SOLUTIONS OF JULY 11th WRITTEN TEST

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July 2022

Solutions

Problem 1

(a) For the given system, it can be written that

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -2x_1 - 3x_2 + u.$$

Hence, the state-space representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(b) From the state space representation found in (a), the transfer function can be obtained using

$$G(s) = C(sI - A)^{-1}B + D,$$

which yields

$$\begin{aligned} G(s) &= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2 + 3s + 2} \\ &= \frac{1}{s^2 + 3s + 2}. \end{aligned}$$

For the impulse response,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[G(s)] \\ &= [e^{-t} - e^{-2t}]\delta_{-1}(t). \end{aligned}$$

The input-output representation can be found by using

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{1}{s^2 + 3s + 2}, \\ s^2Y(s) + 3sY(s) + 2Y(s) &= U(s), \end{aligned}$$

which yields

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t).$$

Comparing denominator of $G(s)$ to the standard form of the second order polynomial with complex conjugate poles

$$s^2 + 2\zeta\omega_n s + \omega_n^2,$$

we can obtain the natural frequency

$$\omega_n = \sqrt{2}\,\mathrm{rad\,s^{-1}}$$

and the damping ratio

$$\begin{aligned}\zeta &= \frac{3}{2\,\omega_n} \\ &= \frac{3}{2\sqrt{2}}.\end{aligned}$$

Notice however that $|\zeta| > 1$, which means the given system is overdamped, the step input %OS is zero, the poles are real and not complex conjugate. Since the poles lie in -1 and -2 the correct factorizations of the denominator are $(s+1)(s+2)$ or $2(1+s)(1+0.5s)$.
(c) For the unity negative feedback,

$$\begin{aligned}G'(s) &= \frac{\frac{1}{s^2+3s+2}}{\frac{1}{s^2+3s+2}+1} \\ &= \frac{1}{s^2+3s+3}.\end{aligned}$$

Problem 2

(a) It is given that the final slope of the Bode plot is -60dB/dec. This is possible with the smallest value of $p = 0$ and $q = 3$.
(b) It is given that the system has a cutoff frequency at 100 rad/s. It can be obtained with infinite combination of coefficients. Here three such combinations are illustrated that include (1) no poles at zero, (2) one pole at zero, and (3) two poles at zero.
(1) Let's choose the transfer function $G_1(s)$ such that the system has a cutoff pulsation at 100 rad/s and no pole at zero. One possible transfer function is

$$\begin{aligned}G_1(s) &= \frac{10}{(1+s)(1+\frac{s}{10})(1+\frac{s}{100})} \\ &= \frac{10000}{s^3+111s^2+1110s+1000}\end{aligned}$$

where we chose the other cutoff pulsations at 1 rad/s and 10 rad/s and a static gain 10. For the above equation of $G_1(s)$, the values $p = 0, q = 3, b_0 = 10000, a_0 = 1, a_1 = 111, a_2 = 1110, a_q = 1000$, and all other coefficients are 0.
The asymptotic Bode plots of the function $G_1(s)$ obtained with pen and paper are given below (see in Fig. 1). You can check the correctness of your plot using MATLAB (see in Fig. 2).

From the $G_1(s)$, due to three real poles, the frequency response of the transfer function breaks downward at $\omega = 1$ rad/s with slope $-20,$dB/decade. Again it will bend downwards at $\omega = 10$ rad/s with slope $-20-20 = -40$ dB/decade. At the end again it will bend downwards at $\omega = 100$ rad/s with slope $-20-40 = -60$ dB/decade.

For the (asymptotic) Bode phase plot, at $\omega = 1$ rad/s the phase value decreases by $-90°$. Further, at the next cut-off pulsation $\omega = 10$ rad/s, the phase value becomes $-90° - 90° = -180°$. At the end of cut-off pulsation $\omega = 100$ rad/s, the phase value becomes $-90° - 180° = -270°$.

