Algorithms & Data Structures

Lesson 9: Binary Heaps

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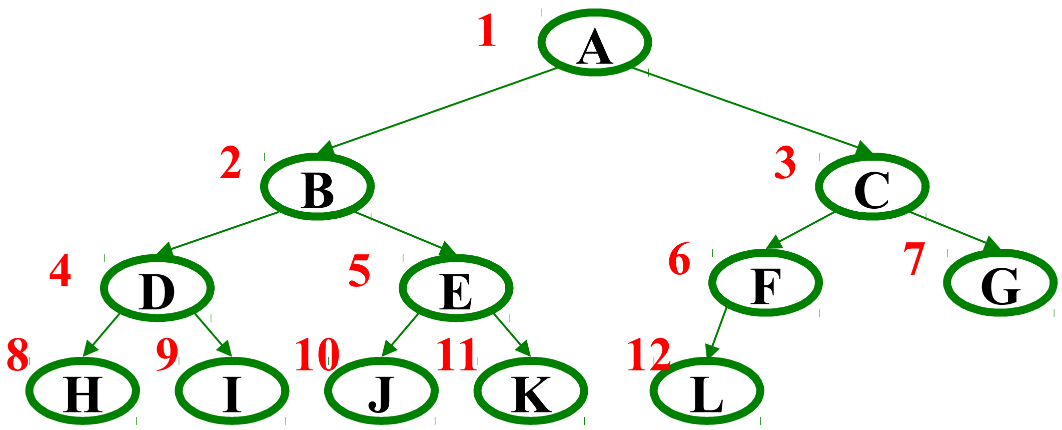
Edition 2017-2018

Array Representation of Binary Trees



From node `i`:

left child: `i*2`
right child: `i*2+1`
parent: `i/2`

(wasting index 0 is convenient for the index arithmetic)

implicit (array) implementation:

	A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Judging the array implementation

Plusses:

- Non-data space: just index 0 and unused space on right
  - In conventional tree representation, one edge per node (except for root), so *n*-1 wasted space (like linked lists)
  - Array would waste more space if tree were not complete
- Multiplying and dividing by 2 is very fast (shift operations in hardware)
- Last used position is just index **`size`**

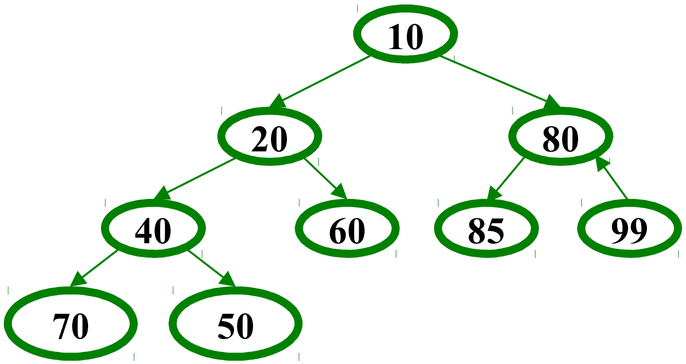
Minuses:

- Same might-be-empty or might-get-full problems we saw with stacks and queues (resize by doubling as necessary)

Pseudocode: insert into binary heap

```
void insert(int val) {
  if(size==arr.length-1)
    resize();
  size++;
  i=percolateUp(size,val);
  arr[i] = val;
}
```

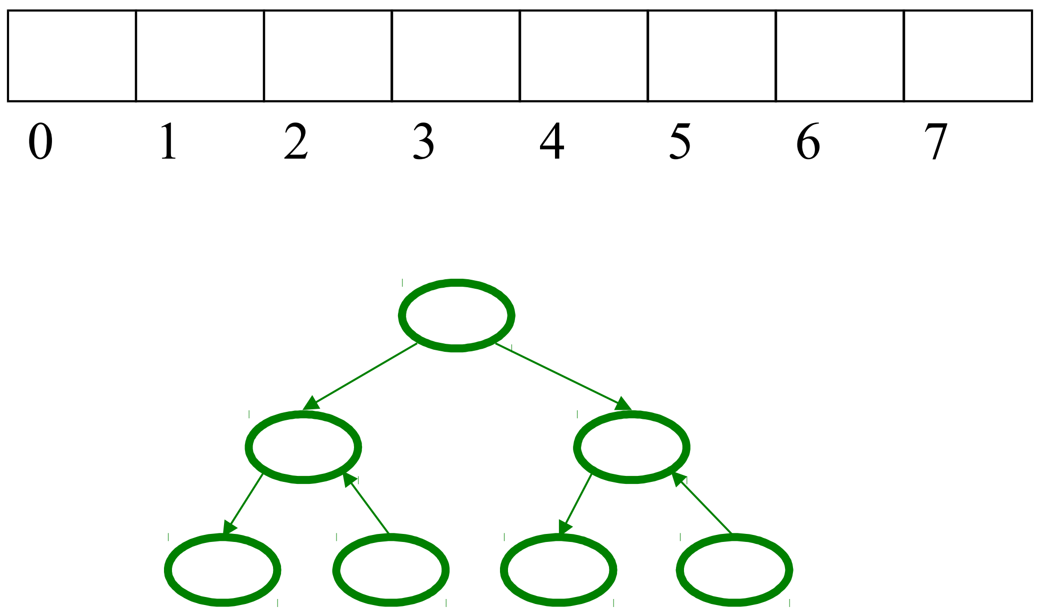
```
int percolateUp(int hole,
                int val) {
  while(hole > 1 &&
         val < arr[hole/2])
    arr[hole] = arr[hole/2];
    hole = hole / 2;
  }
  return hole;
}
```



