2017

NYU | TANDON SCHOOL OF ENGINEERING

FRE-GY 5990

Capstone Project Report

Project Advisor: Prof. Edith Mandel

Project Member: Mengyang Liu, Xingyue Huang

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1. Introduction to SABR model

1.1 SABR model

The SABR model is defined by the two processes:

$$dF_k(t) = \alpha_k(t) F_k(t)^{\beta_k} dW_{F_k}(t),$$

$$d\alpha_k(t) = \nu_k \alpha_k(t) dW_{\alpha_k}(t),$$

where $F_k(t)$ is the forward interest rate, $\alpha_k(t)$ is the stochastic volatility, $\beta_k(t)$ is the elasticity coefficient, $\nu_k(t)$ is the volatility of volatility process and

$E^k[dW_{F_k}(t) dW_{\alpha_k}(t)] = \rho_k dt.$

1.2 SABR parameters

In this section we analyze how SABR parameters influence the level, slope and curvature of implied volatility smile.

1.2.1 The level parameter α_k

The α_k parameter is the initial value of stochastic volatility process $\alpha_k(t)$. It moves up and down the smile curve with almost no changes to the smile shape.

1.2.2 The slope parameter β_k and ρ_k

The β_k parameter represents constant elasticity of variance (CEV), which takes value between 0 and 1. This is because that the SABR model is a martingale only if $0 < \beta_k < 1$ or as long as $\rho_k \leq 0$ for $\beta_k = 1$. It exerts effects on the smile slope. In particular, the slope will get more pronounced as β_k moves from 1 to 0. An intuitive explanation of this change is the fact that the model will switch from lognormal to normal-like behavior when β_k is lowered.

The ρ_k parameter is the correlation between the two Brownian motions governing the forward rate process and the volatility process respectively. It can take any value within -1 and 1. Its effect on the implied volatility smile curve is similar to that of β_k: the smile will get steeper when ρ_k is more negative.

1.2.3 The curvature parameter $\boldsymbol{\nu_k}$

The υ_k parameter is defined as the volatility of the stochastic volatility $\alpha_k(t)$. The effect of υ_k is to change the curvature of the smile curve. Specifically, higher υ_k increases the implied volatility for OTM and ITM options.

2. Hagan et al. Approximation

2.1 Lognormal approximation

2.2 Normal approximation

3. SABR calibration in practice

3.1 Over-specification test for Hagan et al. approximation

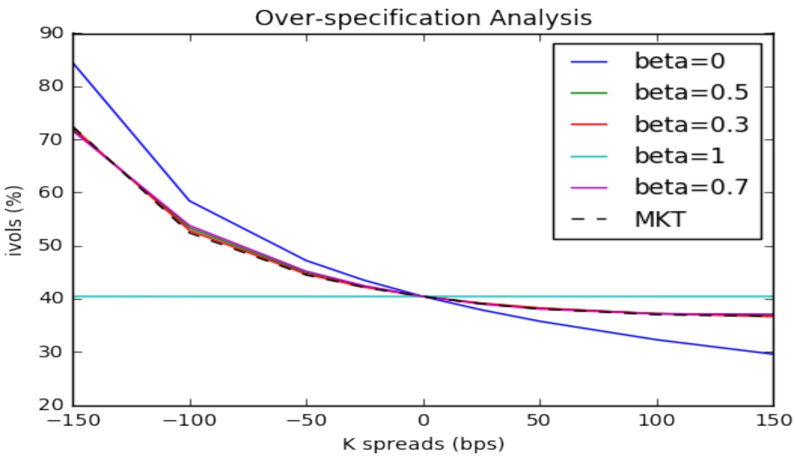
In this section we have conducted an over-specification test for Hagan et al. approximation. This test is aimed to examine the calibration quality of the Hagan approximation, and how the particular β_k will have an effect on it. Our tests are run with different sets of SABR parameters kept fixed and the remaining parameters calibrated based on the minimization algorithm and then ATM volatility recovered. Our subject data here is the data with expiry T_{k-1}=1Y.

3.1.1 fixed β_k

The calibration has been performed with β_k keeping fixed and calibrating the other two parameters ρ_k, ν_k. We have repeated the calibration exercise using:

$$\beta_k = 0,$$
$$\beta_k = 0.3,$$
$$\beta_k = 0.5,$$
$$\beta_k = 0.7,$$
$$\beta_k = 1,$$

According to the plot below, all approximations provide excellent fit to market quotes except $\beta_k = 0, 1$. Generally the smile slope gets more pronounced as β_k moves closer to 1, which represents a switch from normal approximation to lognormal approximation.

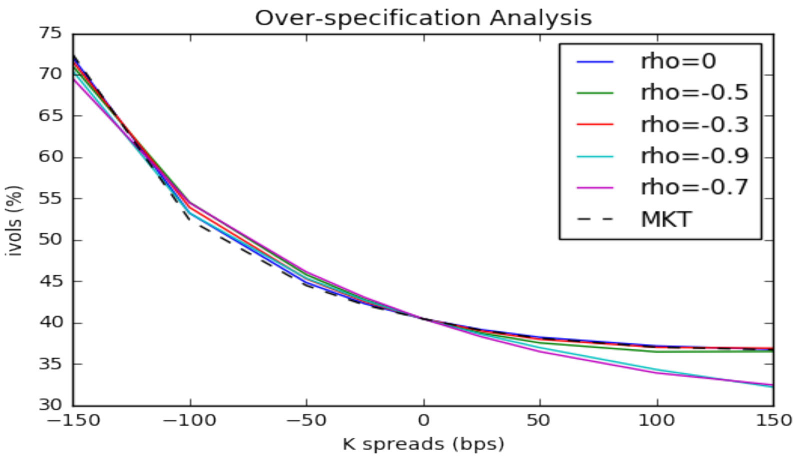


3.1.2 fixed ρ_k

We also present an assessment for the calibration where ρ_k is kept fixed. We have repeated the calibration using:

$$\rho_k = 0,$$
$$\rho_k = -0.3,$$
$$\rho_k = -0.5,$$
$$\rho_k = -0.7,$$
$$\rho_k = -0.9,$$

We can see that ρ_k =0, -0.3, -0.5 all give good approximations from the plot below while ρ_k =-0.7, -0.9 do not fit well for out-of-the-money options. It's also straightforward that ρ_k has a similar effect on the smile shape as β_k does: the smile slope becomes steeper as ρ_k gets more negative.

