SOLUTIONS OF JUNE 13th WRITTEN TEST

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June 2022

Solutions

Problem 1

(a) For the given system, let's take

$$x_1 = y,$$
$$x_2 = \dot{x}_1 = \dot{y}.$$

Hence, the system

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = 8u - 4x_2 - 8x_1,$$

i.e.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

To check the stability of the given system, $\det(sI - A) = s^2 + 4s + 8$. The given system is second order with positive coefficient, hence it is stable.
(b) From the state space representation found in (a), the transfer function can be written as

$$F(s) = C(sI - A)^{-1}B + D,$$

which yields

$$F(s) = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+4 & 1 \\ -8 & s \end{bmatrix} \begin{bmatrix} 0 \\ 8 \end{bmatrix}}{s^2 + 4s + 8},$$
$$= \frac{8}{s^2 + 4s + 8}.$$

Comparing denominator of the above function to the standard form of the second order polynomial with complex conjugate poles

$$s^2 + 2\zeta\omega_n s + \omega_n^2,$$

we can obtain the natural frequency

$$\omega_n = 2\sqrt{2}\,\mathrm{rad\,s^{-1}}$$

and the damping ratio

$$\zeta = \frac{4}{2\,\omega_n},$$
$$= \frac{1}{\sqrt{2}}.$$

Notice that $|\zeta| < 1$ as required for the polynomial to have complex conjugate poles.
Also, the time constant

$$\tau = \frac{1}{\zeta\,\omega_n}, \\ = \frac{1}{2}\,\mathrm{s}.$$

(c) The response of the system taking rectangular input with unit amplitude and duration twice the time constant (i.e., $u(t) = \delta_{-1}(t) - \delta_{-1}(t-1)$) can be written as

$$Y(s) = \frac{8}{s^2+4s+8}\left(\frac{1}{s} - \frac{e^{-s}}{s}\right), \\ = \frac{8(1-e^{-s})}{s(s^2+4s+8)},$$

which yields,

$$Y(s) = \frac{1}{s} - \frac{s+4}{s^2+4s+8} - \frac{e^{-s}}{s} + \frac{e^{-s}(s+4)}{s^2+4s+8}.$$

After taking Laplace inverse of the function, the response of the given system can be written as,

$$y(t) = \left[1 - e^{-2t}(\sin 2t + \cos 2t)\right]\delta_{-1}(t) - \left[1 - e^{-2(t-1)}(\sin 2(t-1) + \cos 2(t-1))\right]\delta_{-1}(t-1).$$

Problem 2

(a) For the odd function, below mentioned diagram can be obtained. The function $f(t)$ can be

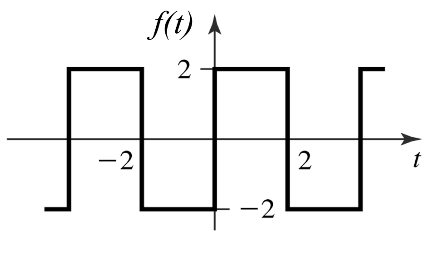

Figure 1: Periodic forcing function.

written as,

$$f(t) = \begin{cases} -2, & -2 < t < 0, \\ 2, & 0 < t < 2. \end{cases}$$

For an odd function we know that $a_n = 0$.
The coefficients b_n can be found, given the period $T_p = 2$,

$$b_n = \frac{2}{T_p}\int_0^{T_p} f(t)\sin\frac{n\pi\,t}{T_p}dt,\ n = 1, 2, \ldots \\ = \frac{2}{2}\left[\int_0^2 2\sin\frac{n\pi\,t}{2}dt\right] \\ = -\frac{4}{n\pi}(\cos n\pi - 1).$$

The average value is clearly zero.
The Fourier sine series can be written by substituting $\cos n\pi = (-1)^n$:

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} \frac{4[1-(-1)^n]}{n\pi} \sin \frac{n\pi t}{2} \\ &= \frac{8}{\pi} \sin \frac{\pi t}{2} - \frac{8}{3\pi} \sin \frac{3\pi t}{2} + \frac{8}{5\pi} \sin \frac{5\pi t}{2} - \cdots \end{aligned}$$

