# Invertibility Conditions for the Admittance Matrices of Balanced Power Systems

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***Abstract*—The admittance matrix encodes the network topology and electrical parameters of a power system in order to relate the current injection and voltage phasors. Since admittance matrices are central to many power engineering analyses, their characteristics are important subjects of theoretical studies. This paper focuses on the key characteristic of *invertibility*. Previous literature has presented an invertibility condition for admittance matrices. This paper first identifies and fixes a technical issue in the proof of this previously presented invertibility condition. This paper then extends this previous work by deriving new conditions that are applicable to a broader class of systems with lossless branches and transformers with off-nominal tap ratios.**

***Index Terms*—Admittance matrix, circuit analysis.**

## NOTATION

| | |
|---|---|
| $j$ | The imaginary unit ($j^2+1=0$) |
| $a, A$ | (No boldface letter) scalar |
| $\mathbf{a}$ | (Boldface lowercase letter) column vector |
| $\mathbf{A}$ | (Boldface uppercase letter) matrix |
| $\mathcal{A}$ | (Calligraphic font uppercase letter) set |
| $\mathrm{Re}(\,\cdot\,)$ | Element-wise real part operator |
| $\mathrm{Im}(\,\cdot\,)$ | Element-wise imaginary part operator |
| $(\,\cdot\,)^*$ | Element-wise conjugate operator |
| $(\,\cdot\,)^T$ | Transpose operator |
| $(\,\cdot\,)^H$ | Conjugate transpose operator |
| $\mathbf{0}_{n\times m}$ | Zero matrix of size $n \times m$ |
| $\mathbf{0}$ | Zero matrix of appropriate size, determined from context |
| $\{\mathbf{a}\}_k$ | $k$-th element of vector $\mathbf{a}$ (scalar) |
| $\{\mathbf{A}\}_k$ | $k$-th row of matrix $\mathbf{A}$ (row vector) |
| $\{\mathbf{A}\}_{ik}$ | Element of matrix $\mathbf{A}$ in row $i$, column $k$ (scalar) |
| $\lvert a\rvert$ | Absolute value of scalar $a$ |
| $\lvert\mathcal{A}\rvert$ | Cardinality of set $\mathcal{A}$ |
| $\lVert\mathbf{a}\rVert_1$ | 1-norm of vector $\mathbf{a}$: $\lVert\mathbf{a}\rVert_1 = \sum_k \lvert\{\mathbf{a}\}_k\rvert$ |
| $\lVert\mathbf{a}\rVert$ | Euclidean norm of vector $\mathbf{a}$: $\lVert\mathbf{a}\rVert = (\sum_k \lvert\{\mathbf{a}\}_k\rvert^2)^{1/2}$ |
| $\mathrm{diag}\,(\mathbf{a})$ | Diagonal matrix such that $\{\mathrm{diag}\,(\mathbf{a})\}_{kk} = \{\mathbf{a}\}_k$. $\mathrm{diag}\,(\mathbf{a})$ has as rows and columns as the size of $\mathbf{a}$ |
| $\mathrm{rank}\,(\mathbf{A})$ | Rank of matrix $\mathbf{A}$ (scalar) |
| $\mathrm{Null}\,(\mathbf{A})$ | Null space (kernel) of matrix $\mathbf{A}$ (vector space) |
| $\dim(\,\cdot\,)$ | Dimension of a vector space (scalar) |
| $\mathrm{Sym}\,(\mathbf{B})$ | Symmetric part of square matrix $\mathbf{B}$: $\mathrm{Sym}\,(\mathbf{B}) = \left(\mathbf{B}+\mathbf{B}^T\right)/2$ |
| $\mathbf{B} \succeq \mathbf{0}$ | Square matrix $\mathbf{B}$ is positive-semidefinite (for all $\mathbf{x} \neq \mathbf{0}$, $\mathrm{Re}\left(\mathbf{x}^H\mathbf{B}\mathbf{x}\right) \geq 0$), but not necessarily Hermitian |
| $\mathbf{B} \succ \mathbf{0}$ | Square matrix $\mathbf{B}$ is positive-definite (for all $\mathbf{x} \neq \mathbf{0}$, $\mathrm{Re}\left(\mathbf{x}^H\mathbf{B}\mathbf{x}\right) > 0$), but not necessarily Hermitian |

## I. INTRODUCTION

The admittance matrix, which relates the current injections to the bus voltages, is one of the most fundamental concepts in power engineering. In the phasor domain, admittance matrices are complex-valued square matrices. These matrices are used in many applications, including system modeling, power flow, optimal power flow, state estimation, stability analyses, etc. [1], [2]. This paper thoroughly characterizes the invertibility of admittance matrices, which is a fundamental property for many power system applications.

