Lecture 9

Introduction to Graphs

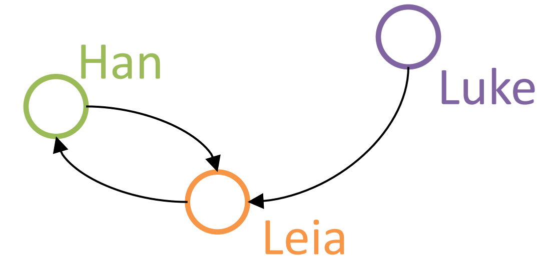
Graphs

- A graph is a formalism for representing relationships among items. One way to write graphs:

- A graph `G = (V,E)`
  - A set of vertices, also known as nodes

    $\mathbf{V} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$
  - A set of edges

    $\mathbf{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$
    - Each edge $\mathbf{e}_i$ is a pair of vertices $(\mathbf{v}_j, \mathbf{v}_k)$
    - An edge "connects" the vertices




```
V = {Han,Leia,Luke}
E = {(Luke,Leia),
     (Han,Leia),
     (Leia,Han)}
```

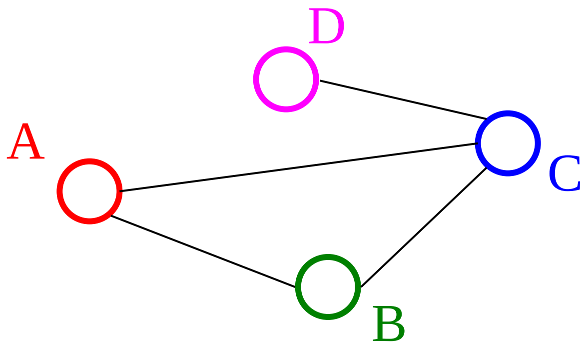
- Graphs can be directed or undirected

Are Graphs An ADT?

- Can think of graphs as an ADT with operations like **`isEdge`**$((v_j, v_k))$**,** **`addVertex`**(v_{new})**,** …
- But it is unclear what the "standard operations" are
- Instead we tend to develop algorithms over graphs and then use data structures that are efficient for those algorithms
- Many important problems can be solved by:
  1. Formulating them in terms of graphs
  2. Applying a standard graph algorithm
- To make the formulation easy and standard, we have a lot of *standard terminology* about graphs

Undirected Graphs

- In undirected graphs, edges have no specific direction
  - Edges are always “two-way”

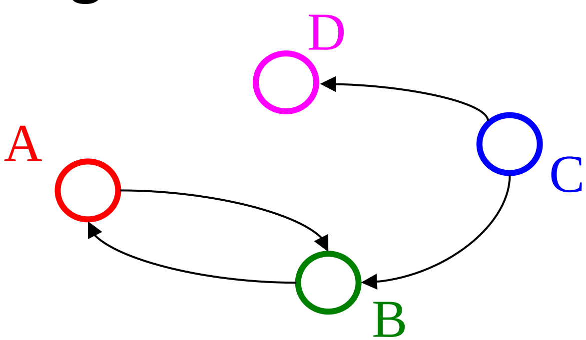


- Thus, $(u,v) \in E$ implies $(v,u) \in E$
  - Only one of these edges needs to be in the set
  - The other is implicit, so normalize how you check for it

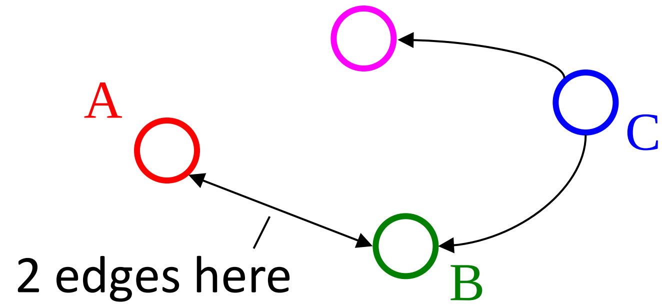
- Degree of a vertex: number of edges containing that vertex
  - Put another way: the number of adjacent vertices

Directed Graphs

- In directed graphs (sometimes called digraphs), edges have a direction



or



- Thus, `(u,v) ∈ E` does *not* imply `(v,u) ∈ E`.
  - Let `(u,v) ∈ E` mean u → v
  - Call `u` the source and `v` the destination
- In-degree of a vertex: number of in-bound edges,

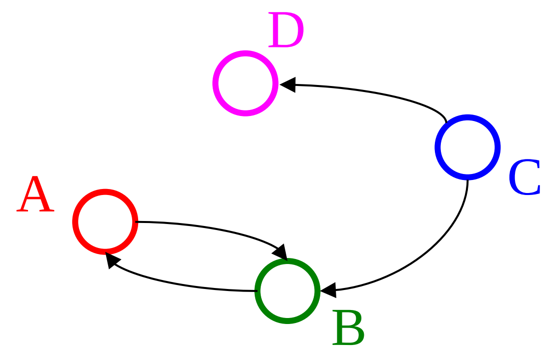
  i.e., edges where the vertex is the destination
- Out-degree of a vertex: number of out-bound edges

  i.e., edges where the vertex is the source

Self-Edges, Connectedness

- A self-edge a.k.a. a loop is an edge of the form **`(u,u)`**
  - Depending on the use/algorithm, a graph may have:
    - No self edges
    - Some self edges
    - All self edges (often therefore implicit, but we will be explicit)
- A node can have a degree / in-degree / out-degree of zero
- A graph does not have to be connected
  - Even if every node has non-zero degree