From the given series, the fifth harmonic is $\frac{8}{5\pi} \sin \frac{5\pi t}{2}$.
(b) The standard form of the first order system is

$$F(s) = \frac{\mu}{1+sT}.$$

From the given static gain 20dB, we can write the value of $\mu = 10$. Also, it is given that the cutoff frequency is equal to the frequency of the second harmonic of the square wave above. For the given square wave signal, the fundamental pulsation is $\omega_0 = \frac{2\pi}{T_p} = \pi$ and hence the second harmonic has pulsation $2\omega_0 = 2\pi$ (the corresponding amplitude is zero). The cutoff pulsation is given by $1/T$ resulting in $T = \frac{1}{2\pi}$.

Problem 3

(a) The transfer function of the given system can be obtained using z-transform,

$$\begin{aligned} z^2Y(z) - \frac{3}{2}zY(z) + \frac{1}{2}Y(z) &= U(z) \\ Y(z)\left[z^2 - \frac{3}{2}z + \frac{1}{2}\right] &= U(z) \\ \frac{Y(z)}{U(z)} &= \frac{1}{z^2 - \frac{3}{2}z + \frac{1}{2}}. \end{aligned}$$

For the given system, poles are $z = \frac{1}{2}$, and $z = 1$. Since the dominant pole is on the unit circle with unit multiplicity, the system is marginally stable.
(b) For the input $u[k] = \left(\frac{1}{3}\right)^k \delta_{-1}(k)$, given system equation can be written as

$$y[k+2] - \frac{3}{2}y[k+1] + \frac{1}{2}y[k] = \left(\frac{1}{3}\right)^k \delta_{-1}(k).$$

By taking the z-transform both the sides and putting the given initial value,

$$\begin{aligned} z^2[Y(z) - y(0) - z^{-1}y(1)] - \frac{3}{2}z[Y(z) - y(0)] + \frac{1}{2}Y(z) &= \frac{z}{z-\frac{1}{3}}, \\ Y(z)\left[z^2 - \frac{3}{2}z + \frac{1}{2}\right] &= \frac{z}{z-\frac{1}{3}} + 4z^2 - 6z. \end{aligned}$$

The two separate terms of the above function can be written as,

$$Y(z) = T_1(z) + T_2(z)$$

For the term $T_1(z) = \frac{z}{(z^2-\frac{3}{2}z+\frac{1}{2})(z-\frac{1}{3})}$,

$$\frac{T_1(z)}{z} = \frac{A_1}{z-\frac{1}{2}} + \frac{A_2}{z-1} + \frac{A_3}{z-\frac{1}{3}}.$$

This leads to $A_1 = -12$, $A_2 = 3$, and $A_3 = 9$. Hence,

$$T_1(z) = \frac{-12z}{z - \frac{1}{2}} + \frac{3z}{z-1} + \frac{9z}{z - \frac{1}{3}}.$$

The resulting response can be written as

$$t_1(k) = \left[-12\left(\frac{1}{2}\right)^k + 3 + 9\left(\frac{1}{3}\right)^k \right] \delta_{-1}(k).$$

For the second term $T_2(z) = \frac{4z^2-6z}{z^2-\frac{3}{2}z+\frac{1}{2}}$,

$$\frac{T_2(z)}{z} = \frac{B_1}{z - \frac{1}{2}} + \frac{B_2}{z-1}$$

which yields $B_1 = 8$, $B_2 = -4$, which gives

$$T_2(z) = \frac{8z}{z - \frac{1}{2}} - \frac{4z}{z-1}.$$

The resulting response can be written as

$$t_2(k) = \left[8\left(\frac{1}{2}\right)^k - 4 \right] \delta_{-1}(k).$$

Now putting both the response together the final expression can be written as

$$y(k) = t_1(k) + t_2(k) = \left[-4\left(\frac{1}{2}\right)^k - 1 + 9\left(\frac{1}{3}\right)^k \right] \delta_{-1}(k).$$

Notice that $y(0) = 4$ and $y(1) = 0$ as required.