

Algorithms & Data Structures

Lesson 11: Introduction to graphs

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Graphs

- A graph is a formalism for representing relationships among items
- A graph is a pair

$$G = (V, E)$$

- A set of vertices, also known as nodes

$$V = \{v_1, v_2, \dots, v_n\}$$

- A set of edges

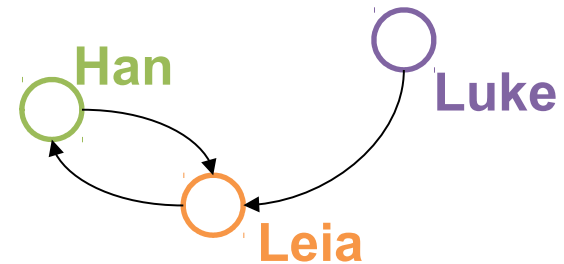
$$E = \{e_1, e_2, \dots, e_m\}$$

- Each edge e_i is a pair of vertices

$$(v_j, v_k)$$

- An edge “connects” the vertices

- Graphs can be directed or undirected



$$V = \{\text{Han}, \text{Leia}, \text{Luke}\}$$

$$E = \{(\text{Luke}, \text{Leia}), (\text{Han}, \text{Leia}), (\text{Leia}, \text{Han})\}$$

An ADT?

- Can think of graphs as an ADT with operations like `isEdge(v_j, v_k)`, `addVertex(v_{new})`, ...
- But it is unclear what the “standard operations” are
- Instead we tend to develop algorithms over graphs and then use data structures that are efficient for those algorithms
- Many important problems can be solved by:
 1. Formulating them in terms of graphs
 2. Applying a standard graph algorithm
- To make the formulation easy and standard, we have a lot of *standard terminology* about graphs

Some Graphs

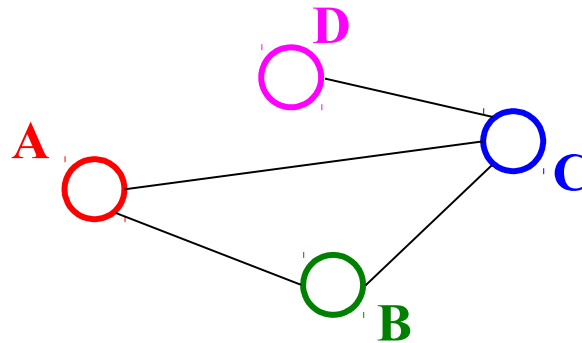
For each, what are the **vertices** and what are the **edges**?

- Web pages with links
- Facebook friends
- “Input data” for the “7 degrees of separation from Kevin Bacon game”
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites

Using the same algorithms for diverse problems across so many domains sounds like “core computer science and engineering”

Undirected Graphs

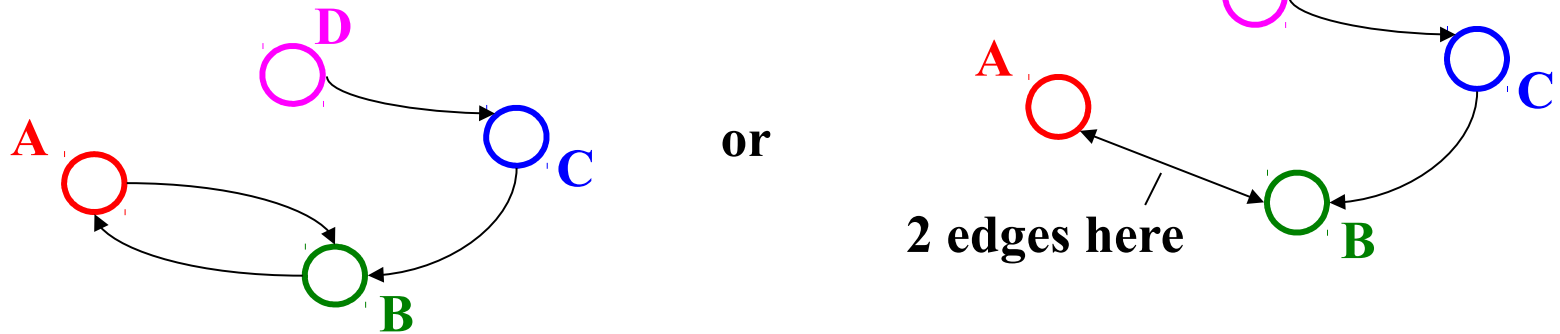
- In **undirected graphs**, edges have no specific direction
 - Edges are always “two-way”