More Notation



For a graph **G = (V,E)**

- **|V|** is the number of vertices
- **|E|** is the number of edges (assuming no self loops)
  - Minimum? 0
  - Maximum for directed? |V|*(|V|-1) ∈ *O*(|V|²)
  - Maximum for undirected? (|V|*(|V|-1))/2 ∈ *O*(|V|²)

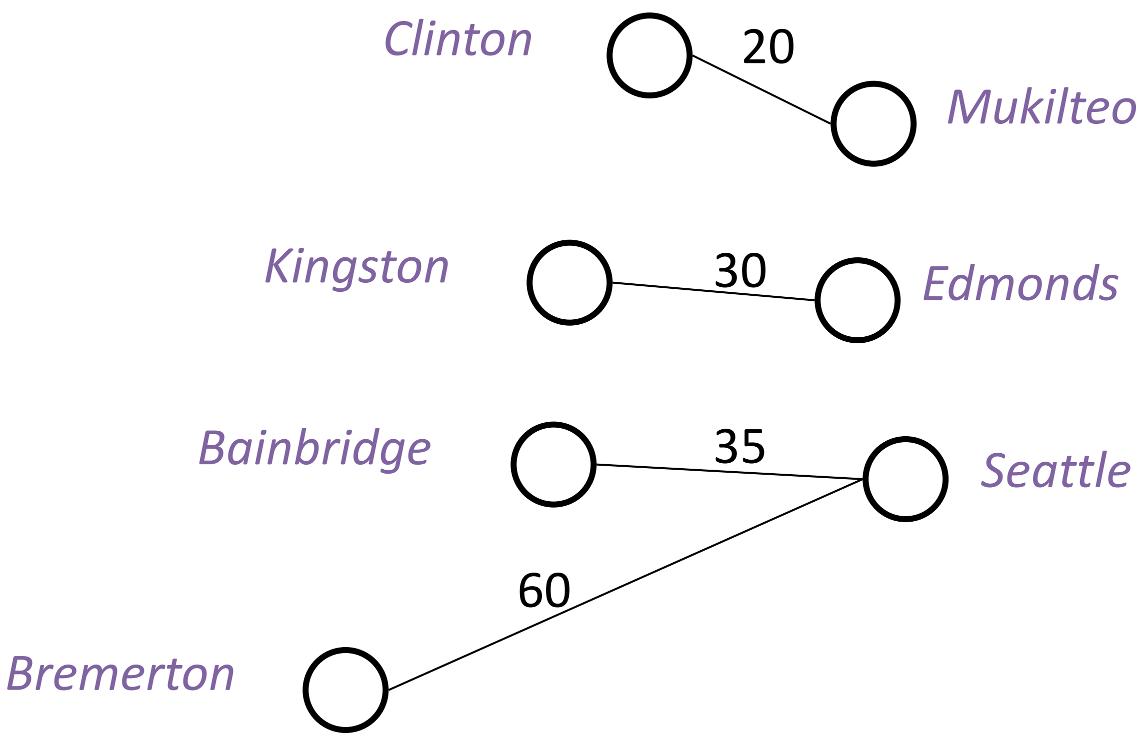
V = {A, B, C, D}

E = {(C, B),
(A, B),
(B, A)
(C, D)}

- If **(u,v) ∈ E**
  - Then **v** is a neighbor of **u**, i.e., **v** is adjacent to **u**
  - Order matters for directed edges
    - **u** is not adjacent to **v** unless **(v,u) ∈ E**