Several applications directly rely on the invertibility of the admittance matrix. For instance, Kron reduction [3] is a popular technique for reducing the number of independent bus voltages modeled in a power system. The feasibility of Kron reduction is contingent on the invertibility of an appropriate sub-block of the admittance matrix. Many applications of Kron reduction assume that this procedure is feasible without performing further verification (e.g., [4]–[6]). Additionally, various fault analysis techniques require the explicit computation of the inverse of the admittance matrix (the impedance matrix) [7]. The DC power flow [8] and its derivative applications [9], [10] also require the invertibility of admittance matrices for purely inductive systems. The invertibility of the admittance matrix is a requirement seen in both classical literature and recent research efforts (see, e.g., [11], [12]).

Checking invertibility of a matrix can be accomplished via rank-revealing factorizations [13], [14]. However, this approach is computationally costly for large matrices. Invertibility can also be checked approximately by computing the condition number via iterative algorithms that have lower complexity than matrix factorizations [15]. However, iterative estimation of the condition number can be inaccurate [16]. In some applications, such as transmission switching [17] and topology reconfiguration [18], [19], the admittance matrix changes as part of the problem and checking invertibility for every case is intractable. Recent research has studied the theoretical characteristics of the admittance matrix in order to guarantee invertibility without the need for computationally expensive explicit checks [20]–[22].

One of the most important results regarding theoretical invertibility guarantees comes from [20]. The authors of [20] show that the admittance matrix is invertible for connected networks consisting of reciprocal branches without mutual coupling and at least one shunt element. (See [23] for the definition and properties of reciprocal branches.) This result relies on additional modeling assumptions requiring that all admittances have positive conductances and prohibiting transformers with off-nominal tap ratios (including on-load tap changers which control the voltage magnitudes or phase shifters which control the voltage angles).

These requirements can be restrictive for practical power system models. While perfectly lossless branches do not exist in physical circuits, power system datasets often approximate certain branches as lossless. For instance, out of the 41 systems with more than 1000 buses in the PGLib test case repository [24], zero-conductance branches exist in 26 systems (63.4%). We further note that transformers with off-nominal

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tap ratios and non-zero phase shifts are also present in many practical datasets (e.g., 39 of the 41 PGLib systems (95.1%)).

In addition to these modeling restrictions, there is a technical issue with the proof presented in [20]. This paper demonstrates that the result of [20] can still be achieved and generalized to a broader class of power system models. We first detail the technical issue in the proof in [20]. We then prove invertibility of the admittance matrix under a condition that generalizes the requirements in [20]. The condition holds for a broad class of realistic systems, including systems with lossless branches and transformers with off-nominal tap ratios. Next we show that the theorem condition holds for networks that can be decomposed into reactive components with simple structure. Finally, we present a proof-of-concept program that implements the theorem, and we show through numerical experiments that the theorem can be applied to a wide variety of realistic power systems.

The rest of the paper is organized as follows. Section II describes the result of previous research and the technical issue in their proof. Section III states the modifications and additional lemmas required to amend and generalize the previous result to systems with purely reactive elements and more general transformer models. Section IV describes the implementation and numerical experiments. Section V concludes the paper.

## II. Claims from Previous Literature and Limitations

Borrowing the notation of [20], the admittance matrix is (see [25]):

$$\mathbf{Y}_{\mathcal{N}} = \mathbf{A}_{\mathcal{L},\mathcal{N}}^T \mathbf{Y}_{\mathcal{L}} \mathbf{A}_{\mathcal{L},\mathcal{N}} + \mathbf{Y}_{\mathcal{T}}, \tag{1}$$

where $\mathbf{A}_{\mathcal{L},\mathcal{N}}$ is the oriented incidence matrix of the network graph (excluding ground), $\mathbf{Y}_{\mathcal{L}} = \operatorname{diag}(\mathbf{y}_{\mathcal{L}})$ is the diagonal matrix with the series admittances of each branch, and $\mathbf{Y}_{\mathcal{T}} = \operatorname{diag}(\mathbf{y}_{\mathcal{T}})$ is the diagonal matrix with the total shunt admittances at each node. $\mathcal{N}$ is the set of nodes (excluding ground) and $\mathcal{L}$ is the set of branches. Reference [20] states the following assumption and lemmas (presented here with some minor extensions as described below):

**Assumption 1.** *The branches are not electromagnetically coupled and have nonzero admittance, hence $\mathbf{Y}_{\mathcal{L}}$ is full-rank.*

**Lemma 1.** *The rank of the oriented incidence matrix of a connected graph with $|\mathcal{N}|$ nodes, $\mathbf{A}_{\mathcal{L},\mathcal{N}}$, is $|\mathcal{N}|-1$. The vector of ones $\mathbf{1}$ forms a basis of the null space of $\mathbf{A}_{\mathcal{L},\mathcal{N}}$.*

While the second statement regarding the basis of the null space is not included in Lemma 1 as presented in [20], it is a well-known characteristic of oriented incidence matrices[1] that we will use later in this paper.