	10	20	80	40	60	85	99	70	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Example of insert

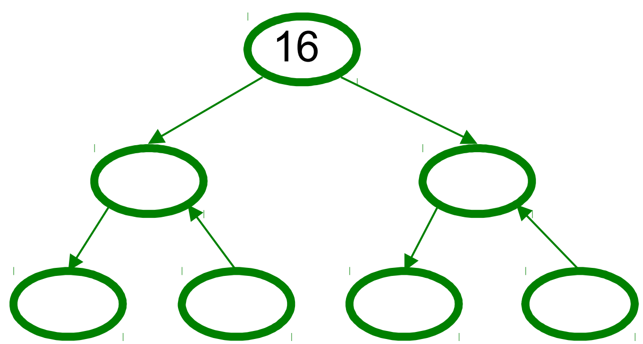
insert: 16, 32, 4, 67, 105, 43, 2



Example of insert

insert: **<u>16</u>**, 32, 4, 67, 105, 43, 2

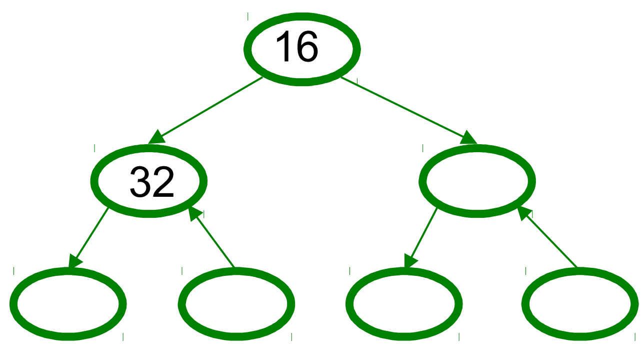
	16						
0	1	2	3	4	5	6	7



Example of insert

insert: **16**, **32**, 4, 67, 105, 43, 2

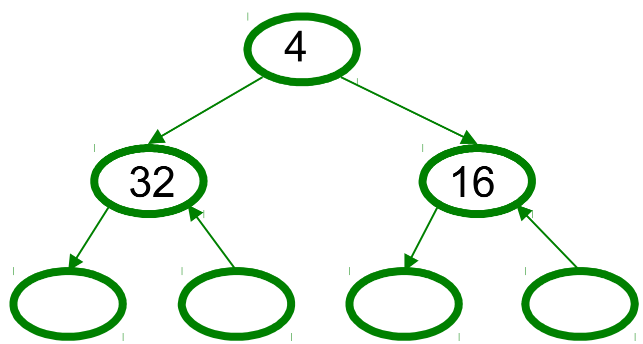
	16	**32**					
0	1	2	3	4	5	6	7



Example of insert

insert: **16**, **32**, **4**, 67, 105, 43, 2

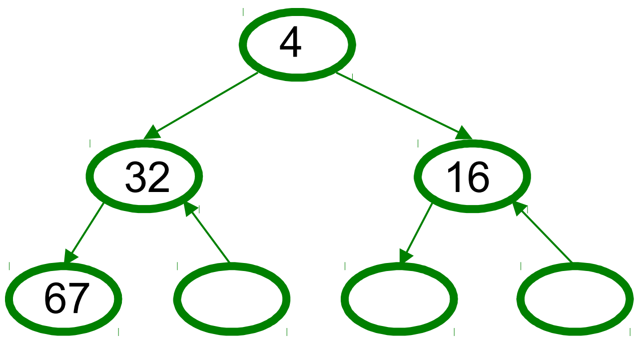
	4	**32**	**16**				
0	1	2	3	4	5	6	7



Example of insert

insert: **<u>16</u>**, **<u>32</u>**, **<u>4</u>**, **<u>67</u>**, 105, 43, 2

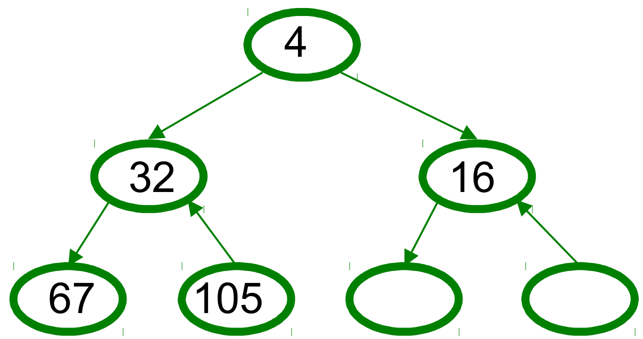
	4	**32**	**16**	**67**			
0	1	2	3	4	5	6	7