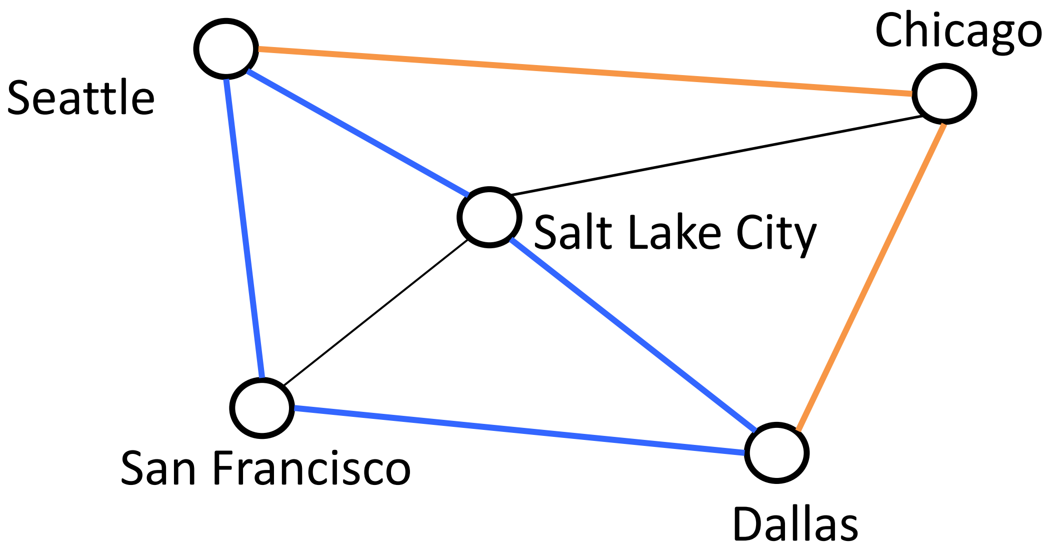
Weighted Graphs

- In a weighed graph, each edge has a weight a.k.a. cost
  - Typically numeric (most examples use ints)
  - *Orthogonal* to whether graph is directed
  - Some graphs allow *negative weights*; many do not



Paths and Cycles

- A path is a list of vertices $[v_0, v_1, \ldots, v_n]$ such that $(v_i, v_{i+1}) \in E$ for all $0 \leq i < n$. Say *"a path from* v_0 *to* v_n*"*
- A cycle is a path that begins and ends at the same node ($v_0 == v_n$)



Path: [Seattle, Chicago, Dallas]

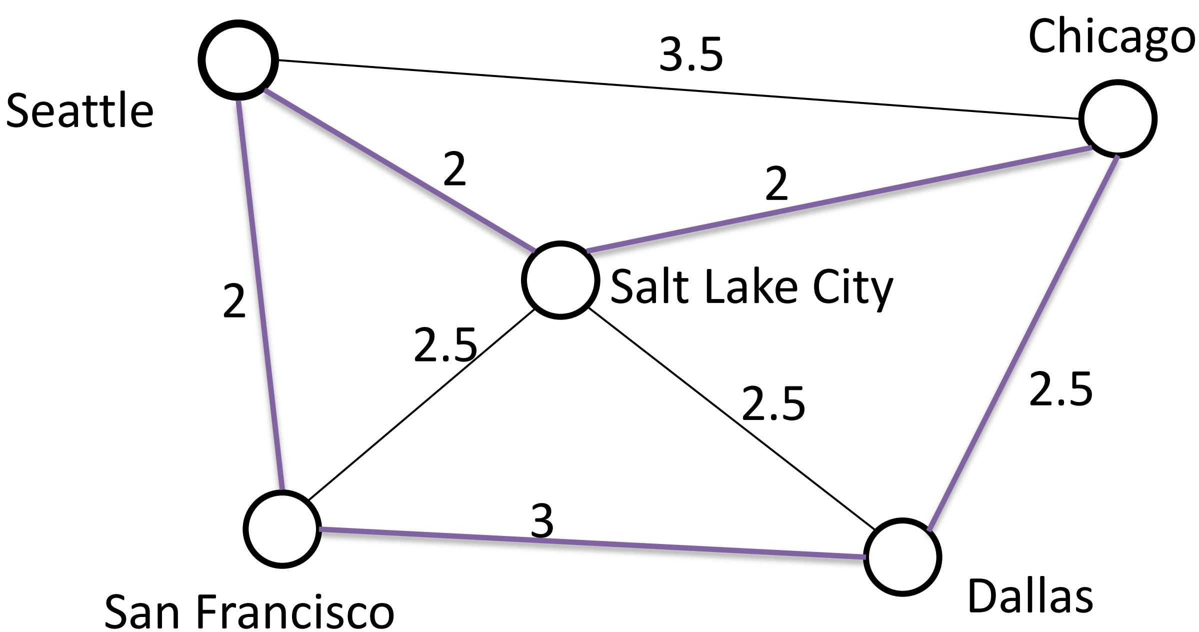
Cycle: [Seattle, Salt Lake City, Dallas, San Francisco, Seattle]

Path Length and Cost

- Path length: Number of *edges* in a path
- Path cost: Sum of *weights* of edges in a path

Example:

P= [Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]



length(P) = 5

cost(P) = 11.5

Simple Paths and Cycles

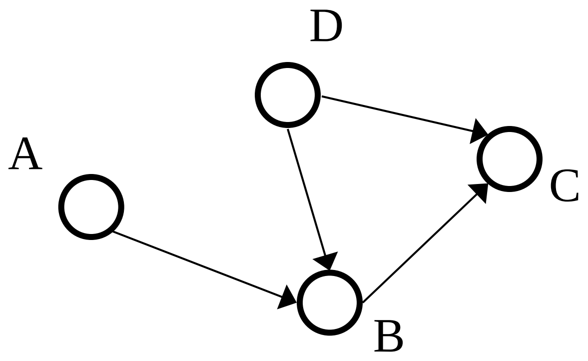
- A simple path repeats no vertices, except the first might be the last
  [Seattle, Salt Lake City, San Francisco, Dallas]
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

- Recall, a cycle is a path that ends where it begins
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]
  [Seattle, Salt Lake City, Seattle, Dallas, Seattle]

