

Algorithms & Data Structures II

Lesson 15: Shortest Paths

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Single source shortest paths

- Goal: finding the minimum path length from \mathbf{v} to \mathbf{u} in $O(|E|+|V|)$
- Actually, can find the minimum path length from \mathbf{v} to *every node*
 - Still $O(|E|+|V|)$
 - No faster way for a “distinguished” destination in the worst-case
- Weighted graphs

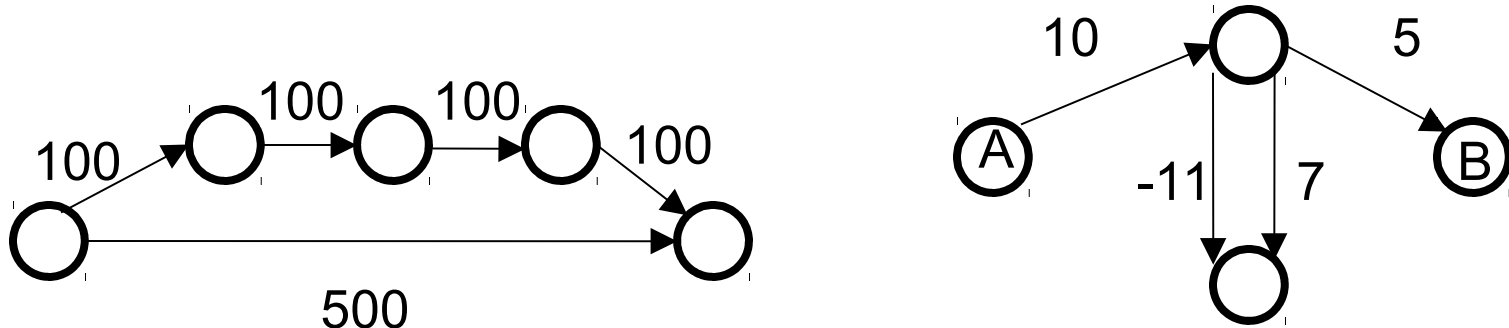
Given a weighted graph and node \mathbf{v} ,
find the minimum-cost path from \mathbf{v} to every node

- Asymptotically no harder than for one destination

Applications

- Driving directions
- Cheap flight itineraries
- Network routing
- Critical paths in project management

Not as easy



Shortest path may not have the fewest edges

- Annoying when this happens with costs of flights

We will assume there are no negative weights

- *Problem is ill-defined* if there are negative-cost cycles
- Today's *algorithm is wrong* if edges can be negative
 - There are other, slower (but not terrible) algorithms

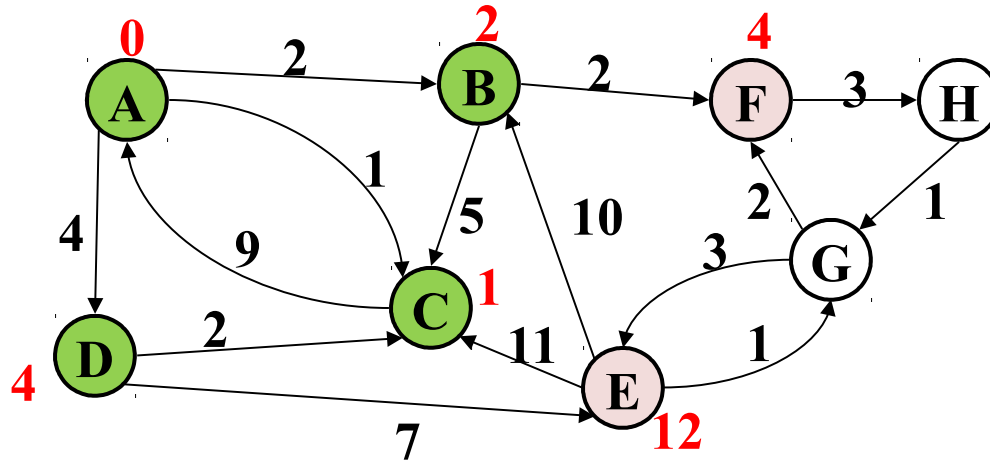
Dijkstra

- Algorithm named after its inventor Edsger Dijkstra (1930-2002)
 - Truly one of the “founders” of computer science; this is just one of his many contributions
 - My favorite Dijkstra quote: “computer science is no more about computers than astronomy is about telescopes”

Dijkstra's algorithm

- The idea
 - Grow the set of nodes whose shortest distance has been computed
 - Nodes not in the set will have a “best distance so far”
 - A priority queue will turn out to be useful for efficiency

Dijkstra's Algorithm: Idea



- Initially, start node has cost 0 and all other nodes have cost ∞
- At each step:
 - Pick closest unknown vertex **v**
 - Add it to the “cloud” of known vertices
 - Update distances for nodes with edges from **v**
- That's it! (But we need to prove it produces correct answers)

The Algorithm

1. For each node v , set $v.cost = \infty$ and $v.known = false$
2. Set $source.cost = 0$
3. While there are unknown nodes in the graph
 - a) Select the unknown node v with lowest cost
 - b) Mark v as known
 - c) For each edge (v, u) with weight w ,

$c1 = v.cost + w$ // cost of best path through v to u
 $c2 = u.cost$ // cost of best path to u previously known
 $if (c1 < c2) \{$ // if the path through v is better