**Lemma 2.** *The sum of the columns of $\mathbf{Y}_{\mathcal{N}}$ equals the transpose of the sum of its rows, which also equals the vector of shunt elements $\mathbf{y}_{\mathcal{T}}$ (see [27]).*

**Lemma 3.** *For any matrix $\mathbf{M}$, $\operatorname{rank}\left(\mathbf{M}^T\mathbf{M}\right) = \operatorname{rank}(\mathbf{M})$.*

As we will discuss shortly, *Lemma 3 as stated above is incorrect*. This is the technical issue in [20] mentioned above.

**Lemma 4.** *For square matrices $\mathbf{N}_L$ and $\mathbf{N}_R$ with full rank and matching size, $\operatorname{rank}(\mathbf{N}_L\mathbf{M}) = \operatorname{rank}(\mathbf{M}) = \operatorname{rank}(\mathbf{M}\mathbf{N}_R)$. Furthermore, $\operatorname{Null}(\mathbf{N}_L\mathbf{M}) = \operatorname{Null}(\mathbf{M})$.*

While the second statement regarding the relationship between the null spaces is not included in Lemma 4 as presented in [20], it is a well-known result from matrix theory.[2]

One of the main results of [20] is the following theorem:

**Theorem 1.** *If the graph $(\mathcal{N},\mathcal{L})$ defines a connected network and Assumption 1 holds, then:*

$$\operatorname{rank}(\mathbf{Y}_{\mathcal{N}}) = \begin{cases} |\mathcal{N}|-1 & \text{if } \mathbf{y}_{\mathcal{T}} = \mathbf{0}, \\ |\mathcal{N}| & \text{otherwise.} \end{cases} \tag{2}$$

The authors of [20] prove Theorem 1 by cases. They first assume $\mathbf{y}_{\mathcal{T}} = \mathbf{0}$ and use the fact that $\mathbf{Y}_{\mathcal{L}}$ is diagonal to write it as

$$\mathbf{Y}_{\mathcal{L}} = \mathbf{B}^T\mathbf{B}, \tag{3}$$

where $\mathbf{B} \in \mathbb{C}^{|\mathcal{N}|\times|\mathcal{N}|}$ is full-rank. Therefore:

$$\mathbf{Y}_{\mathcal{N}} = \mathbf{A}_{\mathcal{L},\mathcal{N}}^T\mathbf{B}^T\mathbf{B}\mathbf{A}_{\mathcal{L},\mathcal{N}}, \tag{4a}$$

$$\mathbf{Y}_{\mathcal{N}} = \left(\mathbf{B}\mathbf{A}_{\mathcal{L},\mathcal{N}}\right)^T\mathbf{B}\mathbf{A}_{\mathcal{L},\mathcal{N}}, \tag{4b}$$

$$\mathbf{Y}_{\mathcal{N}} = \mathbf{M}^T\mathbf{M}, \tag{4c}$$

where $\mathbf{M} = \mathbf{B}\mathbf{A}_{\mathcal{L},\mathcal{N}}$. According to Lemma 1, $\mathbf{A}_{\mathcal{L},\mathcal{N}}$ has rank $|\mathcal{N}|-1$. According to Lemma 4, $\operatorname{rank}(\mathbf{B}\mathbf{A}_{\mathcal{L},\mathcal{N}}) = \operatorname{rank}(\mathbf{A}_{\mathcal{L},\mathcal{N}})$, so $\operatorname{rank}(\mathbf{M}) = |\mathcal{N}|-1$. Finally, according to Lemma 3, $\operatorname{rank}(\mathbf{Y}_{\mathcal{N}}) = |\mathcal{N}|-1$.

There is a technical issue in the proof of Theorem 1 resulting from the fact that Lemma 3 only holds for real-valued matrices. A complex-valued counterexample is the following:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ j & 0 \end{bmatrix}, \qquad \operatorname{rank}(\mathbf{M}) = 1, \tag{5}$$

$$\mathbf{M}^T\mathbf{M} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \operatorname{rank}\left(\mathbf{M}^T\mathbf{M}\right) = 0. \tag{6}$$

However, there is a very similar property to Lemma 3 that holds for complex-valued matrices:

**Lemma 3 (Corrected).** *For any matrix $\mathbf{M}$ with complex entries, $\operatorname{rank}\left(\mathbf{M}^H\mathbf{M}\right) = \operatorname{rank}(\mathbf{M})$. Furthermore, $\operatorname{Null}\left(\mathbf{M}^H\mathbf{M}\right) = \operatorname{Null}(\mathbf{M})$.*

