SOLUTIONS OF APRIL 26TH'S WRITTEN TEST

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Solutions

Problem 1

(a) For the given system

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = \mu(1 - x_1^2(t))x_2(t) - x_1(t),$$

At equilibrium, the equations can be written

$$0 = \bar{x}_2,$$
$$0 = \mu(1 - \bar{x}_1^2)\bar{x}_2 - \bar{x}_1,$$

which gives equilibrium point

$$\bar{x}_1 = 0, \bar{x}_2 = 0.$$

(b) For the linearization of the given system, let's take

$$f_1(x) = x_2,$$
$$f_2(x) = \mu(1 - x_1^2)x_2 - x_1.$$

After variable substitution, the linearized system can be written

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x=\bar{x}} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix},$$

where

$$\delta x_1 = x_1 - \bar{x}_1 = x_1,$$
$$\delta x_2 = x_2 - \bar{x}_2 = x_2.$$

From the above equation, matrix A is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}.$$

(c) From the equation $\det(\lambda I - A) = 0$, one can obtain,

$$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda - \mu \end{vmatrix} = 0,$$

$$\lambda^2 - \mu\lambda + 1 = 0.$$

For $\mu < 0$, the system is stable.

Problem 2

(a) The given system is

$$G(s) = \frac{K}{s(s+1)(1+0.1s)(1+0.01s)}.$$

The magnitude of the above system is given 0 db at $\omega = 3$ rad/sec, hence it can be written as

$$|G(j3)| = \left| \frac{K}{(j3)(j3+1)(1+0.1(j3))(1+0.01(j3))} \right| = 1,$$

which yields

$$K = 9.9 \approx 10.$$

The asymptotic Bode plots of the function $G(s)$ obtained with pencil and paper are given below. You can check the correctness of your plot using MATLAB (see in Fig. 2).

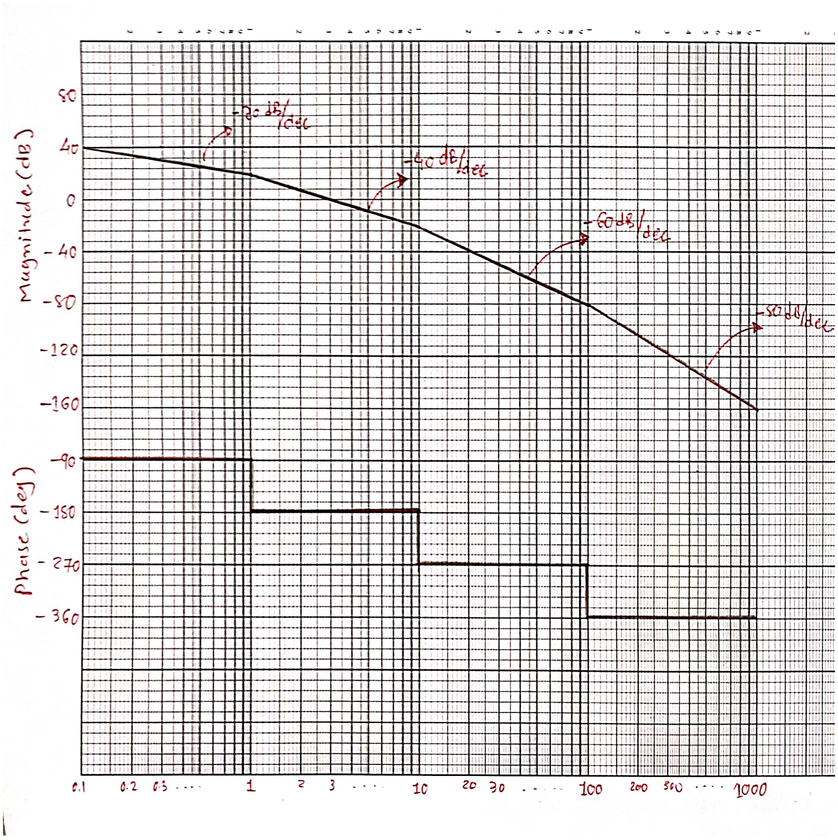

Figure 1: Bode plots for $K = 10$

From the $G(s)$, due to one pole at origin and three real poles, the frequency response of the transfer function breaks downward at $\omega = 1$ rad/s with slope $-20-20 = -40$ dB/decade. Again it will bend downwards at $\omega = 10$ rad/s with slope $-20-40 = -60$ dB/decade. At the end again it will bend downwards at $\omega = 100$ rad/s with slope $-20-60 = -80$ dB/decade.