- Thus, $(u, v) \in E$ implies $(v, u) \in E$
 - Only one of these edges needs to be in the set
 - The other is implicit, so normalize how you check for it
- **Degree** of a vertex: number of edges containing that vertex
 - Put another way: the number of adjacent vertices

Directed Graphs

- In **directed graphs** (sometimes called **digraphs**), edges have a direction

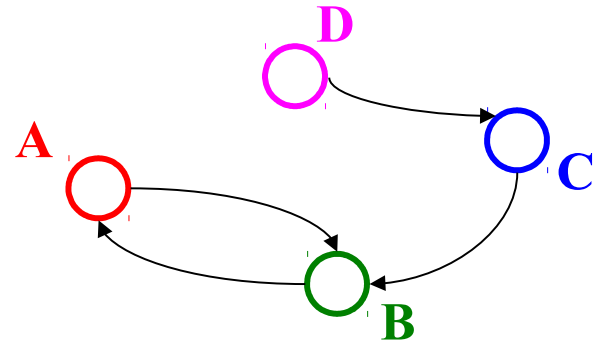


- Thus, $(u, v) \in E$ does *not* imply $(v, u) \in E$.
 - Let $(u, v) \in E$ mean $u \rightarrow v$
 - Call u the **source** and v the **destination**
- In-degree** of a vertex: number of in-bound edges, i.e., edges where the vertex is the destination
- Out-degree** of a vertex: number of out-bound edges i.e., edges where the vertex is the source

Self-Edges, Connectedness

- A **self-edge** a.k.a. a **loop** is an edge of the form (u, u)
 - Depending on the use/algorithm, a graph may have:
 - No self edges
 - Some self edges
 - All self edges (often therefore implicit, but we will be explicit)
- A node can have a degree / in-degree / out-degree of **zero**
- A graph does not have to be **connected**
 - Even if every node has non-zero degree

More Notation



For a graph $G = (V, E)$

- $|V|$ is the number of vertices
- $|E|$ is the number of edges

$$V = \{A, B, C, D\}$$

$$E = \{ (C, B),$$

– Minimum?

– Maximum for undirected? $|V|(|V|+1)/2 \in O(|V|^2)$

– Maximum for directed? $|V|^2 - |V| \in O(|V|^2)$

- If $(u, v) \in E$

– Then v is a **neighbor** of u , i.e., v is **adjacent** to u

– Order matters for directed edges

- u is not **adjacent** to v unless $(v, u) \in E$

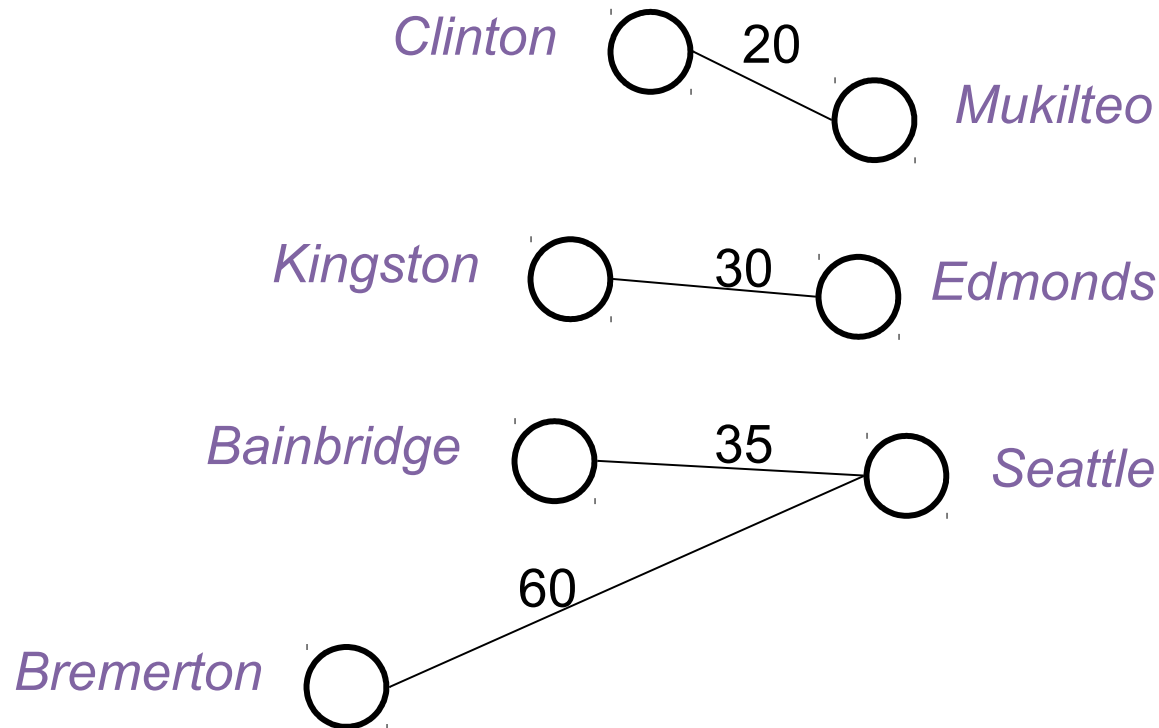
Examples again

Which would use **directed edges**? Which would have **self-edges**?
Which would be **connected**? Which could have **0-degree nodes**?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites
- ...