Example of insert

insert: **<u>16</u>**, **<u>32</u>**, **<u>4</u>**, **<u>67</u>**, **<u>105</u>**, 43, 2

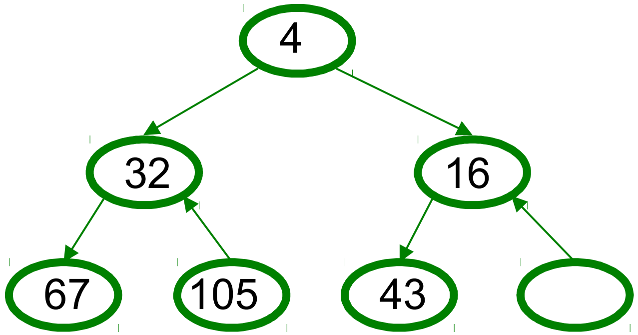
	4	**32**	**16**	**67**	**105**		
0	1	2	3	4	5	6	7



Example of insert

insert: **<u>16</u>**, **<u>32</u>**, **<u>4</u>**, **<u>67</u>**, **<u>105</u>**, **<u>43</u>**, 2

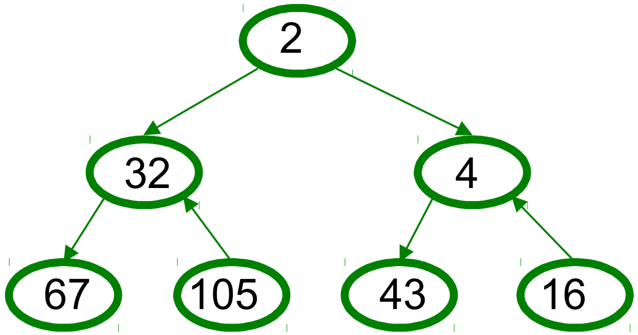
	4	**32**	**16**	**67**	**105**	**43**	
0	1	2	3	4	5	6	7



Example of insert

insert: <u>**16**</u>, <u>**32**</u>, <u>**4**</u>, <u>**67**</u>, <u>**105**</u>, <u>**43**</u>, <u>**2**</u>

	2	**32**	**4**	**67**	**105**	**43**	**16**
0	1	2	3	4	5	6	7



Other operations

- **`decreaseKey`**: given pointer to object in priority queue (e.g., its array index), lower its priority value by *p*
  - Change priority and percolate up

- **`increaseKey`**: given pointer to object in priority queue (e.g., its array index), raise its priority value by *p*
  - Change priority and percolate down

- **`remove`**: given pointer to object in priority queue (e.g., its array index), remove it from the queue
  - **`decreaseKey`** with $p = \infty$, then **`deleteMin`**

Running time for all these operations?

Build Heap

- Suppose you have *n* items to put in a new (empty) priority queue
  - Call this operation **`buildHeap`**

- *n* **`insert`**s works
  - Only choice if ADT doesn't provide **`buildHeap`** explicitly
  - $O(n \log n)$

- Why would an ADT provide this unnecessary operation?
  - Convenience
  - Efficiency: an $O(n)$ algorithm called Floyd's Method
  - Common issue in ADT design: how many specialized operations

Floyd's Method

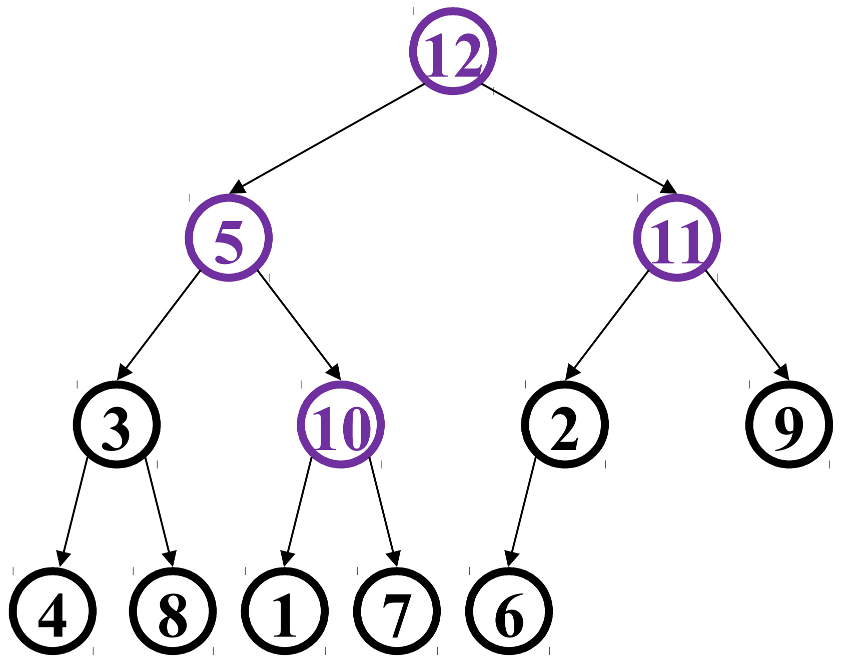
1. Use *n* items to make any complete tree you want
   – That is, put them in array indices 1,…,*n*

