SOLUTIONS OF EXAM PAPER 3

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Solutions

Problem 1

(a) For the given system we can write

$$I(s) = I_L(s) = \frac{V(s)}{Z_{\text{total}}},$$

and hence, through current partitioning,

$$\begin{aligned} I_C(s) &= I_L(s)\,\frac{R}{R+\frac{1}{sC}} \\ &= V(s)\,\frac{1}{sL+\frac{R}{sRC+1}}\,\frac{R}{R+\frac{1}{sC}} \\ &= V(s)\,\frac{\frac{s}{L}}{s^2+\frac{1}{RC}s+\frac{1}{LC}}. \end{aligned}$$

Hence, we can write the transfer function for the given system:

$$F(s) = \frac{I_C}{V(s)} = \frac{\frac{s}{L}}{s^2+\frac{1}{RC}s+\frac{1}{LC}}.$$

From above equation, we can write, by comparison with the standard second order polynomial with complex conjugate poles,

$$\omega_n = \frac{1}{\sqrt{LC}}$$

and

$$2\zeta\,\omega_n = \frac{1}{RC},$$

i.e.,

$$\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

(and remember that must be $\zeta < 1$ to have complex conjugate poles).
(b) By comparing the poles of the transfer function $F(s)$ obtained in (a) with their standard form

$$-\zeta\,\omega_n \pm j\,\omega\,\sqrt{1-\zeta^2},$$

we can obtain the desired time constant $\tau = 1.25$ s as follows

$$\begin{aligned}\zeta\,\omega_n &= \frac{1}{2RC},\\ \tau &= \frac{1}{\zeta\,\omega_n}\\ &= 2RC = 1.25\end{aligned}$$

i.e.,

$$R = 0.625\,\Omega.$$

(c) For the given values of L and C, and obtained value of R, we can write the transfer function of the system

$$F(s) = \frac{s}{s^2 + \frac{8}{5}s + 1}.$$

As known, the impulse response $f(t) = \mathcal{L}^{-1}(F(s))$. In view of the initial value theorem for Laplace transforms, we can write

$$f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s^2}{s^2 + \frac{8}{5}s + 1} = 1.$$

Further,

$$\begin{aligned}\dot{f}(0) &= \lim_{s\to\infty} s\mathcal{L}(\dot{f}(t)) = \lim_{s\to\infty} s[s\mathcal{L}(f(t)) - f(0)]\\ &= \lim_{s\to\infty} s[sF(s) - f(0)]\\ &= \lim_{s\to\infty} s\left[\frac{s^2 - s^2 - \frac{8}{5}s - 1}{s^2 + \frac{8}{5}s + 1}\right] = -\frac{8}{5}.\end{aligned}$$

Problem 2

(a) Given $G(s) = \frac{\mu\,\omega_n}{s^2 + 2\zeta\,\omega_n s + \omega_n^2}$, where $\omega_n = 10\,\mathrm{rad/s}$ and $\zeta = 0.5$. Using the formula for computing the closed loop transfer function after *unit* negative feedback, we can write the characteristics equation

$$s^2 + 10s + 100 + 100\mu = 0.$$

For the stability of this 2$^{\text{nd}}$ order polynomial, all coefficients need to have equal sign and then the term $100 + 100\mu$ should be greater than zero, i.e.,

$$\mu > -1.$$

(b) Another system $G_2(s) = \frac{1}{s+2}$ is connected in cascade to $G_1(s)$, then the new system $G_{\text{new}}(s)$ is

$$\begin{aligned}G_{\text{new}}(s) &= G_1(s)G_2(s)\\ &= \left(\frac{1}{s+2}\right)\left(\frac{\mu\,\omega_n}{s^2 + 2\zeta\,\omega_n s + \omega_n^2}\right).\end{aligned}$$

Let's put the given value of $\omega_n = 10$ and $\zeta = 0.5$, one can obtain,

$$G_{\text{new}}(s) = \frac{100\mu}{(s+2)(s^2+10s+100)}. \tag{1}$$

After the unit feedback, the characteristics equation of the closed loop system is

$$s^3 + 12s^2 + 120s + 200 + 100\mu = 0.$$

From R-H criteria (now needed, we cannot use only the signs of the coefficients to determine stability of a 3rd order polynomial!),

$$\begin{array}{c|cc} s^3 & 1 & 120 \\ s^2 & 12 & 200+100\mu \\ s & \frac{1440-200-100\mu}{12} & 0 \\ s^0 & 200+100\mu & \end{array}$$

Hence, after simple computations we find that asymptotic stability of the closed loop system is ensured by

$$-2 < \mu < 12.4.$$

(c) For a given amplification factor 3 at the pulsation $\omega = 1\,\text{rad/s}$, we can find the value of μ

$$|G_{\text{new}}(j\omega)|_{\omega=1} = \frac{100\mu}{\sqrt{2^2+1}\sqrt{10^2+99^2}} = 3, \tag{2}$$

which yields,

$$\mu = 6.675.$$

Putting the value of μ in the equation (1),

$$G_{\text{new}}(s) = \frac{667.5}{(s+2)(s^2+10s+100)}$$

The asymptotic Bode plots of the function $G_{\text{new}}(s)$ obtained with pencil and paper are given below. You can check the correctness of your plot using MATLAB (see in Fig. 2).