Weighted Graphs

- In a weighed graph, each edge has a **weight** a.k.a. **cost**
 - Typically numeric (most examples use ints)
 - *Orthogonal* to whether graph is directed
 - Some graphs allow *negative weights*; many do not



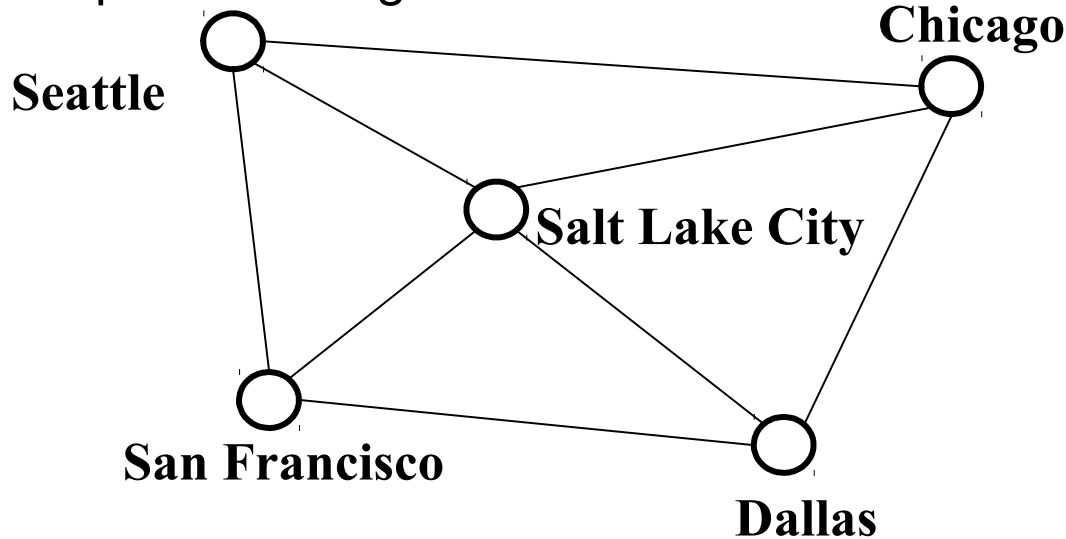
Examples

What, if anything, might weights represent for each of these?
Do negative weights make sense?

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Paths and Cycles

- A **path** is a list of vertices $[v_0, v_1, \dots, v_n]$ such that $(v_i, v_{i+1}) \in E$ for all $0 \leq i < n$. Say “a path from v_0 to v_n ”
- A **cycle** is a path that begins and ends at the same node ($v_0 = v_n$)



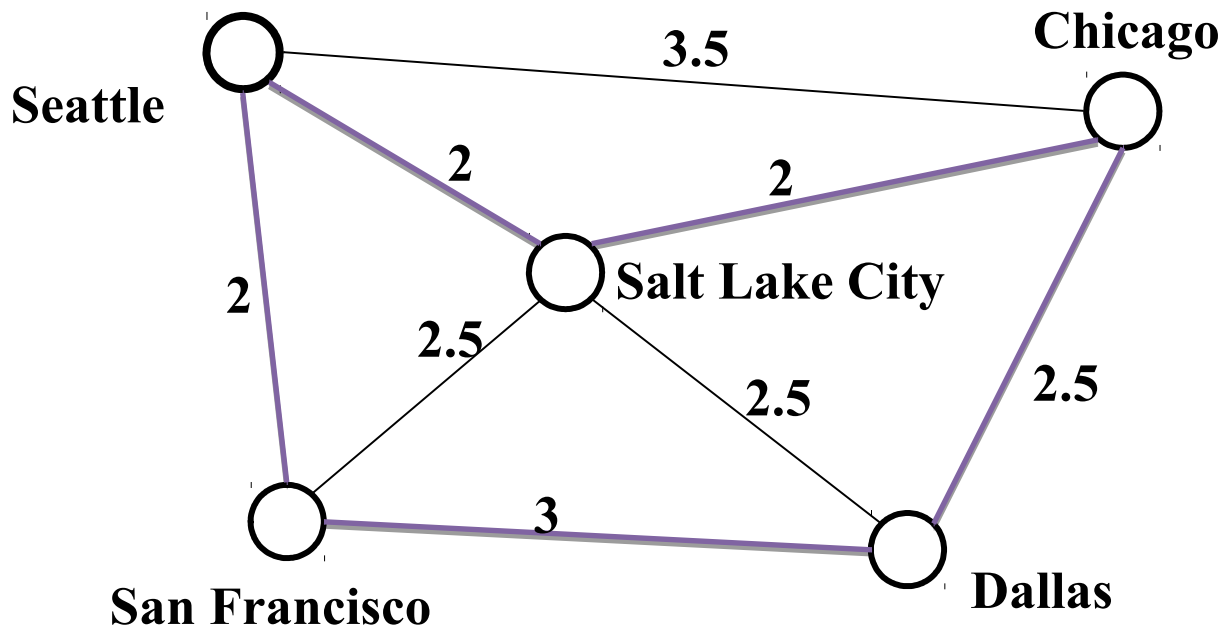
Example: [Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]

