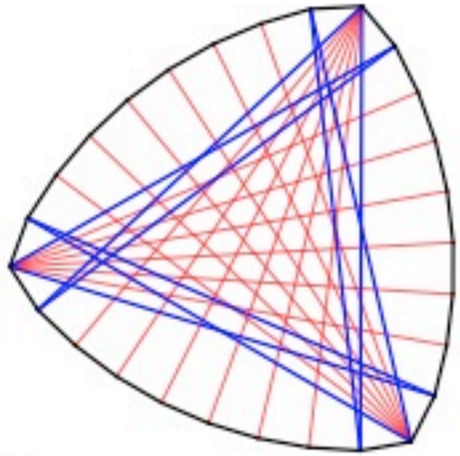
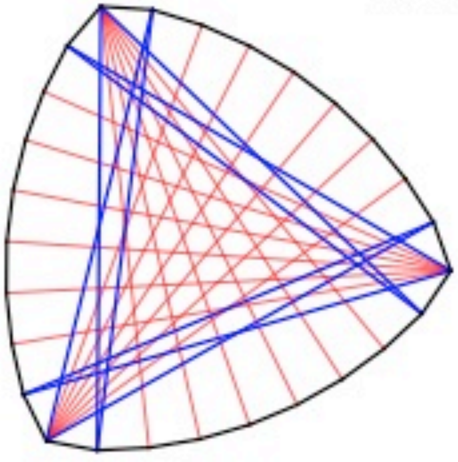


Computation and Analysis on Reinhardt Polygons with Multiple Prime Divisors

Molly Feldman, Robert Kenyon, Jiahui Liu



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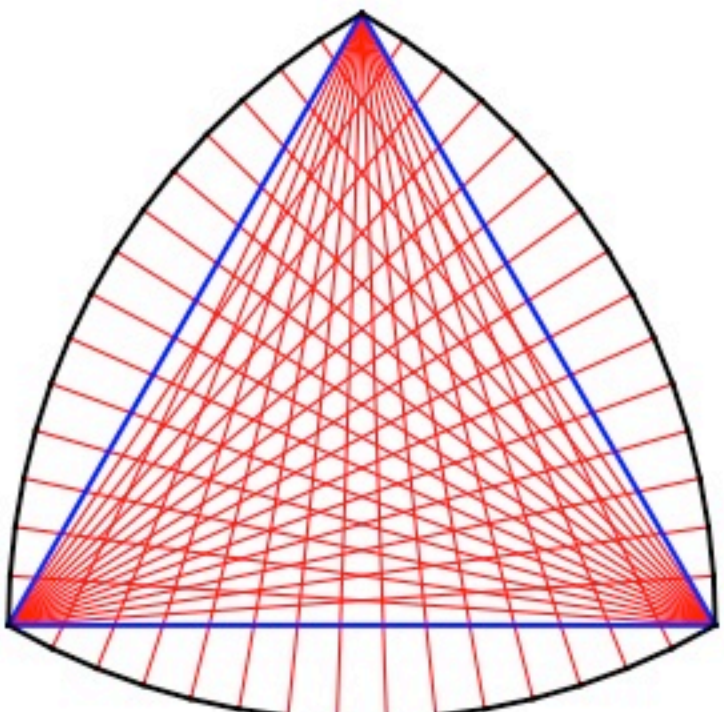
Reinhardt Polygons

- P is a convex polygon in the plane with n sides
- P has four relevant properties
  - Area
  - Perimeter
  - Diameter
  - Width
- There are 3 extremal problems
- Reinhardt polygons are optimal in all three problems

Reinhardt Polygons (cont.)

Polynomial: $F(z) = 1 - z^{15} + z^{30}$



Polygon

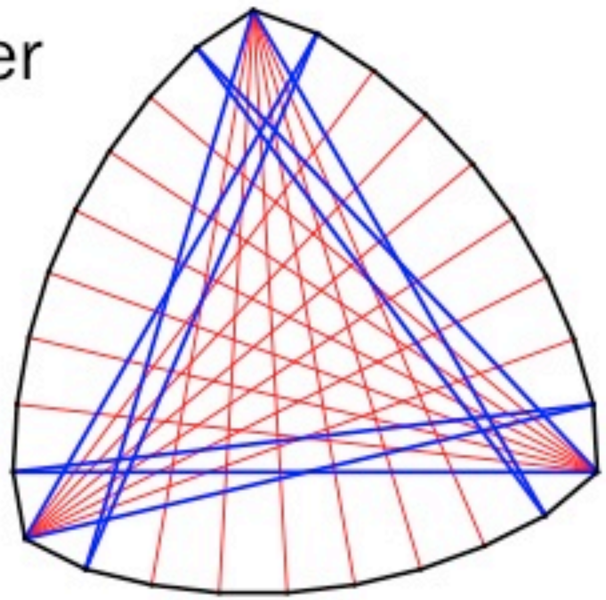
Coefficient array:

$[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$
$0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$
$0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$
$0, 0, 0, 0]$

Composition: $[\ 15\ 15\ 15\ ]$

Representations

1. Polygons
   - Must have a “star” cycle in it’s center (shown in blue)
   - Each angle in the star can be represented as a multiple of the angle $\frac{\pi}{n}$
   - All diameters cross one another



Representations (cont.)

2. Polynomials

- $F(z)$ is divisible by $\Phi_{2n}(z) = \Phi_n(-z)$
- $F(z)$ has alternating sign and all coefficients are elements of $\{1, 0, -1\}$
- $F(z)$ has an odd number of nonzero coefficient
- In most cases, we restrict $F(0) = 1$

Representations (cont.)