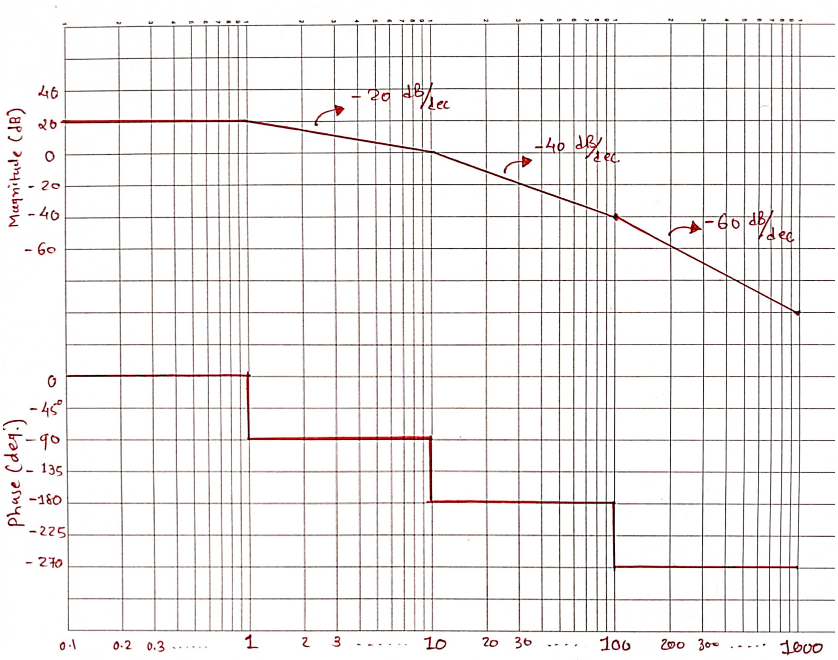

Figure 1: Bode plots with no pole at zero.

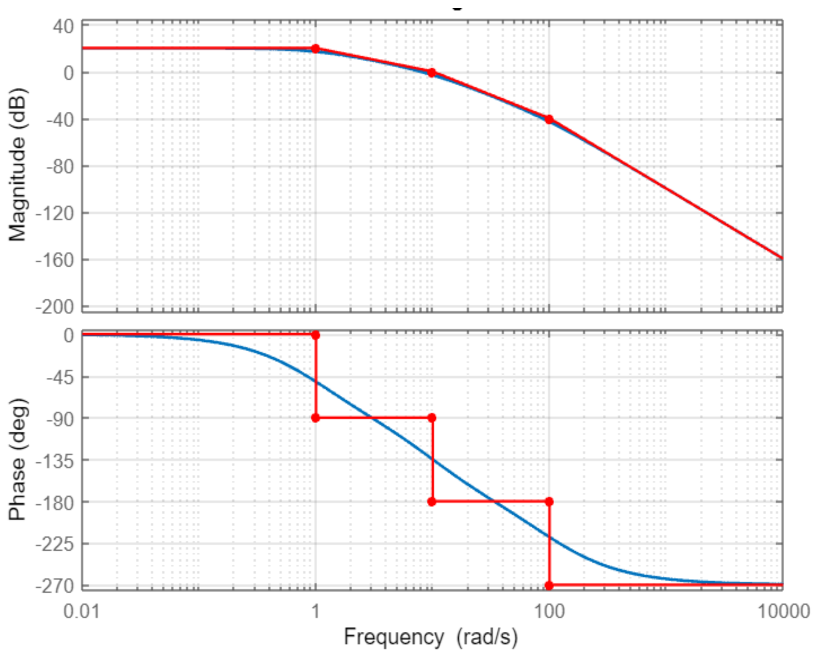

Figure 2: Bode plots with no pole at zero.

(2) Let's choose the transfer function $G_2(s)$ such that the system has a cutoff pulsation at 100 rad/s and one pole at zero. The equation for the $G_2(s)$ can be written as

$$
\begin{aligned}
G_2(s) &= \frac{10}{s(1+\frac{s}{10})(1+\frac{s}{100})} \\
&= \frac{10000}{s^3+110s^2+1000s}.
\end{aligned}
$$

Now we chose the other cutoff pulsation at 10 rad/s. Notice that now the static gain is infinite since there is an integrator in the system. For the above equation of $G_2(s)$, the values $p = 0, q = 3, b_0 = 10000, a_0 = 1, a_1 = 110, a_2 = 1000$, and all other coefficients are 0.
The asymptotic Bode plots of the function $G_2(s)$ obtained with MATLAB illustrated in Fig. 3.