- A simple cycle is a cycle and a simple path
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

Paths and Cycles in Directed Graphs

Example:

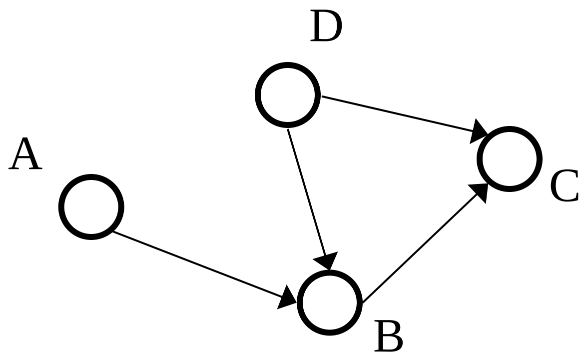


Is there a path from A to D?

Does the graph contain any cycles?

Paths and Cycles in Directed Graphs

Example:

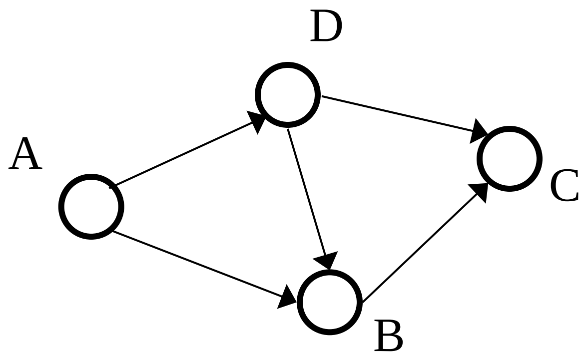


Is there a path from A to D?   No

Does the graph contain any cycles?   No

Paths and Cycles in Directed Graphs

Example:



Is there a path from A to D?

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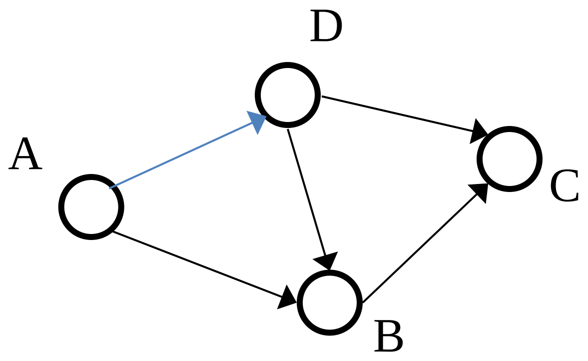
Paths and Cycles in Directed Graphs

Example:

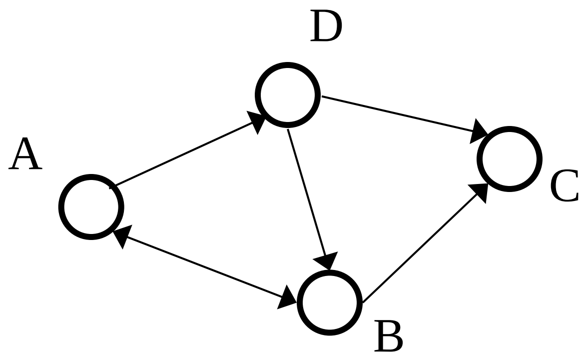


Is there a path from A to D? Yes

Does the graph contain any cycles? No

Paths and Cycles in Directed Graphs

Example:

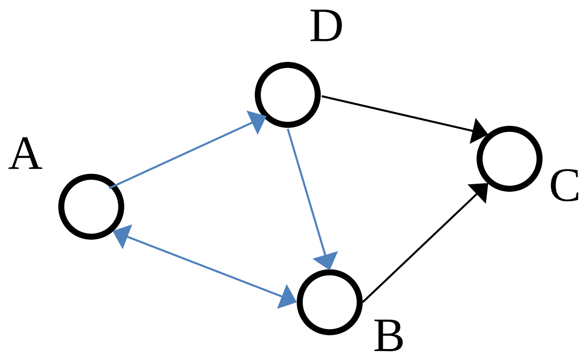


Is there a path from A to D?

Does the graph contain any cycles?

Paths and Cycles in Directed Graphs

Example:

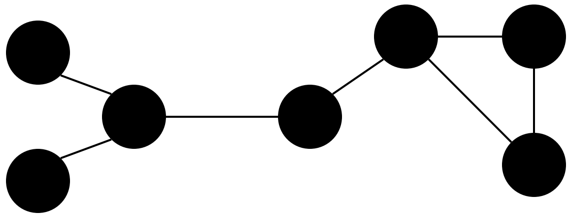


Is there a path from A to D? Yes

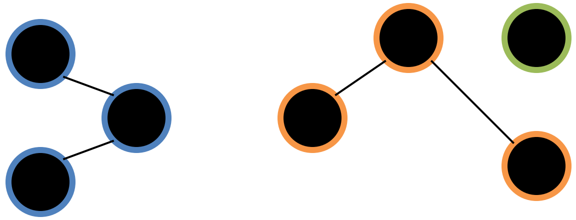
Does the graph contain any cycles? Yes

Undirected-Graph Connectivity

- An undirected graph is connected if for all pairs of vertices **u**, **v**, there exists a *path* from **u** to **v**