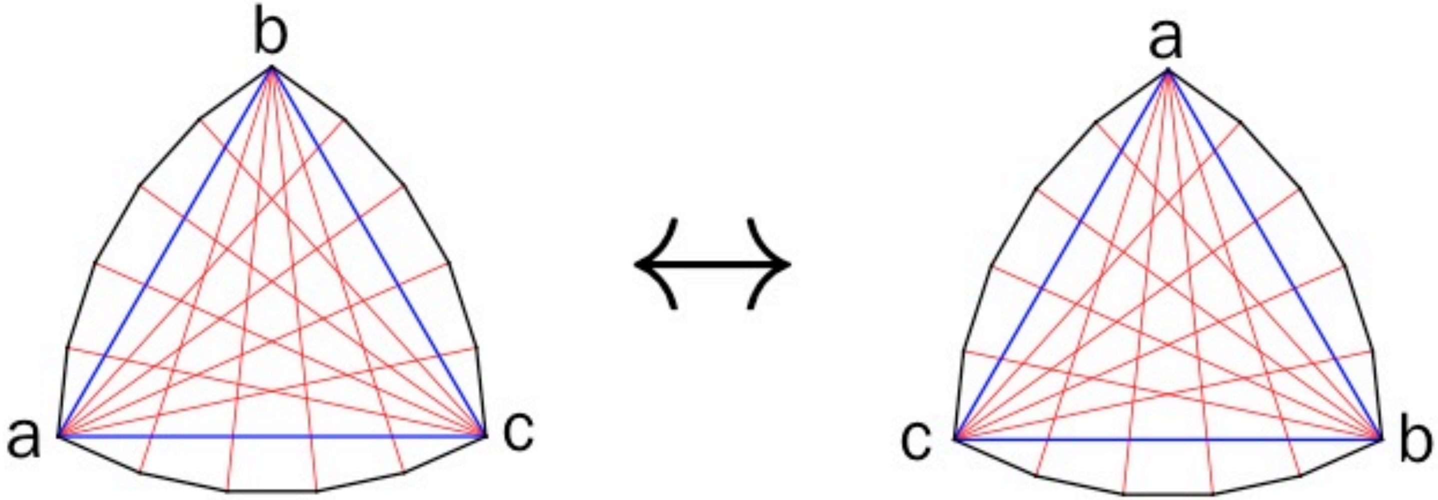
3. Compositions

- Composition of n is a sequence of positive integers whose sum is n
- We represent a Reinhardt polynomial as a composition of the number of “gaps” between nonzero terms
  - Must always have an odd number of terms

For *n* = 30,

$$1 - z^4 + z^7 - z^8 + z^9 \Rightarrow [4\ 3\ 1\ 1\ 21\ ]$$

Dihedral Equivalence



$$
\begin{array}{cc}
[\ 4\ 21\ 1\ 1\ 3\ ] & [\ 4\ 3\ 1\ 1\ 21\ ] \\
[\ 3\ 4\ 21\ 1\ 1\ ] & [\ 3\ 1\ 1\ 21\ 4\ ] \\
[\ 1\ 3\ 4\ 21\ 1\ ] & [\ 1\ 1\ 21\ 4\ 3\ ] \\
[\ 1\ 1\ 3\ 4\ 21\ ] & [\ 1\ 21\ 4\ 3\ 1\ ] \\
[\ 21\ 1\ 1\ 3\ 4\ ] & [\ 21\ 4\ 3\ 1\ 1\ ]
\end{array}
$$

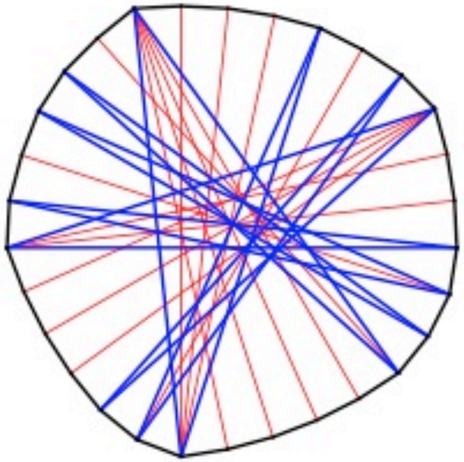
Normalization

- Method to create one unique representation of a polygon

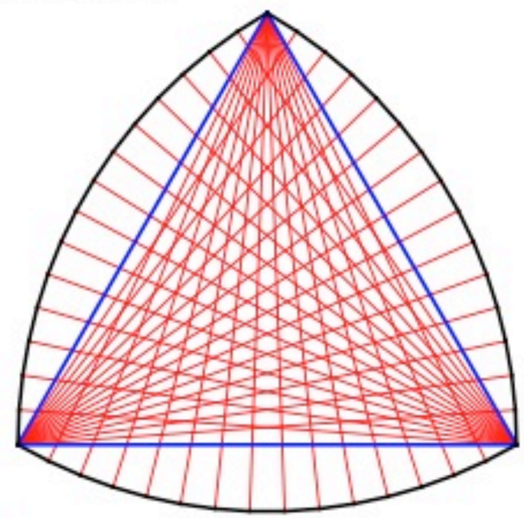
Start: [3 4 1 1 2 1 4 5 2 1 1 1 2 1 1]

[3 4 1 1 2 1 4 **5** 2 1 1 1 2 1 1]

End: [5 4 1 2 1 1 4 3 1 1 2 1 1 1 2]

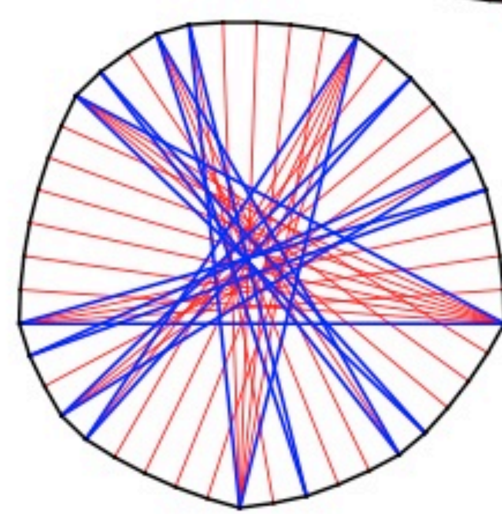


Sporadic vs. Periodic



- *Periodic* Reinhardt polygons are those where the composition repeats periodically

  e.g. [15 15 15] = $[(15)^3]$



- *Sporadic* polygons are Reinhardt polygons which are not periodic

  e.g. [7 4 1 1 2 3 1 2 5 5 2 1 3 2 1 1 4]

Alternate Representation

- From previous work, we can define

$$F(z) = f_1(z)\Phi_q(-z^{pr}) + f_2(z)\Phi_p(-z^{qr})$$

$$g_1(z) = f_1(z)\Phi_q(-z^{pr})$$

e.g.

f_1 + 0 0 − + 0

g_1 + 0 0 − + 0 − 0 0 + − 0 + 0 0 − + 0 ...

- This representation restricts the problem of finding valid Reinhardt polynomials (i.e. $F(z)$) to finding valid f_1 and f_2

Previous Results

- The number of periodics is known for all n
- For almost all $n \geq 105$, the number of sporadic polygons exceeds the number of periodic ones
- Let $E_1(n)$ be the number of sporadic Reinhardt polygons for a given n
- $E_1(2pq) = \frac{2^{p-1}-1}{p} \cdot \frac{2^{q-1}-1}{q}$

Previous Data

n	$E_1(n)$
30	3
45	144
60	4,392
66	93
75	153,660
78	315
84	161,028
90	5,385,768
110	279
117	2,587,284

Second Construction

- Goal: find valid f_1 and f_2 such that they satisfy the alternate representation,

$$F(z) = f_1(z)\Phi_q(-z^{pr}) + f_2(z)\Phi_p(-z^{qr})$$

and the other necessary properties

Second Construction

- We know $F(z) = f_1(z)\Phi_q(-z^{pr}) + f_2(z)\Phi_p(-z^{qr})$
- Goal: find specific f_1 and f_2 such that this equation holds
- To do so, we utilize the following algorithm, also known as “the second construction”

Remember: p and q are distinct odd primes

$r \geq 2, r \in \mathbb{Z}$

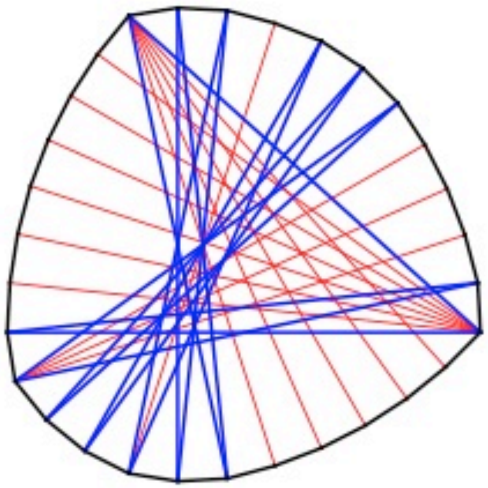
$$n = 30;\ p = 3,\ q = 5,\ r = 2$$

$$A_1 = +0\ \ B_1 = 00\ \ A_2 = 0-\ \ B_2 = +-$$

$$g_1\ \overbrace{+\,0\,0\,-\,+\,0}^{f_1}\ \boxed{-\,0\,0\,+\,-\,0}\ +\,0\,0\,-\,+\,0\ \boxed{-\,0\,0\,+\,-\,0}\ +\,0\,0\,-\,+\,0$$

$$g_2\ 0\,0\,0\,+\,-\,0\,0\,+\,-\,0\ \boxed{0\,0\,0\,-\,+\,0\,0\,-\,+\,0}\ 0\,0\,0\,+\,-\,0\,0\,+\,-\,0$$

$$F\ +\,0\,0\,0\,0\,0\,-\,+\,-\,+\,-\,0\,+\,-\,+\,-\,+\,-\,0\,0\,0\,+\,-\,+\,0\,0\,0\,0\,0\,0$$



[7 6 1 1 1 1 2 1 1 1 1 1 4 1 1]

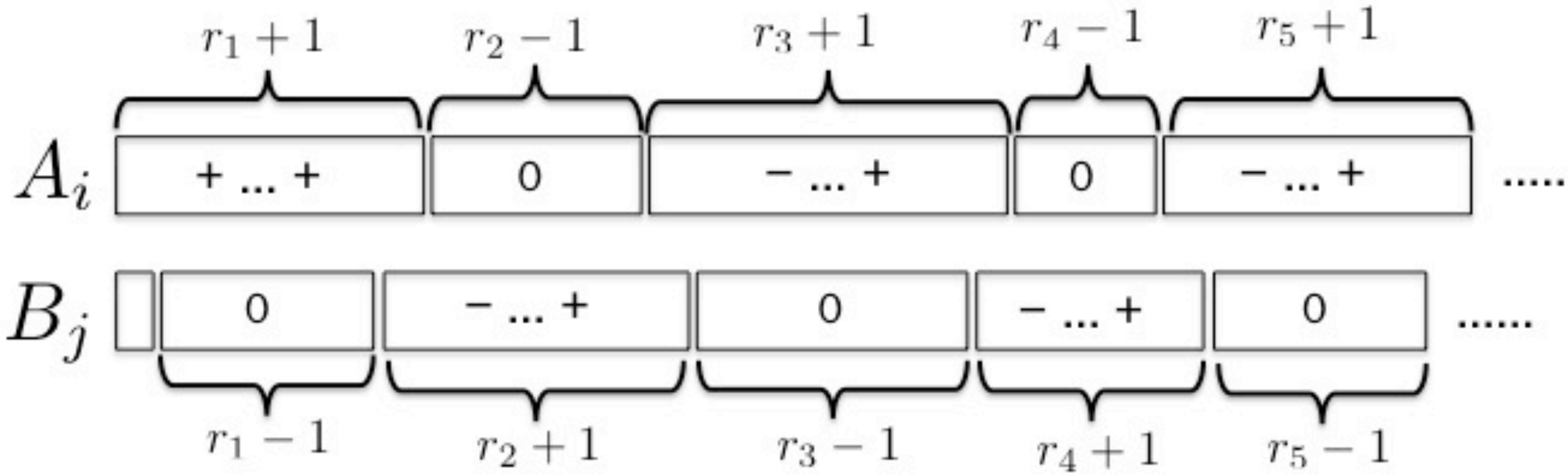
Second Construction (cont.)

- Let c be a composition of r into an even number of parts: $c = (r_1, r_2, \ldots, r_{2m})$
- We decompose f_1 and f_2 into blocks of r terms, i.e.

