\mathcal{P}_{-j,-j'}}\left[M(\mathcal{I}')\sum_{\substack{\mathcal{I}\supseteq\mathcal{I}'\\ \mathcal{I}\in\mathcal{P}_{-j,-j'}}}\frac{\Theta(\mathcal{I}'\to\mathcal{I})}{1-\rho_{j'}}\underbrace{\left(\frac{\partial V_j}{\partial M'(\mathcal{I})} - \frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j'\})} - \frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j\})} + \frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j\}\cup\{j'\})}\right)}_{<0} + \right.\right.$$
$$\left.\left. + \Big(M(\mathcal{I}') + M(\mathcal{I}'\cup\{j\})\Big)\sum_{\substack{\mathcal{I}\supseteq\mathcal{I}'\cup\{j\}\\ j'\notin\mathcal{I}}}\frac{\Theta(\mathcal{I}'\cup\{j\}\to\mathcal{I})}{1-\rho_{j'}}\left(\frac{\partial V_j}{\partial M'(\mathcal{I})} - \frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j'\})}\right)\right]\right)\left(-\frac{\partial\rho_{j'}}{\partial\alpha_j}\right)$$

12. Finally, since $\left(\frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j\})} - \frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j\}\cup\{j'\})}\right) > \left(\frac{\partial V_j}{\partial M'(\mathcal{I})} - \frac{\partial V_j}{\partial M'(\mathcal{I}\cup\{j'\})}\right)$, because the firm value clearly is more affected if it is a customer who learns about another good (since she will reduce the spending in j) rather than if it is a non-customer who learns about another good. So, we see that the anticompetitive motive increases more if $M(\mathcal{I}'\cup\{j\})$ increases rather than $M(\mathcal{I}')$ (direct from taking the derivatives with respect to $M(\mathcal{I}'\cup\{j\})$ and $M(\mathcal{I}')$ in the previous expression.

7.6 Derivative of a function of next period vector of masses

Lemma 1 *If $f : \vec{\hat{M}}' \to \mathbb{R}$, then we have:*

1. $\frac{\partial f}{\partial \rho_j} = \sum\limits_{\mathcal{I}\in\mathcal{P}_{-j}} M(\mathcal{I}) \sum\limits_{\mathcal{I}'\in\mathcal{P}_{-j},\mathcal{I}'\supseteq\mathcal{I}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right]$

2. *For the anticompettive motive, it will be useful:*
$\frac{\partial f}{\partial \rho_j} = \sum\limits_{\mathcal{I}\in\mathcal{P}_{-k,-j}} M(\mathcal{I}) \sum\limits_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}\left[\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right] - \left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k,j\})} - \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k\})}\right]\right]$
$+ \sum\limits_{\mathcal{I}\in\mathcal{P}_{-k,-j}} (M(\mathcal{I}) + M(\mathcal{I}\cup\{k\})) \sum\limits_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\cup\{k\}\to\mathcal{I}'\cup\{k\})}{1-\rho_j}\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k,j\})} - \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k\})}\right]$

Proof. First, recall that $\hat{M}'(\mathcal{I}') = \sum_{\mathcal{I}\in\mathcal{P}(\mathcal{J})} M(\mathcal{I})\Theta(\mathcal{I}\to\mathcal{I}')$.
Next, the derivatives of $\Theta(\mathcal{I}\to\mathcal{I}')$ wrt ρ_j are:

1. If $\mathcal{I} \nsubseteq \mathcal{I}'$: $\frac{\partial\Theta(\mathcal{I}\to\mathcal{I}')}{\partial\rho_j} = 0$
2. If $\mathcal{I} \subseteq \mathcal{I}'$:
   (a) If $j \in \mathcal{I}$: $\frac{\partial\Theta(\mathcal{I}\to\mathcal{I}')}{\partial\rho_j} = 0$
   (b) If $j \in \mathcal{I}'\setminus\mathcal{I}$: $\frac{\partial\Theta(\mathcal{I}\to\mathcal{I}')}{\partial\rho_j} = (1-\delta)\prod\limits_{k\in\mathcal{I}'\setminus(\mathcal{I}\cup\{j\})}\rho_k\prod\limits_{k\notin\mathcal{I}'}(1-\rho_k) = \frac{\Theta(\mathcal{I}\to\mathcal{I}'\setminus\{j\})}{1-\rho_j}$
   (c) If $j \notin \mathcal{I}'$: $\frac{\partial\Theta(\mathcal{I}\to\mathcal{I}')}{\partial\rho_j} = -(1-\delta)\prod\limits_{k\in\mathcal{I}'\setminus\mathcal{I}}\rho_k\prod\limits_{k\notin(\mathcal{I}'\cup\{j\})}(1-\rho_k) = -\frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}$

Using this, the derivative of $\hat{M}'(\mathcal{I}')$ wrt ρ_j is:

1. If $j\in\mathcal{I}'$: $\frac{\partial\hat{M}'(\mathcal{I}')}{\partial\rho_j} = \sum\limits_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subset\mathcal{I}'} M(\mathcal{I})\frac{\partial\Theta(\mathcal{I}\to\mathcal{I}')}{\partial\rho_j} = \sum\limits_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subset\mathcal{I}'} M(\mathcal{I})\frac{\Theta(\mathcal{I}\to\mathcal{I}'\setminus\{j\})}{1-\rho_j}$
2. If $j\notin\mathcal{I}'$: $\frac{\partial\hat{M}'(\mathcal{I}')}{\partial\rho_j} = \sum\limits_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subseteq\mathcal{I}'} M(\mathcal{I})\frac{\partial\Theta(\mathcal{I}\to\mathcal{I}')}{\partial\rho_j} = -\sum\limits_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subseteq\mathcal{I}'} M(\mathcal{I})\frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}$

And the derivative of a generic function $f : \vec{\hat{M}}' \to \mathbb{R}$ wrt ρ_j is:

$$\begin{aligned}\frac{\partial f}{\partial\rho_j} &= \sum_{\mathcal{I}'\in\mathcal{P}}\frac{\partial f}{\partial M'(\mathcal{I}')}\frac{\partial M'(\mathcal{I}')}{\partial\rho_j} = \sum_{\mathcal{I}'\in\mathcal{P}_j}\frac{\partial f}{\partial M'(\mathcal{I}')}\frac{\partial M'(\mathcal{I}')}{\partial\rho_j} + \sum_{\mathcal{I}'\in\mathcal{P}_{-j}}\frac{\partial f}{\partial M'(\mathcal{I}')}\frac{\partial M'(\mathcal{I}')}{\partial\rho_j}\\ &= \sum_{\mathcal{I}'\in\mathcal{P}_j}\frac{\partial f}{\partial M'(\mathcal{I}')}\sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subset\mathcal{I}'}M(\mathcal{I})\frac{\Theta(\mathcal{I}\to\mathcal{I}'\setminus\{j\})}{1-\rho_j} - \sum_{\mathcal{I}'\in\mathcal{P}_{-j}}\frac{\partial f}{\partial M'(\mathcal{I}')}\sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subseteq\mathcal{I}'}M(\mathcal{I})\frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}\end{aligned}$$

Now we are going to merge the two summations using that $\mathcal{P}_j = \{\mathcal{I}\cup\{j\}|\mathcal{I}\in\mathcal{P}_{-j}\}$ [**Proof:** from any set $\mathcal{I}$ that doesn't contain j we can build one by adding j to $\mathcal{I}$ (that is, $\{\mathcal{I}\cup\{j\}|\mathcal{I}\in\mathcal{P}_{-j}\}\subseteq\mathcal{P}_j$), and that from any $\mathcal{I}'$ that contains j we can build another one that doesn't contain j by removing j from $\mathcal{I}'$ (that is, $\mathcal{P}_j = \{(\mathcal{I}'\setminus\{j\})\cup\{j\}|\mathcal{I}'\in\mathcal{P}_j\}\subseteq\{\mathcal{I}\cup\{j\}|\mathcal{I}\in\mathcal{P}_{-j}\}$].
Using this in the previous expression, we get:

$$\frac{\partial f}{\partial \rho_j} = \sum_{\mathcal{I}'\in\mathcal{P}_{-j}} \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} \sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subseteq\mathcal{I}'} M(\mathcal{I})\frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j} - \sum_{\mathcal{I}'\in\mathcal{P}_{-j}} \frac{\partial f}{\partial M'(\mathcal{I}')} \sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subseteq\mathcal{I}'} M(\mathcal{I})\frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}$$
$$= \sum_{\mathcal{I}'\in\mathcal{P}_{-j}} \sum_{\mathcal{I}\in\mathcal{P}_{-j},\mathcal{I}\subseteq\mathcal{I}'} M(\mathcal{I})\frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right]$$
$$= \sum_{\mathcal{I}\in\mathcal{P}_{-j}} M(\mathcal{I}) \sum_{\mathcal{I}'\in\mathcal{P}_{-j},\mathcal{I}'\supseteq\mathcal{I}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right]$$

where for the last equality, I have used that $\{(\mathcal{I},\mathcal{I}')|\mathcal{I}\in\mathcal{P},\mathcal{I}'\subseteq\mathcal{I}\} = \{(\mathcal{I},\mathcal{I}')|\mathcal{I}'\in\mathcal{P},\mathcal{I}\supseteq\mathcal{I}'\}$.