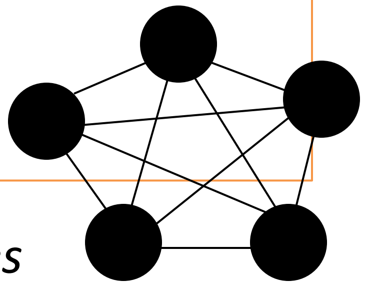
Connected graph



Disconnected graph

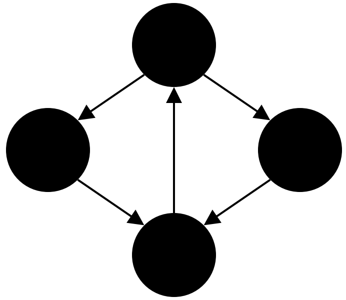
- An undirected graph is complete, a.k.a. fully connected if for *all* pairs of vertices **u**, **v**, there exists an *edge* from **u** to **v**

plus self edges

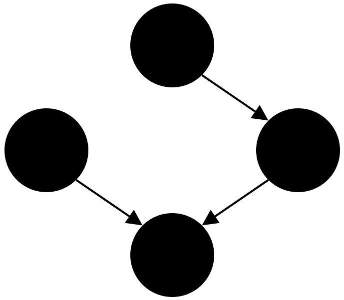


Directed-Graph Connectivity

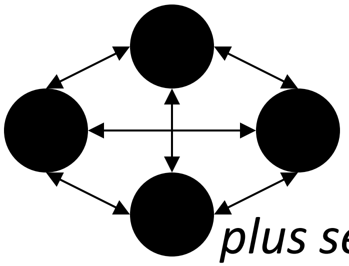
- A directed graph is strongly connected if there is a path from every vertex to every other vertex



- A directed graph is weakly connected if there is a path from every vertex to every other vertex *ignoring direction of edges*



- A complete a.k.a. fully connected directed graph has an edge from every vertex to every other vertex



plus self edges

Trees as Graphs

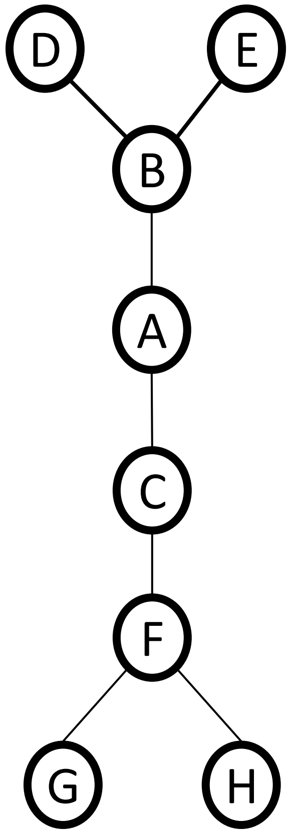
When talking about graphs,

we say a tree is a graph that is:

- Acyclic (no cycles)
- Connected

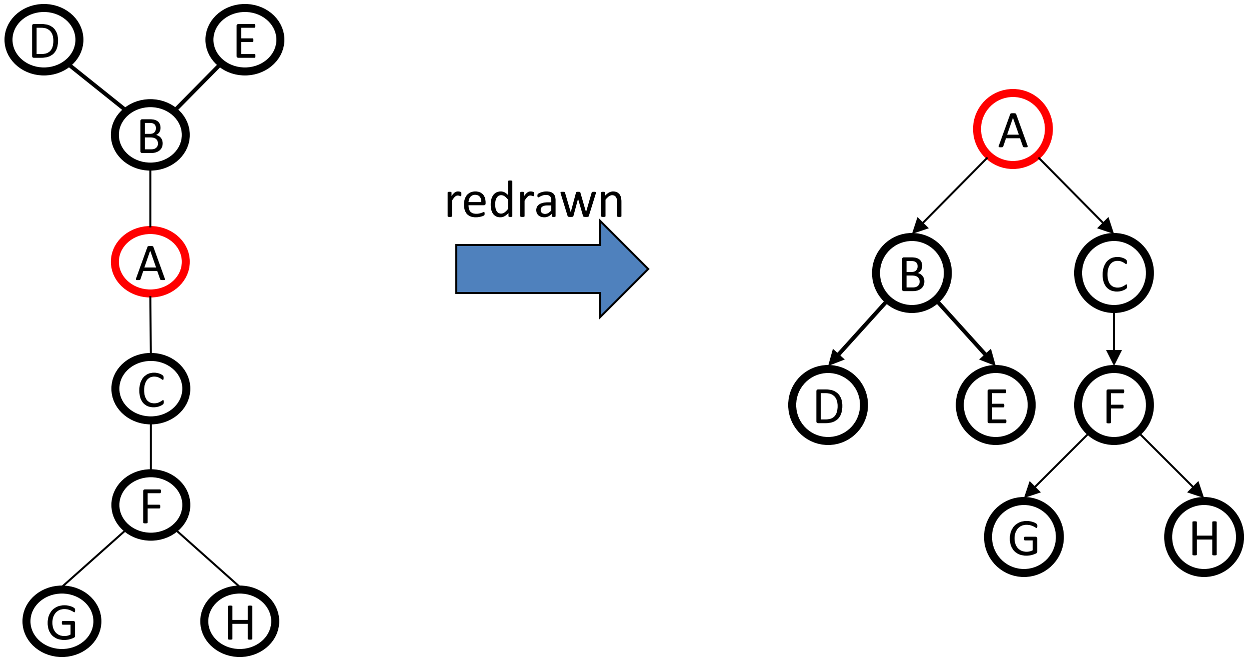
So all trees are graphs, but not all graphs are trees