2. Treat it as a heap and fix the heap-order property
   – Bottom-up: leaves are already in heap order, work up toward the root one level at a time

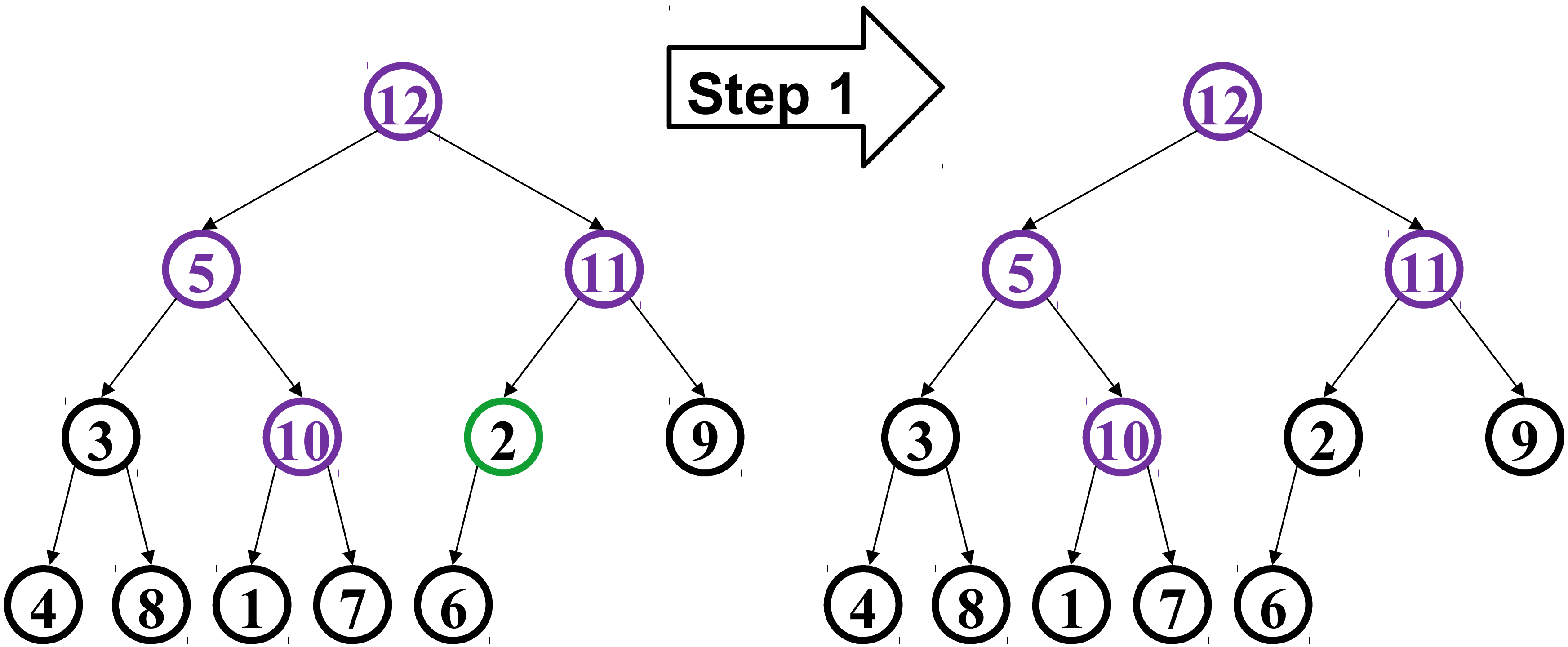
```
void buildHeap() {
   for(i = size/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i,val);
     arr[hole] = val;
   }
}
```

Example

- In tree form for readability
  - Purple for node not less than descendants
    - heap-order problem
  - Notice no leaves are purple
  - Check/fix each non-leaf bottom-up (6 steps here)