**Proof.** Suppose a vector $\mathbf{z}$ is in the null space of $\mathbf{M}$, then:

$$\mathbf{0} = \mathbf{M}\mathbf{z}, \quad \Longrightarrow \quad \mathbf{0} = \mathbf{M}^H\mathbf{M}\mathbf{z}, \tag{7}$$

so $\mathbf{z}$ is also in the null space of $\mathbf{M}^H\mathbf{M}$. Moreover, suppose a vector $\mathbf{z}$ is in the null space of $\mathbf{M}^H\mathbf{M}$. Then, we have

$$\mathbf{0} = \mathbf{M}^H\mathbf{M}\mathbf{z}, \quad \Longrightarrow \quad 0 = \mathbf{z}^H\mathbf{M}^H\mathbf{M}\mathbf{z} = \|\mathbf{M}\mathbf{z}\|^2 \tag{8a}$$

$$\Longrightarrow \quad \mathbf{0} = \mathbf{M}\mathbf{z}, \tag{8b}$$

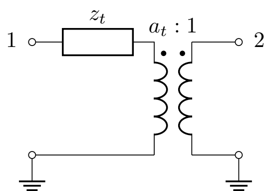
so $\mathbf{z}$ is also in the null space of $\mathbf{M}$. In conclusion, $\mathbf{z}$ is in the null space of $\mathbf{M}$ if and only if it is in the null space $\mathbf{M}^H\mathbf{M}$; this means that $\operatorname{Null}\left(\mathbf{M}^H\mathbf{M}\right) = \operatorname{Null}(\mathbf{M})$. Now we apply the rank-nullity theorem (see [28]) to complete the proof. □

With the corrected version of Lemma 3 and a modeling restriction to systems where all branches are strictly lossy (have positive conductances), we can fix the proof of Theorem 1 as stated above. More specifically, the assumptions of [20] imply Theorem 2 stated in the next section.

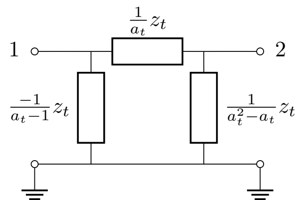
We now turn our attention to the modeling restrictions of [20]. Before generalizing Theorem 1, we need to understand why a system that violates the modeling restrictions may

[1] The sum of the elements of each row of $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ is always zero since every row has exactly one entry of 1 and one entry of -1 with the rest of the entries equal to zero; see [26].

[2] Since the only solution of $\mathbf{N}_L\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, we make $\mathbf{x} = \mathbf{M}\mathbf{z}$ for some vector $\mathbf{z}$ and the result follows.




(a) Transformer circuit providing a counterexample to Theorem 1 in [20].




(b) $\pi$-equivalent circuit for the transformer in Fig. 1a for a real-valued turns ratio $a_t$.

Fig. 1. Transformer circuits.

not satisfy the theorem. Consider the circuit modeling a transformer with an off-nominal tap ratio shown in Fig. 1a. Let $y_t = 1/z_t$. The transformer's turns ratio $a_t$ is an arbitrary complex number. The transformer's admittance matrix is:

$$\mathbf{Y}_t = \begin{bmatrix} y_t & -a_t y_t \\ -a_t^* y_t & |a_t|^2 y_t \end{bmatrix} = y_t \, \mathbf{a}_t \, \mathbf{a}_t^H, \tag{9}$$

where $\mathbf{a}_t^H = [1, -a_t]$. If $a_t$ is purely real, then $a_t^* = a_t$ and we can model the transformer with the $\pi$ circuit in Fig. 1b [1].

The transformer's $\pi$ circuit is a two-port network with $\operatorname{rank}(\mathbf{Y}_t) = 1$. This $\pi$ circuit violates the requirement of strictly lossy branches, as one one of the shunts will always have non-positive conductance. Notice that the impedances around the loop in the $\pi$ circuit have the sum $\frac{-1}{a_t-1} z_t + \frac{1}{a_t} z_t + \frac{1}{a_t^2 - a_t} z_t = 0$. With a zero-impedance loop (i.e., a closed path through the circuit where the sum of the impedances along the path equals zero), it is mathematically possible to have non-zero voltages even in the case of zero current injections. This means that the admittance matrix is singular. More generally, admittance matrix singularity can result from other power system models with zero-impedance loops besides those associated with transformers.

The strict-lossiness restriction in [20] requires all impedances in the power systems to have strictly positive real part. This means that the sum of the impedances over any possible loop will always have positive real part, thus being different from zero. Hence, the strict-lossiness restriction forbids the existence of zero-impedance loops. However, this also restricts the presence of transformers with off-nominal tap ratios and branches modeled as purely reactive elements, both of which appear in practical power system datasets as discussed in Section I. To circumvent this issue, we will treat transformers as general series elements while modeling the shunt elements of the transformer $\pi$ circuit by employing an appropriate representation of the admittance matrix. With this approach, the conditions we derive in this paper only forbid the existence of *non-transformer* zero-impedance loops. Further, the new representation allows us to generalize Theorem 1 to systems with purely reactive elements and transformers with off-nominal tap ratios.

## III. Main Results

This section describes our process for fixing and generalizing the main theorem. We first state and prove all necessary lemmas that will be used to prove the main result. We also declare an additional reasonable assumption that allow us to extend the result to systems with general transformer models. We then state the generalized version of the main theorem, which requires a relaxed condition in order to hold. We close this section by proving that the relaxed condition the generalized Theorem 1 holds for power systems with reasonably common structures.

