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$$\overline{(\mathcal{U}x_k)}^T = \overline{(\lambda_k x_k)}^T \Rightarrow x_k^* \mathcal{U}^* = \lambda_k^* \vec{x}_k^* \quad eq(2)$$

Multiplying eq.(1) and eq(2):

$$\vec{x}_k^* \mathcal{U}^* \mathcal{U} \vec{x}_k = \lambda_k^* x_k^* \lambda_k x_k \qquad (3)$$

Applying the definition of a unitary matrix:

$$\vec{x}_k^* I \vec{x}_k = \lambda_k^* \lambda_k x_k^* x_k$$
$$(\vec{x}_k^* \vec{x}_k) = (\lambda_k^* \lambda_k) \vec{x}_k^* \vec{x}_k \qquad (4)$$

Because the modulus of a complex number is equal to that of its conjugate, we can write:

$$\|\vec{x}_k\|_2 = (\lambda_k^* \lambda_k) \|\vec{x}_k\|_2 \qquad (5)$$

Because $\mathcal{U}^{-1} = \mathcal{U}^*$ (proved in 3.2), the values along the diagonal of $\mathcal{U}$ must be real and λ_k must be real. As such eq. (5) can be written as:

$$\|\vec{x}_k\|_2 = |\lambda_k|^2 \|\vec{x}_k\|_2 \qquad (6)$$

Simple solving of eq. 6 shows

$$\sqrt{1} = \sqrt{|\lambda_k|^2}$$

$$\boxed{|\lambda_k| = 1}$$ qed.