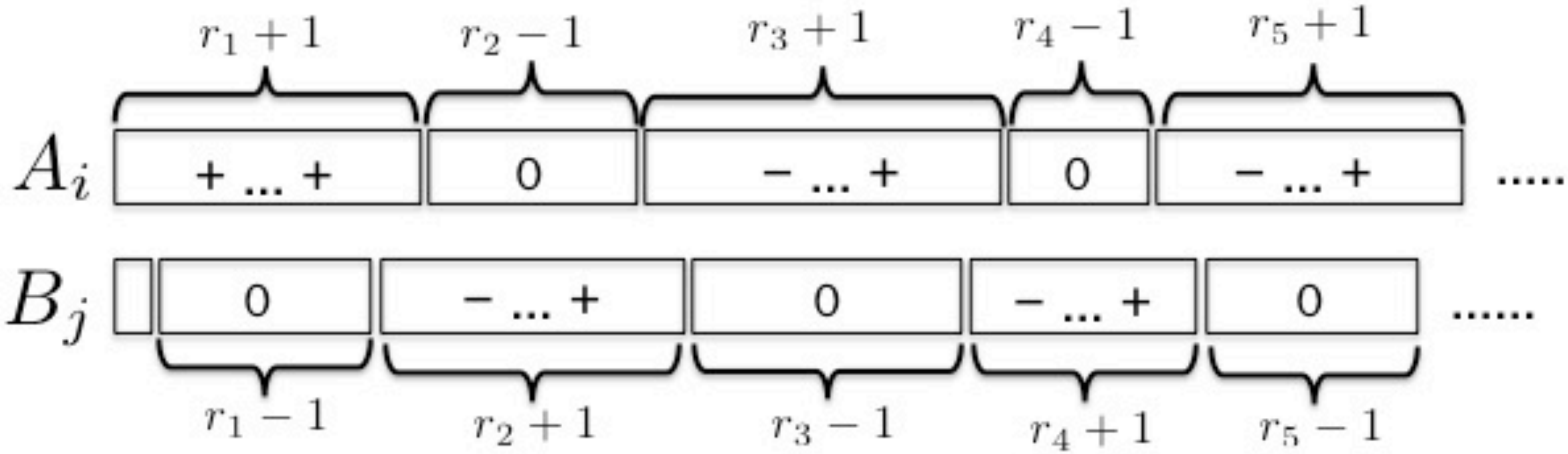
$$f_1 = A_1 A_2 A_3 \ldots A_p \qquad \text{degree}(f_1) < rp$$

$$f_2 = B_1 B_2 B_3 \ldots B_q \qquad \text{degree}(f_2) < rq$$

Second Construction (cont.)



Second Construction (cont.)



$$A_{i,j} \in \begin{cases} S_o(r_1+1, (-1)^{i+1}s), & j=1, \\ S_e(r_j+1, (-1)^{i}s), & j \geq 3 \text{ odd}, \\ Z(r_j-1), & j \text{ even} \end{cases}$$

$$B_{i,j} \in \begin{cases} Z(r_j-1), & j \text{ odd}, \\ S_e(r_j+1, (-1)^{i}s), & j \text{ even} \end{cases}$$

$$n = 30;\ p = 3,\ q = 5,\ r = 2$$

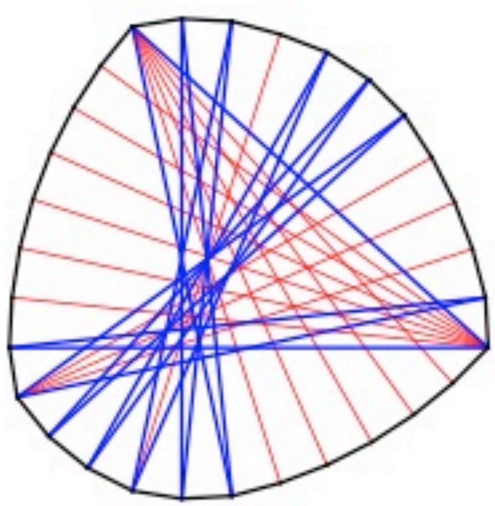
$$A_1 = 0+ \quad B_1 = 0- \quad A_2 = +0 \quad B_2 = 0-$$

$$C = +$$

f_1 + − 0 0 0 0 − + 0 0 0 0 + − 0 0 0 0 − + 0 0 0 0 + − 0 0 0 0

f_2 0 0 + 0 − + 0 0 − + 0 0 − 0 + − 0 0 + − 0 0 + 0 − + 0 0 − +

F + − + 0 − + − + − + 0 0 0 − + − 0 0 0 0 0 0 + 0 0 0 0 0 − +



[7 6 1 1 1 1 2 1 1 1 1 4 1 1]

Analysis of Second Construction

- It has been proven that this algorithm only gives us Reinhardt polygons (although many dihedrals)
- Counting the number of periodics and sporadics we obtain a lower bound on $E_1(n)$ by dividing by the total number of possible dihedrals

$$E_1(n) \geq \frac{v}{r}\left(2^{r-2} \cdot \frac{2^{r_o(p-1)}-1}{p} \cdot \frac{2^{r_e(q-1)}-1}{q} - \frac{U}{4pq}\right)$$

EXTENSIONS TO 3 DISTINCT PRIME DIVISORS

Third Construction

- We generalized the second construction to include a third distinct prime factor, l , so now we have $n = pqlr$ with $r \geq 3$

- Goal is to determine nontrivial $f_1(z),\ f_2(z),$ and $f_3(z)$ such that

$$F(z) = f_1(z)\Phi_q(-z^{lpr}) + f_2(z)\Phi_p(-z^{lqr}) + f_3(z)\Phi_l(-z^{pqr})$$

is a valid Reinhardt polynomial

Third Construction (cont.)