### A. Preliminaries

We start by introducing the following assumption:

**Assumption 2.** *For any series branch $l \in \mathcal{L}$ from node $i$ to node $k$, the admittance matrix associated with just this element can be written as $\mathbf{Y}_l = y_l \mathbf{a}_l \mathbf{a}_l^H \in \mathbb{C}^{|\mathcal{N}| \times |\mathcal{N}|}$, where $\{\mathbf{a}_l\}_i = 1$, $\{\mathbf{a}_l\}_k = -a_l^*$ ($a_l$ is a non-zero complex number) and all other entries of $\mathbf{a}_l$ are zero.*

Transmission lines and transformers (including transformers with off-nominal tap ratios) satisfy Assumption 2. Transmission lines can be modeled as transformers with $a_l = 1$ along with some shunt elements. This permits modeling, for instance, $\Pi$-circuit models of transmission lines. Using Assumption 2, the admittance matrix of the full system is:

$$\mathbf{Y}_{\mathcal{N}} = \sum_{l \in \mathcal{L}} \mathbf{Y}_l + \mathbf{Y}_{\mathcal{T}}. \tag{10}$$

In (10), note that $\mathbf{Y}_{\mathcal{T}}$ does not include the shunt elements in the transformers' $\pi$ circuits as these elements are instead included in $\mathbf{Y}_l$. The sum of the matrices can be rewritten as:

$$\mathbf{Y}_{\mathcal{N}} = \mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{Y}_{\mathcal{L}} \mathbf{A}_{\mathcal{L},\mathcal{N}} + \mathbf{Y}_{\mathcal{T}}, \tag{11}$$

where, in a slight abuse of notation relative to Section II, $\mathbf{A}_{\mathcal{L},\mathcal{N}}$ is the *generalized incidence matrix*, whose $l$-th row is $\{\mathbf{A}_{\mathcal{L},\mathcal{N}}\}_l = \mathbf{a}_l^H$; $\mathbf{Y}_{\mathcal{L}} = \operatorname{diag}(\mathbf{y}_{\mathcal{L}})$ is the diagonal matrix containing the series admittances for each branch; and $\mathbf{Y}_{\mathcal{T}} = \operatorname{diag}(\mathbf{y}_{\mathcal{T}})$ is the diagonal matrix containing the total shunt admittances at each node. This is the default representation for the rest of the paper.

Parallel shunts or branches with the same tap ratio can be reduced to a single branch or shunt by adding the admittances, so we assume that this reduction is always performed:

**Remark 1.** *There are no parallel shunts or parallel branches with the same tap ratio.*

The connectedness condition of the network is evaluated considering its representation with parallel branches reduced. Parallel transformers with different tap ratios cannot be represented as single branch in the form state by Assumption 2, so they are not reduced (each parallel branch individually satisfies Assumption 2, so our results are also applicable to those cases). We next state the rank-nullity theorem as we will use it several times in the paper:

**Rank-nullity theorem ([Theorem 4.4.15] in [28]).** *Let $\mathbf{M} \in \mathbb{C}^{m \times n}$ be an arbitrary matrix, then:*

$$\operatorname{rank}(\mathbf{M}) + \dim(\operatorname{Null}(\mathbf{M})) = n. \tag{12}$$

Now we extend Lemma 1 to generalized incidence matrices:

**Lemma 1 (Extended).** *The rank of the generalized incidence matrix of an arbitrary connected network with $|\mathcal{N}|$ nodes, $\mathbf{A}_{\mathcal{L},\mathcal{N}} \in \mathbb{C}^{|\mathcal{L}| \times |\mathcal{N}|}$, is at least $|\mathcal{N}| - 1$. If $\mathbf{A}_{\mathcal{L},\mathcal{N}}$ is not full column rank, then none of the basis vectors of its null space have null entries.*

**Proof.** Let $\mathcal{S} \subseteq \mathcal{L}$ be a set of branches forming a spanning tree of the network graph (as the network is connected, such a set always exists, see [29]). We can order the branches of $\mathcal{L}$

by numbering all the branches of $\mathcal{S}$ first. Thus we can write $\mathbf{A}_{\mathcal{L},\mathcal{N}}$ in blocks as follows:

$$\mathbf{A}_{\mathcal{L},\mathcal{N}} = \left[ \begin{array}{c} \mathbf{A}_{\mathcal{S},\mathcal{N}} \\ \mathbf{A}_{\mathcal{L}\setminus\mathcal{S},\mathcal{N}} \end{array} \right], \tag{13}$$

where $\mathbf{A}_{\mathcal{S},\mathcal{N}}$ is the generalized incidence matrix of the branches in $\mathcal{S}$ and $\mathbf{A}_{\mathcal{L}\setminus\mathcal{S},\mathcal{N}}$ is the generalized incidence matrix of the remaining branches. For any vector $\mathbf{x}$ in the null space of $\mathbf{A}_{\mathcal{L},\mathcal{N}}$, $\mathbf{x}$ must be orthogonal to all rows of $\mathbf{A}_{\mathcal{L},\mathcal{N}}$:

$$\mathbf{a}_s^H \mathbf{x} = 0, \qquad \forall s \in \mathcal{S}. \tag{14}$$

Take an arbitrary branch $s$ that goes from node $i$ to node $k$, then from (14) we have:

$$\{\mathbf{x}\}_i - a_s \{\mathbf{x}\}_k = 0, \tag{15}$$

where $a_s$ is the tap ratio of branch $s$. We can write:

$$\{\mathbf{x}\}_i = a_s \{\mathbf{x}\}_k, \tag{16a}$$
$$\{\mathbf{x}\}_k = a_s^{-1} \{\mathbf{x}\}_i. \tag{16b}$$