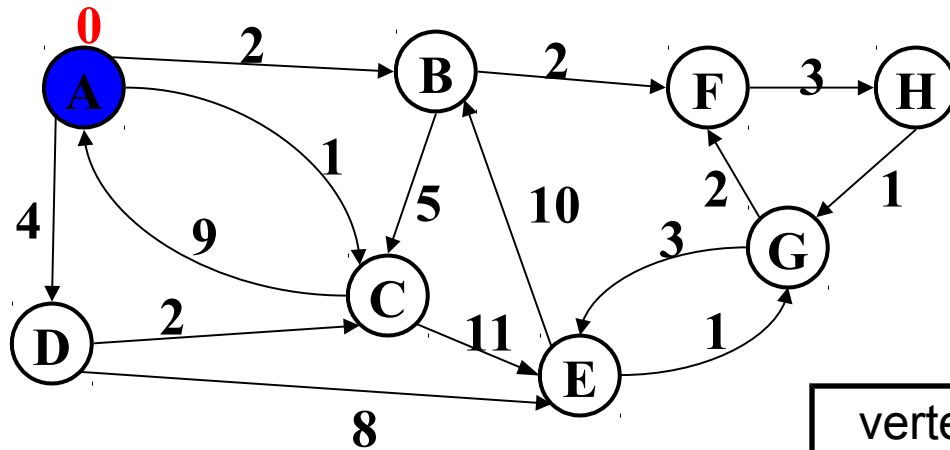
$u.cost = c1$
 $u.path = v$ // for computing actual paths

 $\}$

Important features

- When a vertex is marked known, the cost of the shortest path to that node is known
 - The path is also known by following back-pointers
- While a vertex is still not known, another shorter path to it *might* still be found

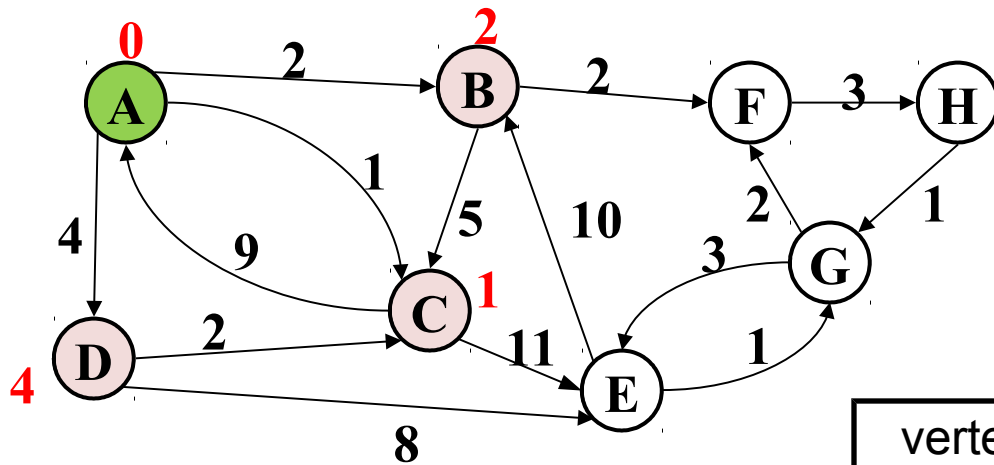
Example #1



vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	
H		??	

Order Added to Known Set:

Example #1

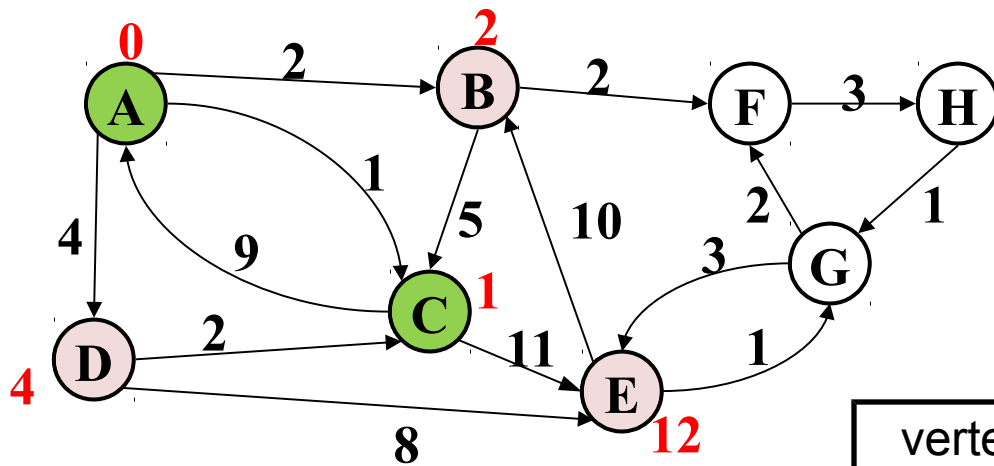


vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C		≤ 1	A
D		≤ 4	A
E		??	
F		??	
G		??	
H		??	

Order Added to Known Set:

A

Example #1

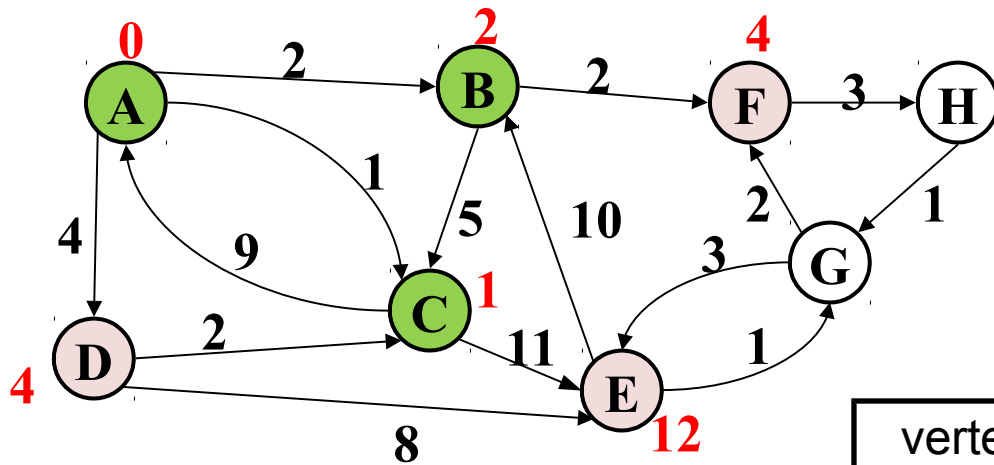


vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		??	
G		??	
H		??	