Example:



Rooted Trees

- We are more accustomed to rooted trees where:
  - We identify a unique root
  - We think of edges as directed: parent to children
- Given a graph that is a tree, picking a root gives a unique rooted tree



Rooted Trees

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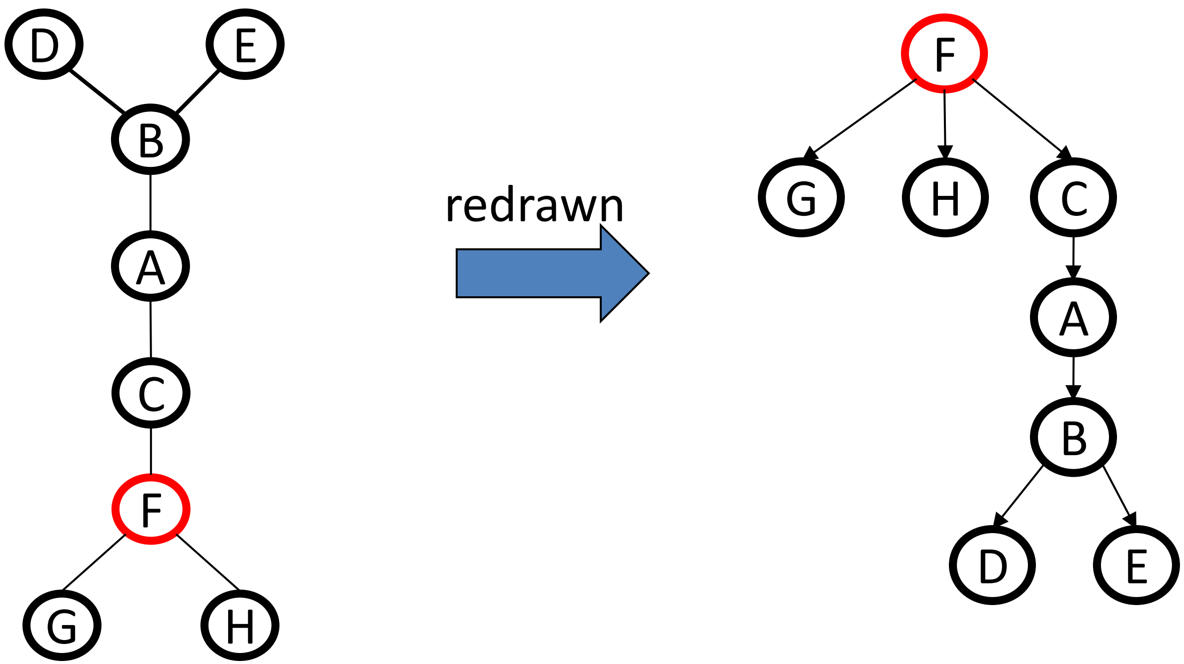
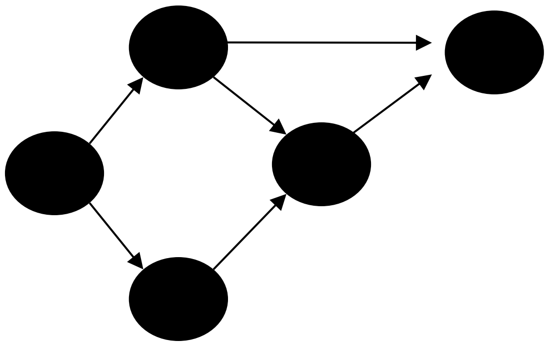
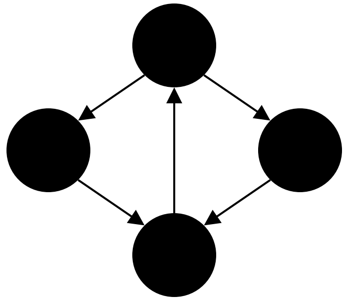


Directed Acyclic Graphs (DAGs)

- A DAG is a directed graph with no (directed) cycles
  - Every rooted directed tree is a DAG
  - But not every DAG is a rooted directed tree