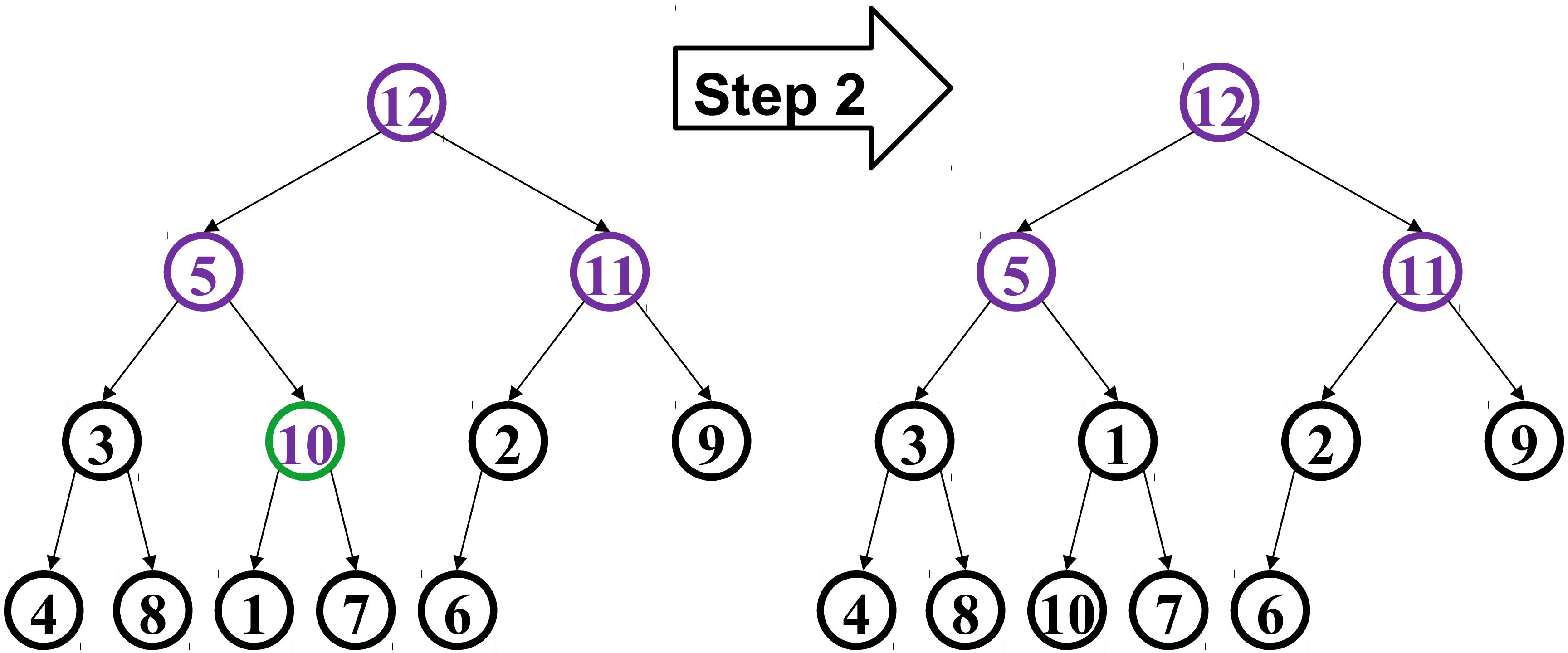


Example



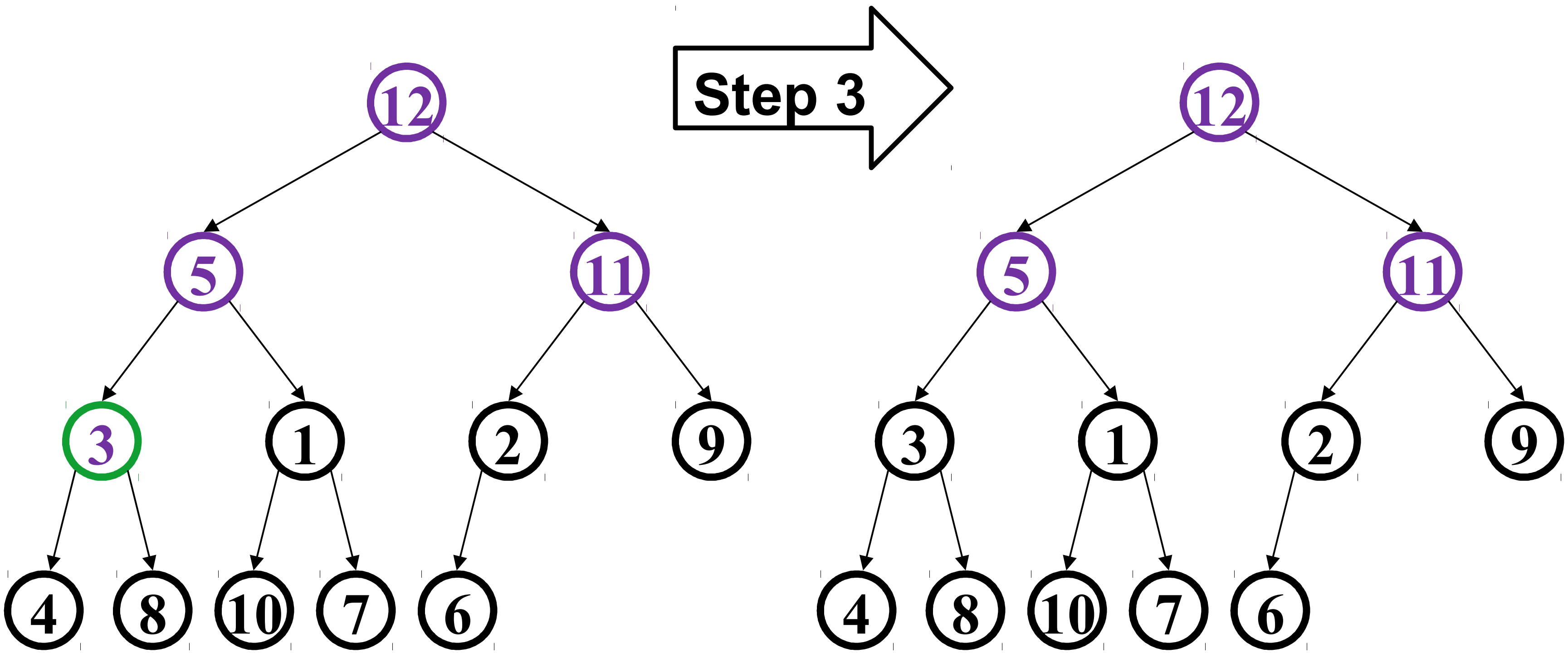
- Happens to already be less