Order Added to Known Set:

A, C

Example #1

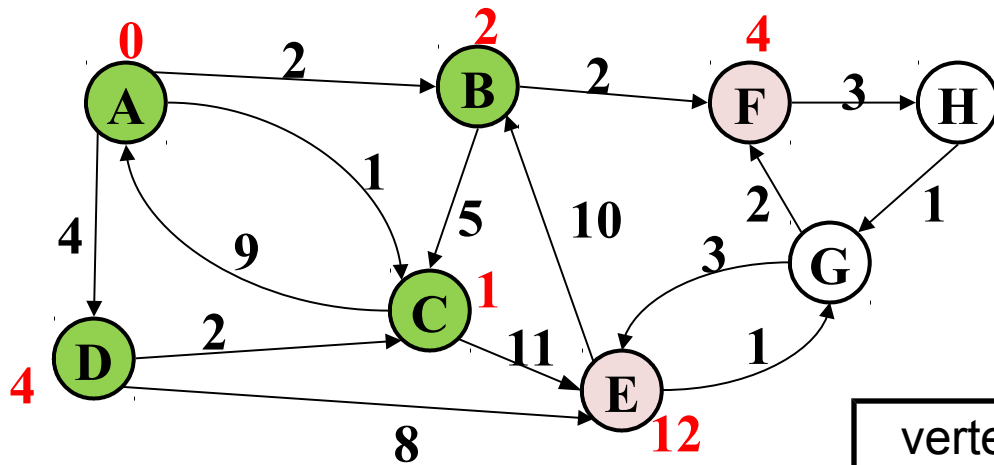


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

Order Added to Known Set:

A, C, B

Example #1

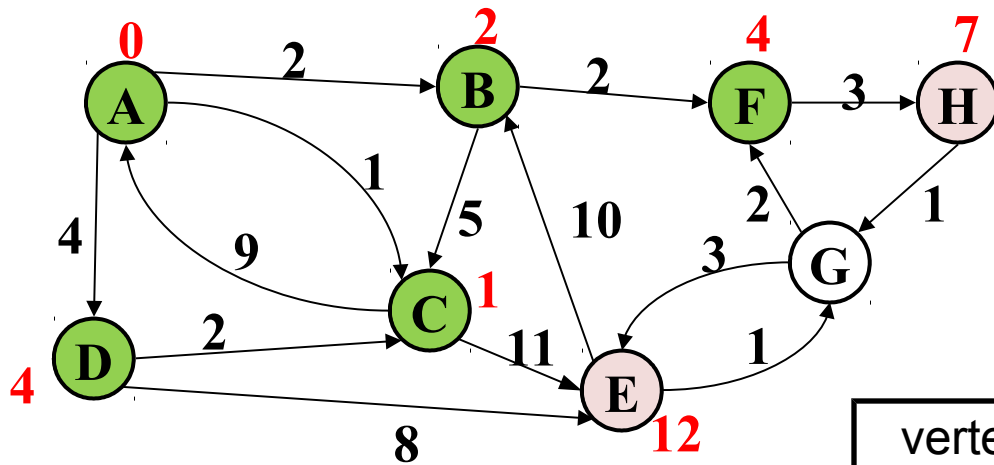


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

Order Added to Known Set:

A, C, B, D

Example #1

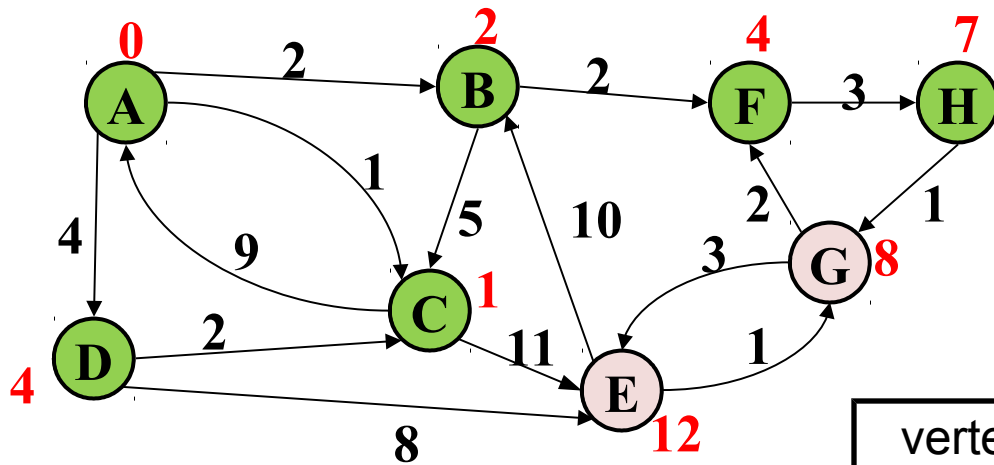


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		??	
H		≤ 7	F

Order Added to Known Set:

A, C, B, D, F

Example #1

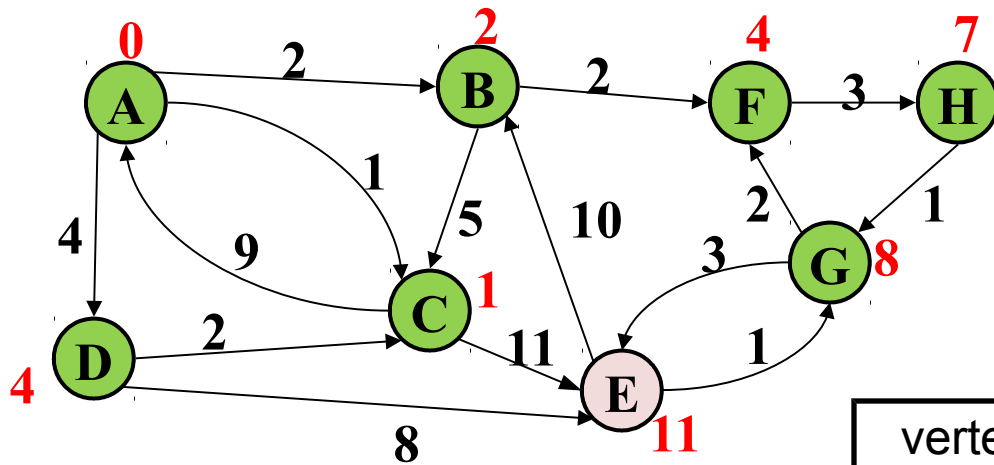


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		≤ 8	H
H	Y	7	F

Order Added to Known Set:

A, C, B, D, F, H

Example #1

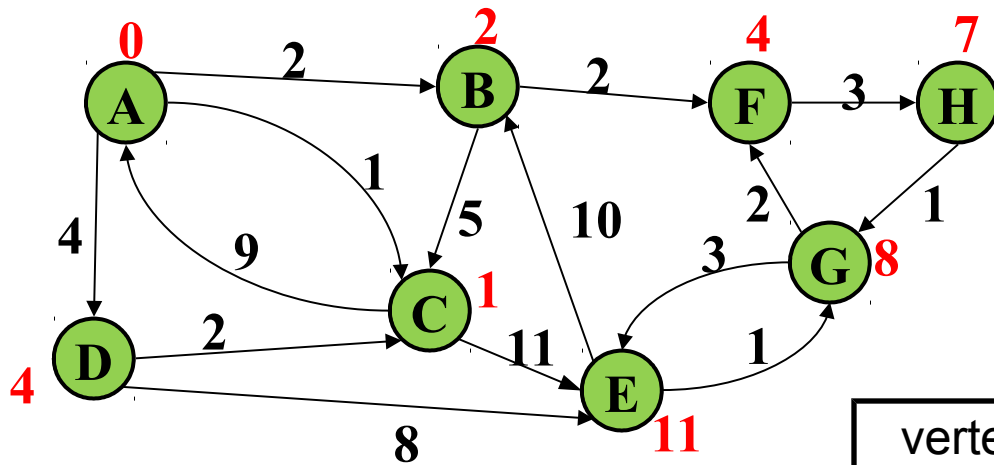


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Order Added to Known Set:

A, C, B, D, F, H, G

Example #1



vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Order Added to Known Set:

A, C, B, D, F, H, G, E

Features

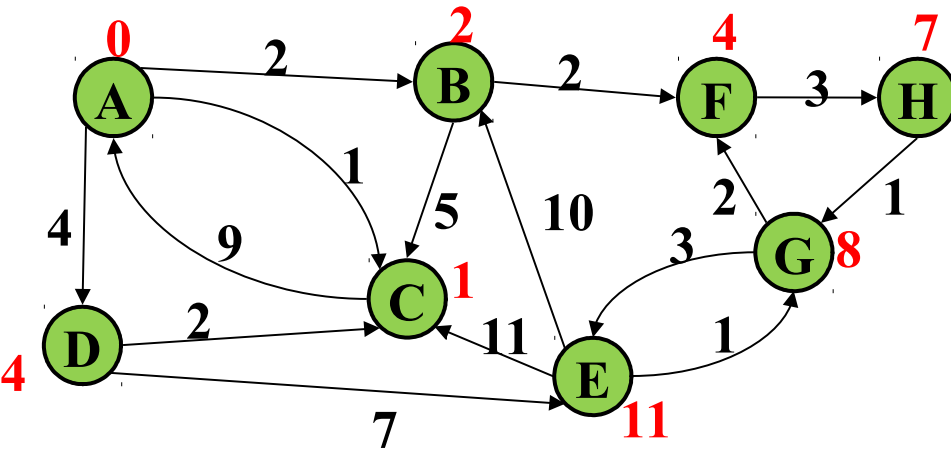
- When a vertex is marked known,
the cost of the shortest path to that node is known
 - The path is also known by following back-pointers
- While a vertex is still not known,
another shorter path to it **might** still be found

Note: The “Order Added to Known Set” is not important

- A detail about how the algorithm works (client doesn't care)
- Not used by the algorithm (implementation doesn't care)
- It is sorted by path-cost, resolving ties in some way
 - Helps give intuition of why the algorithm works

Interpreting the Results

- Now that we're done, how do we get the path from, say, A to E?