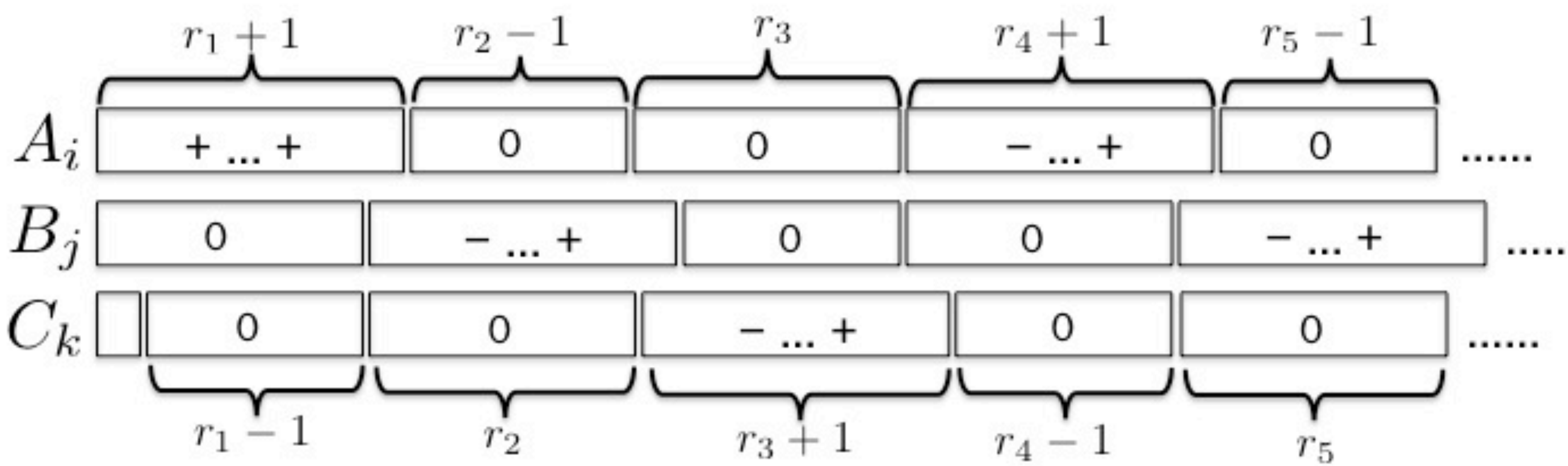
- Let c be a composition of r into m parts such that $m \equiv 0 \mod 3$: $c = (r_1, r_2, \ldots r_m)$
- We decompose f_1, f_2 , and f_3 into blocks of r terms, i.e.

$$f_1 = A_1 A_2 A_3 \ldots A_{pq} \qquad \text{degree}(f_1) < rpq$$

$$f_2 = B_1 B_2 B_3 \ldots B_{pl} \qquad \text{degree}(f_2) < rpl$$

$$f_3 = C_1 C_2 C_3 \ldots C_{ql} \qquad \text{degree}(f_3) < rql$$

Third Construction (cont.)



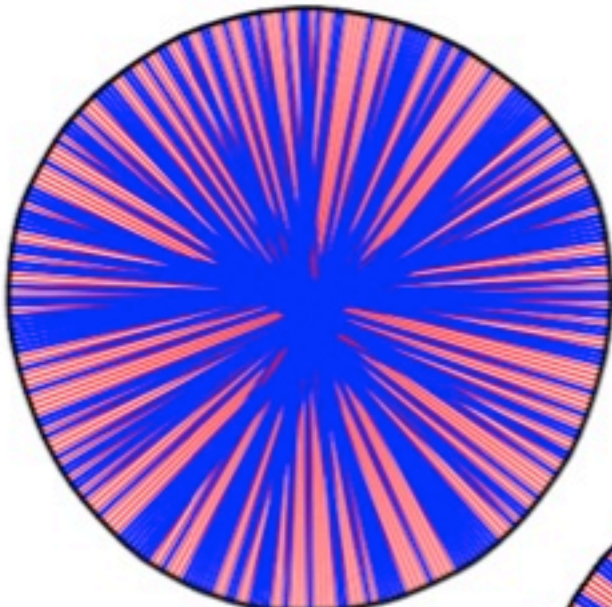
Third Construction (cont.)

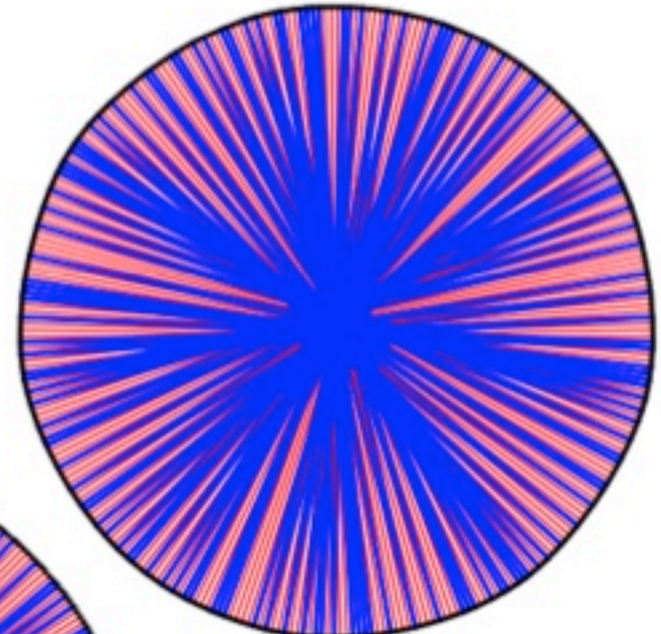
$$A_{i,j} \in \begin{cases} S_o(r_j+1,(-1)^{i+1}) & \text{if } j=1 \\ S_e(r_j+1,(-1)^{i}) & \text{if } j=1 \bmod 3 \\ Z(r_j-1) & \text{if } j=0 \bmod 3 \\ Z(r_j) & \text{if } j=2 \bmod 3 \end{cases}$$