This proves the first expression of the lemma. For the second:
First, note that above we have shown that $\{\mathcal{I}\cup\{j\}|\mathcal{I}\in\mathcal{P}_{-j}\} = \mathcal{P}_j$, which implies that $\mathcal{P} = \mathcal{P}_{-j}\cup\{\mathcal{I}\cup\{j\}|\mathcal{I}\in\mathcal{P}_{-j}\}$. Analogously, defining $\mathcal{P}_{-k,-j} = \{\mathcal{I}\in\mathcal{P}|j,k\notin\mathcal{I}\}$, we have: $\mathcal{P}_{-j} = \mathcal{P}_{-k,-j}\cup\{\mathcal{I}\cup\{k\}|\mathcal{I}\in\mathcal{P}_{-k,-j}\}$, so the previous expression becomes

$$\frac{\partial f}{\partial \rho_j} = \sum_{\mathcal{I}\in\mathcal{P}_{-k,-j}} M(\mathcal{I}) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right]$$
$$+ M(\mathcal{I}\cup\{k\}) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-j}\\ \mathcal{I}'\supseteq\mathcal{I}\cup\{k\}}} \frac{\Theta(\mathcal{I}\cup\{k\}\to\mathcal{I}')}{1-\rho_j}\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right]$$

Now, for the first line, I use the following equivalence: for each $\mathcal{I}\in\mathcal{P}_{-k,-j}$ we have $\{\mathcal{I}'\in\mathcal{P}_{-j}|\mathcal{I}'\supseteq\mathcal{I}\} = \{\mathcal{I}'\in\mathcal{P}_{-k,-j}|\mathcal{I}'\supseteq\mathcal{I}\}\cup\{\mathcal{I}'\cup\{k\}|\mathcal{I}'\in\mathcal{P}_{-k,-j},\mathcal{I}'\supseteq\mathcal{I}\}$. And for the second line, I use the equivalence: for $\mathcal{I}\cup\{k\}$ we have $\{\mathcal{I}'\in\mathcal{P}_{-j}|\mathcal{I}'\supseteq\mathcal{I}\cup\{k\}\} = \{\mathcal{I}'\cup\{k\}|\mathcal{I}'\in\mathcal{P}_{-k,-j},\mathcal{I}'\supseteq\mathcal{I}\}$.

$$\frac{\partial f}{\partial \rho_j} = \sum_{\mathcal{I}\in\mathcal{P}_{-k,-j}} M(\mathcal{I}) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \left[\frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right] + \frac{\Theta(\mathcal{I}\to\mathcal{I}'\cup\{k\})}{1-\rho_j}\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k,j\})} - \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k\})}\right]\right]$$
$$+ M(\mathcal{I}\cup\{k\}) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\cup\{k\}\to\mathcal{I}'\cup\{k\})}{1-\rho_j}\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k,j\})} - \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k\})}\right]$$

Finally, I use that for $\mathcal{I},\mathcal{I}'\in\mathcal{P}_{-k,-j}$ with $\mathcal{I}'\supseteq\mathcal{I}$, we have $\Theta(\mathcal{I}\cup\{k\}\to\mathcal{I}'\cup\{k\}) = \prod_{h\in\mathcal{I}'\setminus\mathcal{I}}\rho_h \prod_{h\notin\mathcal{I}'\cup\{k\}}(1-\rho_h)\cdot(\rho_k+1-\rho_k) = \Theta(\mathcal{I}\to\mathcal{I}'\cup\{k\}) + \Theta(\mathcal{I}\to\mathcal{I}')$. So, I substitute in the first line $\Theta(\mathcal{I}\to\mathcal{I}'\cup\{k\}) = \Theta(\mathcal{I}\cup\{k\}\to\mathcal{I}'\cup\{k\}) - \Theta(\mathcal{I}\to\mathcal{I}')$.

$$\frac{\partial f}{\partial \rho_j} = \sum_{\mathcal{I}\in\mathcal{P}_{-k,-j}} M(\mathcal{I}) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}\left[\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{j\})} - \frac{\partial f}{\partial M'(\mathcal{I}')}\right] - \left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k,j\})} - \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k\})}\right]\right]$$
$$+ (M(\mathcal{I}) + M(\mathcal{I}\cup\{k\})) \sum_{\substack{\mathcal{I}'\in\mathcal{P}_{-k,-j}\\ \mathcal{I}'\supseteq\mathcal{I}}} \frac{\Theta(\mathcal{I}\cup\{k\}\to\mathcal{I}'\cup\{k\})}{1-\rho_j}\left[\frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k,j\})} - \frac{\partial f}{\partial M'(\mathcal{I}'\cup\{k\})}\right]$$

■

7.6.1 With uncertainty:

In the model, there is uncertainty on $\mathcal{J}'$, so I am more interested in finding the derivative of the expected value of a function $g : \vec{M}' \to \mathbb{R}$ rather than the derivative of a function $f : \vec{\hat{M}}' \to \mathbb{R}$ (note that f is defined on $\vec{\hat{M}}'$, that is, the next period distribution if there weren't entry and exit, whereas g is defined on the actual next period distribution after the uncertainty has been resolved). Recall that, given the set of exiters $\mathcal{E}$ and the indicator 1_e equal to 1 if there is an entrant, the mapping between $\vec{\hat{M}}'$ and $\vec{M}'$ is given by $F_{\mathcal{E},1_e} : \vec{\hat{M}}' \to \vec{M}'$. And the expected value is defined as:

$$\mathbb{E}g\Big(\vec{M}\Big) = \sum_{1_e=0}^{1} z_e^{1_e}(1-z_e)^{1-1_e} \sum_{\mathcal{E}\subseteq\mathcal{J}} \kappa^{\#\mathcal{E}}(1-\kappa)^{\#(\mathcal{J}\setminus\mathcal{E})} g\Big(F_{\mathcal{E},1_e}(\vec{\hat{M}}')\Big)$$

And $F_{\mathcal{E},1_e}(\vec{\hat{M}}')$ is given by:

$$F_{\mathcal{E},1_e}(\vec{\hat{M}}') = \begin{cases} M'(\mathcal{I}) = \sum_{\hat{\mathcal{E}}\subseteq\mathcal{E}} \hat{M}'(\mathcal{I}\cup\hat{\mathcal{E}}) & \text{for } \mathcal{I}\in\mathcal{P}(\mathcal{J}\setminus\mathcal{E}) \\ M'(\mathcal{I}) = 0 & \text{for } \mathcal{I}\in\mathcal{P}(\mathcal{J}')\setminus\mathcal{P}(\mathcal{J}) \text{ (that is, } \{e\}\in\mathcal{I}) \end{cases}$$

Going in the reverse order, each $\mathcal{I}\in\mathcal{P}(\mathcal{J})$ is associated to $\mathcal{I}' = \mathcal{I}\setminus\mathcal{E}\in\mathcal{P}(\mathcal{J}\setminus\mathcal{E})$; therefore $\frac{\partial g\left(F_{\mathcal{E},1_e}(\vec{\hat{M}}')\right)}{\partial\hat{M}'(\mathcal{I})} = \frac{\partial g(\vec{M}')}{\partial\hat{M}'(\mathcal{I})} = \frac{\partial g(\vec{M}')}{\partial M'(\mathcal{I}\setminus\mathcal{E})}\frac{\partial M'(\mathcal{I}\setminus\mathcal{E})}{\partial\hat{M}'(\mathcal{I})} = \frac{\partial g(\vec{M}')}{\partial M'(\mathcal{I}\setminus\mathcal{E})}$. Then, we can apply this to the result of the case without entry and exit and we have:

$$\frac{\partial g\left(F_{\mathcal{E},1_e}(\vec{\hat{M}}')\right)}{\partial\rho_j} = \sum_{\mathcal{I}\in\mathcal{P}_{-j}} M(\mathcal{I}) \sum_{\mathcal{I}'\in\mathcal{P}_{-j},\mathcal{I}'\supseteq\mathcal{I}} \frac{\Theta(\mathcal{I}\to\mathcal{I}')}{1-\rho_j}\left[\frac{\partial g}{\partial M'(\mathcal{I}'\cup\{j\}\setminus\mathcal{E})} - \frac{\partial g}{\partial M'(\mathcal{I}'\setminus\mathcal{E})}\right]$$

Note that if $j\in\mathcal{E}$, then this derivative is 0 since in this case $\frac{\partial g}{\partial M'(\mathcal{I}'\cup\{j\}\setminus\mathcal{E})} = \frac{\partial g}{\partial M'(\mathcal{I}'\setminus\mathcal{E})}$.

