Proof the Eighth for CS250

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This document will prove the following statement, by induction ...

$$\sum_{i=1}^{n} i^3 = \left(\frac{(n(n+1))}{2}\right)^2 \tag{1}$$

Proof. We begin our induction proof by identifying P_n ...

$$P_n = \sum_{i=1}^{n} i^3 = \left(\frac{(n(n+1))}{2}\right)^2 \tag{2}$$

The sum of the values of $i \in \mathbb{N}$ from 1 to n, raised to the third power, is equal to n, times $n+1$, divided by two, all squared.

Next, we illustrate the truth of $P(1)$, where $n = 1$ is true.

$$P(1) = \sum_{i=1}^{1} i^3 = 1^3 = 1 \tag{3}$$

$$\equiv P(1) = \left(\frac{(1(1+1))}{2}\right)^2 \tag{4}$$

$$P(1) = \left(\frac{2}{2}\right)^2 = 1^2 = 1 \tag{5}$$

The value of the summation for $P(1)$ and the value of the expression we wish to prove is equivalent are both 1 for $n = 1$.

In the inductive step, we shall assume that $P(k)$ is true, for a particular but arbitrarily chosen $k \in \mathbb{N}$.

$P(k)$ is obtained by substituting k for every n. This is our inductive hypothesis. ...

$$P_k = \sum_{i=1}^{k} i^3 = \left(\frac{(k(k+1))}{2}\right)^2 = 1^3 + 2^3 + ... + (k-1)^3 + k^3 \tag{6}$$

Similarly, $P(k+1)$ is obtained by substituting the quantity $(k+1)$ for every n in $P(n)$ ...

$$P_{k+1} = \sum_{i=1}^{k+1} i^3 = \left(\frac{(k+1)(k+2))}{2}\right)^2 = 1^3 + 2^3 + ... + k^3 + (k+1)^3 \tag{7}$$

Notice that the extended summation of $P(k)$ is identical to that of $P(k+1)$, save for the addition of the final expression, $(k+1)^3$. By algebra, we can therefore substitute the expression which is equal to $P(k)$ into the extended summation of $P(k+1)$, which accounts for all values except for $(k+1)^3$.

Solving algebraically, if the resulting expression is equivalent to the expression representing the summation of $P(k+1)$, we will have proven that this expression holds as an accurate equivalence of the summation in all cases.

By substitution, therefore, from our inductive hypothesis ...

$$P_{k+1} = \sum_{i=1}^{k+1} i^3 = \left(\frac{(k(k+1))}{2}\right)^2 + (k+1)^3 \tag{8}$$

By "foiling" the two expressions, we obtain the following ...

$$P_{k+1} = \sum_{i=1}^{k+1} i^3 = \left(\frac{k^4 + 2k^3 + k^2}{4}\right) + (k^3 + 3k^2 + 3k + 1) \tag{9}$$

Multiplying the second term by $\frac{4}{4}$, we obtain a common denominator ...

$$P_{k+1} = \sum_{i=1}^{k+1} i^3 = \left(\frac{k^4 + 2k^3 + k^2}{4}\right) + \left(\frac{4(k^3 + 3k^2 + 3k + 1)}{4}\right) \tag{10}$$

By adding fractions with a common denominator ...

$$= \sum_{i=1}^{k+1} i^3 = \left(\frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}\right) \tag{11}$$

Finally, we will "foil" out our original $P(k+1)$ expression, to see that they're the same ...

$$P_{k+1} = \sum_{i=1}^{k+1} i^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2 \tag{12}$$

$$= \sum_{i=1}^{k+1} i^3 = \left(\frac{(k^2+3k+2)}{2}\right)^2 \tag{13}$$

$$= \sum_{i=1}^{k+1} i^3 = \left(\frac{(k^2+3k^3+2k^2+3k^3+9k^2+6k+2k^2+6k+4}{4}\right) \tag{14}$$

$$= \sum_{i=1}^{k+1} i^3 = \left(\frac{k^4+6k^3+13k^2+12k+4}{4}\right) \tag{15}$$

As has clearly been demonstrated, expressions (11) and (16) are identical. Therefore, the original expression has been proven to be an accurate representation of our summation for all $n \in \mathbb{N}$.

$\square$