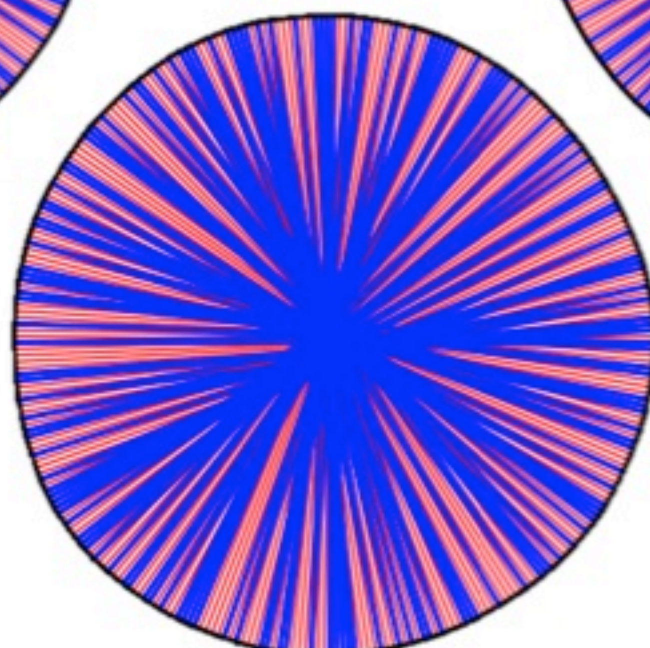
$$B_{i,j} \in \begin{cases} S_e(r_j+1,(-1)^{i}) & \text{if } j=2 \bmod 3 \\ Z(r_j-1) & \text{if } j=0 \bmod 3 \\ Z(r_j) & \text{if } j=1 \bmod 3 \end{cases}$$

$$C_{i,j} \in \begin{cases} S_e(r_j+1,(-1)^{i}) & \text{if } j=0 \bmod 3 \\ Z(r_j-1) & \text{if } j=1 \bmod 3 \\ Z(r_j) & \text{if } j=2 \bmod 3 \end{cases}$$

Computational Approach







Analysis of the Third Construction

- The total number of Reinhardt polygons produced by the third construction is $2^{(pqr_1+plr_2+qlr_3)}$
- If $r=3$, i.e. $r_1=r_2=r_3=1$, then the number of periodics produced is

$$2^{(ql+q+l+p-1)}+2^{(pl+p+l+q-1)}+2^{pq+p+q+l-1)}-2^{(2l+p+q-1)}-2^{(2q+l+p-1)}-2^{(2p+p+l-1)}$$

Analysis (cont.)

$$E_1(pqlr) \geq \tfrac{v}{4pqlr}(2^{(pqr_1+plr_2+qlr_3)} -$$
$$2^{(qlr_3+qr_1+lr_2+(p-1)\max(r_1,r_2))} -$$
$$2^{(plr_2+pr_1+lr_2+(q-1)\max(r_1,r_3))} -$$
$$2^{(pqr_1+pr_2+qr_3+(l-1)\max(r_2,r_3))} +$$
$$2^{(l(r_2+r_3)+\min[(p+q-1)r_1,pr_1+(q-1)r_4,qr_1+(p-1)r_2])} +$$
$$2^{(q(r_1+r_3)+\min[(l+p-1)r_2,lr_2+(p-1)r_1,pr_2+(l-1)r_3])} +$$
$$2^{(p(r_1+r_2)+\min[(q+l-1)r_3,qr_3+(l-1)r_2,lr_3+(q-1)r_1])})$$

Analysis (cont.)

- If we let $r = 3$, i.e. $r_1 = r_2 = r_3 = 1$ in the previous equation, and $n = pqlr$, we obtain

$$E_1(n) \geq \tfrac{v}{4pqlr}\left(2^{(p+1+l-1)}\right) \cdot \left[2^{(pq+lp+qp-p-q-l+1)} - 2^{ql} - 2^{pq} - 2^{pl} + 2^{l} + 2^{q} + 2^{p}\right]$$

Comparisons

- For the second construction, the lower bound gives us approximately
$$E_1(315) \geq 1.13 \cdot 10^{16}$$

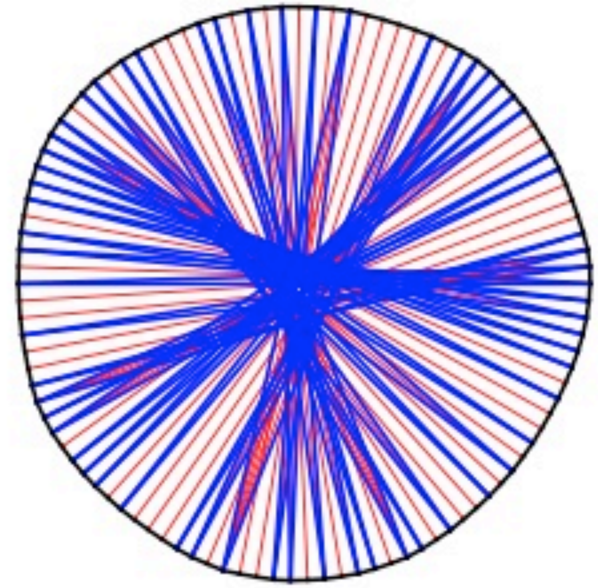
- The third construction gives us approximately
$$E_1(315) \geq 5.62 \cdot 10^{18}$$

What's Next?

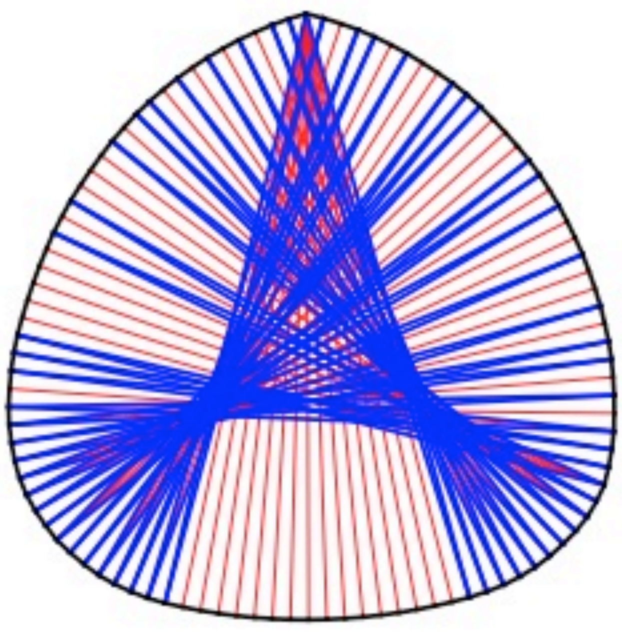
- Can we combine different compositions and obtain a better lower bound?