7.7 Social planner problem

$$\max_{\{N_{M,t},N_{j,i,t},h_{e,i,t},p_{j,i,t},\alpha_{j,i,t}\}} U = \sum_{t=0}^{\infty}\beta^t \int_0^1 [\ln C_{\ell t} + L_{\ell t}]\, d\ell$$

s.t. $C_{\ell,t}$ from (3), $C_{\ell,i,t}$ from (4), $c_{\ell,j,i,t}$ from (8), and $L_{\ell,t}$ from 2, with $T_t = Q_t$

$$y_{j,i,t} = N_{j,i,t}, \quad y_{0,i,t} = A_0 N_{0,i,t}, \quad Q_t = AN_{m,t}^{\varphi} \qquad \text{(Production functions)}$$

$$1 = N_{m,t} + \int_0^1 [N_{0,i,t} + \sum_{j\in\mathcal{J}_{i,t}} N_{j,i,t}]di \quad , \quad w_t + \int_0^1 h_{e,i,t}di = E_t \qquad \text{(Resource constraints)}$$

$$\sum_{j\in\mathcal{J}_{i,t}} \alpha_{j,i,t} = \alpha_i, (6), (7), (11), (12), (13) \qquad \text{(Learning process)}$$

$$z_{e,i,t} = \phi\left(\frac{h_{e,i,t}}{E_t}\right)^{\frac{1}{2}}, (12), (13) \qquad \text{(Entry and exit)}$$

Plugging $C_{\ell,t}$ and $L_{\ell,t}$ with $T_t = Q_t$ into the objective function and interchanging the integrals over ℓ and i:

$$\max_{\{N_{M,t},N_{e,i,t},N_{j,i,t},p_{j,i,t},\alpha_{j,i,t}\}} U = \int_0^1 \int_0^1 \sum_{t=0}^{\infty} \beta^t \ln C_{\ell,i,t} d\ell di + \sum_{t=0}^{\infty} \beta^t \upsilon \frac{Q_t^2}{2}$$

s.t. $C_{\ell,i,t}$ from (4), $c_{\ell,j,i,t}$ from (8)

$$y_{j,i,t} = N_{j,i,t}, \quad y_{0,i,t} = A_0 N_{0,i,t}, \quad Q_t = A N_{m,t}^{\varphi} \qquad \text{(Production functions)}$$

$$1 = N_{m,t} + \int_0^1 [N_{0,i,t} + \sum_{j \in \mathcal{J}_{i,t}} N_{j,i,t}] di \quad , \quad w_t + \int_0^1 h_{e,i,t} di = E_t \qquad \text{(Resource constraints)}$$

$$\sum_{j \in \mathcal{J}_{i,t}} \alpha_{j,i,t} = \alpha_i, (6), (7), (11), (12), (13) \qquad \text{(Learning process)}$$

$$z_{e,i,t} = \phi \left(\frac{h_{e,i,t}}{E_t} \right)^{\frac{1}{2}}, (12), (13) \qquad \text{(Entry and exit)}$$

The planner decides how much to produce for each individual and, accordingly, sets the prices that induce the consumers to consume these quantities. Let N_t^P be the labor used to produce all the goods in the production sector. Then, The FOC for $c_{\mathcal{I},j,i,t}$ writes:

$$[c_{\mathcal{I},j,i,t}] : \quad \beta^t \frac{1}{C_t} \frac{\partial C_t}{\partial C_{i,t}} \frac{\partial C_{i,t}}{\partial C_{\mathcal{I},i,t}} \frac{\partial C_{\mathcal{I},i,t}}{\partial c_{\mathcal{I},j,i,t}} = \beta^t \lambda \frac{\partial N_t^P}{\partial C_t} \frac{\partial C_t}{\partial C_{i,t}} \frac{\partial C_{i,t}}{\partial C_{\mathcal{I},i,t}} \frac{\partial C_{\mathcal{I},i,t}}{\partial c_{\mathcal{I},j,i,t}} \tag{23}$$

1. Dividing both sides $\frac{\partial C_t}{\partial C_{i,t}} \frac{\partial C_{i,t}}{\partial C_{\mathcal{I},i,t}} \frac{\partial C_{\mathcal{I},i,t}}{\partial c_{\mathcal{I},j,i,t}} > 0$, and defining $\hat{P}_t = \frac{w_t N_t^P}{C_t}$, we get:

$$\frac{1}{C_t} = \lambda \frac{\hat{P}_t}{w_t} \implies \lambda = \frac{w_t}{\hat{P}_t C_t} \tag{24}$$

2. $\ln C_t = \int_0^1 \ln C_{i,t} di$. Dividing both sides of 23 by $\frac{\partial C_{i,t}}{\partial C_{\mathcal{I},i,t}} \frac{\partial C_{\mathcal{I},i,t}}{\partial c_{\mathcal{I},j,i,t}} > 0$, letting $N_{i,t}$ be the the labor used in sector i, $\frac{\partial N_t^P}{\partial C_t} \frac{\partial C_t}{\partial C_{i,t}} = \frac{\partial N_t^P}{\partial C_{i,t}}$ and defining $\hat{P}_{i,t} = \frac{w_t N_{i,t}}{C_{i,t}}$, we get:

$$\frac{1}{C_{i,t}} = \lambda \frac{\hat{P}_{i,t}}{w_t} \implies \lambda = \frac{w_t}{\hat{P}_{i,t} C_{i,t}} \implies C_{i,t} = C_t \frac{\hat{P}_t}{\hat{P}_{i,t}} \tag{25}$$

where for the last expression I have used 24. Plugging $C_{i,t}$ into the definition of C_t, we get:

$$\ln \hat{P}_t = \int_0^1 \ln \hat{P}_{i,t} di \tag{26}$$

3. $\ln C_{i,t} = \int_0^1 \ln C_{\ell,i,t} d\ell$. Dividing both sides of 23 by $\frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} > 0$, letting $N_{\ell,i,t}$ be the the labor used in sector i by ℓ, $\frac{\partial N_{i,t}}{\partial C_{i,t}} \frac{\partial C_{i,t}}{\partial C_{\ell,i,t}} = \frac{\partial N_{i,t}}{\partial C_{\ell,i,t}}$ and defining $\hat{P}_{\ell,i,t} = \frac{w_t N_{\ell,i,t}}{C_{\ell,i,t}}$, we get:

$$\frac{1}{C_{\ell,i,t}} = \lambda \frac{\hat{P}_{\ell,i,t}}{w_t} \implies \lambda = \frac{w_t}{\hat{P}_{\ell,i,t} C_{\ell,i,t}} \implies C_{\ell,i,t} = C_{i,t} \frac{\hat{P}_{i,t}}{\hat{P}_{\ell,i,t}} \tag{27}$$

where for the last expression I have used 25. Plugging $C_{\ell,i,t}$ into the definition of $C_{i,t}$, we get:

$$\ln \hat{P}_{i,t} = \int_0^1 \ln \hat{P}_{\ell,i,t} di \tag{28}$$

4. $C_{\ell,i,t}$ given by 4. Letting $N_{\ell,j,i,t}$ be the the labor used in good j in sector i by ℓ,

$\frac{\partial N_{\ell,i,t}}{\partial C_{\ell,i,t}}\frac{\partial C_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = \frac{\partial N_{\ell,i,t}}{\partial c_{\ell,j,i,t}} = \frac{1}{A_j}$, we get:

$$\frac{1}{C_{\ell,i,t}}\left(\frac{C_{\ell,i,t}}{c_{\ell,j,i,t}}\right)^{\frac{1}{\sigma}}\omega_{j,i,t} = \lambda\frac{1}{A_j} \implies \left(\frac{C_{\ell,i,t}}{c_{\ell,j,i,t}}\right)^{\frac{1}{\sigma}}\omega_{j,i,t} = \frac{w_t}{\hat{P}_{\ell,i,t}A_j} \implies c_{\ell,j,i,t} = C_{\ell,i,t}\hat{P}^{\sigma}_{\mathcal{I},i,t}\left(\omega_{j,i,t}\frac{A_j}{w_t}\right)^{\sigma} \tag{29}$$

where I have used λ from 27. Plugging $c_{\ell,j,i,t}$ into the definition of $C_{\ell,i,t}$, we get:

$$\hat{P}_{\mathcal{I},i,t} = \left(\left(\frac{A_0}{w_t}\right)^{\sigma-1} + \sum_{j\in\mathcal{I}}\omega^{\sigma}_{j,i,t}\left(\frac{1}{w_t}\right)^{\sigma-1}\right)^{\frac{1}{1-\sigma}} \tag{30}$$

Since we have $\hat{P}_{\ell,i,t}C_{\ell,i,t} = \hat{P}_{i,t}C_{i,t} = \hat{P}_tC_t = w_tN^P_t$ and λ from 24, then we have:

$$c_{\mathcal{I},j,i,t} = w_tN^P_t\hat{P}^{\sigma-1}_{\mathcal{I},i,t}\left(\omega_{j,i,t}\frac{A_j}{w_t}\right)^{\sigma}, \quad N^P_t = \frac{1}{\lambda} \tag{31}$$

Comparing this with the consumer choices:

$$c_{\mathcal{I},j,i,t} = E_tP^{\sigma-1}_{\mathcal{I},i,t}p^{-\sigma}_{j,i,t}\omega^{\sigma}_{j,i,t}, \quad P_{\mathcal{I},i,t} = \left(p^{1-\sigma}_{0,i,t} + \sum_{j\in\mathcal{I}}\omega^{\sigma}_{j,i,t}p^{1-\sigma}_{j,i,t}\right)^{\frac{1}{1-\sigma}}$$

It is straightforward to check that the planner can induce the consumer to consume the quantities in 31 by setting prices equal to the marginal cost times a markup (or a tax) equal to the ratio of expenditure to the production costs; i.e. $p_{j,i,t} = \frac{w_t}{A_j}\tau_t$, where $\tau_t = \frac{E_t}{w_tN^P_t}$. And also, we have $P_{\mathcal{I},i,t} = \tau\hat{P}_{\mathcal{I},i,t}$, $P_{i,t} = \tau\hat{P}_{i,t}$, and (given that P_t is the numeraire) $1 = \tau\hat{P}_t$. This gives us an expression for τ, for which we need to know first the distribution of industry states.