  - Not every directed graph is acyclic



Density / Sparsity

- Recall: In an undirected graph, $0 \leq |E| < |V|^2$
- Recall: In a directed graph: $0 \leq |E| \leq |V|^2$
- So for any graph, $O(|E|+|V|^2)$ is $O(|V|^2)$
- Because $|E|$ is often much smaller than its maximum size, we do not always approximate $|E|$ as $O(|V|^2)$
  - This is a correct upper bound, it just is often not tight
  - If it is tight, i.e., $|E|$ is $\Theta(|V|^2)$ we say the graph is dense
  - If $|E|$ is $O(|V|)$ we say the graph is sparse

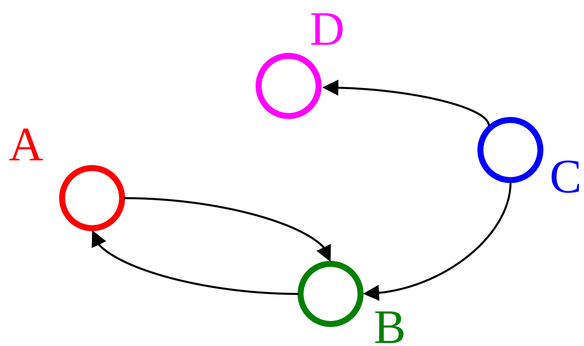
How do we implement this?

- The “best” implementation can depend on:
  - Properties of the graph (e.g., dense vs sparse)
  - The common queries (e.g., “is **`(u,v)`** an edge?” vs “what are the neighbors of node **`u`**?”)

- We’ll discuss the two standard graph representations
  - Adjacency Matrix and Adjacency List
  - Different trade-offs, particularly time versus space

Adjacency Matrix

- Assign each vertex/node a number from **`0`** to **`|V|-1`**
- A **`|V|`** x **`|V|`** matrix (i.e., 2-D array) of Booleans (or 1 vs. 0)
  - If **`M`** is the matrix, then **`M[u][v]`** being **`true`** means there is an edge from **`u`** to **`v`**



	A	B	C	D
A	F	**T**	F	F
B	**T**	F	F	F
C	F	**T**	F	**T**
D	F	F	F	F

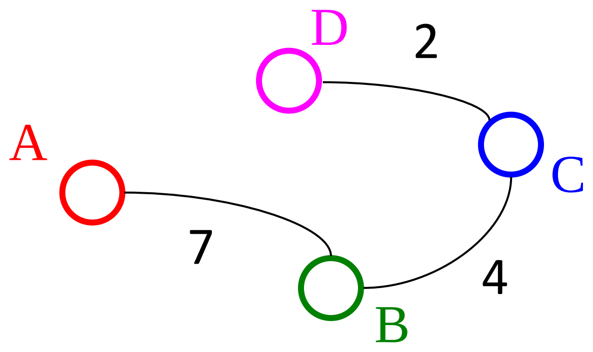
Adjacency Matrix Properties

- Running time to:
  - Get a vertex's out-edges: $O(|V|)$
  - Get a vertex's in-edges: $O(|V|)$
  - Decide if some edge exists: $O(1)$
  - Insert an edge: $O(1)$
  - Delete an edge: $O(1)$
- Space requirements:
  - $|V|^2$ bits
- Better for sparse or dense graphs?
  - Better for dense graphs

	A	B	C	D
A	F	**T**	F	F
B	**T**	F	F	F
C	F	**T**	F	**T**
D	F	F	F	F

Adjacency Matrix Properties

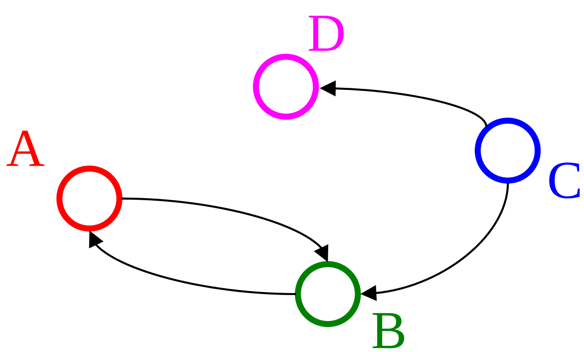
- How will the adjacency matrix vary for an *undirected graph*?
  - Undirected will be symmetric around the diagonal
- How can we adapt the representation for *weighted graphs*?
  - Instead of a Boolean, store a number in each cell
  - Need some value to represent ‘not an edge’
    - In *some* situations, 0 or -1 works