We generalize this result and say that if nodes $i$ and $k$ are connected through a branch $b \in \mathcal{S}$ we can write:

$$\{\mathbf{x}\}_k = a_b^{d(b,\mathcal{S}(i,k))} \{\mathbf{x}\}_i, \tag{17}$$

where $a_b$ is the tap ratio of branch $b$, $\mathcal{S}(i,k) \subseteq \mathcal{S}$ is the (unique) set of branches in $\mathcal{S}$ forming a path from node $i$ to node $k$ (in this case the only member of $\mathcal{S}(i,k)$ is $b$), and $d(b,\mathcal{S}(i,k))$ is a function that returns either 1 or $-1$ depending on the direction of branch $b$ relative to the path defined by $\mathcal{S}(i,k)$ (if branch $b$ goes from node $i$ to node $k$ then $d(b,\mathcal{S}(i,k)) = -1$, otherwise $d(b,\mathcal{S}(i,k)) = 1$). As $\mathcal{S}$ is a spanning tree, there exists a unique path from node 1 to every other node $k \neq 1$. Define $p(i,\mathcal{S}(i,k))$ as a function returning the node in the $i$-th position along the path from node 1 to node $k$ ($p(1,\mathcal{S}(i,k)) = i$ and $p(1+|\mathcal{S}(i,k)|,\mathcal{S}(i,k)) = k$), and let $b(i,\mathcal{S}(i,k)) \in \mathcal{S}(i,k)$ be the branch connecting nodes $p(i,\mathcal{S}(i,k))$ and $p(i+1,\mathcal{S}(i,k))$. Let $D(k) = |\mathcal{S}(1,k)|$. We write $\{\mathbf{x}\}_k$ in terms of $\{\mathbf{x}\}_1$ by chaining (17) for each pair of consecutive nodes in the path between nodes 1 and $k$:

$$1 \xrightarrow{b(1,\mathcal{S}(1,k))} p(2,\mathcal{S}(1,k)) \xrightarrow{b(2,\mathcal{S}(1,k))} \cdots$$
$$\cdots p(D(k),\mathcal{S}(1,k)) \xrightarrow{b(D(k),\mathcal{S}(1,k))} k.$$

We backtrack the chain of equations starting from node $k$ until we reach node 1:

$$\{\mathbf{x}\}_k = a_{b(D(k),\mathcal{S}(1,k))}^{d(b(D(k),\mathcal{S}(1,k)),\mathcal{S}(1,k))} \{\mathbf{x}\}_{p(D(k),\mathcal{S}(1,k))}, \tag{18a}$$
$$\{\mathbf{x}\}_k = a_{b(D(k),\mathcal{S}(1,k))}^{d(b(D(k),\mathcal{S}(1,k)),\mathcal{S}(1,k))} \cdot a_{b(D(k)-1,\mathcal{S}(1,k))}^{d(b(D(k)-1,\mathcal{S}(1,k)),\mathcal{S}(1,k))} \cdot \{\mathbf{x}\}_{p(D(k)-1,\mathcal{S}(1,k))}, \tag{18b}$$
$$\vdots$$
$$\{\mathbf{x}\}_k = \{\mathbf{x}\}_1 \prod_{i=1}^{D(k)} a_{b(i,\mathcal{S}(1,k))}^{d(b(i,\mathcal{S}(1,k)),\mathcal{S}(1,k))}, \tag{18c}$$

or written more succinctly (as the product is commutative):

$$\{\mathbf{x}\}_k = \{\mathbf{x}\}_1 \prod_{s\in\mathcal{S}(1,k)} a_s^{d(s,\mathcal{S}(1,k))}. \tag{19}$$

Let $\{\mathbf{x}\}_1 = \alpha$, for an arbitrary $\alpha$. We can then write $\mathbf{x}$ as:

$$\mathbf{x} = \alpha \mathbf{v}, \tag{20a}$$
$$\{\mathbf{v}\}_1 = 1, \tag{20b}$$
$$\{\mathbf{v}\}_k = \prod_{s\in\mathcal{S}(1,k)} a_s^{d(s,\mathcal{S}(1,k))}, \qquad k = 2,\ldots,|\mathcal{N}|. \tag{20c}$$

Since $\mathbf{x}$ has only one free parameter ($\alpha$), the rank-nullity theorem implies that $\operatorname{rank}(\mathbf{A}_{\mathcal{S},\mathcal{N}}) = |\mathcal{N}| - 1$. Furthermore, as $a_l \neq 0$ for all $l \in \mathcal{L}$, then all entries of $\mathbf{v}$ are non-zero.