than children (er, child)

Example



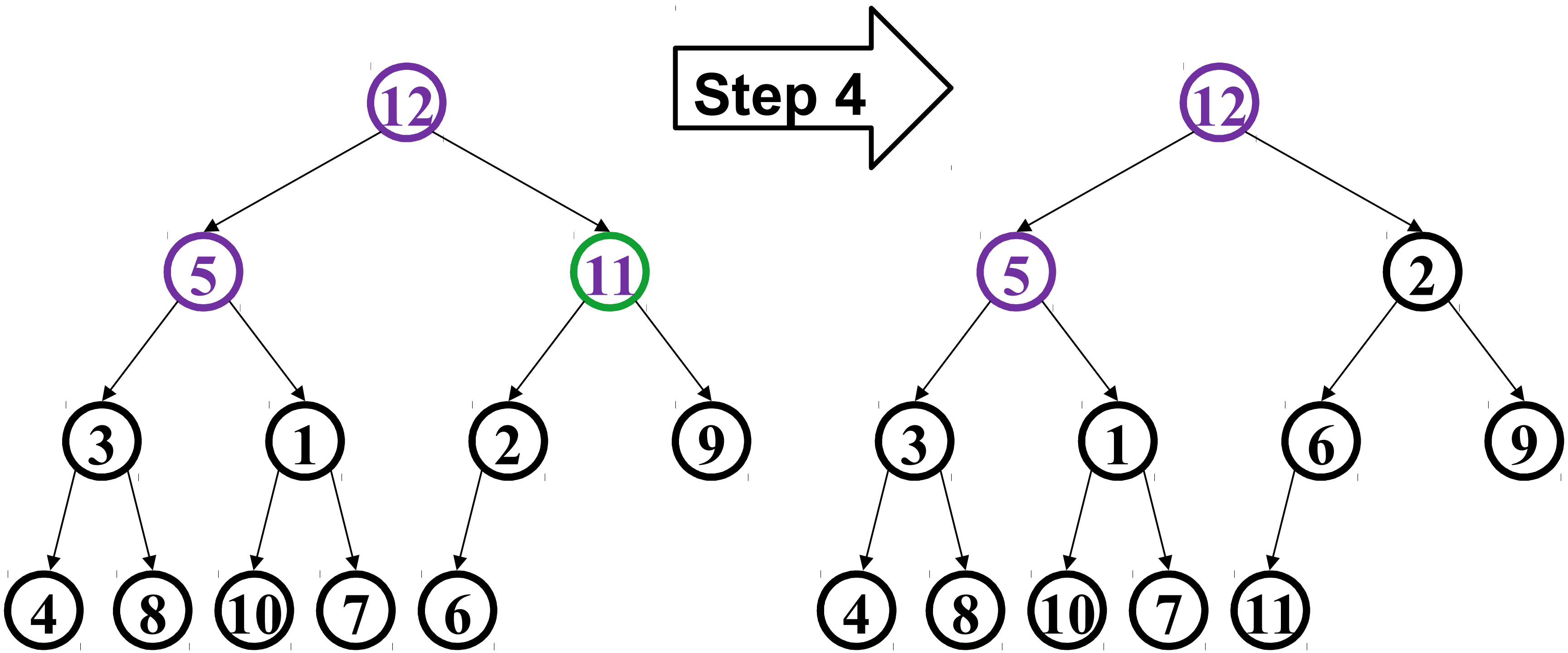
- Percolate down (notice that moves 1 up)