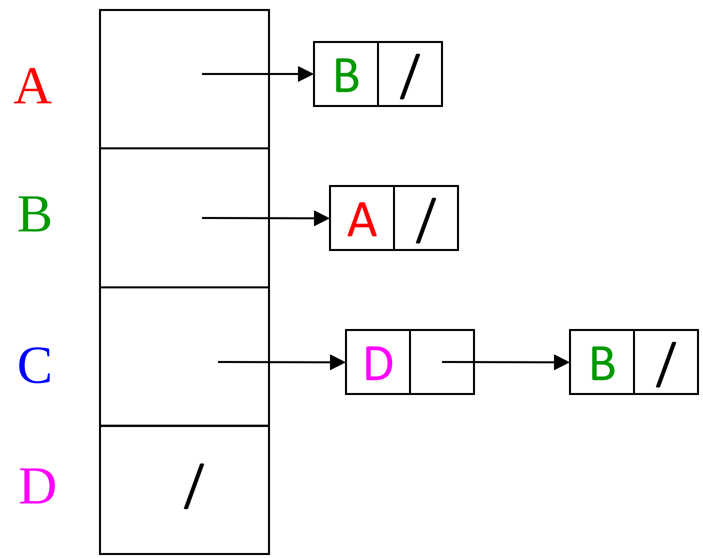


	A	B	C	D
A	-1	**7**	-1	-1
B	**7**	-1	**4**	-1
C	-1	**4**	-1	**2**
D	-1	-1	**2**	-1

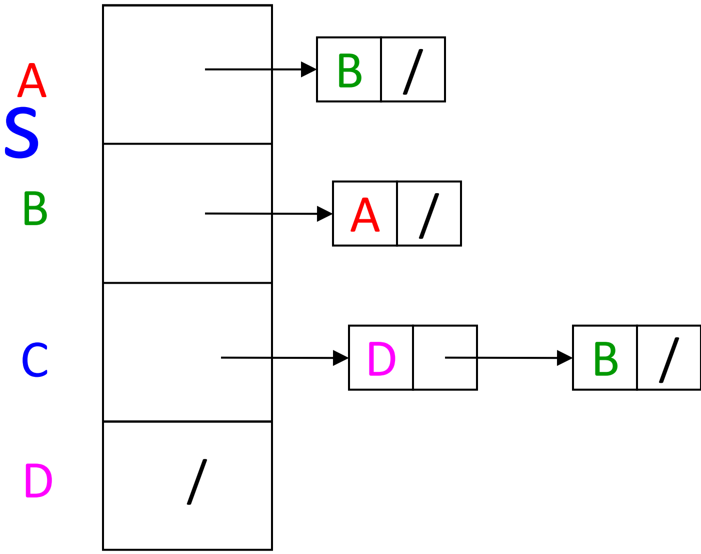
Adjacency List

- Assign each node a number from **`0`** to **`|V|-1`**
- An array of length **`|V|`** in which each entry stores a list of all adjacent vertices (e.g., linked list)





Adjacency List Properties



- Running time to:
  - Get all of a vertex's out-edges:

    $O(d)$ where d is out-degree of vertex
  - Get all of a vertex's in-edges:

    O(|E| + |V|) (but could keep a second adjacency list for this!)
  - Decide if some edge exists:

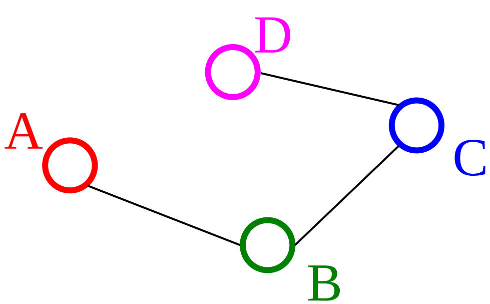
    $O(d)$ where d is out-degree of source
  - Insert an edge: $O(1)$ (unless you need to check if it's there)
  - Delete an edge: $O(d)$ where d is out-degree of source
- Space requirements:
  - $O(|V|+|E|)$
- Better for dense or sparse graphs?
  - Better for sparse graphs

Undirected Graphs

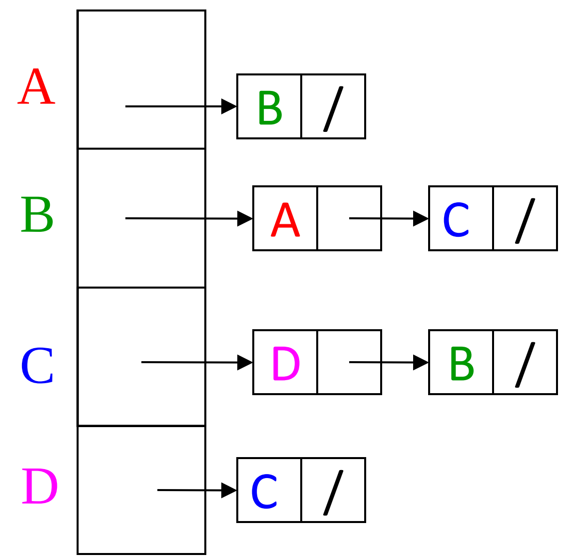
Adjacency matrices & adjacency lists both do fine for undirected graphs

- Matrix: Can save roughly 2x space
  - But may slow down operations in languages with “proper” 2D arrays (not Java, which has only arrays of arrays)
  - How would you “get all neighbors”?
- Lists: Each edge in two lists to support efficient “get all neighbors”

Example:



	A	B	C	D
A	F			
B	T	F		
C	F	T	F	
D	F	F	T	F



Some Applications as Graphs

For each of the following examples:

- what are the vertices and what are the edges?
- would you use directed edges? Would they have self-edges?
- Are there 0-degree nodes? Is it strongly or weakly connected?
- Does it have weights? Do negative weights make sense?
- Does it have cycles? Is it a DAG?

- Web pages with links
- Facebook friends
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites
- Political donations to candidates