- Can we find a formula for $E_1(3pql)$?

- Does the second/third construction give us all Reinhardt polygons?



FINDING SPORADIC 105-GONS



Relevant Challenges

1. Computational Limits
   - *Current solution:* restricting search space

2. Determining Uniqueness
   - *Current solution:* normalization

Previously Known Results

- There are approximately 245 million *periodic* Reinhardt 105-gons
- We believe there are over 350 million *sporadic* 105-gons
- We can only determine **around 60%** of all sporadic Reinhardt 105-gons using the second construction

New Generation Methods

- Enumerating all possible f_1, f_2, and f_3 is computationally impossible
  - This is on order of $O(2^k)$

- Requires a new approach: if we can restrict one of f_1, f_2, or f_3, then we can obtain a more attainable runtime

First Program

- For $n = 105$, f_1 has degree 35, f_2 has degree 21, and f_3 has degree 15

- We can cycle through all possibilities for f_1 and f_2 and then restrict the possible f_3's

e.g.

$$\begin{array}{ll} f_1 & \ldots + - \ldots \\ f_2 & \ldots + - \ldots \\ f_3 & \ldots - + \ldots \\ \hline F & \ldots + - \ldots \end{array}$$

Not Enough Restrictions

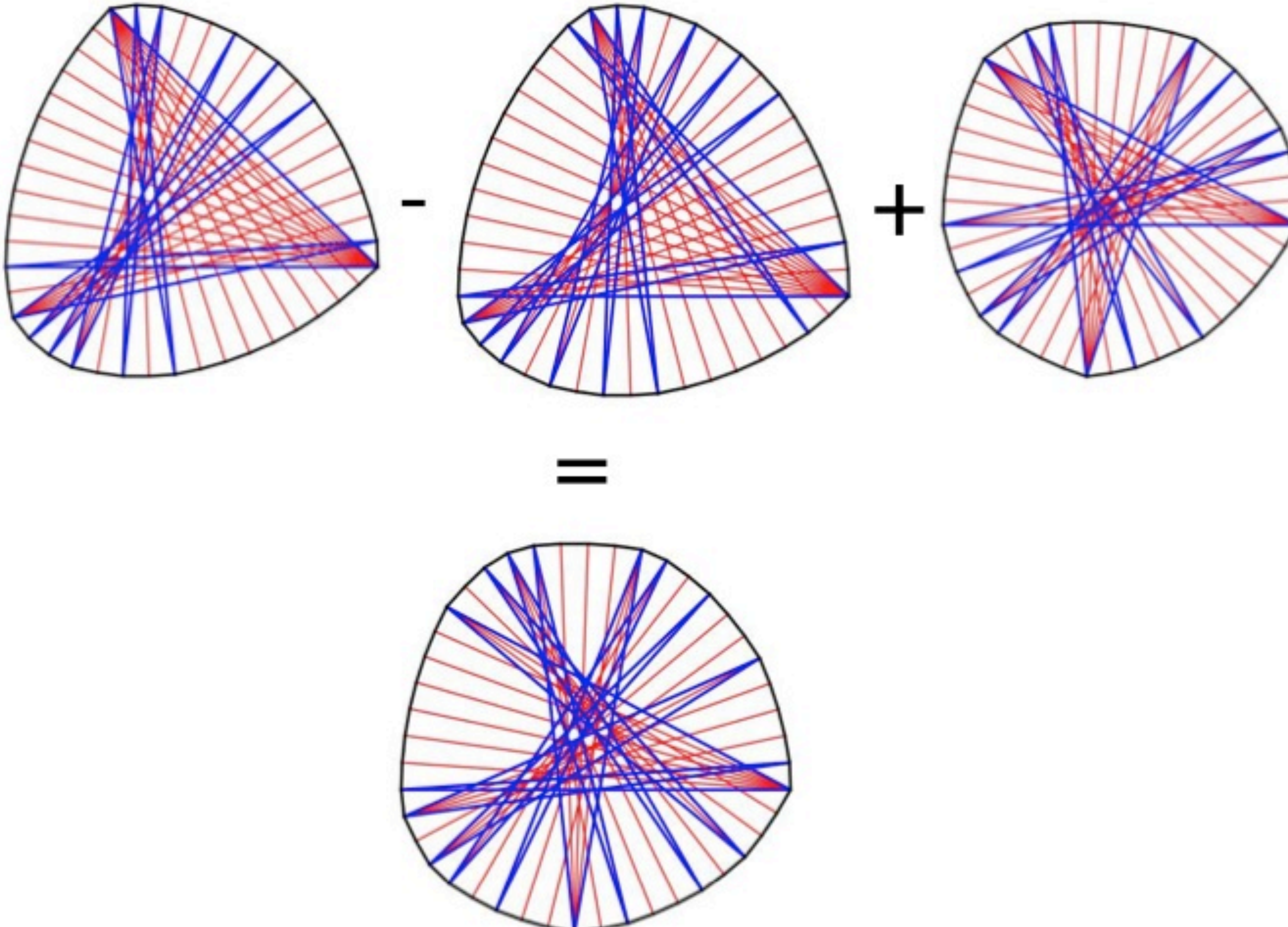
- The previous technique is still exponential
  - But we can specify unique base cases and obtain some results

- How else can we generate more sporadic 105-gons?