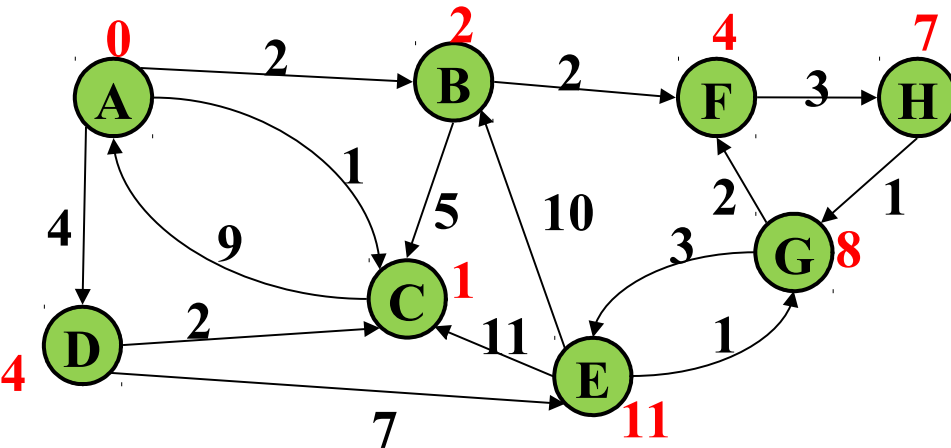
Order Added to Known Set:

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Stopping Short

- How would this have worked differently if we were only interested in:
 - The path from A to G?
 - The path from A to E?

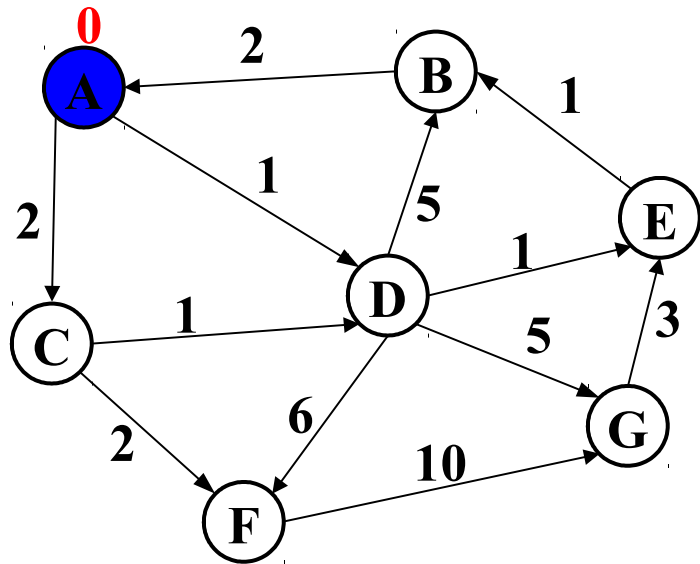


Order Added to Known Set:

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

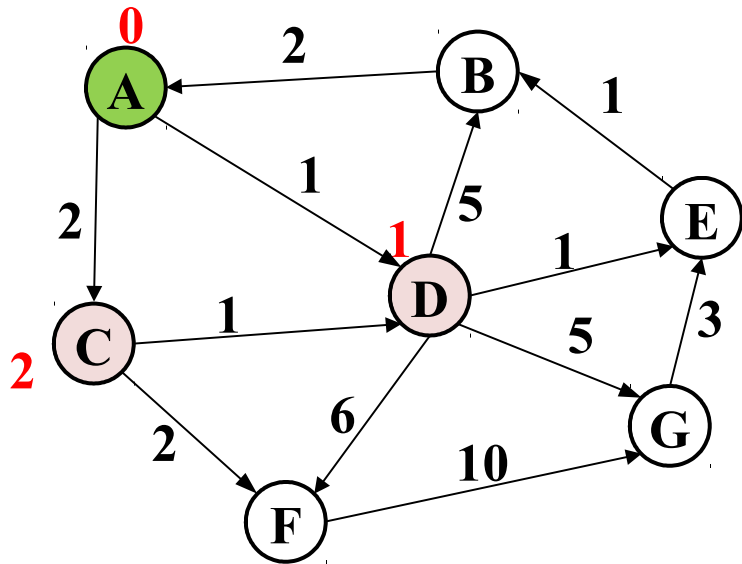
Example #2



Order Added to Known Set:

vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	

Example #2

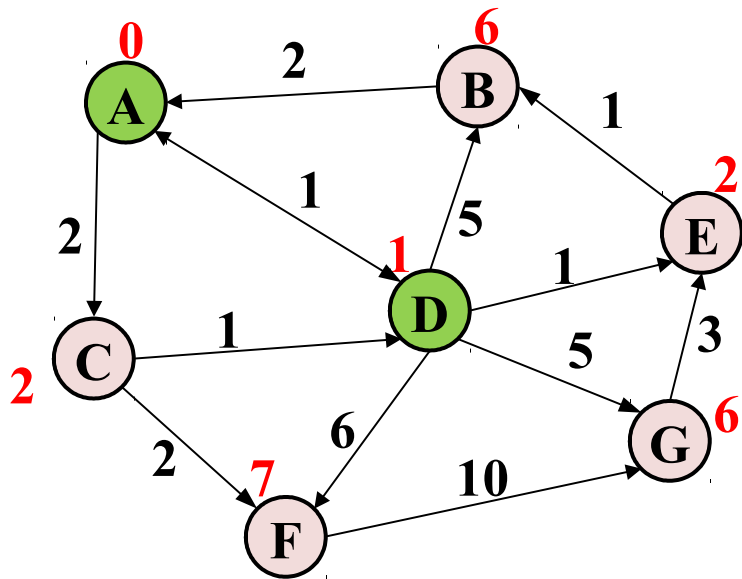


Order Added to Known Set:

A

vertex	known?	cost	path
A	Y	0	
B		??	
C		≤ 2	A
D		≤ 1	A
E		??	
F		??	
G		??	