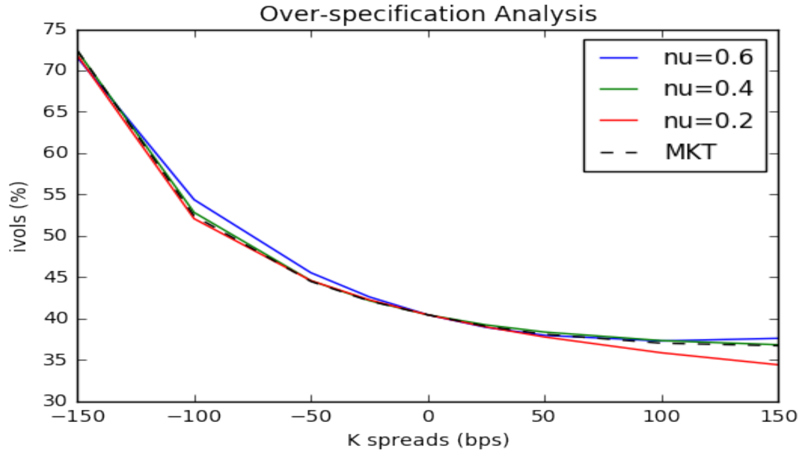


3.1.3 fixed ν_k

The same calibration procedure is run for parameter ν_k using:

$$\upsilon_k = 0.2,$$
$$\upsilon_k = 0.4,$$
$$\upsilon_k = 0.6,$$

And it can be seen from the plot below that the effect of υ_k is to increase or decrease its curvature: higher υ_k leads to increased volatility for out of the money (OTM) and in the money (ITM) options. Of these three υ_k values, the best performance is given by $\upsilon_k = 0.4$ and $\upsilon_k = 0.2{,}0.6$ also give good approximations.

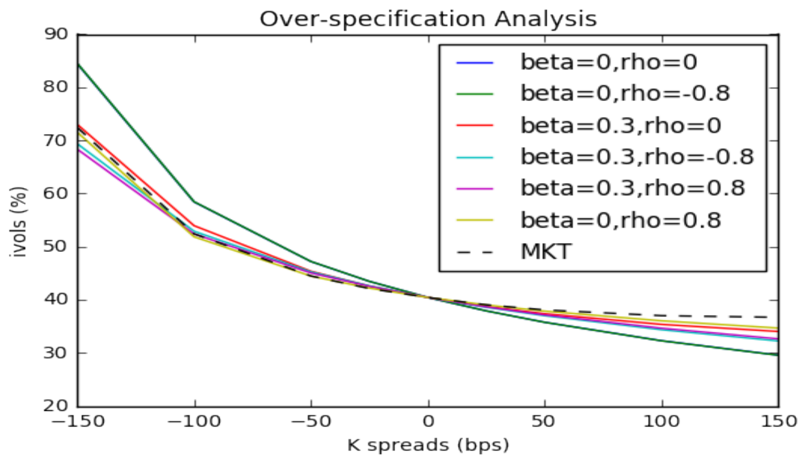


3.1.4 fixed β_k and ρ_k

We repeat the calibration procedure where parameter β_k and ρ_k kept fixed and υ_k calibrated using:

$$\beta_k = 0,\ \rho_k = 0,$$
$$\beta_k = 0,\ \rho_k = 0.8,$$
$$\beta_k = 0,\ \rho_k = -0.8,$$
$$\beta_k = 0.3,\ \rho_k = 0,$$
$$\beta_k = 0.3,\ \rho_k = 0.8,$$
$$\beta_k = 0.3,\ \rho_k = -0.8,$$

All combinations of β_k and ρ_k yield good approximations except β_k =0, ρ_k =-0.8, as can be seen from the plot below. In other words, a sound approximation of Hagan et al. lognormal SABR model does not require all three parameters β_k, ρ_k and υ_k to be calibrated at once. Setting two of them fixed and calibrating the remaining one can be more computationally efficient without harming the quality of calibration.

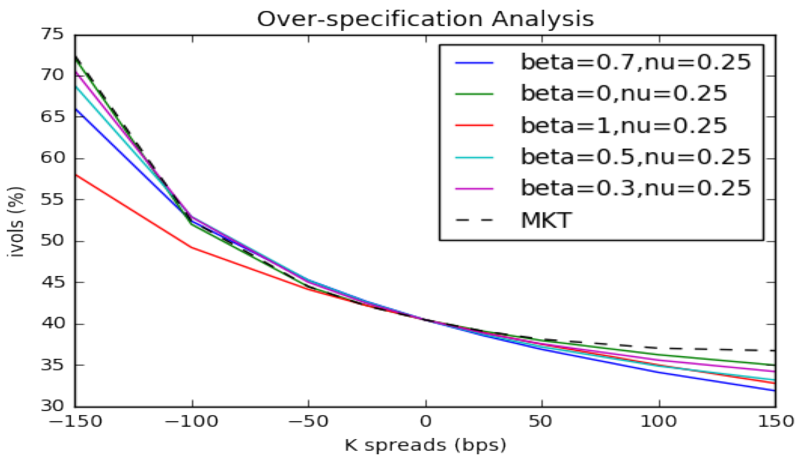


3.1.5 fixed β_k and ν_k

Then again we repeat the calibration procedure where parameter β_k and υ_k kept fixed and ρ_k calibrated using:

$$\beta_k = 0,\ \upsilon_k = 0.25,$$
$$\beta_k = 0.3,\ \upsilon_k = 0.25,$$
$$\beta_k = 0.5,\ \upsilon_k = 0.25,$$
$$\beta_k = 0.7,\ \upsilon_k = 0.25,$$
$$\beta_k = 1,\ \upsilon_k = 0.25,$$

When υ_k is fixed to 0.25, β_k closer to 0 gives better performance while a high β_k such as 0.7 and 1 do not fit well either for in-the-money options or for out-of-the-money options. For other values of β_k, in general their ability to fit market data does not vary too much from each other.

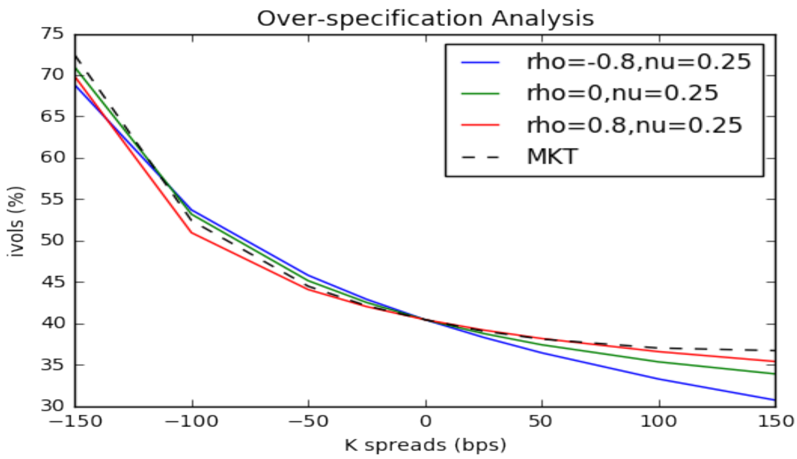


3.1.6 fixed ρ_k and ν_k

We repeat the calibration procedure where parameter ρ_k and υ_k kept fixed and β_k calibrated using:

$$\rho_k = 0,\ \upsilon_k = 0.25,$$
$$\rho_k = 0.3,\ \upsilon_k = 0.25,$$
$$\rho_k = 0.5,\ \upsilon_k = 0.25,$$

Of the three combinations given here, ρ_k=0.8, υ_k=0.25 has the best performance for out-of-the-money options while ρ_k=-0.8, υ_k=0.25 gives the worst.