Adding Polynomials

$$F(z) = F'(z) + F''(z) - F'''(z)$$

- Maintains divisibility by $\Phi_{2n}(z)$ implicitly
- Check for alternating, an odd number of nonzero terms, and $F(0) = 1$

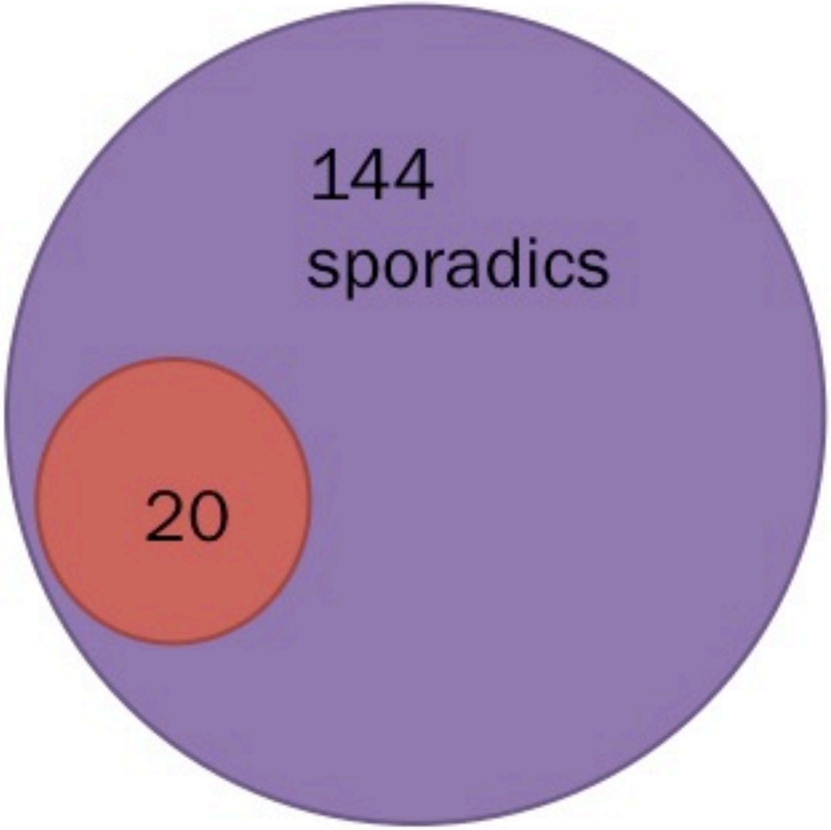


Applications

We applied addition to two problems:

1. Determining a generating set for $n = 45$
   - Helps test the validity of the method
     - All $n = 45$ sporadic polygons are known
2. Finding previously unknown 105-gons

Generating Sets for *n* = 45



- We found a generating set
- Example: 20 sporadic polygons, the G_i which generate the entire sporadic set, S

Let $F \in S$, we obtain $F = G_1 - G_3 + G_9 - G_{15} + G_{17}$

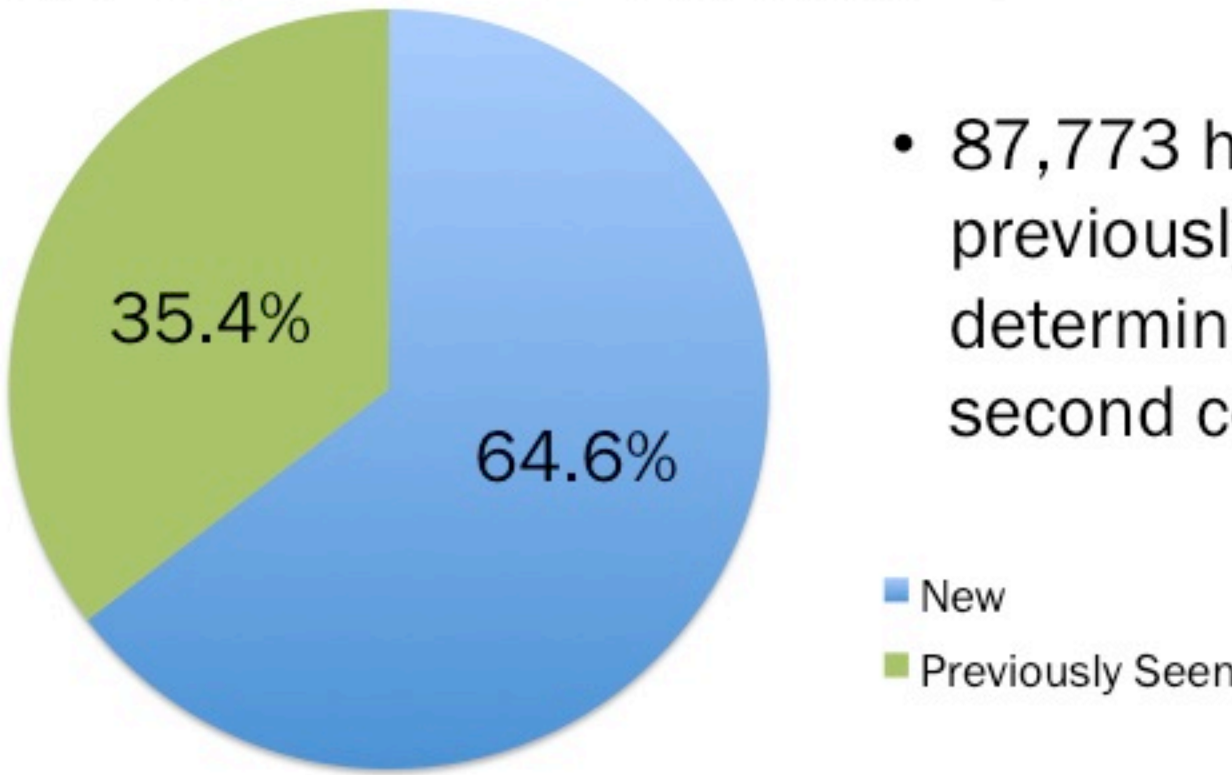
Adding to Obtain More 105-gons

- Start by using the first program to generate a small set of 105-gons

- Add the 105-gons together, combine the results and then iterate

New Results for $n = 105$

- We've been able to generate over 135,823 sporadic Reinhardt polygons



- 87,773 had not been previously determined by the second construction

What's Next?

- Maximizing computational power and obtaining adequate space management to find all sporadic Reinhardt 105-gons
- Can we find a generating set for $n \geq 45$?
- What properties define elements in the generating set from others?

Acknowledgements

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Questions?