Example #2

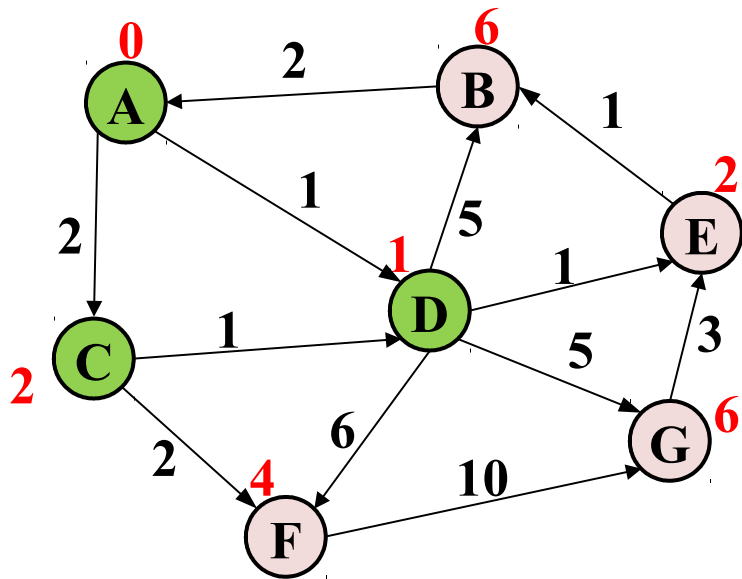


Order Added to Known Set:

A, D

vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C		≤ 2	A
D	Y	1	A
E		≤ 2	D
F		≤ 7	D
G		≤ 6	D

Example #2

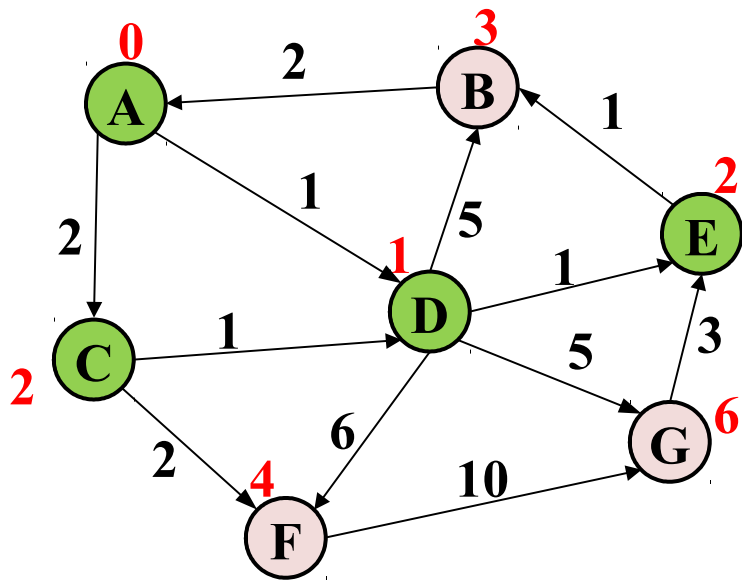


Order Added to Known Set:

A, D, C

vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C	Y	2	A
D	Y	1	A
E		≤ 2	D
F		≤ 4	C
G		≤ 6	D

Example #2

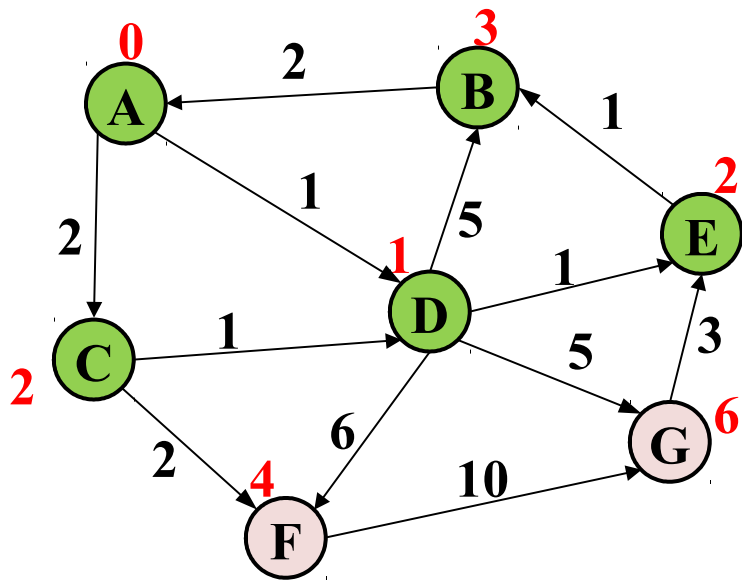


Order Added to Known Set:

A, D, C, E

vertex	known?	cost	path
A	Y	0	
B		≤ 3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

Example #2

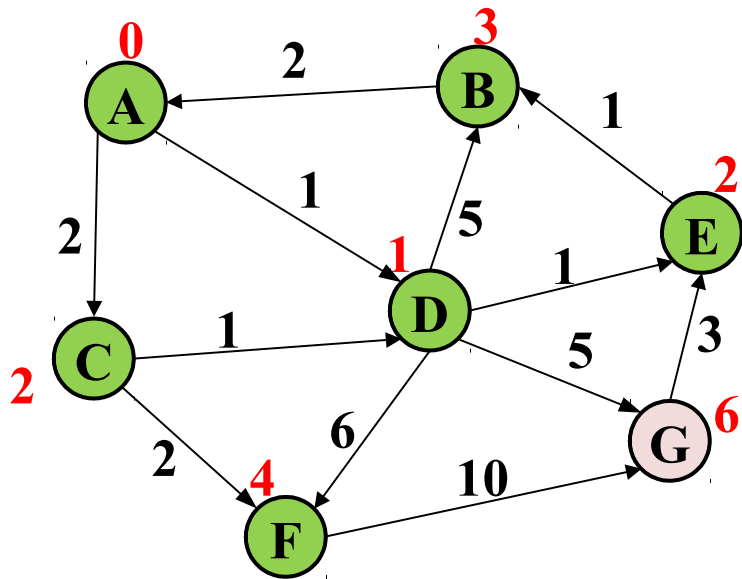


Order Added to Known Set:

A, D, C, E, B

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

Example #2

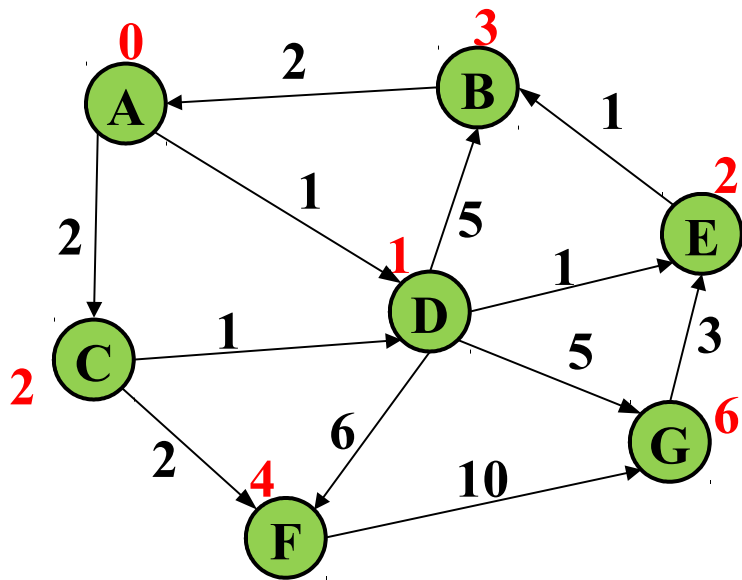


Order Added to Known Set:

A, D, C, E, B, F

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G		≤ 6	D

Example #2

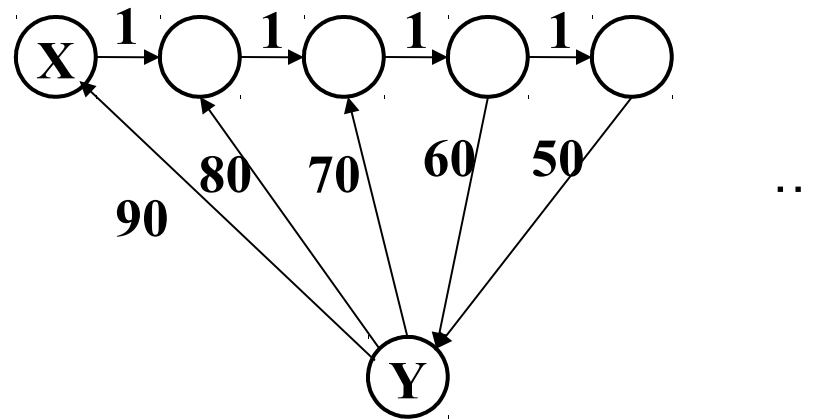


Order Added to Known Set:

A, D, C, E, B, F, G

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G	Y	6	D

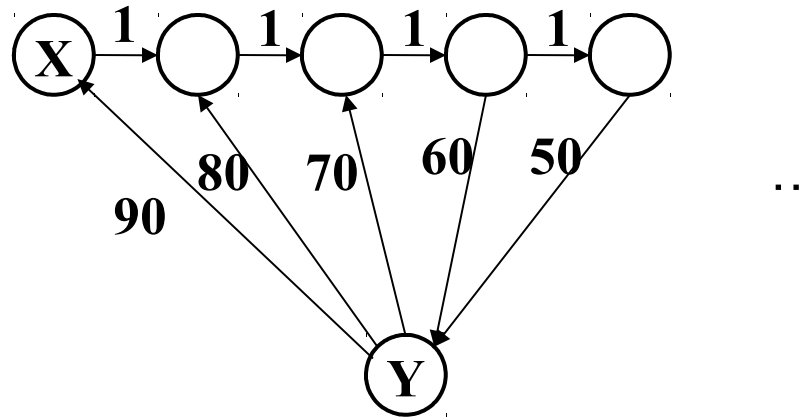
Example #3



How will the best-cost-so-far for Y proceed?

Is this expensive?

Example #3



How will the best-cost-so-far for Y proceed? 90, 81, 72, 63, 54, ...

Is this expensive? No, each *edge* is processed only once

A Greedy Algorithm

- Dijkstra's algorithm
 - For single-source shortest paths in a weighted graph (directed or undirected) with no negative-weight edges
- An example of a *greedy algorithm*:
 - At each step, irrevocably does what seems best at that step
 - A locally optimal step, not necessarily globally optimal
 - Once a vertex is known, it is not revisited
 - Turns out to be globally optimal

Correctness: Intuition

Rough intuition:

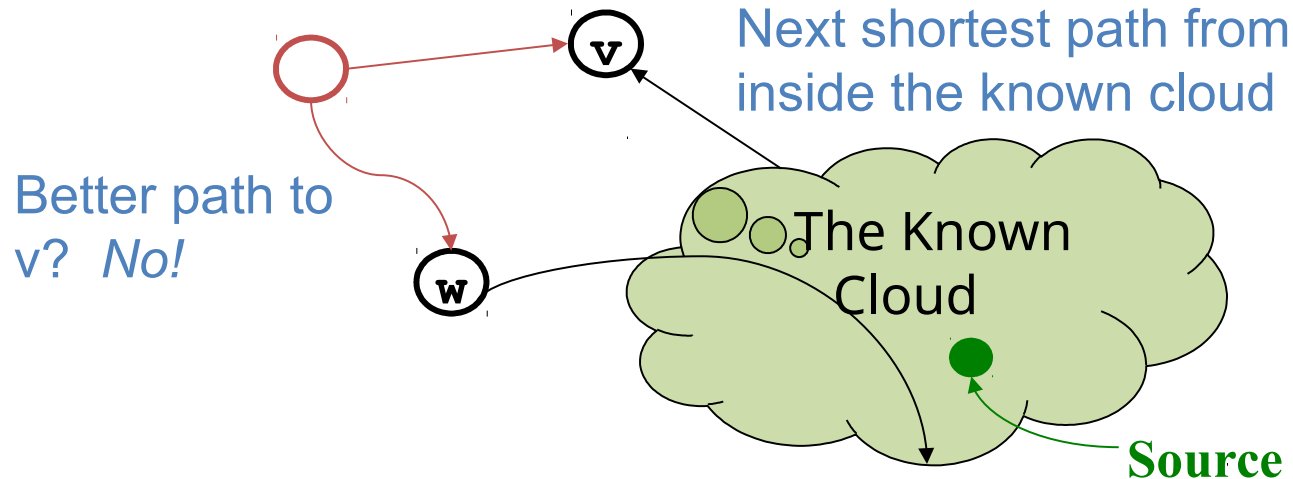
All the “known” vertices have the correct shortest path

- True initially: shortest path to start node has cost 0
- If it stays true every time we mark a node “known”, then by induction this holds and eventually everything is “known”

Key fact we need: When we mark a vertex “known” we won’t discover a shorter path later!

- This holds only because Dijkstra’s algorithm picks the node with the next shortest path-so-far
- The proof is by contradiction...

Correctness: The Cloud (Rough Sketch)



Suppose v is the next node to be marked known (“added to the cloud”)

- The **best-known path** to v must have only nodes “in the cloud”
 - Else we would have picked a node closer to the cloud than v
- Suppose the **actual shortest path** to v is different
 - It won’t use only cloud nodes, or we would know about it
 - So it must use non-cloud nodes. Let w be the *first* non-cloud node on this path. The part of the path up to w is **already known** and must be shorter than the best-known path to v . So v would not have been picked. Contradiction.

Naïve asymptotic running time

- So far: $O(|V|^2)$
- We had a similar “problem” with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next
 - We solved it with a queue of zero-degree nodes
 - But here we need the lowest-cost node and costs can change as we process edges
- Solution?

Improving asymptotic running time

- So far: $O(|V|^2)$
- We had a similar “problem” with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next
 - We solved it with a queue of zero-degree nodes
 - But here we need the lowest-cost node and costs can change as we process edges
- Solution?
 - A priority queue holding all unknown nodes, sorted by cost
 - But must support **decreaseKey** operation
 - Must maintain a reference from each node to its current position in the priority queue
 - Conceptually simple, but can be a pain to code up

Efficiency, second approach

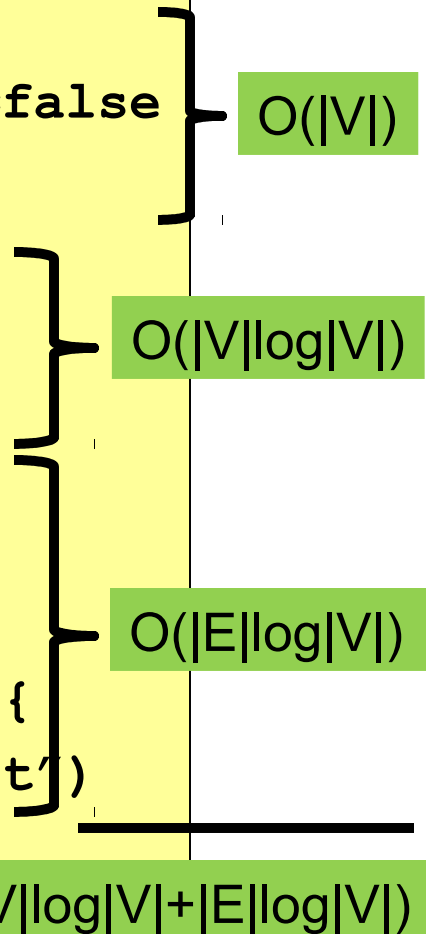
Use pseudocode to determine asymptotic run-time

```
dijkstra(Graph G, Node start) {  
  for each node: x.cost=infinity, x.known=false  
  start.cost = 0  
  build-heap with all nodes  
  while(heap is not empty) {  
    b = deleteMin()  
    b.known = true  
    for each edge (b,a) in G  
      if(!a.known)  
        if(b.cost + weight((b,a)) < a.cost){  
          decreaseKey(a,"new cost - old cost")  
          a.path = b  
        }  
  }  
}
```

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
dijkstra(Graph G, Node start) {  
  for each node: x.cost=infinity, x.known=false  
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    for each edge (b,a) in G  
      if(!a.known)  
        if(b.cost + weight((b,a)) < a.cost) {  
          decreaseKey(a, "new cost - old cost")  
          a.path = b  
        }  
  }  
}
```



$O(|V|)$

$O(|V|\log|V|)$

$O(|E|\log|V|)$

$O(|V|\log|V| + |E|\log|V|)$