SOLUTIONS OF EXAM PAPER 2

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Jan 2022

Solutions

Problem 1

(a) From the given graph, peak time, peak overshoot, and static gain are $t_p = 3.4\,\mathrm{s}$, $M_p = 25.4\%$, and $\mu = 1$. From peak overshoot, we can find the value of ζ

$$M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}},$$

i.e.,

$$0.254 = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}.$$

Therefore, after some passages we can find

$$\zeta = 0.4.$$

Also, from the peak time we can find the value of natural frequency ω_n: since

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}},$$

hence,

$$\omega_n = 1.$$

From the obtained value of ζ and ω_n, we can write the transfer function

$$T(s) = \frac{1}{s^2 + 0.8s + 1}.$$

Comparing the above equation with the transfer function of the mass-spring system, i.e.

$$T(s) = \frac{\mu}{ms^2 + bs + k}$$

yields $m = 1\,\mathrm{kg}$, $b = 0.8\,\mathrm{kg/s}$, $k = 1\,\mathrm{N/m}$.
(b) When the damper is faulty then we can take the value $b = 0$, hence the equation

$$T(s) = \frac{1}{s^2 + 1}.$$

Then the impulse response

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) \\ &= \sin t\,\delta_{-1}(t). \end{aligned}$$

The other impulse $3\,\delta_{-1}(t-5)$ is applied and its Laplace transform is $3\,e^{-5s}$, hence we can write the total response of the system

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left(\frac{1}{s^2+1} + \frac{3\,e^{-5s}}{s^2+1}\right) \\ &= \sin t\,\delta_{-1}(t) + 3\,\sin(t-5)\,\delta_{-1}(t-5). \end{aligned}$$

Said in different terms, since the system is linear and stationary, time-shifting the impulse at the input and multiplying it by 3, determines a time-shift of the output and its multiplication by 3.

Problem 2

(a) It is given that $G(s) = \frac{k}{s(s^2+18s+77)}$ and because of unity feedback we have $H(s) = 1$, so we can write the closed loop transfer function

$$\begin{aligned} T(s) &= \frac{G(s)H(s)}{1+G(s)H(s)} \\ &= \frac{k}{s^3+18s^2+77s+k}. \end{aligned} \tag{1}$$

From R-H criteria,

$$\begin{array}{c|cc} s^3 & 1 & 77 \\ s^2 & 18 & k \\ s & \frac{1386-k}{18} & \\ s^0 & k & \end{array}$$

Hence, asymptotic stability of the closed loop system is ensured by

$$0 < k < 1386.$$

(b) Let's put the value $k = 77$ in equation (1):

$$\begin{aligned} T(s) &= \frac{77}{s^3+18\,s^2+77\,s+77} \\ &= \frac{77}{(s+12.21)(s+4.33)(s+1.45)} \\ &= \frac{1}{(\frac{s}{12.21}+1)(\frac{s}{4.33}+1)(\frac{s}{1.45}+1)}. \end{aligned} \tag{2}$$

Due to the poles, from the equation (2) the frequency response of the transfer function breaks downward at $\omega = 1.45$ with slope $-20\,\mathrm{dB/decade}$. Again it will bend downwards at $\omega = 4.33$ with slope $-20-20 = -40\,\mathrm{dB/decade}$. When we reach the next break point $\omega = 12.21$, the slope of the curve is $-20-20-20 = -60\,\mathrm{dB/decade}$.
For the (asymptotic) Bode phase plot, at $\omega = 1.45$ the phase value decreases by $-90°$. Further, at the next cut-off frequency $\omega = 4.33$, the phase value becomes $-90° - 90° = -180°$. At the last corner frequency $\omega = 12.21$, the phase value decreases again $-90° - 90° - 90° = -270°$.
Using MATLAB, Bode plot of this system is illustrated below:

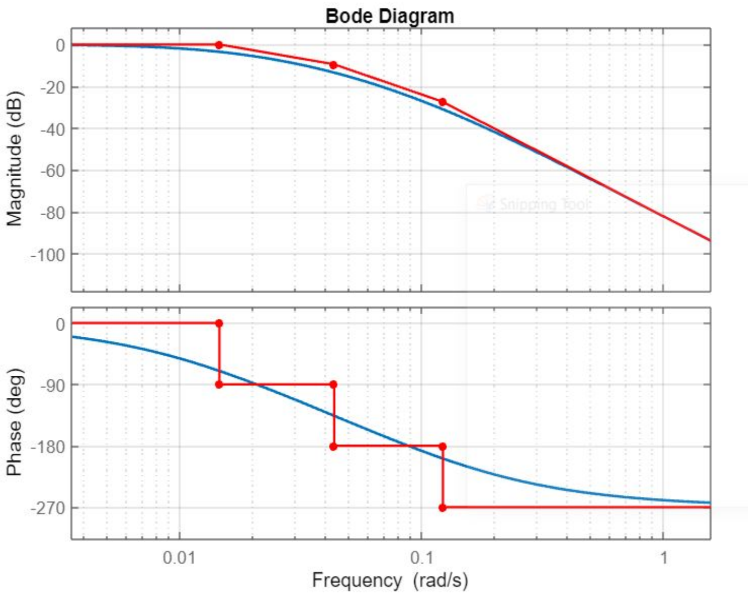
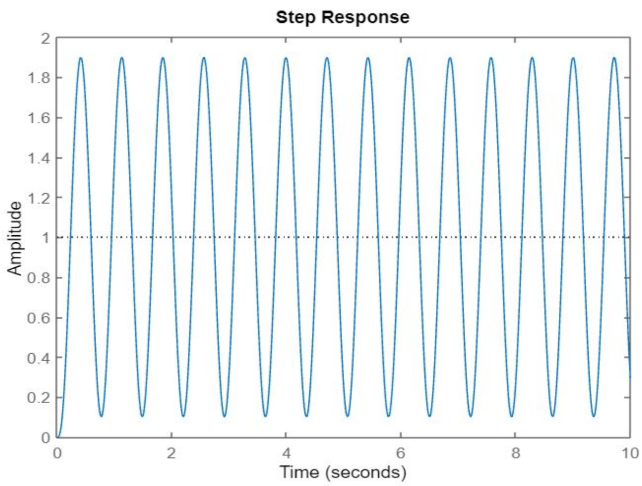


Figure 1: Bode plot

(c) For the equation derived in (1) let's put the value of k =1386, and roots -18, $8.775j$ and $-8.775j$ can be found using MATLAB. It can be seen that the two poles are on the imaginary axis, and the resulting system is marginally stable. Also, it can be confirmed using the MATLAB simulation shown below by applying step input to the equation 1 for the value $k = 1386$.



Problem 3

(a) From given block diagram we can write the equations,

$$x_1(k+1) = -ax_1(k) + u(k),$$
$$x_2(k+1) = -bx_1(k) + u(k),$$
$$y(k) = \frac{5}{3}x_1(k) - \frac{2}{3}x_2(k),$$

i.e.,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix},$$
$$y(k) = \begin{bmatrix} \frac{5}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}.$$

(b) The transfer function of a linear system in (A, B, C, D) form is given by the formula

$$G(z) = C(zI - A)^{-1}B + D.$$

Since

$$(zI - A)^{-1} = \frac{\begin{bmatrix} z+b & 0 \\ 0 & z+a \end{bmatrix}}{(z+a)(z+b)},$$

the transfer function of the given system will be

$$\begin{aligned} G(z) &= \frac{\begin{bmatrix} \frac{5}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} z+b & 0 \\ 0 & z+a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{(z+a)(z+b)} \\ &= \frac{z + \frac{(5b-2a)}{3}}{(z+a)(z+b)}. \end{aligned}$$

By comparing this equation with the given transfer function

$$\begin{aligned} G(z) &= \frac{z+1}{z^2 + 1.3z + 0.4} \\ &= \frac{z+1}{(z+0.5)(z+0.8)}, \end{aligned}$$

one obtains $a = 0.5$ and $b = 0.8$.

(c) Given $y(k) = \frac{10}{3}[(-a)^k - (-b)^k]\delta_{-1}(k)$, and knowing that

$$G(z) = \frac{Y(z)}{U(z)},$$

we can write

$$U(z) = \frac{Y(z)}{G(z)}.$$

For the given value of a and b, we can write $Y(z)$

$$\begin{aligned} Y(z) &= \frac{10}{3}\left(\frac{1}{z+0.5} - \frac{1}{z+0.8}\right) \\ &= \frac{z}{(z+0.5)(z+0.8)}; \end{aligned}$$

hence,

$$\begin{aligned} U(z) &= \frac{\frac{z}{(z+0.5)(z+0.8)}}{\frac{z+1}{(z+0.5)(z+0.8)}} \\ &= \frac{z}{z+1}. \end{aligned}$$

By taking the inverse z-transform, one obtains

$$\begin{aligned} u(k) &= \mathcal{Z}^{-1}\left(\frac{z}{z+1}\right) \\ &= (-1)^k \delta_{-1}(k). \end{aligned}$$