The particular level of τ affects the level of consumption, but not the share of expenditure allocated to each good, since, as seen in the following expression, $s_{\mathcal{I},j,i,t}$ is independent of τ (that is, τ doesn't distort how N^P_t is allocated among the production goods):

$$s_{\mathcal{I},j,i,t} = \frac{p_{j,i,t}c_{\mathcal{I},j,i,t}}{E_{\mathcal{I},i,t}} = \frac{\tau\frac{w_t}{A_j}c_{\mathcal{I},j,i,t}}{\tau w_tN_{\mathcal{I},i,t}} = \omega^{\sigma}_{j,i,t}\left(\frac{w_t}{A_j\hat{P}_{\mathcal{I},i,t}}\right)^{1-\sigma} \implies s_{\mathcal{I},j,i} = \frac{\omega^{\sigma}_{j,i}}{A_0^{\sigma-1} + \sum_{k\in\mathcal{I}}\omega^{\sigma}_{k,i}} \tag{32}$$

Using that $c_{\mathcal{I},j,i} = A_jN_{\mathcal{I},j,i} = A_j\frac{N_{\mathcal{I},j,i}}{N_{\mathcal{I},i}}N_{\mathcal{I},i} = A_js_{\mathcal{I},j,i}\frac{E_t}{w_t}\frac{1}{\tau}$; then, we can write $C_{\mathcal{I},i}$ as:

$$C_{\mathcal{I},i,t} = \frac{1}{\tau}\frac{E_t}{w_t}\left((A_0s_{\mathcal{I},0,i,t})^{\frac{\sigma-1}{\sigma}} + \sum_{k\in\mathcal{I}}\omega_{k,i,t}(s_{\mathcal{I},k,i,t})^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{33}$$

and combining this with $\frac{E_t}{\tau} = w_tN^P_t = C_{\mathcal{I},i,t}\hat{P}_{\mathcal{I},i,t}$, we get (where for the second equality I use 32):

$$\left((A_0s_{\mathcal{I},0,i,t})^{\frac{\sigma-1}{\sigma}} + \sum_{k\in\mathcal{I}}\omega_{k,i,t}(s_{\mathcal{I},k,i,t})^{\frac{\sigma-1}{\sigma}}\right) = \left(\frac{w_t}{\hat{P}_{\mathcal{I},i,t}}\right)^{\frac{\sigma-1}{\sigma}} = \omega_{j,i,t}s^{-\frac{1}{\sigma}}_{\mathcal{I},j,i,t} \tag{34}$$

Next, I move to the advertising part of the planner problem. We will use the following derivatives:

$$\frac{\partial C_{\mathcal{I},i,t}}{\partial \omega_{j,i}} = \frac{\sigma}{\sigma-1} C_{\mathcal{I},i,t}^{\frac{1}{\sigma}} \left[s_{\mathcal{I},j,i,t}^{\frac{\sigma-1}{\sigma}} + \frac{\sigma-1}{\sigma} \left(\sum_{k\in\mathcal{I}} \omega_{k,i,t} s_{\mathcal{I},k,i,t}^{\frac{-1}{\sigma}} \frac{\partial s_{\mathcal{I},k,i,t}}{\partial \omega_{j,i}} + A_0^{\frac{\sigma-1}{\sigma}} s_{\mathcal{I},0,i,t}^{\frac{-1}{\sigma}} \frac{\partial s_{\mathcal{I},0,i,t}}{\partial \omega_{j,i}} \right) \right]$$

$$\frac{\partial s_{\mathcal{I},j,i,t}}{\partial \omega_{j,i}} = s_{\mathcal{I},j,i,t}(1-s_{\mathcal{I},j,i,t}) \frac{\sigma}{\omega_{j,i,t}}, \qquad \frac{\partial s_{\mathcal{I},k,i,t}}{\partial \omega_{j,i}} = -s_{\mathcal{I},j,i,t} s_{\mathcal{I},k,i,t} \frac{\sigma}{\omega_{j,i,t}}$$

$$\frac{\partial \omega_{j,i}}{\partial \alpha_{j,i,t}} = \nu_c \nu_s T^{\nu_c} \alpha_{j,i,t}^{\nu_c - 1} = \frac{T}{\alpha_{j,i,t}} \frac{\partial \omega_{j,i}}{\partial T}$$

The term in parenthesis of the first line can be rewritten as (in the second expression, I use 34):

$$\left(\omega_{j,i,t} - \sum_{k\in\mathcal{I}} \omega_{k,i,t} s_{\mathcal{I},k,i,t}^{\frac{\sigma-1}{\sigma}} - A_0^{\frac{\sigma-1}{\sigma}} s_{\mathcal{I},0,i,t}^{\frac{\sigma-1}{\sigma}} \right) s_{\mathcal{I},j,i,t} \frac{\sigma}{\omega_{j,i,t}} = \left(\omega_{j,i,t} - \omega_{j,i,t} s_{\mathcal{I},j,i,t}^{-\frac{1}{\sigma}} \right) s_{\mathcal{I},j,i,t} \frac{\sigma}{\omega_{j,i,t}} < 0$$

so, the term in the parenthesis is negative. And we have:

$$\frac{\partial \ln C_{\mathcal{I},i,t}}{\partial \omega_{j,i}} = \frac{\sigma}{\sigma-1} C_{\mathcal{I},i,t}^{\frac{1-\sigma}{\sigma}} \left[s_{\mathcal{I},j,i,t}^{\frac{\sigma-1}{\sigma}} + (\sigma-1)\left(1 - s_{\mathcal{I},j,i,t}^{-\frac{1}{\sigma}}\right) s_{\mathcal{I},j,i,t} \right] = \frac{\sigma}{\sigma-1} \left(\frac{s_{\mathcal{I},j,i,t}}{C_{\mathcal{I},i,t}} \right)^{\frac{\sigma-1}{\sigma}} \left[1 + (\sigma-1)\left(s_{\mathcal{I},j,i,t}^{\frac{1}{\sigma}} - 1 \right) \right]$$

So:

$$\frac{\partial \ln C_{\mathcal{I},i,t}}{\partial \alpha_{j,i}} = \left(\frac{s_{\mathcal{I},j,i,t}}{C_{\mathcal{I},i,t}} \right)^{\frac{\sigma-1}{\sigma}} \left[\frac{\sigma}{\sigma-1} + \sigma \left(s_{\mathcal{I},j,i,t}^{\frac{1}{\sigma}} - 1 \right) \right] \nu_c \nu_s T^{\nu_c} \alpha_{j,i,t}^{\nu_c - 1}$$

For the dynamic problem of advertising/media, it is useful to define $U_X = \int_0^1 \sum_{t=0}^{\infty} \beta^t \ln C_{\ell,i,t} d\ell$ as the expected life-time industry-consumption utility of an industry with the current industry state being X.

The social planner has to decide on (i) how much labor to allocate to the media sector, $N_{m,t}$, and (ii) how to allocate the ad space among the differentiated firms of each industry, $\alpha_{j,i,t}$.

First, let's see the social planner choice of $\alpha_{j,i,t}$. The allocation of the ad space has to be such that the marginal social gain of increasing the ad space given to each firm is the same, since otherwise we could improve the allocation. Formally, it must be $\beta \frac{\partial \mathbb{E} U_{X'}}{\partial \rho_{j,i}} \frac{\partial \rho_{j,i}}{\partial \alpha_{j,i}} + \frac{\partial \ln C_{X,t}}{\partial \alpha_{j,X}} = \hat{h}_X$ for some $\hat{h}_X$ and all $j \in \mathcal{J}_X$, together with $\sum_{j\in\mathcal{J}_X} \alpha_{j,X} = \alpha_X$.

Second, let's see the social planner choice of N_m.

$$\left[\frac{\partial L}{\partial Q} + \sum_{X\in\Omega} \mu_t(X) \sum_{j\in\mathcal{J}_X} \left[\beta \frac{\partial \mathbb{E} U_{X'}}{\partial \rho_{j,X}} \frac{\partial \rho_{j,X}}{\partial T} + \frac{\partial \ln C_{X,t}}{\partial \omega_{j,X}} \frac{\partial \omega_{j,X}}{\partial T} \right] \frac{\partial T}{\partial Q} \right] \frac{\partial Q}{\partial N_m} = \lambda$$

where $\frac{\partial L}{\partial \tilde{Q}} = \upsilon Q$, $\frac{\partial T}{\partial Q} = 1$ (if $Q < 1$, otherwise it is 0). Also, using that $\frac{\partial \rho_{j,X}}{\partial T} = \frac{\alpha_{j,X}}{T} \frac{\partial \rho_{j,X}}{\partial \alpha_{j,X}}$, $\frac{\partial \omega_{j,X}}{\partial T} = \frac{\alpha_{j,X}}{T} \frac{\partial \omega_{j,X}}{\partial \alpha_{j,X}}$, and $\frac{\partial Q}{\partial N_m} = \varphi \frac{Q}{N_m}$

$$\left[\upsilon Q + \sum_{X\in\Omega} \mu_t(X) \sum_{j\in\mathcal{J}_X} \left[\beta \frac{\partial \mathbb{E} U_{X'}}{\partial \rho_{j,X}} \frac{\partial \rho_{j,X}}{\partial \alpha_{j,X}} + \frac{\partial \ln C_{X,t}}{\partial \alpha_{j,X}} \right] \frac{\alpha_{j,X}}{T} \right] \varphi \frac{Q}{N_m} = \lambda$$

and using that $\frac{\partial \mathbb{E} U_{X'}}{\partial \rho_{j,X}} \frac{\partial \rho_{j,X}}{\partial \alpha_{j,X}} + \frac{\partial \ln C_{X,t}}{\partial \alpha_{j,X}} = \hat{h}_X$ for some value $\hat{h}_X$ and all j, that $\sum_j \alpha_{j,X} = \alpha_X$, and $Q = T$, then the condition for N_m writes:

$$\upsilon Q^2 + \sum_{X\in\Omega} \mu(X) \hat{h}_X \alpha_X = \frac{\lambda}{\varphi} N_m \tag{35}$$