Path Length and Cost

- **Path length**: Number of edges in a path
- **Path cost**: Sum of *weights* of edges in a path

Example where

$P = [\text{Seattle}, \text{Salt Lake City}, \text{Chicago}, \text{Dallas}, \text{San Francisco}, \text{Seattle}]$



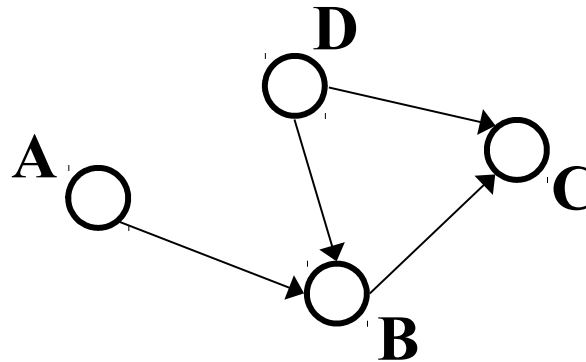
$\text{length}(P) = 5$
 $\text{cost}(P) = 11.5$

Simple Paths and Cycles

- A **simple path** repeats no vertices, except the first might be the last
[Seattle, Salt Lake City, San Francisco, Dallas]
[Seattle, Salt Lake City, San Francisco, Dallas, Seattle]
- Recall, a **cycle** is a path that ends where it begins
[Seattle, Salt Lake City, San Francisco, Dallas, Seattle]
[Seattle, Salt Lake City, Seattle, Dallas, Seattle]
- A **simple cycle** is a cycle and a simple path
[Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

Paths and Cycles in Directed Graphs

Example:

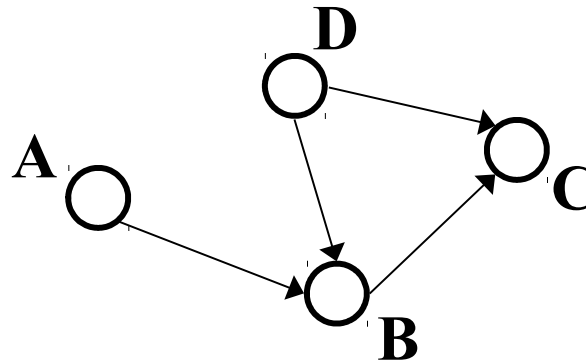


Is there a path from A to D?

Does the graph contain any cycles?

Paths and Cycles in Directed Graphs

Example:

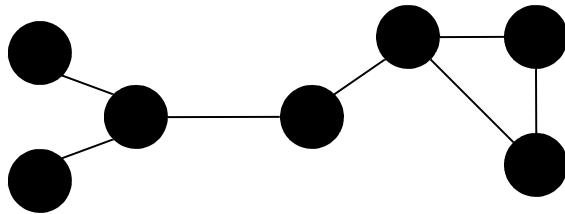


Is there a path from A to D? **No**

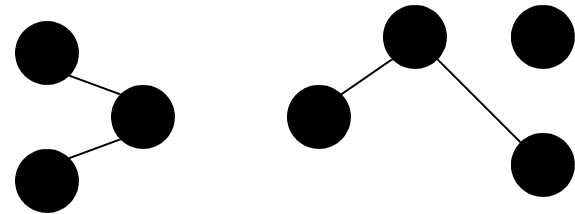
Does the graph contain any cycles? **No**

Undirected-Graph Connectivity

- An undirected graph is **connected** if for all pairs of vertices u, v , there exists a *path* from u to v