3.2 Collinearity test for Hagan et.al approximation

To further explore the quality of Hagan SABR approximation, we calculate the condition number of the calibration Jacobian matrix to detect collinearity.

Table: Collinearity test for Hagan et al. implementation

Hagan et al. SABR	Condition number of Jacobian matrix
Calibrate β_k, ρ_k and υ_k	1277
Fix $\beta_k = 0.7$, calibrate ρ_k and υ_k	438
Fix $\rho_k =$ -0.5, calibrate β_k and υ_k	878
Fix $\upsilon_k = 0.4$, calibrate β_k and ρ_k	25969
Fix β_k=0 and ρ_k=0.8, calibrate υ_k	72
Fix β_k=0 and υ_k=0.25, calibrate ρ_k	1087
Fix ρ_k=0.8 and υ_k=0.25, calibrate β_k	29

We can see from the table above that 1) with one or two parameters fixed in calibration, SABR model has less collinearity as the condition number of the transposed Jacobian matrix of calibration has reduced from 1277 to around 1000 or even 100 below; 2) Of these different calibrations, fixing ρ_k and one more factor β_k or υ_k can reduce collinearity most. However, keeping aspecific parameters fixed to different values can give very different condition numbers. For example, the condition number rises to 14767 when we calibrate with β_k fixed to 0.

4. Monte Carlo simulation for SABR

After having analyzed the fitting performances of Hagan et al. SABR model in terms of calibration fitting, we would like to use Monte Carlo simulation to investigate if they are able to correctly approximate the evolution of the SABR processes.

4.1 Monte Carlo standard error

Let's denote Monte Carlo average estimator $\overline{F_k}$ as

$$\overline{F_k} = \frac{1}{n_{sim}} \sum_{i=1}^{n_{sim}} F_k^i(T_{k-1})$$

where n_{sim} is the total number of paths simulated and $F_k^i(T_{k-1})$ is the forward interest rate $F_k(T_{k-1})$ generated by the i-th simulation. The quantity

$$\varsigma_{n_{sim}}^2 = \frac{1}{n_{sim}-1} \sum_{i=1}^{n_{sim}} (F_k^i(T_{k-1}) - \overline{F_k})^2$$

is used to determine the Monte Carlo standard error. The lower the standard error the better the accuracy of the tested Monte Carlo scheme.

4.2 Monte Carlo schemes

In this section we discuss two of most commonly used Monte Carlo schemes: Euler scheme and Milstein scheme and their simulations of Hagan et al. lognormal approximation.

4.2.1 Euler scheme

In Euler scheme, the SABR process can be rewritten as

$$\widehat{F_k}(t_{i+1}) = \widehat{F_k}(t_i) + \widehat{\alpha_k}(t_i)\widehat{F_k}(t_i)^{\beta_k}\Delta W_{\widehat{F_k}}(t_{i+1}),$$

$$\widehat{\alpha_k}(t_{i+1}) = \widehat{\alpha_k}(t_i) + \nu_k\widehat{\alpha_k}(t_i)\Delta W_{\widehat{\alpha_k}}(t_{i+1}),$$

where $\widehat{F_k}$ and $\widehat{\alpha_k}$ are discrete versions of F_k and α_k respectively.

Here we implement a zero absorbing boundary for the forward process when $0 < \beta_k < 1$ as only in this case will SABR remain a martingale.

There is a risk that Euler scheme may fail to reach convergence in simulating the implied volatility. Therefore we have performed Monte Carlo simulations with different combinations of time step size and SABR parameters. The tests shown below are classified in three different groups based on ρ_k:

$$\rho_k = 0,$$
$$\rho_k = 0.8,$$
$$\rho_k = -0.8,$$

For each of them we have tested five different values of β_k:

$$\beta_k = 0,$$
$$\beta_k = 0.3,$$
$$\beta_k = 0.5,$$
$$\beta_k = 0.7,$$
$$\beta_k = 1,$$

and five different time step numbers:

$$n_{step} = 1,$$
$$n_{step} = 40,$$
$$n_{step} = 240,$$
$$n_{step} = 480,$$
$$n_{step} = 960,$$

for all cases: n_{sim}=100000, T_{k-1}=10Y and v_k=0.25. And time steps are chosen based on how long we want the discrete steps be. For example, if we choose $n_{step} = 40$, we are dealing with discrete steps that are about 60 trading days long considering we have 252 trading days in a year. We put a cap at $n_{step} = 960$ as we have not seen any considerable improvement in the convergence for higher values.

Table: Equivalence between the total n_{step} and the actual n_{step} per year

Simulation time step n_{step}	Equivalent time step per year
1	0.1
40	4
240	24
480	48
960	96

Black implied volatilities (%)by Euler scheme for ρ_k=0 and various combinations of β_k. For all cases: T_{expiry}=10Y, υ_k=0.25

	Steps	Strike spreads (bps) -150	-100	-50	-25	0	25	50	100	150
β_k=0	1	30.11	26.70	24.14	23.10	22.18	21.35	20.59	19.28	18.17
ρ_k=0	40	32.31	28.10	25.15	24.01	23.03	22.20	21.49	20.38	19.60

υ_k=0.25	240	32.72	28.43	25.43	24.27	23.28	22.44	21.73	20.62	19.83
	480	32.67	28.38	25.37	14.19	23.20	22.35	21.64	20.53	19.74
	960	33.18	28.75	25.65	24.46	23.45	22.59	21.87	20.72	19.93
β_k=0.3	1	29.83	26.52	24.05	23.04	22.13	21.31	20.56	19.26	18.15
ρ_k=0	40	28.67	26.00	24.06	23.30	22.67	22.13	21.67	20.97	20.48
υ_k=0.25	240	28.54	23.86	23.95	23.21	22.60	22.08	21.64	20.97	20.52
	480	28.41	25.76	23.85	23.12	22.50	21.98	21.55	20.90	20.45
	960	28.30	25.68	23.78	23.04	22.41	21.89	21.45	20.78	20.33
β_k=0.5	1	29.76	26.51	24.08	23.08	22.18	21.37	20.63	19.33	18.22
ρ_k=0	40	27.78	25.50	23.92	23.32	22.94	22.44	22.10	21.64	21.34
υ_k=0.25	240	26.84	24.84	23.41	22.86	22.41	22.05	21.77	21.36	21.13
	480	27.08	24.97	23.51	22.96	22.50	22.14	21.84	21.42	21.16
	960	26.68	24.69	23.32	22.80	22.37	22.02	21.73	21.32	21.06
β_k=0.7	1	30.31	26.91	24.41	23.36	22.43	21.59	20.82	19.51	18.39
ρ_k=0	40	25.96	24.34	23.27	22.89	22.59	22.36	22.19	21.97	21.88
υ_k=0.25	240	25.91	24.30	23.26	22.89	22.61	22.39	22.23	22.05	21.99
	480	25.76	24.19	23.18	22.82	22.55	22.35	22.20	22.05	22.02
	960	25.88	24.29	23.26	22.91	22.64	22.44	22.29	22.11	22.06
β_k=1	1	30.60	27.11	24.53	23.47	22.52	21.67	20.90	19.56	18.43
ρ_k=0	40	24.62	23.66	23.16	23.04	22.98	23.00	23.00	23.16	23.41
υ_k=0.25	240	23.82	22.97	22.55	22.45	22.42	22.49	22.49	22.67	22.93
	480	22.57	22.11	21.90	21.86	21.87	21.99	21.99	22.21	22.48
	960	22.58	22.15	21.93	21.90	21.91	22.05	22.05	22.27	22.54

Black implied volatilities (%)by Euler scheme for ρ_k=0.8 and various combinations of β_k. For all cases: T_{expiry}=10Y, υ_k=0.25

		Strike spreads (bps)								
	Steps	-150	-100	-50	-25	0	25	50	100	150
β_k=0	1	30.72	27.35	24.86	23.83	22.90	22.06	21.30	19.96	18.81
ρ_k=0.8	40	25.87	24.53	23.75	23.49	23.28	23.12	22.99	22.81	22.69
υ_k=0.25	240	25.31	24.16	23.52	23.31	23.15	23.03	22.94	22.81	22.73
	480	25.77	24.45	23.72	23.49	23.31	23.16	23.05	22.91	22.83
	960	24.89	23.84	23.25	23.05	23.05	22.79	22.71	22.59	22.51
β_k=0.3	1	30.52	27.08	24.53	23.48	22.54	21.70	20.94	19.59	18.45
ρ_k=0.8	40	23.55	23.26	23.29	23.36	22.45	23.56	23.66	23.87	20.08
υ_k=0.25	240	22.27	22.33	22.56	22.69	22.83	22.98	23.12	23.39	23.63
	480	23.25	23.00	23.08	23.18	23.29	23.41	23.53	23.78	24.02
	960	23.10	22.94	23.08	23.19	23.31	23.44	23.57	23.82	24.06

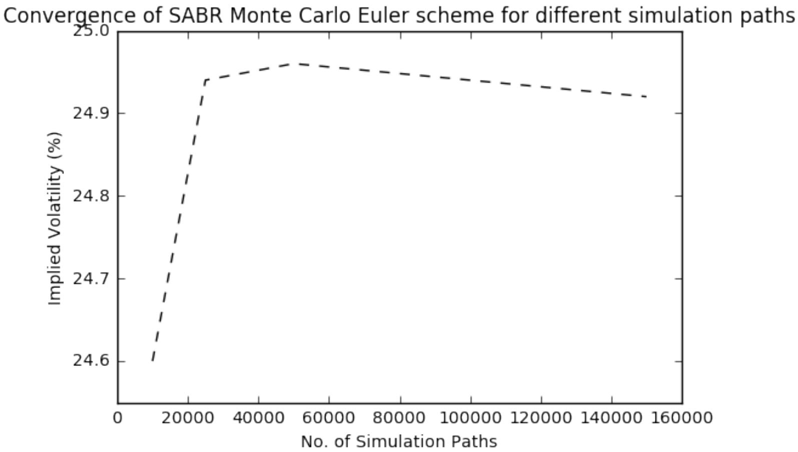
β_k=0.5	1	29.68	26.43	24.02	23.03	22.14	21.33	20.59	19.31	18.22
ρ_k=0.8	40	22.30	22.56	23.06	23.33	23.60	23.85	24.10	24.56	24.96
υ_k=0.25	240	21.30	21.83	22.49	22.81	23.12	23.41	23.71	24.24	24.73
	480	22.86	22.88	23.32	23.59	23.85	24.12	24.37	24.84	25.29
	960	22.76	22.77	23.18	23.44	23.70	23.96	24.21	24.70	25.15
β_k=0.7	1	29.62	26.28	23.82	22.80	21.90	21.08	20.33	19.03	17.94
ρ_k=0.8	40	22.05	22.57	23.43	23.88	24.31	24.73	25.13	25.86	26.53
υ_k=0.25	240	20.08	21.16	22.29	22.81	23.31	23.78	24.41	25.00	25.73
	480	20.46	21.40	22.48	23.01	23.51	23.99	24.44	25.26	25.29
	960	21.53	22.12	23.06	23.54	24.00	24.44	24.86	25.65	25.15
β_k=1	1	28.74	25.50	23.11	22.13	21.24	20.45	19.73	18.47	17.40
ρ_k=0.8	40	31.95	30.36	30.66	31.07	31.55	32.08	32.61	33.67	34.68
υ_k=0.25	240	13.97	17.89	20.25	21.22	22.09	22.89	23.62	24.93	26.07
	480	12.10	17.14	19.65	20.66	21.56	22.38	22.13	24.46	25.62
	960	10.00	15.45	18.36	19.44	20.38	21.22	21.98	23.30	24.45

Black implied volatilities (%) by Euler scheme for ρ_k=-0.8 and various combinations of β_k. For all cases: T_{expiry}=10Y, υ_k=0.25

		Strike spreads (bps)								
	Steps	-150	-100	-50	-25	0	25	50	100	150
β_k=0	1	31.16	27.63	25.03	23.96	23.02	22.16	21.39	20.03	18.88
ρ_k=-0.8	40	37.46	31.32	26.73	24.82	23.10	21.54	20.13	17.66	15.61
υ_k=0.25	240	37.68	31.45	26.79	24.86	23.12	21.55	20.12	17.63	15.57
	480	37.86	31.58	26.88	24.93	23.17	21.57	20.12	17.60	15.50
	960	37.89	31.59	26.89	24.94	23.19	21.60	20.16	17.64	15.57
β_k=0.3	1	32.89	29.14	26.38	25.24	24.23	23.33	22.51	21.07	18.85
ρ_k=-0.8	40	32.60	28.55	25.31	23.90	22.61	21.41	20.30	18.34	16.66
υ_k=0.25	240	32.49	28.51	25.29	23.87	22.57	21.37	20.25	18.27	16.58
	480	32.05	28.20	25.06	23.69	22.42	21.24	20.15	18.19	16.53
	960	32.02	18.18	25.05	23.67	22.40	21.23	20.14	18.19	16.52
β_k=0.5	1	33.28	29.56	26.81	25.68	24.67	23.76	22.94	21.49	20.27
ρ_k=-0.8	40	30.81	27.55	24.89	23.72	22.64	21.64	20.71	19.02	17.57
υ_k=0.25	240	30.77	27.46	24.78	23.62	22.53	21.53	20.60	18.93	17.51
	480	31.02	27.64	24.91	23.72	22.62	21.61	20.68	19.00	17.54
	960	30.92	27.59	24.87	23.68	22.58	21.57	20.63	18.93	17.47
β_k=0.7	1	34.73	30.86	28.00	26.81	25.75	24.80	23.92	22.39	21.08
ρ_k=-0.8	40	29.90	27.12	24.84	23.84	22.91	22.05	21.26	19.82	18.57
υ_k=0.25	240	29.73	26.96	24.70	23.71	22.79	21.94	21.15	19.73	18.49
	480	30.01	27.15	24.82	23.81	22.87	22.01	21.20	19.76	18.51

	960	29.62	26.86	24.60	23.61	22.70	21.86	21.07	19.65	18.43
β_k=1	1	37.28	32.99	29.83	28.54	27.38	26.35	25.41	23.77	22.39
ρ_k=-0.8	40	28.56	26.50	24.83	24.10	23.43	22.80	22.22	21.18	20.28
υ_k=0.25	240	28.49	26.41	24.73	24.00	23..32	22.70	22.13	21.10	20.22
	480	27.86	25.98	24.42	23.73	23.09	22.50	21.95	20.96	20.11
	960	27.72	25.88	24.35	23.68	23.05	22.46	21.91	20.91	20.04

It's evident from the tables above that the case ρ_k=-0.8 shows a generally good convergence of the Monte Carlo simulation under Euler scheme: the implied volatilities with different values of n_{step} enjoy a low variance of 9.86%. The convergence is excellent especially for β_k=0 and β_k=0.5. For ρ_k=0, the results are good for β_k=0.7; we have the worst performance for ρ_k=0.8, especially when β_k=1.



4.2.2 Milstein scheme

Compared with Euler scheme, Milstein scheme increases the accuracy of a stochastic process discrete approximation by adding higher order terms. The Milstein scheme for a stochastic differential equation of the type

$$dX(t) = aX(t) + bX(t)dW(t)$$

is

$$\hat{X}(t_{i+1}) = \hat{X}(t_i) + a(t_i, \hat{X}(t_i))\hat{X}(t_i)\Delta t + b(t_i, \hat{X}(t_i))\hat{X}(t_i)\Delta W_{\hat{X}}(t_{i+1}) + \frac{1}{2}b(t_i, \hat{X}(t_i))b'(t_i, \hat{X}(t_i))((\Delta W_{\hat{X}}(t_{i+1}))^2 - \Delta t)$$

where $b'$is the first derivative of the term b with respect to x. For the SABR forward process we take x=$\widehat{F_k}(t_i)$ and we have

$$a = 0$$
$$b = \widehat{a_k}(t_i)x^{\beta_k}$$
$$b' = \widehat{a_k}(t_i)\beta_k x^{(\beta_k - 1)}$$

Its Milstein discretization is

$$\widehat{F_k}(t_{i+1}) = \widehat{F_k}(t_i) + \widehat{\alpha_k}(t_i)^{\beta_k}\Delta W_{\widehat{F_k}}(t_{i+1})$$

$$+\frac{1}{2}\beta_k\widehat{\alpha_k}(t_i)^2\widehat{F_k}(t_i)^{(2\beta_k-1)}((\Delta W_{\widehat{F_k}}(t_{i+1}))^2 - \Delta t)$$

For the SABR volatility process we take x=$\widehat{\alpha_k}(t_i)$ and we have

$$a = 0$$
$$b = \nu_k x$$
$$b' = \nu_k$$

which leads to the following Milstein discretization equation:

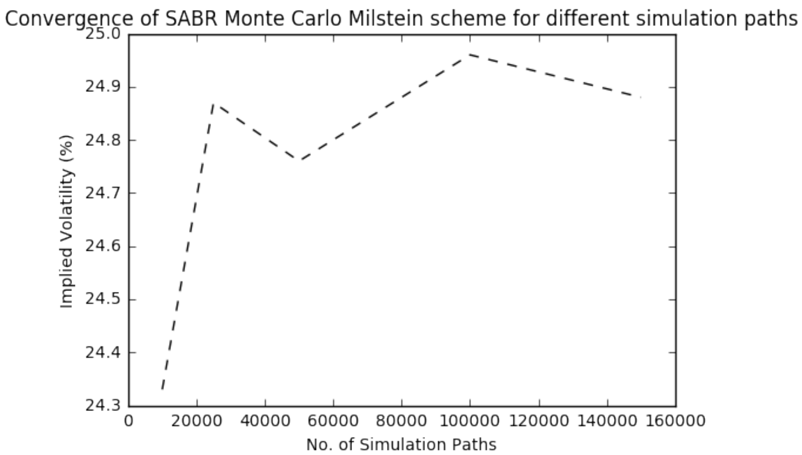
$$\widehat{\alpha_k}(t_{i+1}) = \widehat{\alpha_k}(t_i) + \nu_k\widehat{\alpha_k}(t_i)\Delta W_{\widehat{\alpha_k}}(t_{i+1}) + \frac{1}{2}\nu_k^2\widehat{\alpha_k}(t_i)((\Delta W_{\widehat{\alpha_k}}(t_{i+1}))^2 - \Delta t)$$

For Milstein scheme, we don't repeat the discussion of simulation results for different simulation time step sizes and sets of ρ_k. Here we only provide simulation results for n_{step}=40 and ρ_k=-0.8.

Table: Black implied volatilities (%) by Milstein scheme for various combinations of β_k. For all cases: T_{expiry}=10Y, ν_k=0.25, ρ_k=0.8, n_{step}=40

	Strike Spreads (bps)								
	-150	-100	-50	-25	0	25	50	100	150
β_k=0	37.69	31.50	26.86	24.93	23.20	21.63	20.21	17.75	15.71
β_k=0.3	32.35	28.42	25.24	23.86	22.59	21.42	20.33	18.39	16.73
β_k=0.5	30.88	27.61	24.95	23.78	22.71	21.72	20.80	19.16	17.75
β_k=0.7	29.44	26.80	24.63	23.68	22.80	21.99	21.23	19.85	18.66
β_k=1	28.12	26.20	24.63	23.94	23.31	22.72	22.17	21.19	20.35

Compared with Euler scheme, Milstein scheme enjoys a gain in accuracy of simulation and lower Monte Carlo standard error but it has much longer computation time. In general it doesn't have much benefit over Euler scheme.



5. Validation of Hagan et al. approximation

5.1 Validation of c.d.f and p.d.f

5.1.1 Using lognormal Hagan

5.1.2 Using Monte Carlo simulation

5.2 Validation of lognormal implied volatility

5.2.1 Using lognormal Hagan

5.2.2 Using Monte Carlo simulation

We first investigate simulation results in ν_k=0 case with T_{expiry}=10Y, ρ_k={0, 0.8, -0.8}. Then we explore more general cases with β_k={0, 0.3, 0.5, 0.7, 1}, ρ_k={0, 0.8, -0.8}, ν_k={0.25, 0.5}.

Table: Black implied volatilities (%) by Euler scheme for various combinations of β_k. For all cases: T_{expiry}=10Y, υ_k=0, ρ_k=0, n_{step}=40, n_{sim}=100000

	Strike Spreads (bps)								
	-150	-100	-50	-25	0	25	50	100	150
β_k=0	31.78	28.09	25.41	24.31	23.34	22.45	21.65	20.25	19.07
β_k=0.3	28.11	25.91	24.22	23.51	22.87	22.29	21.76	20.82	20.01
β_k=0.5	26.72	25.18	23.98	23.47	23.01	22.58	22.19	21.50	20.90
β_k=0.7	25.86	24.78	23.97	23.62	23.32	23.03	22.77	22.30	21.89
β_k=1	23.98	23.72	23.57	23.51	23.47	23.43	23.40	23.33	23.28

Table: Black implied volatilities (%) by Euler scheme for various combinations of β_k. For all cases: T_{expiry}=10Y, υ_k=0, ρ_k=0.8, n_{step}=40, n_{sim}=100000

	Strike Spreads (bps)								
	-150	-100	-50	-25	0	25	50	100	150
β_k=0	31.30	27.58	25.04	23.99	23.04	22.19	21.42	20.07	18.92
β_k=0.3	28.49	26.21	24.46	23.72	23.06	22.45	21.90	20.92	20.07
β_k=0.5	27.37	25.65	24.36	23.81	23.32	22.87	22.46	21.73	21.09
β_k=0.7	25.17	24.28	23.57	23.26	22.98	22.71	22.47	22.04	21.65
β_k=1	23.07	23.09	23.07	23.05	23.04	23.02	23.00	22.99	22.97

Table: Black implied volatilities (%) by Euler scheme for various combinations of β_k. For all cases: T_{expiry}=10Y, υ_k=0, ρ_k=-0.8, n_{step}=40, n_{sim}=100000

	Strike Spreads (bps)

	-150	-100	-50	-25	0	25	50	100	150
β_k=0	31.26	27.73	25.12	24.04	23.08	22.21	21.43	20.07	18.91
β_k=0.3	28.59	26.27	24.49	23.75	23.09	22.48	21.93	20.94	20.10
β_k=0.5	26.56	25.08	23.92	23.43	22.98	22.56	22.18	21.50	20.90
β_k=0.7	25.57	24.56	23.80	23.48	23.18	22.91	22.66	22.20	21.80
β_k=1	22.58	22.76	22.82	22.83	22.83	22.83	22.83	22.82	22.81

Table: Black implied volatilities (%) by Euler scheme for various combinations of β_k, υ_k, ρ_k. For all cases: $T_{expiry} = 10$Y, n_{step}=40, n_{sim}=100000

α_k	β_k	ρ_k	υ_k	-150	-100	-50	-25	0	25	50	100	150
0.007	0	0	0.25	32.54	28.29	25.31	24.14	23.15	22.32	21.61	20.51	19.72
0.006	0	0	0.5	35.75	29.77	25.74	24.31	23.24	22.46	21.95	21.48	21.45
0.020	0.3	0	0.25	28.43	25.79	23.91	23.18	22.56	22.04	21.60	20.93	20.47
0.017	0.3	0	0.5	28.07	24.42	21.94	21.08	20.46	20.06	19.84	19.79	20.04
0.040	0.5	0	0.25	27.67	25.45	23.91	23.33	22.84	22.44	22.11	21.65	21.35
0.035	0.5	0	0.5	28.12	24.61	22.28	21.51	20.99	20.69	20.56	20.68	21.06
0.079	0.7	0	0.25	26.51	24.74	23.59	23.19	22.88	22.64	22.46	22.23	22.12
0.069	0.7	0	0.5	27.26	24.11	22.09	21.46	21.07	20.90	20.88	21.17	21.69
0.219	1	0	0.25	24.70	23.70	23.18	23.05	22.98	22.97	22.99	23.15	23.39
0.191	1	0	0.5	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
0.008	0	0.8	0.25	25.53	24.31	23.62	23.38	23.20	23.05	22.94	22.79	22.70
0.007	0	0.8	0.5	25.78	23.99	23.88	24.13	24.46	24.83	25.21	25.95	26.64
0.020	0.3	0.8	0.25	24.09	23.62	23.57	23.61	23.69	23.78	23.88	24.07	24.26
0.020	0.3	0.8	0.5	23.98	22.79	23.36	23.94	24.58	25.22	25.85	27.00	28.03
0.040	0.5	0.8	0.25	21.74	22.16	22.72	23.01	23.29	23.56	23.82	24.30	24.73
0.038	0.5	0.8	0.5	23.92	22.76	23.63	24.39	25.20	26.00	26.78	28.20	29.47
0.077	0.7	0.8	0.25	21.14	21.93	22.89	23.36	23.81	24.24	24.64	25.40	26.07
0.072	0.7	0.8	0.5	31.96	29.01	29.23	29.91	30.72	31.57	32.41	33.98	35.41
0.208	1	0.8	0.25	35.73	33.39	33.31	33.61	34.02	34.48	34.97	35.96	36.94
0.192	1	0.8	0.5	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN
0.008	0	-0.8	0.25	37.33	31.23	26.65	24.74	23.03	21.48	20.08	17.61	15.57
0.007	0	-0.8	0.5	44.31	35.53	29.05	26.37	23.97	21.83	19.93	16.94	15.18
0.022	0.3	-0.8	0.25	32.39	28.42	25.24	23.86	22.58	21.41	20.33	18.38	16.72
0.023	0.3	-0.8	0.5	33.34	28.49	24.48	22.70	21.06	19.54	18.14	15.77	14.14
0.044	0.5	-0.8	0.25	31.25	27.80	25.02	23.81	22.70	21.67	20.72	19.02	17.56
0.048	0.5	-0.8	0.5	32.39	28.04	24.46	22.87	21.40	20.04	18.78	16.62	15.03
0.091	0.7	-0.8	0.25	29.83	27.04	24.77	23.77	22.86	22.01	21.22	19.79	18.55
0.457	0.7	-0.8	0.5	NaN	96.34	87.74	84.47	81.65	79.17	76.97	73.17	69.99
0.265	1	-0.8	0.25	28.64	26.60	24.93	24.20	23.52	22.90	22.31	21.26	20.35
0.350	1	-0.8	0.5	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN

5.3 Validation of normal implied volatility

5.3.1 Using lognormal Hagan

5.3.2 Using Monte Carlo simulation

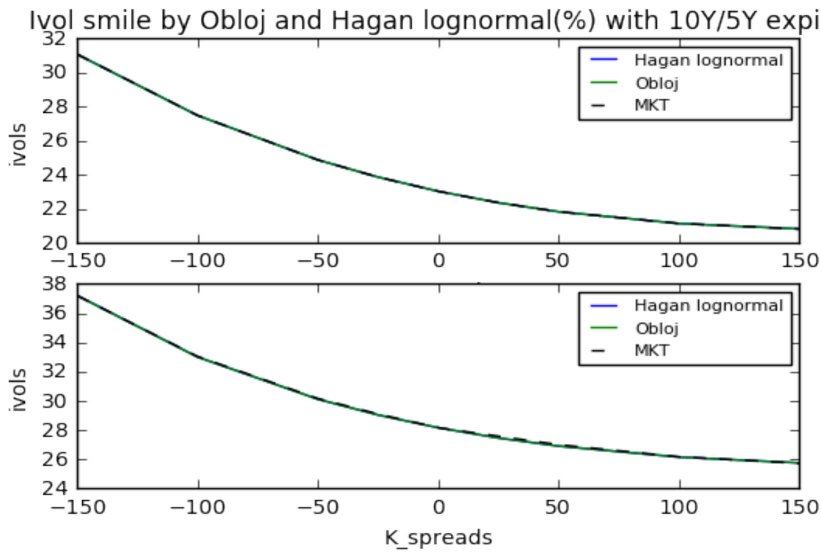
6. The limits of Hagan et al. approximations

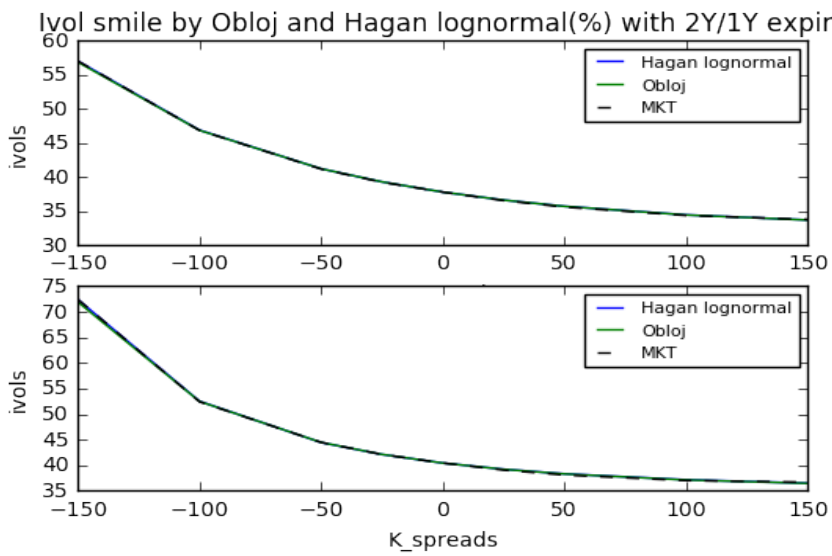
7. Alternative SABR approximations

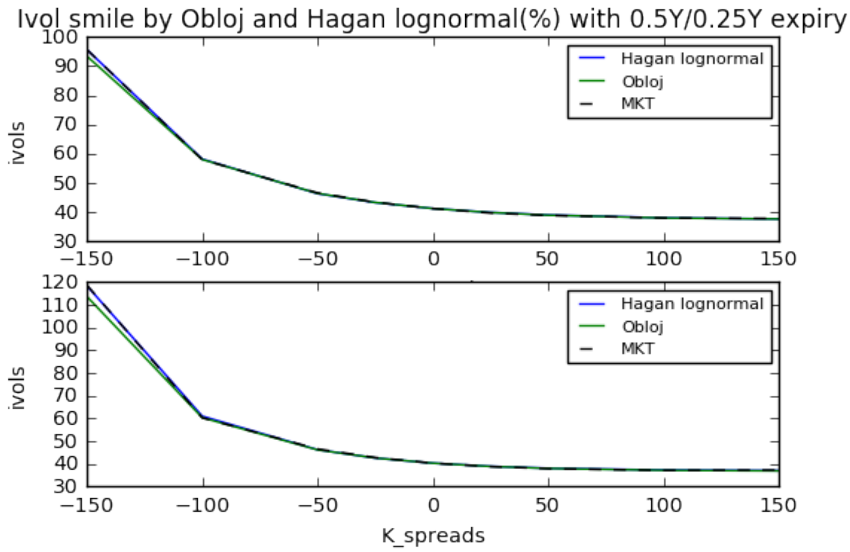
7.1 Obloj SABR

7.1.1 Obloj calibration

We have plotted implied volatility smile curves by Obloj lognormal SABR against Obloj SABR for expiry of 10Y, 5Y, 2Y, 1Y, 0.5Y and 0.25Y. We can see that for options with short maturities within one year, Obloj model provides a slightly lower approximation for deep in-the-money options that Hagan lognormal does while their estimation for at-the-money options and out-of-the-money options are extremely close; they have nearly the same smile curve for maturities over 1 year. Here Hagan lognormal SABR fits better to our market data sample.