Using 31 and 35, the labor market clearing condition writes:

$$1 = N^P + N_m \implies \lambda = 1 + \upsilon Q^2 \varphi + \varphi \sum_{X \in \Omega} \mu(X) \hat{h}_X \alpha_X \tag{36}$$

Finally, the spending in entry in each industry is given by (where η is the Lagrange multiplier of the planner's budget constraint):

$$\eta = \frac{\phi}{2} \hat{h}_{e,X}^{-\frac{1}{2}} \beta \left(\mathbb{E}_e U_{X'} - \mathbb{E}_{-e} U_{X'} \right) \tag{37}$$

where $\mathbb{E}_e U_{X'}$ (resp. $\mathbb{E}_{-e} U_{X'}$) is the expected industry-utility conditional on successfully creating (resp. not creating) a new differentiated good (so the expectation comes from the probabilities the incumbents exit). And from observing that $C_{\mathcal{I},i,t}$ in 33 is linear in $\frac{E_t}{w_t}$; then we see that $\ln C_t$ will be equal to $-\ln\left(\frac{w_t}{E_t}\right)$ plus a term not containing $\frac{w_t}{E_t}$; so taking the first order condition for $\frac{w_t}{E_t}$ we have $\eta = \frac{E_t}{w_t}$. Plugging $\hat{h}_{e,X}$ from 37 with $\eta = \frac{E_t}{w_t}$ into the budget constraint, we have:

$$\frac{w_t}{E_t} + \int_0^1 \hat{h}_{e,i} di = 1$$

which pins down the relative wage $\frac{w_t}{E_t}$. Finally, given λ and $\frac{w}{E}$, we can get τ using that $\frac{w}{E} = \frac{1}{N^P} \frac{w N^P}{E} = \frac{\lambda}{\tau}$.

7.8 Proof of convergence to an ergodic distribution and uniqueness

Uniqueness:

Let τ be the first period that we arrive at state $\mathcal{J} = \emptyset$, and $P_{t,0}(X)$ be he probability that we are at X after t periods starting from $\mathcal{J} = \emptyset$; then the probability we are at state X starting from a given state is:

$P_t\{X\} = \sum_{k=1}^{t} P\{\tau = k\} P_{t-k,0}\{X\} + P\{\tau > t\} P_t\{X | \tau > K\}$

As $t \to \infty$, $P\{\tau > t\} \to 0$ since every period there is a positive probability that all differentiated firms die and we arrive at $\mathcal{J} = \emptyset$. Therefore, this tells us that if $P_{t,0}\{X\}$ converges (which later I prove that this is the case), then, the only stationary distribution we can have is $P_0(X) = \lim_{t\to\infty} P_{t,0}(X)$.

The set of possible states is at most countably infinite

This is a consequence of two things: (i) from a given state you can directly move to a finite number of states; (ii) with probability 1 any industry will pass through the state $\mathcal{J} = \emptyset$ at some point in time. Just as in the proof of Uniqueness, (ii) is telling us that the only stationary distribution we can have (if any, since I haven't proved this yet) is the one we would converge to starting from the state $\mathcal{J} = \emptyset$, which (i) tells us that at most will have a countably infinite number of different states.

Convergence (Existence)

Suppose there are $n \in \mathbb{N} \cup \{\infty\}$ possible states and the probability of moving from state j to state i is $a_{i,j}$, then the transition matrix is

$$Q = \begin{pmatrix} 1-\sum_{j=2}^{n} a_{j,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & 1-\sum_{j\neq 2} a_{j,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & 1-\sum_{j=1}^{n} a_{n,n} \end{pmatrix}$$

Let $m_t = (m_{1,t}, \ldots, m_{n,t})$ be the vector of masses in each state, and call $M_t := m_{t+1} - m_t = (Q - \mathbb{I}_n)m_t$; so $M_{i,t} = \sum_{k\neq i} m_{k,t}a_{i,k} - m_{i,t}\sum_{k\neq i} a_{k,i}$.

Lemma 2 $\sum_{k=1}^{n} M_{k,t} = 0$

Proof. Given that $M_{i,t} = \sum_{k\neq i} m_{k,t}a_{i,k} - m_{i,t}\sum_{k\neq i} a_{k,i}$; then
$\sum_{i=1}^{n} M_{i,t} = \sum_{i=1}^{n}\left[\sum_{k\neq i} m_{k,t}a_{i,k} - m_{i,t}\sum_{k\neq i} a_{k,i}\right] = \sum_{i=1}^{n}\sum_{k\neq i} m_{k,t}a_{i,k} - \sum_{i=1}^{n}\sum_{k\neq i} m_{i,t}a_{k,i}$
$= \sum_{i=1}^{n}\sum_{k\neq i} m_{k,t}a_{i,k} - \sum_{k=1}^{n}\sum_{i\neq k} m_{k,t}a_{i,k}$.
So, we just need to see that $\{k \neq i | i,k \in \{1,\ldots,n\}\} = \{i \neq k | i,k \in \{1,\ldots,n\}\}$, which is clearly satisfied by symmetry of the $\neq$-relationship.
■

And the following lemma expresses M_{t+q} for $q \in \mathbb{N}$ in terms of M_t:

Lemma 3 *For any $q \in \mathbb{N}$, $M_{t+q} = Q^q M_t$, with $M_{i,t+q} = \left(1 - \sum_{k\neq i} a_{k,i}^{(q)}\right) M_{i,t} + \sum_{k\neq i} a_{i,k}^{(q)} M_{k,t}$, where $a_{i,k}^{(q)}$ is the probability of moving from k to i in q periods.*

Proof. By definition, $M_{t+q} = (Q - \mathbb{I}_n)m_{t+q} = (Q - \mathbb{I}_n)Q^q m_t = (Q^{q+1} - Q^q)m_t = Q^q(Q - \mathbb{I}_n)m_t = Q^q M_t$.
■

My main goal here is to study the convergence of M_t towards the null vector; so, we want to establish some result that compares M_t to M_{t+q} for some $q \in \mathbb{N}$. Since M_t is an n-dimensional object, it is important to specify under which metric. To see the importance of this, let's see a counterexample that shows that not necessarily each component of M_t has to monotonically decrease in absolute value:

Lemma 4 *It is not necessarily true that $|m_{t+1}(k) - m_t(k)| \geq |m_{t+2}(k) - m_{t+1}(k)|$ for all k.*

Proof. Suppose that m_t is only non-zero in position i, where $m_{i,t} = 1$. Then

$$M_t = (Q - \mathbb{I}_n)m_t = \begin{pmatrix} a_{1,i} \\ \vdots \\ -\sum_{k\neq i} a_{k,i} \\ \vdots \\ a_{n,i} \end{pmatrix}, \quad m_{t+2} - m_{t+1} = \begin{pmatrix} B_1 \\ \vdots \\ B_i \\ \vdots \\ B_n \end{pmatrix}$$

where $B_j = a_{j,1}\left(1 - \sum_{k\neq i} a_{k,i} - \sum_{k\neq j} a_{k,j}\right) + \sum_{k\notin\{i,j\}} a_{j,k}a_{k,1}$ for $j \neq i$ and $B_i = -\left(\sum_{k\neq i} a_{k,i}\right)\left(1 - \sum_{k\neq i} a_{k,i}\right) + \sum_{k\neq i} a_{i,k}a_{k,i}$.
Then, we can find a counterexample by just supposing $a_{1,i} = 0$ and that there exists $k \notin \{1,i\}$ such that $a_{1,k}a_{k,i} > 0$. Then, we have: $|m_{1,t+1} - m_{1,t}| = a_{1,i} = 0 < a_{1,k}a_{k,i} \leq \sum_{k\notin\{1,i\}} a_{1,k}a_{k,i} = |B_1| = |m_{1,t+2} - m_{1,t+1}|$

■

The norm that will prove useful is $||M_t|| := \max\limits_{\mathcal{A}\subset\{1,\ldots,n\}}\{|\sum_{k\in\mathcal{A}} M_{k,t}|\}$. Define $\mathcal{B}^+ := \{i \in \{1,\ldots,n\}|M_{i,t} > 0\}$ and $\mathcal{B}^- := \{i \in \{1,\ldots,n\}|M_{i,t} < 0\}$

Proposition 4 *It is satisfied that* $\max\limits_{\mathcal{A}\subset\{1,\ldots,n\}}\{|\sum_{k\in\mathcal{A}} M_{k,t+1}|\} \leq \max\limits_{\mathcal{A}\subset\{1,\ldots,n\}}\{|\sum_{k\in\mathcal{A}} M_{k,t}|\} = \sum\limits_{k\in\mathcal{B}^+} M_{k,t}$ *Further, if* $i \in \mathcal{B}^+$*,* $j \in \mathcal{B}^-$ *and there exists* $\ell \in \{1,\ldots,n\}$ *such that* $a^{(q)}_{\ell,i}$*,* $a^{(q)}_{\ell,j} > 0$ *(*ℓ *can be equal to* i *or* j*, which means that this state has period smaller or equal than* q*), then* $\max\limits_{\mathcal{A}\subset\{1,\ldots,n\}}\{|\sum_{k\in\mathcal{A}} M_{k,t+q}|\} < \max\limits_{\mathcal{A}\subset\{1,\ldots,n\}}\{|\sum_{k\in\mathcal{A}} M_{k,t}|\} = \sum_{k\in\mathcal{B}^+} M_{k,t}$