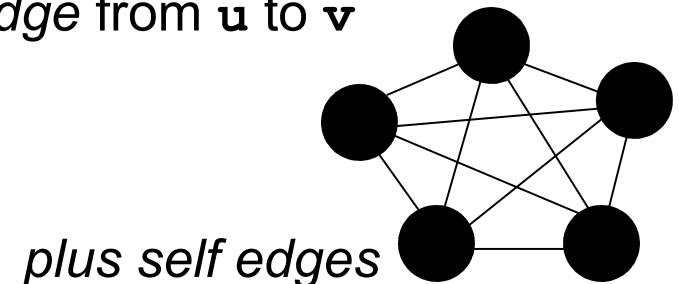


Connected graph



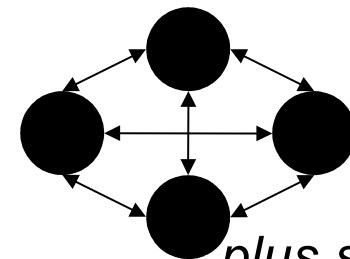
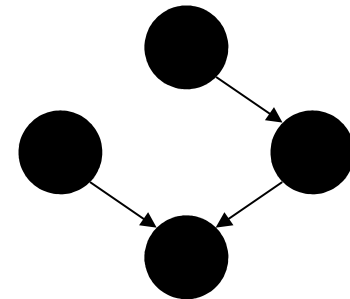
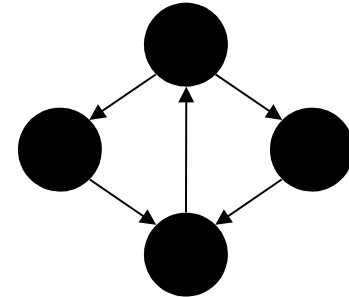
Disconnected graph

- An undirected graph is **complete**, a.k.a. **fully connected** if for *all* pairs of vertices u, v , there exists an *edge* from u to v



Directed-Graph Connectivity

- A directed graph is **strongly connected** if there is a path from every vertex to every other vertex
- A directed graph is **weakly connected** if there is a path from every vertex to every other vertex *ignoring direction of edges*
- A **complete** a.k.a. **fully connected** directed graph has an edge from every vertex to every other vertex



plus self-edges

Examples

For undirected graphs: connected?

For directed graphs: strongly connected? weakly connected?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
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Trees as Graphs

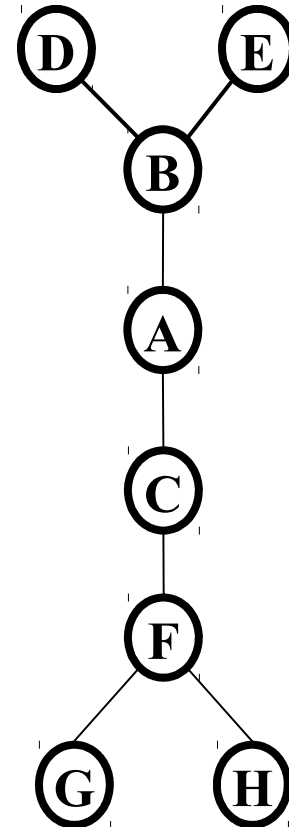
When talking about graphs,
we say a **tree** is a graph that is:

- Acyclic
- Connected

So all trees are graphs, but not
all graphs are trees

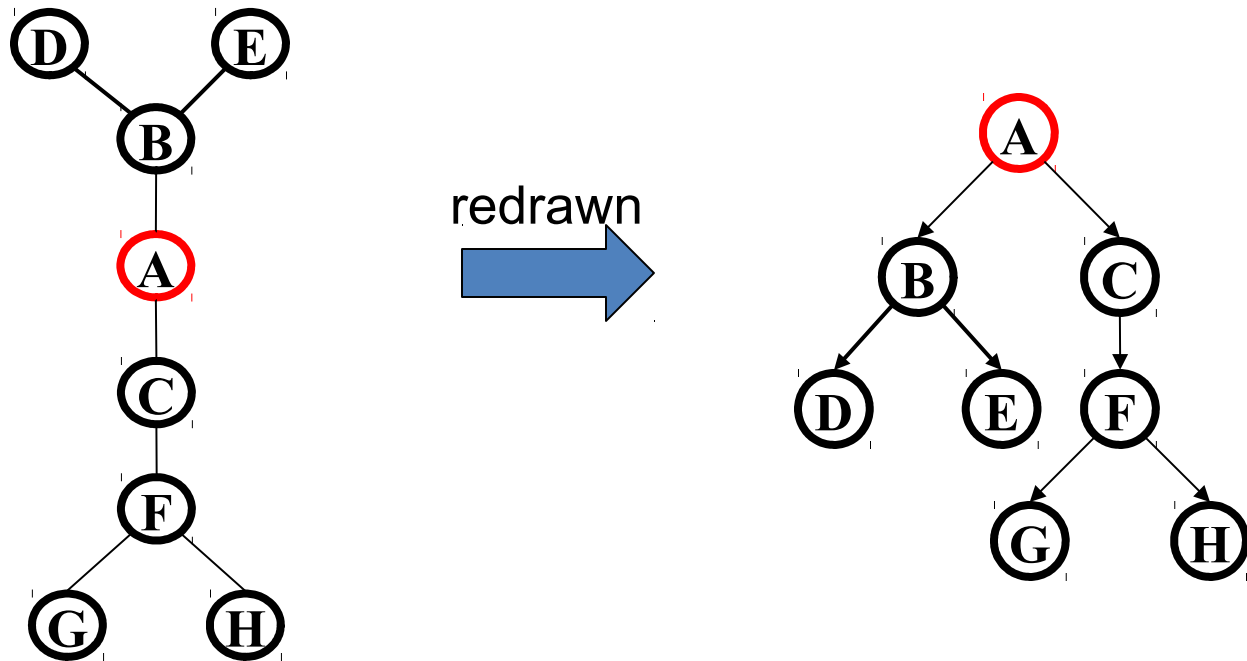
How does this relate to the trees
we know and love?...

Example:



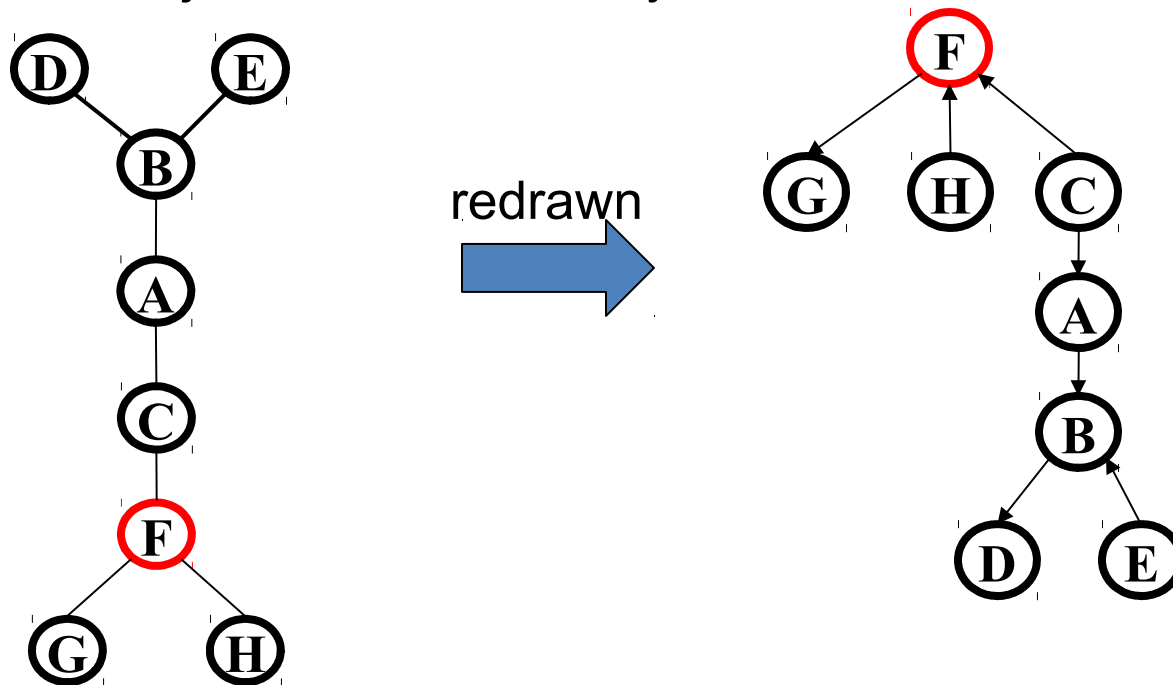
Rooted Trees

- We are more accustomed to **rooted trees** where:
 - We identify a unique root
 - We think of edges as directed: parent to children
- Given a tree, picking a root gives a unique rooted tree
 - The tree is just drawn differently and with undirected edges



