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$\boxed{3.3}$ Prove $\mathcal{U}$ is isometric with respect to the l_2 norm, i.e. $\|\mathcal{U}x\| = \|x\|$.

The l_2 norm is defined as $\sqrt{\sum_{i=1}^{n} (x_i)^2}$. Applying it to both sides of $\|\mathcal{U}x\| = \|x\|$:

1) $$\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} (u_j x_i)^2} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Using the properties of double sums, we can rewrite the lhs of eq. (1) as:

$$\sqrt{\sum_{i=1}^{n}(x_i)^2 \sum_{j=1}^{m}(u_j)^2}$$

However, because $\mathcal{U}$ is unitary, its rows $\vec{u}_j$ have length 1, so the $\sum_{j=1}^{m}(u_j)^2 = 1$.

We're then left with the proof of isometry:

$$\sqrt{\sum_{i=1}^{n}(x_i)^2} = \sqrt{\sum_{i=1}^{n}(x_i)^2}.$$

$$\|\mathcal{U}x\| = \|x\|$$ qed.

$\boxed{3.4}$ Prove that all eigenvalues have modulus identity

By definition $\mathcal{U}\mathcal{U}^* = \mathcal{U}^*\mathcal{U} = I$.
From the eigenvalue problem, $\mathcal{U}\vec{x} = \lambda x$. (1)
If we take the conjugate transpose of eq. (1), we obtain: