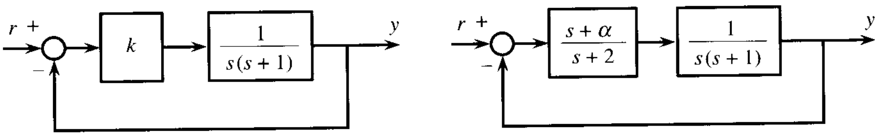
Note that, due to the multiplication by -1, the inequality is reversed in (4.23b). In order to meet (4.23b), k must lie between $(-7.04, 5.54)$. The three conditions in (4.23) are plotted in Figure 4.16(b). In order to meet them simultaneously, k must lie inside $(3.6, 5.54)$. Thus, the system is stable if and only if

$$3.6 < k < 5.54$$

Exercise 4.6.4

Find the stability ranges of the systems shown in Figure 4.17.



(a) (b)

Figure 4.17 Feedback systems.

[**Answers:** (a) $0 < k < \infty$. (b) $0 < \alpha < 9$.]

4.7 STEADY-STATE RESPONSE OF STABLE SYSTEMS—POLYNOMIAL INPUTS

Generally speaking, every control system is designed so that its output $y(t)$ will track a reference signal $r(t)$. For some problems, the reference signal is simply a step function, a polynomial of degree 0. For others, the reference signal may be more complex. For example, the desired altitude of the landing trajectory of a space shuttle may be as shown in Figure 4.18. Such a reference signal can be approximated by

$$r(t) = r_0 + r_1 t + r_2 t^2 + \cdots + r_m t^m$$

a polynomial of t of degree m. Clearly, the larger m, the more complex the reference signal that the system can track. However, the system will also be more complex.

Figure 4.18 Time function.

In practice, control systems are designed to track

$$r(t) = a \qquad \text{(step function)}$$

$$r(t) = at \qquad \text{(ramp function)}$$

or

$$r(t) = at^2 \qquad \text{(acceleration function)}$$

where a is a constant. These are polynomials of degree 0, 1 and 2. The first two are used more often. For example, the temperature setting in a thermostat provides a step reference input, called the *set point* in industry. *To reset the set point* means *to change the amplitude of the step reference input.*

Although it is desirable to design a system so that its output $y(t)$ will track the reference input $r(t)$ immediately—that is, $y(t) = r(t)$, for all $t \geq 0$—it is not possible to achieve this in practice. The best we can hope for is

$$\lim_{t \to \infty} y(t) = r(t)$$

that is, $y(t)$ will track $r(t)$ as t approaches infinity. This is called *asymptotic tracking*, and the response

$$y_s(t) := \lim_{t \to \infty} y(t)$$

is called the *steady-state* response. We will now compute the steady-state response of stable systems due to polynomial inputs.

Consider a system with transfer function

$$G_o(s) = \frac{Y(s)}{R(s)} = \frac{\beta_0 + \beta_1 s + \cdots + \beta_m s^m}{\alpha_0 + \alpha_1 s + \cdots + \alpha_n s^n} \tag{4.24}$$

with $n \geq m$. It is assumed that $G_o(s)$ is stable, or that all the poles of $G_o(s)$ have negative real parts. If we apply the reference input $r(t) = a$, for $t \geq 0$, then the output is given by

$$\begin{aligned} Y(s) = G_o(s)R(s) &= \frac{\beta_0 + \beta_1 s + \cdots + \beta_m s^m}{\alpha_0 + \alpha_1 s + \cdots + \alpha_n s^n} \cdot \frac{a}{s} \\ &= \frac{k}{s} + \text{(terms due to the poles of } G_o(s)) \end{aligned}$$

with k given by, using (A.8b),

$$k = G_o(s) \frac{a}{s} \cdot s \Big|_{s=0} = G_o(0)a = \frac{\beta_0}{\alpha_0} \cdot a$$

If the system is stable, then the response due to every pole of $G_o(s)$ will approach zero as $t \to \infty$. Thus the steady-state response of the system due to $r(t) = a$ is

$$y_s(t) = \lim_{t \to \infty} y(t) = G_o(0)a = \frac{\beta_0}{\alpha_0} \cdot a \tag{4.25}$$

This steady-state response depends only on the coefficients of $G_o(s)$ associated with s^0.

Now we consider the ramp reference input. Let $r(t) = at$. Then

$$R(s) = \frac{a}{s^2}$$

and

$$\begin{aligned} Y(s) &= G_o(s)\frac{a}{s^2} \\ &= \frac{k_1}{s} + \frac{k_2}{s^2} + \text{(terms due to the poles of } G_o(s)) \end{aligned}$$

with, using (A.8c) and (A.8d),

$$k_2 = G_o(s)\frac{a}{s^2}\cdot s^2\bigg|_{s=0} = G_o(0)a$$

and

$$\begin{aligned} k_1 &= \frac{d}{ds}G_o(s)a\bigg|_{s=0} \\ &= a\left[\frac{(\alpha_0 + \alpha_1 s + \cdots + \alpha_n s^n)(\beta_1 + \cdots + m\beta_m s^{m-1})}{(\alpha_0 + \alpha_1 s + \cdots + \alpha_n s^n)^2}\right. \\ &\qquad \left. - \frac{(\beta_0 + \beta_1 s + \cdots + \beta_m s^m)(\alpha_1 + \cdots + n\alpha_n s^{n-1})}{(\alpha_0 + \alpha_1 s + \cdots + \alpha_n s^n)^2}\right]_{s=0} \\ &= a\,\frac{\alpha_0\beta_1 - \beta_0\alpha_1}{\alpha_0^2} \end{aligned}$$

Thus the steady-state response of the system due to $r(t) = at$ equals

$$y_s(t) = G_o(0)at + G_o'(0)a \tag{4.26a}$$

or

$$y_s(t) = \frac{\beta_0}{\alpha_0}\cdot at + a\cdot\frac{\alpha_0\beta_1 - \beta_0\alpha_1}{\alpha_0^2} \tag{4.26b}$$

This steady-state response depends only on the coefficients of $G_o(s)$ associated with s^0 and s.

We discuss now the implications of (4.25) and (4.26). If $G_o(0) = 1$ or $a_0 = \beta_0$ and if $r(t) = a$, $t \geq 0$, then

$$y_s(t) = a = r(t)$$

Thus the output $y(t)$ will track asymptotically any step reference input. If $G_o(0) = 1$ and $G_o'(0) = 0$ or $\alpha_0 = \beta_0$ and $\alpha_1 = \beta_1$, and if $r(t) = at$, $t \geq 0$, then (4.26) reduces to

$$y_s(t) = at$$

that is, $y(t)$ will track asymptotically any ramp reference input. Proceeding forward, if

$$\alpha_0 = \beta_0 \qquad \alpha_1 = \beta_1 \qquad \text{and} \qquad \alpha_2 = \beta_2 \tag{4.27}$$

then the output of $G_o(s)$ will track asymptotically any acceleration reference input at^2. Note that in the preceding discussion, the stability of $G_o(s)$ is essential. If $G_o(s)$ is not stable, the output of $G_o(s)$ will not track any $r(t)$.

Exercise 4.7.1

Find the steady-state responses of

a. $G_o(s) = \dfrac{1}{1 - s^2}$ due to $r(t) = a$

b. $G_o(s) = \dfrac{2}{s + 1}$ due to $r(t) = a$

c. $G_o(s) = \dfrac{2 + 3s}{2 + 3s + s^2}$ due to $r(t) = 2 + t$

d. $G_o(s) = \dfrac{2}{s + 1}$ due to $r(t) = 3t$

e. $G_o(s) = \dfrac{68 + 9s + 9s^2}{68 + 9s + 9s^2 + s^3}$ due to $r(t) = a$

[**Answers:** (a) ∞; (b) $2a$; (c) $y_s(t) = 2 + t$; (d) $6t - 6$; (e) ∞.]

4.7.1 Steady-State Response of Stable Systems—Sinusoidal Inputs

Consider a system with proper transfer function $G_o(s) = Y(s)/R(s)$. It is assumed that $G_o(s)$ is stable. Now we shall show that if $r(t) = a \sin \omega_o t$, then the output $y(t)$ will approach a sinusoidal function with the same frequency as $t \to \infty$.

If $r(t) = a \sin \omega_o t$, then, using Table A.1,

$$R(s) = \frac{a\omega_o}{s^2 + \omega_o^2} \tag{4.28}$$

Hence, we have

$$Y(s) = G_o(s)R(s) = G_o(s) \cdot \frac{a\omega_o}{s^2 + \omega_o^2} = G_o(s) \cdot \frac{a\omega_o}{(s + j\omega_o)(s - j\omega_o)}$$

Because $G_o(s)$ is stable, $s = \pm j\omega_o$ are simple poles of $Y(s)$. Thus $Y(s)$ can be expanded as, using partial fraction expansion,

$$Y(s) = \frac{k_1}{s - j\omega_o} + \frac{k_1^*}{s + j\omega_o} + \text{terms due to the poles of } G_o(s)$$