Solutions

Problem (1)

For the given system we can write the equations

$\ddot{x} + \dot{x} - k\ln(1-\alpha x) = F$

$x_1 = x \Rightarrow \dot{x}_1 = \dot{x}$

$x_2 = \dot{x} \Rightarrow \dot{x}_2 = \ddot{x} = F + k\ln(1-\alpha x) - \dot{x}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \ln(1-\alpha x_1) - x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F$$

At equilibrium.

$-k\ln(1-\alpha\bar{x}_1) = F$

$\ln(1-\alpha\bar{x}_1) = -1 \qquad (\because k=1, F=1)$

$\therefore 1-\alpha\bar{x}_1 = e^{-1}$

Now,

$\delta x = x - \bar{x}$ where $\bar{x} = \begin{pmatrix} \frac{(1-e^{-1})}{\alpha} \\ 0 \end{pmatrix}$; $\bar{x}_1 = \frac{1-e^{-1}}{\alpha}$

$\delta u = u - \bar{u}$ $\qquad \bar{x}_2 = 0$

$\delta y = y - \bar{y}$ $\qquad \bar{y} = \bar{x}_1, \quad \bar{u} = (-1)$

After linearization by taking $f_1 = x_2$, $f_2 = \ln(1-\alpha x_1) - x_2$

$\delta\dot{x} = A\delta x + B\delta u$

$\delta y = C\delta x + 0$

$$A = \frac{\partial f}{\partial x} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x=\bar{x}} = \left.\begin{bmatrix} 0 & 1 \\ \frac{-\alpha}{1-\alpha x_1} & -1 \end{bmatrix}\right|_{x=\bar{x}}$$

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $C = [1 \quad 0]$

$$(sI-A)=\begin{bmatrix} s & -1 \\ 2.71\alpha & s+1 \end{bmatrix} \Rightarrow (sI-A)^{-1} = \frac{\begin{bmatrix} s+1 & 1 \\ -2.71\alpha & s \end{bmatrix}}{s^2+s+2.71s}$$

transfer function for given linearized system

$$T(s) = C(sI-A)^{-1}B + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s+1 & 1 \\ -2.71\alpha & s \end{bmatrix}}{s^2+s+2.71\alpha} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore T(s) = \frac{1}{s^2+s+2.71\alpha}$$

For the stability of given system, all the coefficients should be positive

Hence,

$$2.71\,\alpha > 0 \Rightarrow \alpha > 0$$

$\therefore$ For $\alpha > 0$ the given linearization is always stable.

Problem (2)

(a) Value of static gain K of the given systems

For system (a)

$$20 \log_{10} K = 20$$

$$\therefore \log_{10} K = 1 \Rightarrow \underline{K = 10}$$

For system (b)

$$20 \log_{10} K = -20$$

$$\log_{10} K = -1 \Rightarrow \underline{K = 0.1}$$

(b) For given system 2(b)

At $\omega=1$ change in slope is +60 dB → 3 zeros at $\omega=1$
At $\omega=10$ change in slope is −40 dB → 2 poles at $\omega=10$
At $\omega=100$ change in slope is −20 dB → 1 pole at $\omega=100$

∴ Number of poles = 2+1 = 3
Number of zeros = 3

(c) For the given fig. 2(a) we can write the equation for slope.

$$\frac{26-20}{\log_{10} w_1 - \log_{10} 10} = 20$$

$$\therefore \log_{10} w_1 - 1 = \frac{6}{20}$$

$$\therefore \boxed{w_1 = 19.95 \cong 20}$$

Now

At $\omega = 10$ change in slope is $+20\,dB \rightarrow 1$ zero at $\omega = 10$

At $\omega = 19.95$ change in slope is $-40\,dB \rightarrow 2$ poles at $\omega = 19.95 \cong 20$

Value of K from (a) is 10.

Hence we can write transfer function

$$T.F = T(s) = \frac{10\left(\frac{s}{10}+1\right)}{\left(\frac{s}{19.95}+1\right)^2} \cong \frac{10\left(\frac{s}{10}+1\right)}{\left(\frac{s}{20}+1\right)^2}$$

$$\therefore T(s) = \frac{398\,(s+10)}{(s+20)^2}$$

$\longrightarrow$ Ans.

Problem (3)

(a) From given system we can write the eqn

$x_1(k+1) = x_2(k)$

$x_2(k+1) = -0.21x_1(k) - x_2(k) + u(k)$

$y(k) = x_2(k)$

State-space representation is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.21 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k+1) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

For stability of given system

$$\det(zI - A) = 0 \Rightarrow \left| \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -0.21 & -1 \end{pmatrix} \right| = 0$$

$\Rightarrow z^2 + z + 0.21 = 0$

$\Rightarrow (z+0.3)(z+0.7) = 0$

$\Rightarrow z = -0.3 ; \; z = -0.7$

within the unit circle

$\therefore$ given system is stable

For settling time, we must choose -0.7 over -0.3 since it is the dominant pole, having the largest module

$\therefore \; |0.01| = |0.7|^k$

$\therefore \; k \ln(0.7) = \ln(0.01)$

$\Rightarrow k = \lceil 12.91 \rceil = 13$ sec $\longrightarrow$ Ans.

(b) Transfer function of the given system

$$G(z) = C(zI - A)^{-1}B + D$$

$$= \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z+1 & 1 \\ -0.21 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{z^2 + z + 0.21}$$

$$G(z) = \frac{z}{(z+0.3)(z+0.7)}$$

Static gain of the system

$$G(z)\Big|_{z=1} = \frac{1}{2.21} = 0.452$$

→ Ans

(c) given : $u(k) = (-1)^{k-3} \delta_{-1}(k-3)$

$$\frac{Y(z)}{U(z)} = \frac{z}{(z+0.3)(z+0.7)} \quad ; \quad U(z) = z^{-3} \cdot \frac{z}{z+1}$$

$$\therefore Y(z) = z^{-3} \cdot z \left[\frac{z}{(z+0.3)(z+0.7)(z+1)} \right]$$

$$\therefore Y(z) = z^{-3} \left[\frac{-1.07z}{z+0.3} + \frac{5.83z}{z+0.7} - \frac{4.76z}{z+1} \right]$$

$$\therefore Y(z) = 5.83(-0.7)^{k-3} \delta_{-1}(k-3) - 1.07(-0.3)^{k-3} \delta_{-1}(k-3)$$
$$-4.76(-1)^{k-3} \delta_{-1}(k-3)$$

→ Ans.