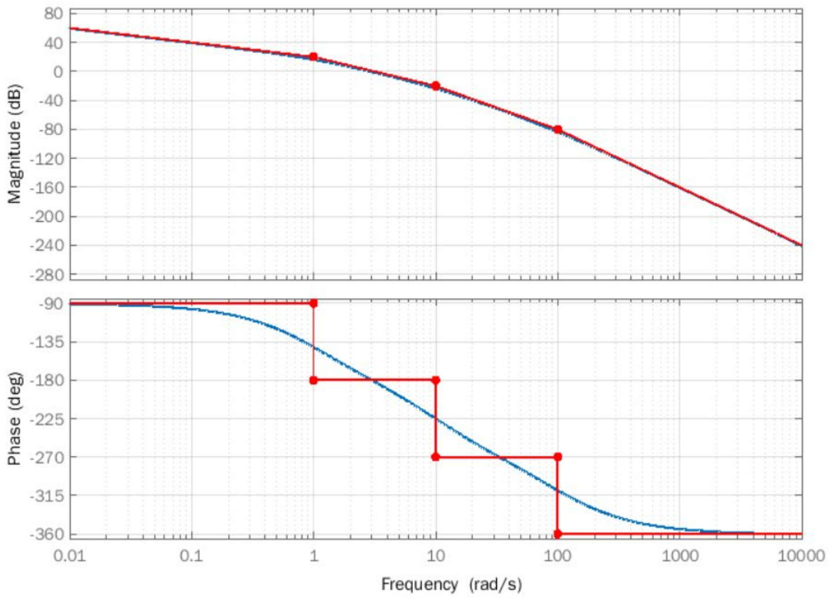

Figure 2: Bode plots for $K = 10$

For the (asymptotic) Bode phase plot, at $\omega = 1$ rad/s the phase value decreases by $-90° - -90° = -180°$. Further, at the next cut-off frequency $\omega = 10$ rad/s, the phase value becomes $-90° - 180° = -270°$. At the end of cut-off frequency $\omega = 100$ rad/s, the phase value becomes $-90° - 270° = -360°$.

(b) From the given response signal $u(t) = 10\sin(0.32\pi t - 10)$, we can write the value of $\omega = 1.0044 \approx 1\,\text{rad/sec}$. Hence, for the given value of ω, and from Bode plot obtained in (a), we can say that the magnitude will multiply by 10 ($20\log(x) = 20$, hence $x = 10$) and it will lag by 180 degree i.e., π rad. As a result final steady state response can be written as

$$\begin{aligned} y(t) &= 10 \times 10\sin(0.32\pi t - 10 - \pi) \\ &= 100\sin(0.32\pi t - 10 - \pi). \end{aligned}$$

Problem 3

(a) The discrete time system that describe the evolution of the population is

$$x(k+1) = 1.5x(k) - 1200$$

(since, $x + 1.2x - 0.7x = 1.5x$).

Let's take z-transform of the system (consider that -1200 is a constant input, a step):

$$\begin{aligned} zX(z) - zx(0) &= 1.5X(z) - \frac{1200z}{z-1}, \\ X(z) &= \frac{12230z}{z-1.5} - \left[\frac{-600z}{z-1} + \frac{600z}{z-1.5}\right]. \end{aligned}$$

The general solution for the lake fish population can be written as

$$\begin{aligned} x(k) &= [12230(1.5)^k + 600 - 600(1.5)^k]\delta_{-1}(k), \\ &= [11630(1.5)^k + 600]\delta_{-1}(k). \end{aligned}$$

(b) To keep the population constant, we would want $x(k+1) = x(k) = \bar{x}$ for all k. Specifically $x(1) = x(0)$. Let h be the harvesting level we wish to find. Thus using $x(1) = x(0)$ in

$$x(1) = 1.5x(0) - h,$$

we want to solve for h:

$$\begin{aligned} x(0) &= 1.5x(0) - h, \\ h &= 6115. \end{aligned}$$

Thus, if the lake resource managers allowed 6115 fish each season to be caught, the fish population size will remain constant from year to year at 12 230.