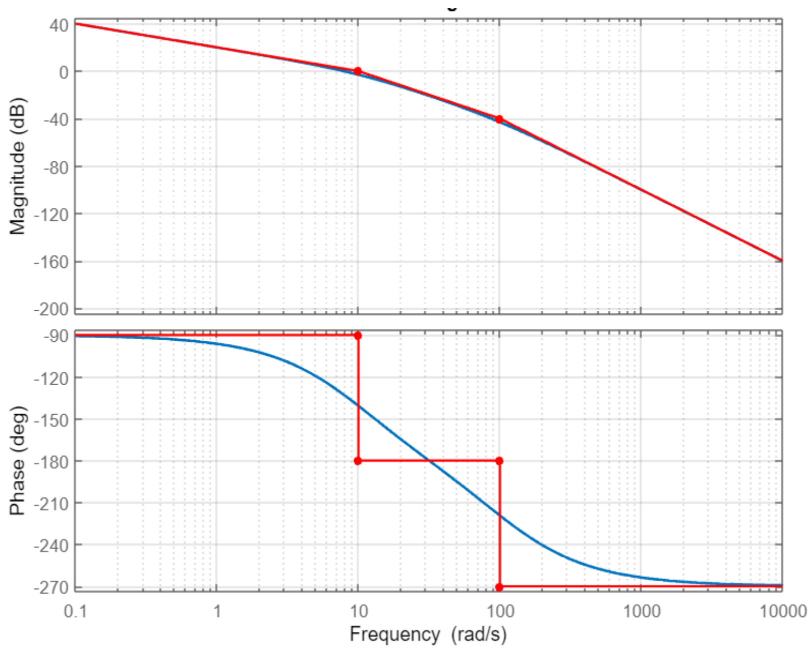

Figure 3: Bode plots with one pole at zero.

From the $G_1(s)$, due to one pole at origin and two real poles, the frequency response of the transfer function at $\omega = 1$ rad/s is with the slope of -20, dB/decade. It will break downwards at $\omega = 10$ rad/s with slope $-20 - 20 = -40$ dB/decade. At the end again it will bend downwards at $\omega = 100$ rad/s with slope $-20 - 40 = -60$ dB/decade.

For the (asymptotic) Bode phase plot, at $\omega = 1$ rad/s the phase value is $-90°$. Further, at the next cut-off pulsation $\omega = 10$ rad/s, the phase value becomes $-90° - 90° = -180°$. At the end of cut-off pulsation $\omega = 100$ rad/s, the phase value becomes $-90° - 180° = -270°$.

(3) Let's choose the transfer function $G_3(s)$ such that the system has a cutoff pulsation at 100 rad/s and two poles at zero. The equation for the $G_3(s)$ can be written as

$$\begin{aligned} G_3(s) &= \frac{10}{s^2(1 + \frac{s}{100})} \\ &= \frac{1000}{s^3 + 100s^2} \end{aligned}$$

Notice again that the static gain is infinite since we have a double integrator in the system. For the above equation of $G_3(s)$, the values $p = 0, q = 3, b_0 = 1000, a_0 = 1, a_1 = 100$, and all other coefficients are 0.
The asymptotic Bode plots of the function $G_3(s)$ obtained with MATLAB illustrated in Fig. 4.

From the $G_3(s)$, due to two poles at origin and one real pole, the frequency response of the

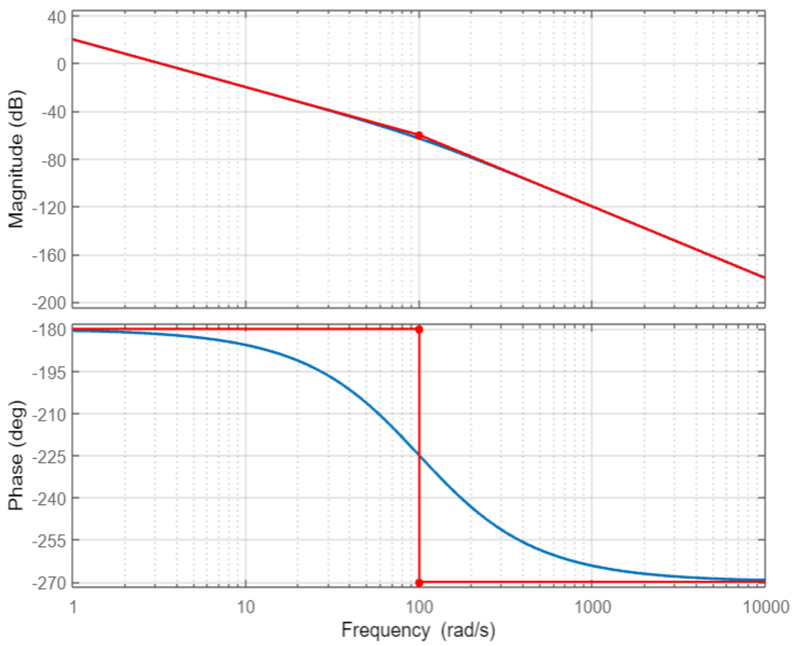

Figure 4: Bode plots with two poles at zero.

transfer function at $\omega = 1$ rad/s is with the slope of $-40, \mathrm{dB/decade}$. It will break downwards at $\omega = 100$ rad/s with slope $-40 - 20 = -60\,\mathrm{dB/decade}$.
For the (asymptotic) Bode phase plot, at $\omega = 1$ rad/s the phase value is $-180°$. Further, at the next cut-off pulsation $\omega = 100$ rad/s, the phase value becomes $-180° - 90° = -270°$.