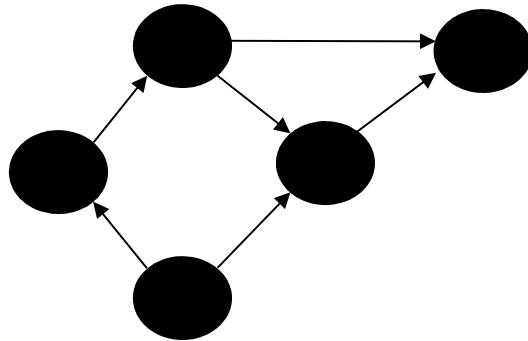
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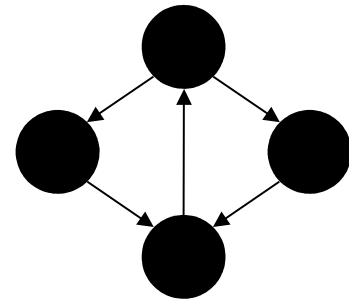


Directed Acyclic Graphs (DAGs)

- A **DAG** is a directed graph with no (directed) cycles
 - Every rooted directed tree is a DAG
 - But not every DAG is a rooted directed tree



- Every DAG is a directed graph
- But not every directed graph is a DAG



Examples

Which of our directed-graph examples do you expect to be a DAG?

- Web pages with links
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Airline routes
- Family trees
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- ...

Density / Sparsity

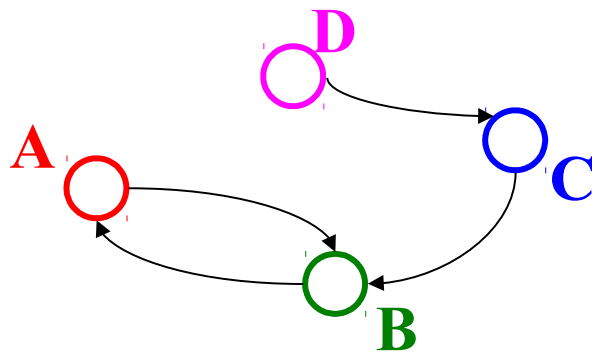
- Recall: In an undirected graph, $0 \leq |E| < |V|^2$
- Recall: In a directed graph: $0 \leq |E| \leq |V|^2$
- So for any graph, $O(|E|+|V|^2)$ is $O(|V|^2)$
- Another fact: If an undirected graph is *connected*, then $|V|-1 \leq |E|$
- Because $|E|$ is often much smaller than its maximum size, we do not always approximate $|E|$ as $O(|V|^2)$
 - This is a correct bound, it just is often not tight
 - If it is tight, i.e., $|E|$ is $\Theta(|V|^2)$ we say the graph is **dense**
 - More sloppily, dense means “lots of edges”
 - If $|E|$ is $O(|V|)$ we say the graph is **sparse**
 - More sloppily, sparse means “most possible edges missing”

What is the Data Structure?

- So graphs are really useful for lots of data and questions
 - For example, “what’s the lowest-cost path from x to y ”
- But we need a data structure that represents graphs
- The “best one” can depend on:
 - Properties of the graph (e.g., dense versus sparse)
 - The common queries (e.g., “is (u, v) an edge?” versus “what are the neighbors of node u ?”)
- So we’ll discuss the two standard graph representations
 - [Adjacency Matrix](#) and [Adjacency List](#)
 - Different trade-offs, particularly time versus space

Adjacency Matrix

- Assign each node a number from 0 to $|V| - 1$
- A $|V| \times |V|$ matrix (i.e., 2-D array) of Booleans (or 1 vs. 0)
 - If M is the matrix, then $M[u][v]$ being **true** means there is an edge from u to v



	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

Adjacency Matrix Properties

- Running time to:
 - Get a vertex's out-edges: $O(|V|)$
 - Get a vertex's in-edges: $O(|V|)$
 - Decide if some edge exists: $O(1)$
 - Insert an edge: $O(1)$
 - Delete an edge: $O(1)$
- Space requirements:
 - $|V|^2$ bits
- Best for sparse or dense graphs?
 - Best for dense graphs