Proof. First, $\max\limits_{\mathcal{A}\subset\{1,\ldots,n\}}\{|\sum_{k\in\mathcal{A}} M_{k,t}|\} = \sum_{k\in\mathcal{B}^+} M_{k,t}$ is because $\max\limits_{\mathcal{A}\subset\{1,\ldots,n\}}\{|\sum_{k\in\mathcal{A}} [m_{t+1}(k) - m_t(k)]\,|\} = \max\{\sum_{k\in\mathcal{B}^+} M_{k,t}, -\sum_{k\notin\mathcal{B}^+} M_{k,t}\}$ and the fact that both terms have the same value by Lemma 2.
Now, from Lemma 3, for any given $q \in \mathbb{N}$ and $\mathcal{A} \subset \{1,\ldots,n\}$, we have:
$\sum\limits_{k\in\mathcal{A}} M_{k,t+q} = \sum\limits_{k\in\mathcal{A}} \left[(1 - \sum\limits_{j\neq k} a^{(q)}_{j,k})M_{k,t} + \sum\limits_{j\neq k} a^{(q)}_{k,j} M_{j,t}\right]$
And I group the terms with same $M_{i,t}$. Let's focus first on the positive terms (i.e. $M_{i,t}$ for $i \in \mathcal{B}^+$):

- If $i \in \mathcal{B}^+\cap\mathcal{A}$, then: (i) for $k = i$ we have the term $(1 - \sum\limits_{j\neq i} a^{(q)}_{j,i})M_{i,t}$; (ii) for each $k \in \mathcal{A}\setminus\{i\}$, we have the term $a^{(q)}_{k,i} M_{i,t}$.
- If $i \in \mathcal{B}^+\setminus\mathcal{A}$: for each $k \in \mathcal{A}$, we have the term $a^{(q)}_{k,i} M_{i,t}$.

Then, the positive terms can be written as:
$\sum\limits_{i\in\mathcal{B}^+\cap\mathcal{A}} M_{i,t}\left(1 - \sum\limits_{j\neq i} a^{(q)}_{j,i} + \sum\limits_{k\in\mathcal{A}\setminus\{i\}} a^{(q)}_{k,i}\right) + \sum\limits_{i\in\mathcal{B}^+\setminus\mathcal{A}} M_{i,t}\left(\sum\limits_{k\in\mathcal{A}} a^{(q)}_{k,i}\right) = \sum\limits_{i\in\mathcal{B}^+\cap\mathcal{A}} M_{i,t}\left(1 - \sum\limits_{k\notin\mathcal{A}} a^{(q)}_{k,i}\right) + \sum\limits_{i\in\mathcal{B}^+\setminus\mathcal{A}} M_{i,t}\left(\sum\limits_{k\in\mathcal{A}} a^{(q)}_{k,i}\right)$
And using that for any i $\sum_{k\neq i} a^{(q)}_{k,i} \in [0,1]$, we have

$$\sum_{i\in\mathcal{B}^+\cap\mathcal{A}} M_{i,t}\left(1 - \sum_{k\notin\mathcal{A}} a^{(q)}_{k,i}\right) + \sum_{i\in\mathcal{B}^+\setminus\mathcal{A}} M_{i,t}\left(\sum_{k\in\mathcal{A}} a^{(q)}_{k,i}\right) \leq \sum_{i\in\mathcal{B}^+} M_{i,t} \tag{38}$$

Analogously, for the negative terms (i.e $M_{i,t}$ for $i \in \mathcal{B}^-$):

- If $i \in \mathcal{A}\cap\mathcal{B}^-$, then: (i) for $k = i$ we have the term $(1 - \sum\limits_{j\neq i} a^{(q)}_{j,i})M_{i,t}$; (ii) for each $k \in \mathcal{A}\setminus\{i\}$, we have the term $a^{(q)}_{k,i} M_{i,t}$
- If $i \in \mathcal{B}^-\setminus\mathcal{A}$: for each $k \in \mathcal{A}$, we have the term $a^{(q)}_{k,i} M_{i,t}$

Then, the negative terms can be written as:
$\sum\limits_{i\in\mathcal{B}^-\cap\mathcal{A}} M_{i,t}\left(1 - \sum\limits_{j\neq i} a^{(q)}_{j,i} + \sum\limits_{k\in\mathcal{A}\setminus\{i\}} a^{(q)}_{k,i}\right) + \sum\limits_{i\in\mathcal{B}^-\setminus\mathcal{A}} M_{i,t}\left(\sum\limits_{k\in\mathcal{A}} a^{(q)}_{k,i}\right) = \sum\limits_{i\in\mathcal{B}^-\cap\mathcal{A}} M_{i,t}\left(1 - \sum\limits_{k\notin\mathcal{A}} a^{(q)}_{k,i}\right) + \sum\limits_{i\in\mathcal{B}^-\setminus\mathcal{A}} M_{i,t}\left(\sum\limits_{k\in\mathcal{A}} a^{(q)}_{k,i}\right)$
So, again using that for any i $\sum_{k\neq i} a^{(q)}_{k,i} \in [0,1]$, we have

$$\sum_{i\in\mathcal{B}^-\cap\mathcal{A}} (-M_{i,t})\left(1 - \sum_{k\notin\mathcal{A}} a^{(q)}_{k,i}\right) + \sum_{i\in\mathcal{B}^-\setminus\mathcal{A}} (-M_{i,t})\left(\sum_{k\in\mathcal{A}} a^{(q)}_{k,i}\right) \leq -\sum_{i\in\mathcal{B}^-} M_{i,t} \tag{39}$$

The first part of the proposition follows directly from the fact that the previous two inequalities for $q = 1$ imply that for any $\mathcal{A} \subset \{1,\ldots,n\}$, we have $|\sum_{k\in\mathcal{A}} M_{k,t+1}| \leq \sum_{k\in\mathcal{B}^+} M_{k,t}$, and so the inequality is also true for the maximum.
For the second part of the proposition, suppose the condition holds and I will show by contradiction that we cannot find any $\mathcal{A}$ such that $|\sum_{k\in\mathcal{A}} M_{k,t+q}| \leq \sum_{k\in\mathcal{B}^+} M_{k,t}$ holds with equality, and so the inequality has

to be strict.
In order for the equality to hold, it must be one of the following two cases:

- Case A: The positive terms are equal to the upper bound, and the negative terms are zero. For this to be the case, we need: (i) for $i \in \mathcal{B}^+ \cap \mathcal{A}$, $\sum_{k \notin \mathcal{A}} a_{k,i}^{(q)} = 0$; (ii) for $i \in \mathcal{B}^+ \setminus \mathcal{A}$, $\sum_{k \in \mathcal{A}} a_{k,i}^{(q)} = 1$; (iii) for $i \in \mathcal{B}^- \cap \mathcal{A}$, $\sum_{k \notin \mathcal{A}} a_{k,i}^{(q)} = 1$; and (iv) for $\mathcal{B}^- \setminus \mathcal{A}$, it must be $\sum_{k \in \mathcal{A}} a_{k,i}^{(q)} = 0$
If $i \in \mathcal{A}$, then condition (i) implies that also $\ell \in \mathcal{A}$, since otherwise we would have the contradiction $0 = \sum_{k \notin \mathcal{A}} a_{k,i}^{(q)} \geq a_{\ell,i}^{(q)} > 0$. If $i \notin \mathcal{A}$, then condition (ii) again implies that $\ell \in \mathcal{A}$, since otherwise we would have the contradiction $1 = \sum_{k \in \mathcal{A}} a_{k,i}^{(q)} \leq 1 - a_{\ell,i}^{(q)} < 0$. Therefore, ℓ must be in $\mathcal{A}$ in order for the positive terms to reach the upper bound.
Next, if $j \in \mathcal{A}$, then condition (iii) implies that $\ell \notin \mathcal{A}$, since otherwise we would have the contradiction $1 = \sum_{k \notin \mathcal{A}} a_{k,j}^{(q)} \leq 1 - a_{\ell,j}^{(q)} < 1$. So, the only possibility is that $j \notin \mathcal{A}$, but then condition (iv) contradicts that $\ell \in \mathcal{A}$, since then we would have the contradiction $0 = \sum_{k \in \mathcal{A}} a_{k,j}^{(q)} \geq a_{\ell,j}^{(q)} > 0$. Therefore, Case A is not possible.

- Case B: The positive terms are equal to zero, and the negative terms are equal to the lower bound. For this to be the case, we need: (i) for $i \in \mathcal{B}^+ \cap \mathcal{A}$, $\sum_{k \notin \mathcal{A}} a_{k,i}^{(q)} = 1$; (ii) for $i \in \mathcal{B}^+ \setminus \mathcal{A}$, $\sum_{k \in \mathcal{A}} a_{k,i}^{(q)} = 0$; (iii) for $i \in \mathcal{B}^- \cap \mathcal{A}$, $\sum_{k \notin \mathcal{A}} a_{k,i}^{(q)} = 0$; and (iv) for $\mathcal{B}^- \setminus \mathcal{A}$, it must be $\sum_{k \in \mathcal{A}} a_{k,i}^{(q)} = 1$
Analogously as in the previous case, we get to the conclusion that this case is not possible.
If $i \in \mathcal{A}$, then (i) implies $\ell \notin \mathcal{A}$, since otherwise $1 = \sum_{k \notin \mathcal{A}} a_{k,i}^{(q)} \leq 1 - a_{\ell,i}^{(q)} < 1$. If $i \notin \mathcal{A}$, then (ii) also implies that $\ell \notin \mathcal{A}$, since otherwise $0 = \sum_{k \in \mathcal{A}} a_{k,i}^{(q)} \geq a_{\ell,i}^{(q)} > 0$. So, it must be $\ell \notin \mathcal{A}$.
If $j \in \mathcal{A}$, (iii) implies the contradiction $0 = \sum_{k \notin \mathcal{A}} a_{k,j}^{(q)} \geq a_{\ell,j} > 0$. But if $j \notin \mathcal{A}$, (iv) also implies the contradiction $1 = \sum_{k \in \mathcal{A}} a_{k,j}^{(q)} \leq 1 - a_{\ell,j} < 1$. So, we conclude that this case is not possible.