Since $\mathbf{x}$ must also be orthogonal to all rows of $\mathbf{A}_{\mathcal{L}\setminus\mathcal{S},\mathcal{N}}$, we have, for each row of $\mathbf{A}_{\mathcal{L}\setminus\mathcal{S},\mathcal{N}}$, the following equation:

$$\alpha \left( \prod_{s\in\mathcal{S}(1,i)} a_s^{d(s,\mathcal{S}(1,i))} - a_l \prod_{s\in\mathcal{S}(1,k)} a_s^{d(s,\mathcal{S}(1,k))} \right) = 0, \tag{21}$$

for any branch $l \in \mathcal{L} \setminus \mathcal{S}$ going from node $i$ to $k$. If the term inside the parentheses is null for all rows, then the (directed) product of tap ratios $a_l$ across branches in a cycle is 1, for all cycles. In that case, $\alpha$ is a free parameter and $\operatorname{rank}(\mathbf{A}_{\mathcal{L},\mathcal{N}}) = |\mathcal{N}| - 1$. Otherwise $\alpha = 0$, and so $\operatorname{rank}(\mathbf{A}_{\mathcal{L},\mathcal{N}}) = |\mathcal{N}|$ (i.e., $\mathbf{A}_{\mathcal{L},\mathcal{N}}$ is full column rank). □

We also require some new lemmas. We start with Lemma 5, which is a simple extension of Lemma 1 from [22]:

**Lemma 5.** *Consider a matrix* $\mathbf{Y} = \mathbf{G} + j\mathbf{B} \in \mathbb{C}^{n\times n}$ *with* $\mathbf{G}, \mathbf{B} \in \mathbb{R}^{n\times n}$. *Suppose* $\mathbf{G} \succeq \mathbf{0}$, *then* $\operatorname{Null}(\mathbf{Y}) \subseteq \operatorname{Null}(\operatorname{Sym}(\mathbf{G}))$ *and* $\operatorname{rank}(\operatorname{Sym}(\mathbf{G})) \leq \operatorname{rank}(\mathbf{Y})$.

**Proof.** Consider a vector $\mathbf{x} \in \mathbb{C}^n$ in the null space of $\mathbf{Y}$. We can write $\mathbf{x}$ in rectangular form as $\mathbf{x} = \mathbf{x}_R + j\mathbf{x}_I$ with $\mathbf{x}_R, \mathbf{x}_I \in \mathbb{R}^n$. Using the definition of the null space, we have:

$$0 = \operatorname{Re}\left(\mathbf{x}^H \mathbf{Y} \mathbf{x}\right), \tag{22a}$$
$$0 = \mathbf{x}_R^T \mathbf{G} \mathbf{x}_R + \mathbf{x}_I^T \mathbf{G} \mathbf{x}_I + \mathbf{x}_I^T \mathbf{B} \mathbf{x}_R - \mathbf{x}_R^T \mathbf{B} \mathbf{x}_I. \tag{22b}$$

The quadratic terms are real, so they only depend on the symmetric part of the matrices[3]:

$$0 = \mathbf{x}_R^T \operatorname{Sym}(\mathbf{G}) \mathbf{x}_R + \mathbf{x}_I^T \operatorname{Sym}(\mathbf{G}) \mathbf{x}_I + \mathbf{x}_I^T \operatorname{Sym}(\mathbf{B}) \mathbf{x}_R - \mathbf{x}_R^T \operatorname{Sym}(\mathbf{B}) \mathbf{x}_I, \tag{23a}$$
$$0 = \mathbf{x}_R^T \operatorname{Sym}(\mathbf{G}) \mathbf{x}_R + \mathbf{x}_I^T \operatorname{Sym}(\mathbf{G}) \mathbf{x}_I + \mathbf{x}_I^T \operatorname{Sym}(\mathbf{B}) \mathbf{x}_R - \mathbf{x}_I^T \operatorname{Sym}(\mathbf{B}) \mathbf{x}_R, \tag{23b}$$
$$0 = \mathbf{x}_R^T \operatorname{Sym}(\mathbf{G}) \mathbf{x}_R + \mathbf{x}_I^T \operatorname{Sym}(\mathbf{G}) \mathbf{x}_I. \tag{23c}$$

As $\operatorname{Sym}(\mathbf{G}) \succeq \mathbf{0}$, both terms must be non-negative. Equality only holds if both terms are zero, and hence both $\mathbf{x}_R$ and $\mathbf{x}_I$ belong to the null space of $\operatorname{Sym}(\mathbf{G})$. Therefore if $\mathbf{Y}\mathbf{x} = \mathbf{0}$ then $\operatorname{Sym}(\mathbf{G})\mathbf{x} = \mathbf{0}$, so $\operatorname{Null}(\mathbf{Y}) \subseteq \operatorname{Null}(\operatorname{Sym}(\mathbf{G}))$. We apply the rank-nullity theorem to conclude the proof. □