Example



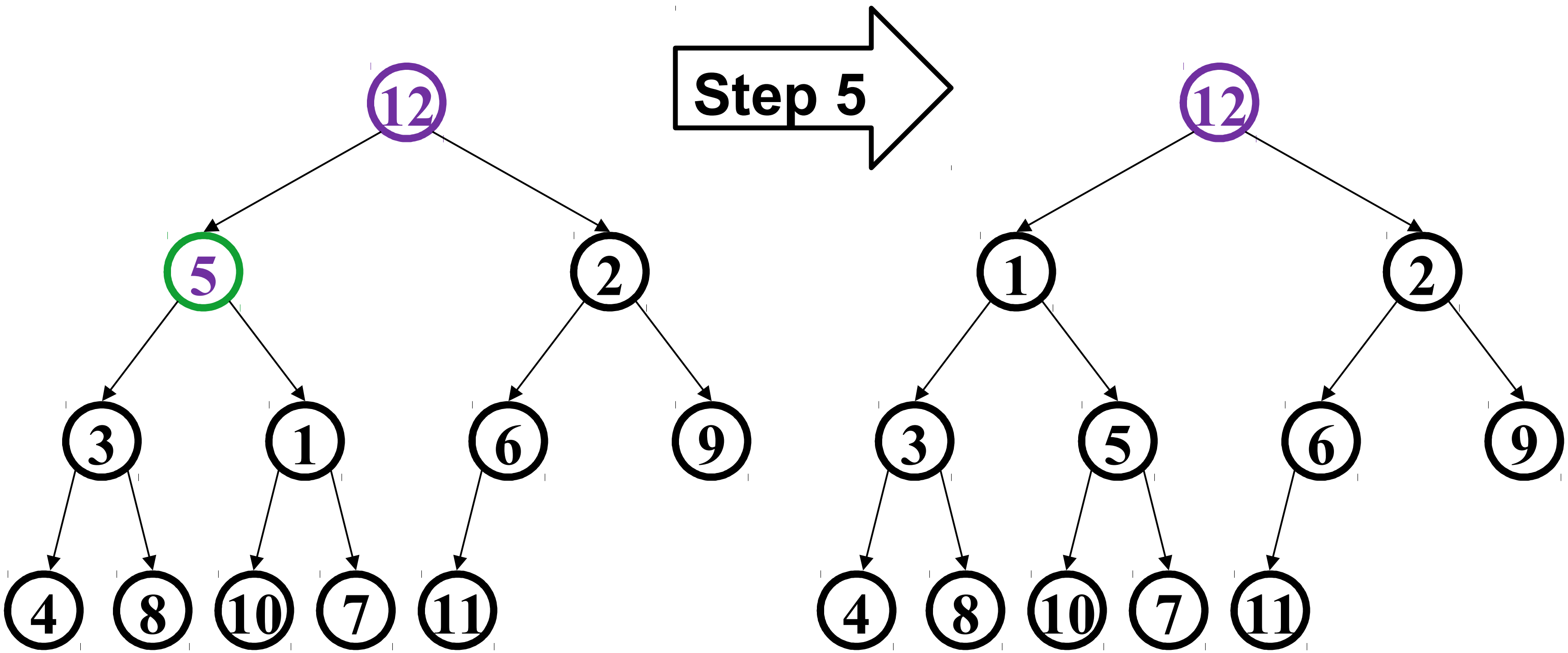
- Another nothing-to-do step

Example

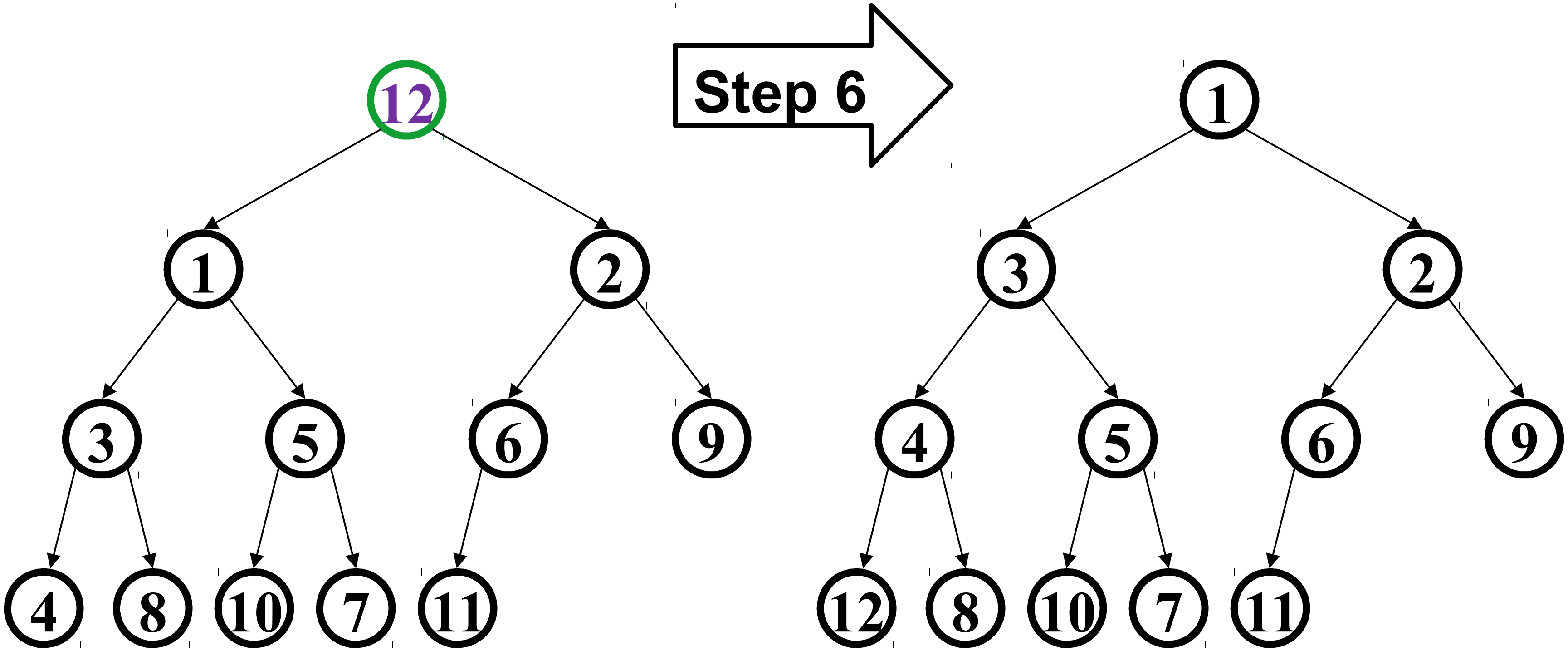


- Percolate down as necessary (steps 4a and 4b)

Example



Example



But is it right?

- "Seems to work"
  - Let's *prove* it restores the heap property (correctness)
  - Then let's *prove* its running time (efficiency)

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Correctness

```
void buildHeap() {
   for(i = size/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i,val);
     arr[hole] = val;
   }
}
```

Loop Invariant: For all `j>i`, `arr[j]` is less than its children

- True initially: If `j > size/2`, then `j` is a leaf
  - Otherwise its left child would be at position > `size`
- True after one more iteration: loop body and `percolateDown` make `arr[i]` less than children without breaking the property for any descendants

So after the loop finishes, all nodes are less than their children

Efficiency

```
void buildHeap() {
   for(i = size/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i,val);
     arr[hole] = val;
   }
}
```

Easy argument: **`buildHeap`** is $O(n \log n)$ where n is **`size`**

- **`size/2`** loop iterations
- Each iteration does one **`percolateDown`**, each is $O(\log n)$

This is correct, but there is a more precise ("tighter") analysis of the algorithm…

Efficiency

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Better argument: **`buildHeap`** is $O(n)$ where n is **`size`**

- **`size/2`** total loop iterations: $O(n)$
- 1/2 the loop iterations percolate at most 1 step
- 1/4 the loop iterations percolate at most 2 steps
- 1/8 the loop iterations percolate at most 3 steps
- …
- $((1/2) + (2/4) + (3/8) + (4/16) + (5/32) + \ldots) < 2$ (page 4 of Weiss)
  - So at most **`2(size/2)`** *total* percolate steps: $O(n)$

Lessons from `buildHeap`

- Without **`buildHeap`**, our ADT already let clients implement their own in $O(n \log n)$ worst case
- By providing a specialized operation internal to the data structure (with access to the internal data), we can do $O(n)$ worst case
  - Intuition: Most data is near a leaf, so better to percolate down
- Can analyze this algorithm for:
  - Correctness:
    - Non-trivial inductive proof using loop invariant
  - Efficiency:
    - First analysis easily proved it was $O(n \log n)$
    - Tighter analysis shows same algorithm is $O(n)$