■

This proposition tells us that a sufficient condition to guarantee convergence to an ergodic distribution is that whenever we are not in a stationary distribution, we can find states that have changed in opposite directions in the previous iteration (period) such that there exists some state which can be reached from each of the two states with positive probability in the same number of periods (in other words, if two points start from state i and j respectively, there is positive probability they will meet at some future period).
The next definitions and proposition show a sufficient condition for this condition to hold:

Definition 1 *A Markov chain is* ***irreducible*** *if for any pair of states i, j, there exists $q \in \mathbb{N}$ such that $a_{j,i}^{(q)} > 0$. (that is, it is possible to get to any state from any other state)*

Definition 2 *Let the* ***longitude of the shortest path between two states*** *i, j be $d_{i,j} = \min\{q \in \mathbb{N} | a_{i,j}^{(q)} > 0\}$ (that is, the smallest number of periods required to go from one state to the other).*

Proposition 5 *In an irreducible Markov chain that contains at least one state i with $d_{i,i} = 1$, as long as we are not in the stationary distribution, it is always possible to find states $j \in \mathcal{B}^+$ and $k \in \mathcal{B}^-$, and a state ℓ such that $a_{\ell,j}^{(q)}, a_{\ell,k}^{(q)} > 0$ for some $q \in \mathbb{N}$ (and so, in such Markov chain we can guarantee convergence to an ergodic distribution).*

Proof. If we are not in a stationary distribution then there are j with $M_{j,t} \neq 0$, and by Lemma 2 there must be $j \in \mathcal{B}^+$ and $k \in \mathcal{B}^-$. .Let i be the state such that $d_{i,i} = 1$. Then, it is sufficient to see that we can find $q \in \mathbb{N}$ such that $a_{i,j}^{(q)}, a_{i,k}^{(q)} > 0$, which is straightforward. We

can check that $q := \max(d_{i,j}, d_{i,k})$ satisfies this (intuitively, the first to arrive from one of the two states then stays with positive probability in i, and at some point the one that started from the other state will also arrive to i). Without loss of generality, assume $\max(d_{i,j}, d_{i,k}) = d_{i,k}$. $a_{i,k}^{d_{i,k}} > 0$ by definition of $d_{i,k}$. But also $a_{i,j}^{d_{i,k}} \geq a_{i,j}^{d_{i,j}} a_{i,i}^{(d_{i,k}-d_{i,j})} \geq a_{i,j}^{d_{i,j}} \left(a_{i,i}^{(1)}\right)^{d_{i,k}-d_{i,j}} > 0$
■

So, in the Uniqueness section I proved that the only possible stationary distribution is the one we would obtain if the initial sate is $\mathcal{J} = \emptyset$ (if this converges). Now, the previous proposition tells us that $P_{t,0}(X)$ converges, since the Markov chain obtained is irreducible (if a state is possible, it means that there was positive probability of arriving to it starting from $\mathcal{J} = \emptyset$; and, from any state, there is probability 1 of eventually going back to $\mathcal{J} = \emptyset$) and the state $\mathcal{J} = \emptyset$ satisfies that the longitude of its shortest path connecting it to itself is 1 (with positive probability there will be no entrant and we stay at $\mathcal{J} = \emptyset$).

7.9 Summary of the method to solve the model

7.10 Multivariate Newton Interpolation

First, as a recap of the univariate Newton interpolation, given $n+1$ different points (nodes) defined as a pair $(x_i, f(x_i))$ with $x_i \neq x_j$ for any $j \neq i$, then the unique n-degree polynomial that passes through these $n+1$ points expressed in the Newton basis polynomials (which are defined as $w_j(x) = \prod_{k=0}^{j-1}(x - x_k)$, $j = 1, 2, \ldots, n$ and $w_0(x) = 1$) is $P_n(x) = \sum_{j=0}^{n} a_j w_j(x)$, where the coefficients a_i are the solutions of the system (note that $w_j(x_i) = 0$ when $i < j$):

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & (x_1 - x_0) & 0 & \ldots & 0 \\ 1 & (x_2 - x_0) & \prod_{k=0}^{1}(x_2 - x_k) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_n - x_0) & \prod_{k=0}^{1}(x_n - x_k) & \ldots & \prod_{k=0}^{n-1}(x_n - x_k) \end{pmatrix}_{(n+1)\times(n+1)} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}$$

This system can be solved reducing the system iteratively. To this purpose, note that a_0 is already solved: $a_0 = f(x_0)$ so we can forget about row 1 and the rest writes:

$$\begin{pmatrix} f(x_0) \\ f(x_0) \\ \vdots \\ f(x_0) \end{pmatrix} + \begin{pmatrix} (x_1 - x_0) & 0 & \ldots & 0 \\ (x_2 - x_0) & \prod_{k=0}^{1}(x_2 - x_k) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (x_n - x_0) & \prod_{k=0}^{1}(x_n - x_k) & \ldots & \prod_{k=0}^{n}(x_n - x_k) \end{pmatrix}_{n\times n} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}$$

Next, note that each element of row i in the $n \times n$ matrix (the Vandermonde matrix) contains $(x_i - x_0)$, so we can divide both sides of row i by $(x_i - x_0)$ and calling $f[x_i, x_0] = \frac{f(x_i) - f(x_0)}{x_i - x_0}$ (divided difference) we obtain an analogous system as the initial one but with one dimension less:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & (x_2-x_1) & 0 & \dots & 0 \\ 1 & (x_3-x_1) & \prod_{k=1}^{2}(x_3-x_k) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_n-x_1) & \prod_{k=1}^{2}(x_n-x_k) & \dots & \prod_{k=1}^{n-1}(x_n-x_k) \end{pmatrix}_{n\times n} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f[x_1,x_0] \\ f[x_2,x_0] \\ f[x_3,x_0] \\ \vdots \\ f[x_n,x_0] \end{pmatrix}$$

Now, $a_1 = f[x_1, x_0]$ is already solved; so, we repeat the procedure: we pass subtracting the $1 \cdot a_i$ of each row to the right hand side and we divide by the common factor of the left side $(x_i - x_1)$, we call $f[x_i, x_1, x_0] = \frac{f[x_i,x_0]-f[x_1,x_0]}{x_i-x_1}$, and we obtain:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & (x_3-x_2) & 0 & \dots & 0 \\ 1 & (x_4-x_2) & \prod_{k=2}^{3}(x_4-x_k) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_n-x_2) & \prod_{k=2}^{3}(x_n-x_k) & \dots & \prod_{k=2}^{n-1}(x_n-x_k) \end{pmatrix}_{(n-1)\times(n-1)} \begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f[x_2,x_1,x_0] \\ f[x_3,x_1,x_0] \\ f[x_4,x_1,x_0] \\ \vdots \\ f[x_n,x_1,x_0] \end{pmatrix}$$

Iterating, in the r-th iteration we will get:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & (x_r-x_{r-1}) & 0 & \dots & 0 \\ 1 & (x_{r+1}-x_{r-1}) & \prod_{k=r-1}^{r}(x_{r+1}-x_k) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_n-x_{r-1}) & \prod_{k=r-1}^{r}(x_n-x_k) & \dots & \prod_{k=r-1}^{n-1}(x_n-x_k) \end{pmatrix}_{(n-r+2)\times(n-r+2)} \begin{pmatrix} a_{r-1} \\ a_r \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f[x_{r-1},x_{r-2}\dots,x_0] \\ f[x_r,x_{r-2}\dots,x_0] \\ f[x_{r+1},x_{r-2}\dots,x_0] \\ \vdots \\ f[x_n,x_{r-2}\dots,x_0] \end{pmatrix}$$

Summarizing, the coefficients of the newton interpolation polynomial are given by the divided differences $a_i = f[x_i, x_{i-1}, \dots, x_0] = \frac{f[x_i,x_{i-2},\dots,x_0]-f[x_{i-1},x_{i-2},\dots,x_0]}{x_i-x_{i-1}}$.

We can extend this to the multivariate case as follows. Suppose we want to interpolate a function $f : \mathbb{R}^m \to \mathbb{R}$ by a polynomial of m variables and degree n.

Definition 3 ***(Generating points):*** *For each dimension $i = 1, \dots, m$, we define $n+1$ points $x_{i,k}$, $k = 0, \dots, n$. $\{\{x_{i,k}\}_{k=0}^{n}\}_{i=1}^{m}$ are called the generating points.*

Definition 4 (Multiindices): *Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_m) \in \Lambda_{m,n} := \{\vec{\alpha} \in \{0, \dots, n\}^m | \sum_{i=1}^{m} \alpha_i \leq n\}$, and $\vec{x}_{\vec{\alpha}} = (x_{1,\alpha_1}, \dots, x_{m,\alpha_m})$.*

The cardinal of $\Lambda_{m,n}$ (i.e. the number of different multiindices) is given by $N(m,n) = \binom{n+m}{n}$ (to see this, you can think of $1^{\alpha_0} x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ with $\sum_{i=0}^{m} \alpha_i = n$, which we can transcribe as $\underbrace{1\dots1}_{\alpha_0}\#\underbrace{x_1\dots x_1}_{\alpha_1}\#\dots\#\underbrace{x_m\dots x_m}_{\alpha_m}$; so the problem of finding the number of different multiindices is equivalent to finding the number of different ways we can choose m boxes from $n+m$ boxes (i.e. the position of the m hashtags), which is $\binom{n+m}{m}$).