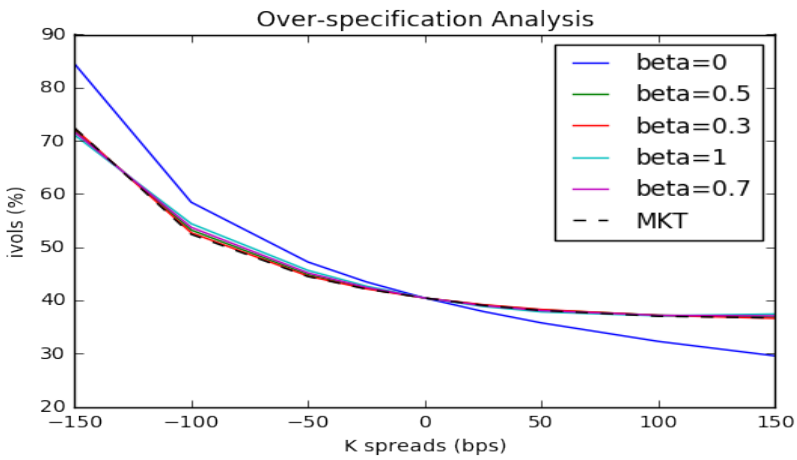


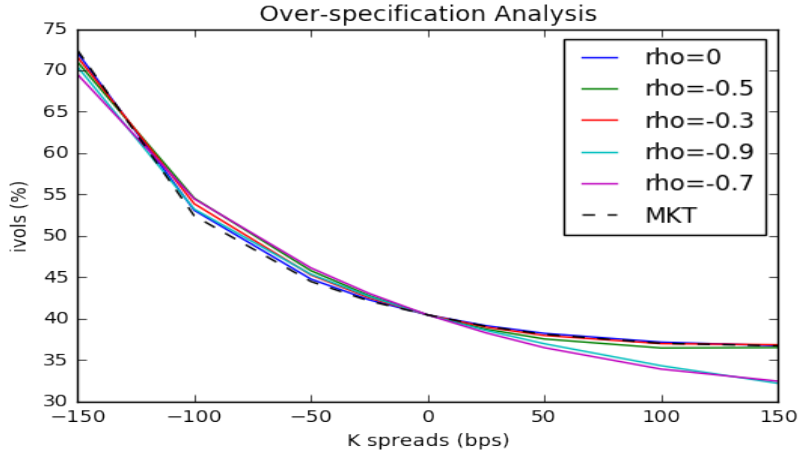


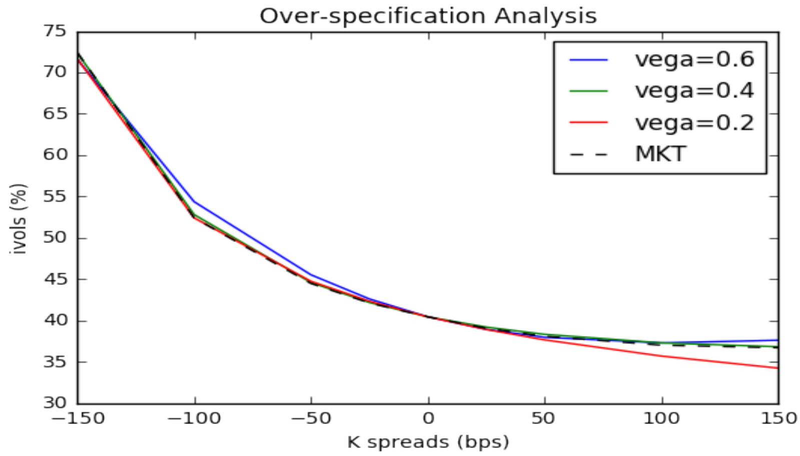
7.1.2 Validation of c.d.f

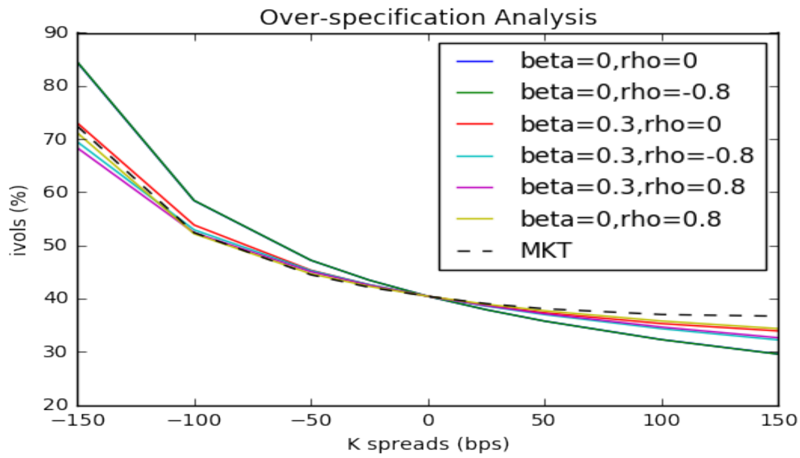
7.1.3 Over-specification and collinearity test

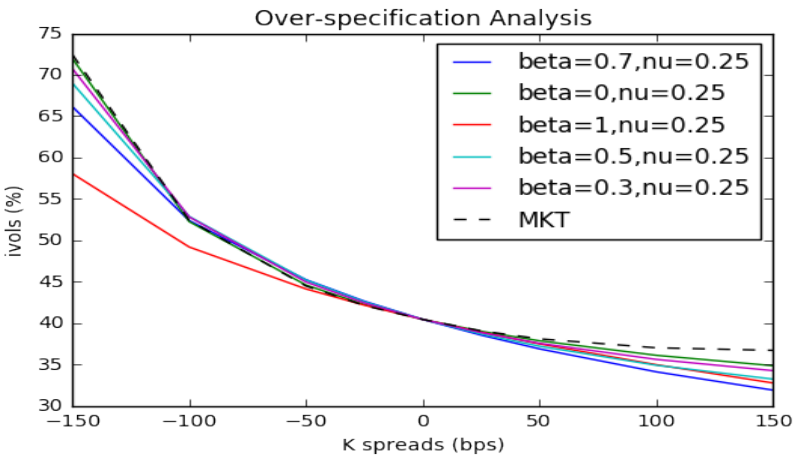
We repeat the same procedure for over-specification and collinearity test for Obloj SABR implementation and the results are listed below. Again our subject data here is the data with expiry T_{k-1}=1Y. It is straightforward that Obloj model and Hagan et al model give very close results for each calibration. What sets Obloj SABR apart from Hagan is that it gives much better approximation when β_k is fixed to 1 than Hagan implementation. Moreover, Obloj model has more potential for collinearity reduction with one or two SABR parameters fixed in calibration.




Over-specification Analysis
ivols (%)
K spreads (bps)
rho=0
rho=-0.5
rho=-0.3
rho=-0.9
rho=-0.7
MKT


Over-specification Analysis
ivols (%)
K spreads (bps)
vega=0.6
vega=0.4
vega=0.2
MKT


Over-specification Analysis
ivols (%)
K spreads (bps)
beta=0,rho=0
beta=0,rho=-0.8
beta=0.3,rho=0
beta=0.3,rho=-0.8
beta=0.3,rho=0.8
beta=0,rho=0.8
MKT



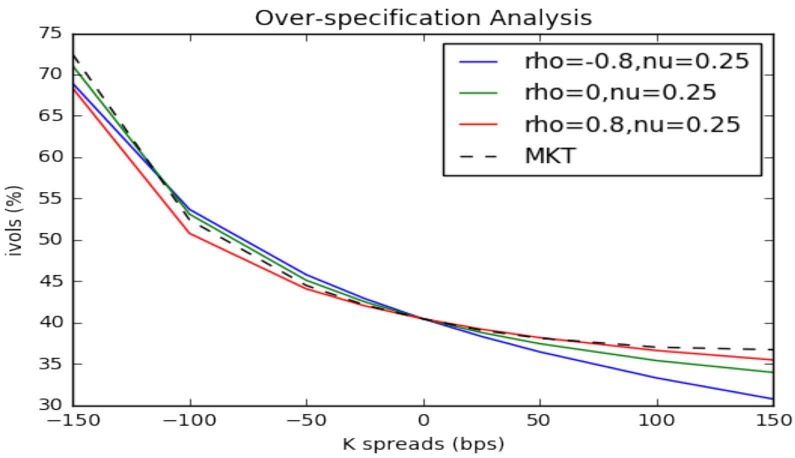


Table: Collinearity test for Obloj implementation

Obloj SABR	Condition number of Jacobian matrix
Calibrate β_k, ρ_k and υ_k	2711
Fix $\beta_k = 0.7$, calibrate ρ_k and υ_k	1496
Fix $\rho_k =$ -0.9, calibrate β_k and υ_k	9
Fix $\upsilon_k = 0.4$, calibrate β_k and ρ_k	2140
Fix β_k=0 and ρ_k=0.8, calibrate υ_k	40
Fix β_k=0 and υ_k=0.25, calibrate ρ_k	878
Fix ρ_k=0.8 and υ_k=0.25, calibrate β_k	22

8. Code structure

We code in Python and manage version controls on Github platform. The full codes are available at https://github.com/gsallc/CapstoneFall2017.git.

For more efficient coding and review, we have structured our codes into six folders: Pricing, Fitter, Bin, Inputs Test and Documentation and below is a summary of them.

- Pricing: library codes for various SABR models including Hagan SABR model and Obloj SABR model, Black Scholes, Monte Carlo simulation
- Fitter: library codes for over-specification test and multi-collinearity test of SABR calibration
- Bin: driver codes for both pricing and fitter parts
- Inputs: market data of options
- Test: unit tests and doc tests
- Documentation: project plans and reports

9. References

[1] Hagan P, D Kumar, A Lesniewski and D Woodward, "Managing smile risk", Wilmott Magazine, pages 84-108 (2002)

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