The drawing should go as follows. From the $G_{\text{new}}(s)$, due to one real and two complex poles, the frequency response of the transfer function breaks downward at $\omega = 2$ rad/s with slope $-20\,\text{dB/decade}$. Again it will bend downwards at $\omega = 10$ rad/s with slope $-20 - 40 = -60\,\text{dB/decade}$.

For the (asymptotic) Bode phase plot, at $\omega = 2$ rad/s the phase value decreases by $-90°$. Further, at the next cut-off frequency $\omega = 10$ rad/s, the phase value becomes $-90° - 180° = -270°$. Notice that since the system is quite *damped*, i.e. $\zeta = 0.5$ is not too small, the difference between the asymptotic diagram and the real diagram, which can be perceived on the plot produced by MATLAB, is negligible. Things would have been much different for smaller ζ. Think about this and check on the book...

Now let's take the value $\mu = 66.75$, then it can be seen in below Fig. 3, that the Bode plot shifted upwards by 20 dB, since the gain value is increased by 10 ($20\log(10) = 20$dB). Further, from the Fig. 4, we can analyse that, for the value $\mu = 0.6675$, Bode plot shifted downwards by 20 dB, since the gain value is decreased by 10 ($20\log(10) = 20$dB). In conclusion, by varying value of gain μ, the entire gain plot moves up and down. This could be used to solve the problem (c) without making the computation in (2).