Definition 5 (Newton polynomial): $w_{\vec{\alpha}}(\vec{x}) = \prod_{i=1}^{m} \prod_{k=0}^{\alpha_i-1} (x_i - x_{i,k})$.

Definition 6 *The m-dimensional* ***Newton interpolating polynomial*** *of degree n of the function f is* $p_{m,n}(\vec{x}) = \sum_{\vec{\alpha}\in\Lambda_{m,n}} a_{\vec{\alpha}} w_{\vec{\alpha}}(\vec{x})$*, satisfying* $f(\vec{x}_{\vec{\alpha}}) = p_{m,n}(\vec{x}_{\vec{\alpha}})$*, for all* $\vec{\alpha} \in \Lambda_{m,n}$*.*

Lemma 5 *Note that given* $\vec{\beta}, \vec{\alpha} \in \Lambda_{m,n}$*, if* $\beta_i - 1 \geq \alpha_i$*, then* $w_{\vec{\beta}}(\vec{x}_{\vec{\alpha}})$ *contains the term* $(x_{i,\alpha_i} - x_{i\alpha_i}) = 0$*.*

Corollary 1 *Then,* $f(\vec{x}_{\vec{\alpha}}) = p_{m,n}(\vec{x}_{\vec{\alpha}}) = \sum_{k_m=-1}^{\alpha_m-1} \cdots \sum_{k_1=-1}^{\alpha_1-1} \prod_{s_m=0}^{k_m} (x_{m,\alpha_m} - x_{m,s_m}) \cdots \prod_{s_m=0}^{k_m} (x_{1,\alpha_1} - x_{1,s_1}) a_{(k_1+1,\ldots,k_m+1)}$

To allow generality, I define:

Definition 7 *Given* $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$*, define:*
(i) $\vec{\alpha}^{(i,k)} = (\alpha_1, \ldots, \alpha_{i-1}, k, \alpha_{i+1}, \ldots, \alpha_m)$ *(i.e.* $\vec{\alpha}^{(i,k)}$ *equals* $\vec{\alpha}$ *except k in position i)*
(ii) $\vec{\alpha}^{(k)} = (\alpha_{k+1}, \ldots, \alpha_m)$*.*
(iii) $\vec{\beta}^{(i,k)} = (\vec{\alpha}^{(i,k-1)}, \ldots, \vec{\alpha}^{(i,0)}, \ldots, \vec{\alpha}^{(m,\alpha_m-1)}, \ldots, \vec{\alpha}^{(m,0)})$*, and let* $\vec{\beta}^{(i,0)} = \vec{\beta}^{(i+1,\alpha_{i+1}-1)}$ *and* $\vec{\beta}^{(m,0)} = \emptyset$*.*

Definition 8 (Divided differences):
$f[\vec{\alpha}, \vec{\beta}^{(i,b)}] = \sum_{k_i=b-1}^{\alpha_i-1} \cdots \sum_{k_1=-1}^{\alpha_1-1} \prod_{s_i=b}^{k_i} (x_{m,\alpha_m} - x_{m,s_m}) \cdots \prod_{s_1=0}^{k_1} (x_{1,\alpha_1} - x_{1,s_1}) a_{(k_1+1,\ldots,k_i+1,\vec{\alpha}^{(i)})}$ *if* $\alpha_i > b$*; and* $f[\vec{\alpha}, \vec{\beta}^{(i,b)}] = f[\vec{\alpha}, \vec{\beta}^{(i-1,0)}]$ *otherwise.*

Note that by Corollary 1, and since $\vec{\alpha}^{(m)} = \emptyset$*, then* $f[\vec{\alpha}, \vec{\beta}^{(m,0)}] = f(\vec{x}_{\vec{\alpha}})$*. The algorithm to find the coefficients* $a_{\vec{\alpha}}$ *is defined as follows:*

1. *Start setting* $i = m$ *and* $b = 0$*.*

2. *If there is some* $\vec{\alpha} \in \Lambda_{m,n}$ *such that* $\alpha_i > b$*, then:*

   (a) *For all the* $\vec{\alpha} \in \Lambda_{m,n}$ *such that* $\alpha_i > b$*: Noting that* $f[\vec{\alpha}^{(i,b)}, \vec{\beta}^{(i,b)}]$ *contains all the terms of* $f[\vec{\alpha}, \vec{\beta}^{(i,b)}]$ *with* $k_i = b - 1$*, and so the remaining terms will all contain* $(x_{i,\alpha_i} - x_{i,b})$*; then:*

   $$\begin{aligned} f[\vec{\alpha}, \vec{\beta}^{(i,b+1)}] &= \frac{f[\vec{\alpha}, \vec{\beta}^{(i,b)}] - f[\vec{\alpha}^{(i,b)}, \vec{\beta}^{(i,b)}]}{x_{i,\alpha_i} - x_{i,b}} \\ &= \sum_{k_i=b}^{\alpha_i-1} \cdots \sum_{k_1=-1}^{\alpha_1-1} \prod_{s_i=b+1}^{k_i} (x_{m,\alpha_m} - x_{m,s_m}) \cdots \prod_{s_1=0}^{k_1} (x_{1,\alpha_1} - x_{1,s_1}) a_{(k_1+1,\ldots,k_i+1,\vec{\alpha}^{(i)})} \end{aligned}$$

   (b) *For all the* $\vec{\alpha} \in \Lambda_{m,n}$ *such that* $\alpha_i \leq b$*, then* $f[\vec{\alpha}, \vec{\beta}^{(i,b+1)}] = f[\vec{\alpha}, \vec{\beta}^{(i,b)}]$ *(satisfies the definition since* $\alpha_i \leq b < b + 1$*, so* $f[\vec{\alpha}, \vec{\beta}^{(i,b+1)}] = f[\vec{\alpha}, \vec{\beta}^{(i,b)}] = f[\vec{\alpha}, \vec{\beta}^{(i-1,0)}]$

   Set $b = b + 1$*, and go back to step 2.*

3. *If* $\alpha_i \leq b$ *for all* $\vec{\alpha} \in \Lambda_{m,n}$ *(which is satisfied if and only if* $b \leq n$*), then make* $f[\vec{\alpha}, \vec{\beta}^{(i-1,0)}] = f[\vec{\alpha}, \vec{\beta}^{(i,b)}]$*, and set* $i = i - 1$ *and* $b = 0$*. If* $i = 0$*, we are done; otherwise, go back to step 2.*

All is left to do is to show that the $f[\vec{\alpha}, \vec{\beta}^{(0,0)}] = a_{\vec{\alpha}}$ for all $\vec{\alpha} \in \Lambda_{m,n}$. Given that the divided difference of $\vec{\alpha}$ just changes when we apply (2a) to it, then it is sufficient to see that in the last time that we select $\vec{\alpha}$ for (2a) it is $f[\vec{\alpha}, \vec{\beta}^{(i,b+1)}] = a_{\vec{\alpha}}$; since then it will be $f[\vec{\alpha}, \vec{\beta}^{(0,0)}] = f[\vec{\alpha}, \vec{\beta}^{(i,b+1)}] = a_{\vec{\alpha}}$.

Proof. *If we have used $a_{\vec{\alpha}}$ in (2a), it means that $\alpha_i > b$, which implies that exactly one of the following is true:*

1. *$\alpha_i > b+1$, in which case $a_{\vec{\alpha}}$ would also be selected in the next iteration, contradicting it was the last time it was selected;*

2. *$\alpha_i = b+1$, in which case $a_{\vec{\alpha}}$ it is the last iteration for variable i that $a_{\vec{\alpha}}$ is selected. In this case there are two possibilities:*

   - *$\alpha_k > 0$ for some $k < i$, in which case in iteration $(k, 0)$ $\vec{\alpha}$ would be selected, contradicting the hypothesis.*

   - *$\alpha_k = 0$ for all $k < i$, in which case we have:*

$$\begin{aligned} f[\vec{\alpha}, \vec{\beta}^{(i,b+1)}] &= \sum_{k_i=\alpha_i-1}^{\alpha_i-1} \cdots \sum_{k_1=-1}^{\alpha_1-1} \prod_{s_i=\alpha_i}^{k_i} (x_{m,\alpha_m} - x_{m,s_m}) \cdots \prod_{s_1=0}^{k_1} (x_{1,\alpha_1} - x_{1,s_1}) a_{(k_1+1,\dots,k_i+1,\vec{\alpha}^{(i)})} \\ &= \prod_{s_i=\alpha_i}^{\alpha_i-1} (x_{m,\alpha_m} - x_{m,s_m}) \cdots \prod_{s_1=0}^{-1} (x_{1,\alpha_1} - x_{1,s_1}) a_{(0,\dots,0,\alpha_i,\vec{\alpha}^{(i)})} \\ &= a_{(0,\dots,0,\alpha_i,\vec{\alpha}^{(i)})} = a_{\vec{\alpha}} \end{aligned}$$

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