Problem 3

(a) The given system $y[k+2] - 5y[k+1] + 6y[k] = u[k]$ can also be written $y(k+2) = 5y(k+1) - 6y(k) + u(k)$ and then the block diagram with delays is illustrated below in Fig. 5:

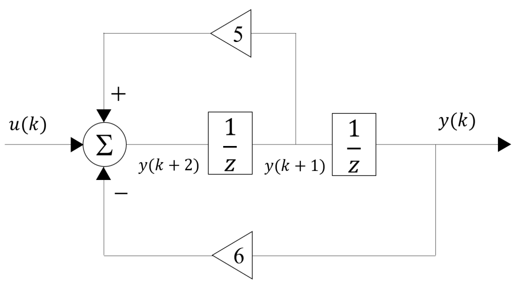

Figure 5: The block diagram with delays.

Now, let's take as states the output of the delays, i.e.

$$x_1(k) = y(k),$$
$$x_2(k) = y(k+1).$$

Hence, the system

$$x_1(k+1) = x_2(k),$$
$$x_2(k+1) = u(k) - 6x_1(k) + 5x_2(k).$$

The above equations can be written as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$
$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + u(k).$$

Using the z-transform, the transfer function for the given system is

$$\begin{aligned} \frac{Y(z)}{U(z)} &= \frac{1}{z^2 - 5z + 6} \\ &= \frac{1}{(z-2)(z-3)}. \end{aligned}$$

From the above transfer function, poles are $z = 2$, and $z = 3$. Since the poles are outside the unit circle, the system is unstable.

(b) For the input $u(k) = \frac{1}{5}[\delta_{-1}(k) - \delta_{-1}(k-1)]$, the given system equation can be written as

$$y[k+2] - 5y[k+1] + 6y[k] = \frac{1}{5}[\delta_{-1}(k) - \delta_{-1}(k-1)].$$

By taking the z-transform of both the sides and putting the given initial values,

$$\begin{aligned} z^2[Y(z) - y(0) - z^{-1}y(1)] - 5z[Y(z) - y(0)] + 6Y(z) &= \frac{1}{5}\left[\frac{z}{z-1} - \frac{z^{-1}z}{z-1}\right], \\ Y(z)\left[z^2 - 5z + 6\right] &= \frac{1}{5} + 4z^2 + 23z. \end{aligned}$$

After taking partial fraction, further it can be written as

$$Y(z) = \frac{1}{5}\left[\frac{-1}{z-2} + \frac{1}{z-3}\right] + \frac{4z}{z-3} + 31\left[\frac{-z}{z-2} + \frac{z}{z-3}\right].$$

The resulting response can be written as

$$y(k) = \left[\frac{-1}{5}(2)^{k-1} + \frac{1}{5}(3)^{k-1}\right]\delta_{-1}(k-1) + \left[4(3)^k - 31(2)^k + 31(3)^k\right]\delta_{-1}(k).$$

Notice that $y(0) = 4$ and $y(1) = 3$ as required.

(c) The input signal given in (b) is illustrated below in Fig. 6. It can be noticed that the resulting

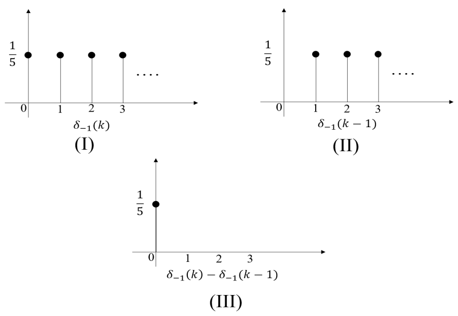

Figure 6: The input signal. In (III) is the difference of (I) and (II).

signal in (III) is an "impulse signal" with a magnitude of $\frac{1}{5}$. Hence, in (b) we have computed the impulse response of the system (scaled by $\frac{1}{5}$).