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

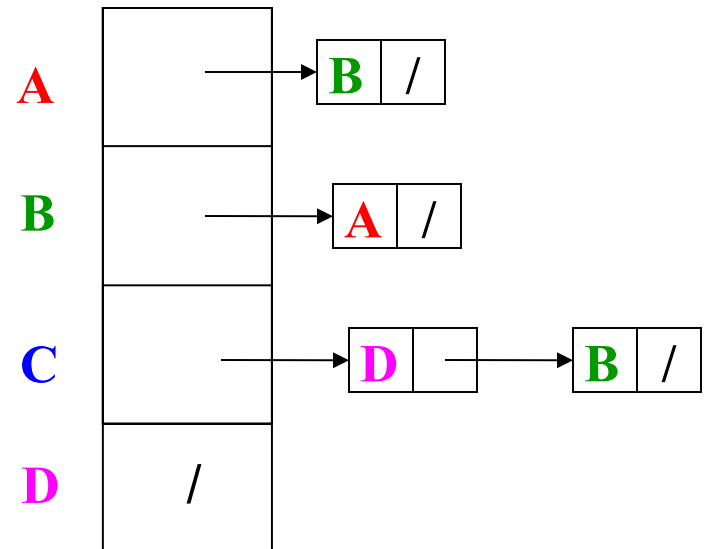
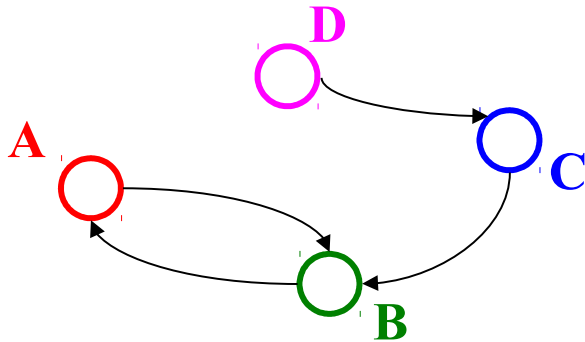
Adjacency Matrix Properties

- How will the adjacency matrix vary for an *undirected graph*?
 - Undirected will be symmetric around the diagonal
- How can we adapt the representation for *weighted graphs*?
 - Instead of a Boolean, store a number in each cell
 - Need some value to represent 'not an edge'
 - In some situations, 0 or -1 works

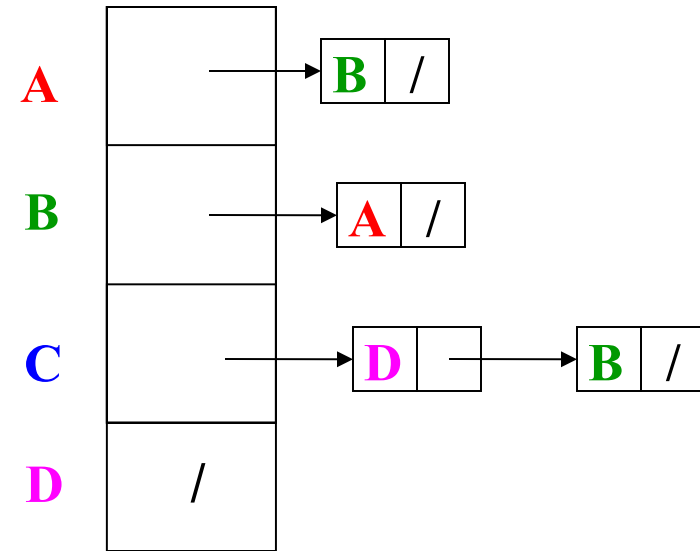
	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

Adjacency List

- Assign each node a number from 0 to $|\mathcal{V}| - 1$
- An array of length $|\mathcal{V}|$ in which each entry stores a list of all adjacent vertices (e.g., linked list)



Adjacency List Properties



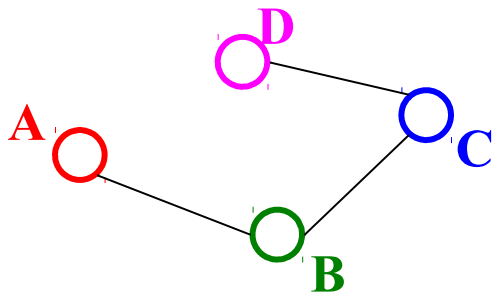
- Running time to:
 - Get all of a vertex's out-edges:
 $O(d)$ where d is out-degree of vertex
 - Get all of a vertex's in-edges:
 $O(|E|)$ (but could keep a second adjacency list for this!)
 - Decide if some edge exists:
 $O(d)$ where d is out-degree of source
 - Insert an edge: $O(1)$ (unless you need to check if it's there)
 - Delete an edge: $O(d)$ where d is out-degree of source
- Space requirements:
 - $O(|V|+|E|)$
- Best for dense or sparse graphs?
 - Best for sparse graphs, so usually just stick with linked lists

Undirected Graphs

Adjacency matrices & adjacency lists both do fine for undirected graphs

- Matrix: Can save roughly 2x space
 - But may slow down operations in languages with “proper” 2D arrays (not Java, which has only arrays of arrays)
 - How would you “get all neighbors”?
- Lists: Each edge in two lists to support efficient “get all neighbors”

Example:



	A	B	C	D
A	F			
B	T	F		
C	F	T	F	
D	F	F	T	F