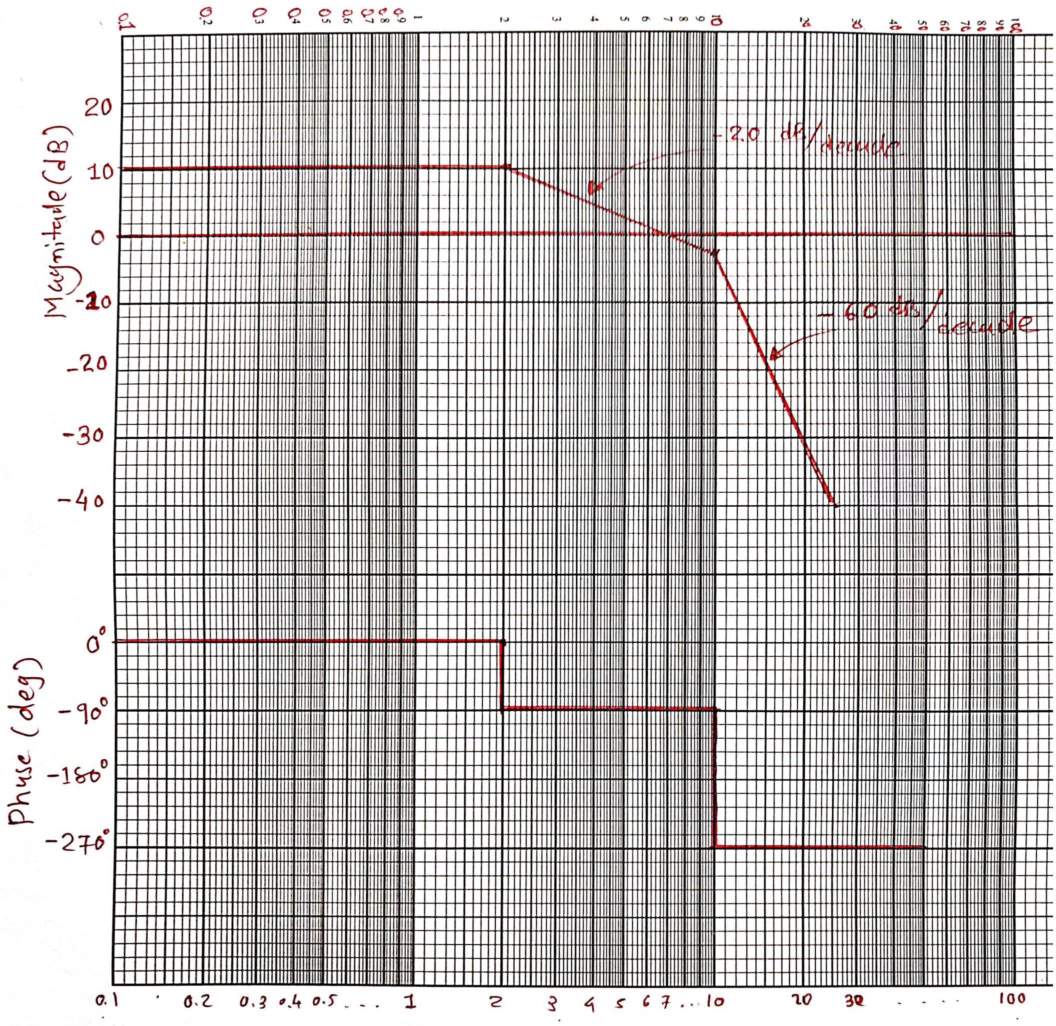

Figure 1: Bode plots for $\mu = 6.675$

Problem 3

(a) The discrete time system that describes the evolution of the population is

$$x(k+1) = 1.1x(k) - u(k).$$

(b) At equilibrium we can write the equation

$$\bar{x} = 1.1\bar{x} - \bar{u}. \tag{3}$$

To have an equilibrium at $\bar{x} = 1000$, equation (3), implies

$$\bar{u} = 100.$$

Hence, 100 bison must be killed every year.
(c) By taking z-transform of the system given in (a), with $u(k) = 20\,\delta_{-1}(k)$,

$$zX(z) - zx(0) - 1.1X(z) = -\frac{20z}{z-1}$$

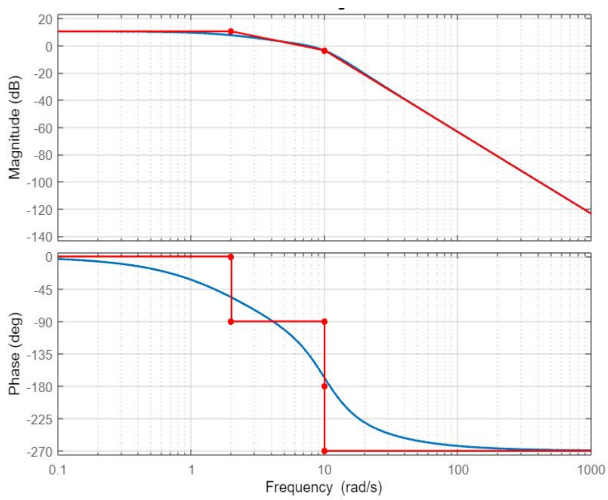

Figure 2: Bode plots for $\mu = 6.675$

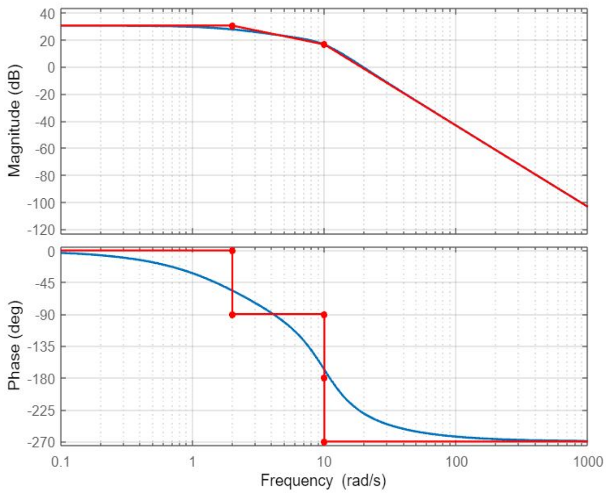

Figure 3: Bode plots for $\mu = 66.75$

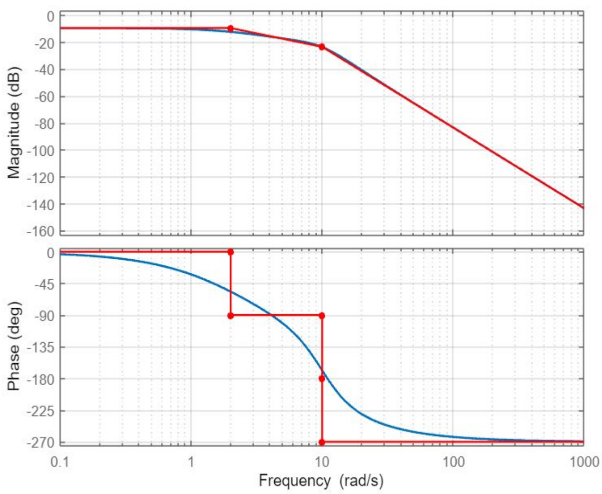

Figure 4: Bode plots for $\mu = 0.6675$

$$(z - 1.1)X(z) = zx(0) - \frac{20z}{z-1};$$

hence,

$$X(z) = \frac{zx(0)}{z-1.1} - \frac{20z}{(z-1)(z-1.1)}.$$

Taking the inverse z-transform and letting $x(0) = 1000$,

$$\begin{aligned} x(k) &= [(1000 - 200)(1.1)^k + 200]\delta_{-1}(k) \\ &= [800(1.1)^k + 200]\delta_{-1}(k). \end{aligned} \tag{4}$$

Imposing that after n years the population doubles, we can employ the equation (4)

$$\begin{gathered} 2000 = 800(1.1)^n + 200, \\ n = \lceil 8.51 \rceil = 9. \end{gathered}$$

So approximately after 9 years the population will double.