**Lemma 6.** *Let* $\mathbf{A} \succeq \mathbf{0}$ *and* $\mathbf{B} \succeq \mathbf{0}$ *be square matrices in* $\mathbb{R}^{n\times n}$. *Then the following equations hold:*

$$\mathbf{A} + \mathbf{B} \succeq \mathbf{0}, \tag{24}$$
$$\operatorname{Null}(\operatorname{Sym}(\mathbf{A}+\mathbf{B})) = \operatorname{Null}(\operatorname{Sym}(\mathbf{A})) \cap \operatorname{Null}(\operatorname{Sym}(\mathbf{B})), \tag{25}$$
$$\operatorname{rank}(\operatorname{Sym}(\mathbf{A})), \operatorname{rank}(\operatorname{Sym}(\mathbf{B})) \leq$$

[3]For any real (possibly non-symmetric) matrix $\mathbf{A}$ and appropriately sized vector $\mathbf{x}$, the following relationships hold: $\mathbf{x}^T\mathbf{A}\mathbf{x} = (\mathbf{x}^T\mathbf{A}\mathbf{x})^T = \mathbf{x}^T\mathbf{A}^T\mathbf{x} = \mathbf{x}^T(\mathbf{A}/2 + \mathbf{A}^T/2)\mathbf{x} = \mathbf{x}^T\operatorname{Sym}(\mathbf{A})\mathbf{x}$.

$$\operatorname{rank}\left(\operatorname{Sym}\left(\mathbf{A}+\mathbf{B}\right)\right). \quad (26)$$

**Proof.** Let us calculate the quadratic form of $\mathbf{A}+\mathbf{B}$:

$$\mathbf{x}^T\left(\mathbf{A}+\mathbf{B}\right)\mathbf{x} = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{B}\mathbf{x}. \quad (27)$$

As both $\mathbf{A}$ and $\mathbf{B}$ are positive semi-definite, then $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ and $\mathbf{x}^T\mathbf{B}\mathbf{x} \geq 0$, thus $\mathbf{x}^T\left(\mathbf{A}+\mathbf{B}\right)\mathbf{x} \geq 0$. Then by definition $\mathbf{A}+\mathbf{B} \succeq \mathbf{0}$. Now let $\mathbf{z}$ be a vector in the null space of $\operatorname{Sym}\left(\mathbf{A}+\mathbf{B}\right)$. This implies that:

$$\mathbf{z}^T\operatorname{Sym}\left(\mathbf{A}+\mathbf{B}\right)\mathbf{z} = 0, \quad (28a)$$

$$\mathbf{z}^T\operatorname{Sym}\left(\mathbf{A}\right)\mathbf{z} + \mathbf{z}^T\operatorname{Sym}\left(\mathbf{B}\right)\mathbf{z} = 0. \quad (28b)$$

As both terms are non-negative:

$$\mathbf{z}^T\operatorname{Sym}\left(\mathbf{A}\right)\mathbf{z} = 0, \qquad \mathbf{z}^T\operatorname{Sym}\left(\mathbf{B}\right)\mathbf{z} = 0, \quad (29)$$

and hence $\mathbf{z}$ belongs to the null spaces of both $\operatorname{Sym}\left(\mathbf{A}\right)$ and $\operatorname{Sym}\left(\mathbf{B}\right)$. The converse can be proved trivially by reversing the steps, so $\operatorname{Null}\left(\operatorname{Sym}\left(\mathbf{A}+\mathbf{B}\right)\right) = \operatorname{Null}\left(\operatorname{Sym}\left(\mathbf{A}\right)\right) \cap \operatorname{Null}\left(\operatorname{Sym}\left(\mathbf{B}\right)\right)$. We then apply the rank-nullity theorem to conclude the proof of Lemma 6. □

**Lemma 7.** *Let $\mathbf{A} \succ \mathbf{0}$ and $\mathbf{B} \succeq \mathbf{0}$ be square matrices in $\mathbb{R}^{n\times n}$. Then $\mathbf{A}+\mathbf{B} \succ \mathbf{0}$.*

**Proof.** Notice that $\mathbf{A} \succ \mathbf{0}$ implies that $\operatorname{Sym}\left(\mathbf{A}\right) \succ \mathbf{0}$. Apply Lemma 6 and Lemma 5 with $\mathbf{Y} = \mathbf{A}+\mathbf{B}$. □

**Lemma 8.** *Let $\mathbf{M} \succeq \mathbf{0}$, $\mathbf{M} \in \mathbb{C}^{n\times n}$ be a Hermitian matrix, then $\operatorname{Re}\left(\mathbf{M}\right) \succeq \mathbf{0}$, $\operatorname{Null}\left(\operatorname{Re}\left(\mathbf{M}\right)\right) = \operatorname{Null}\left(\mathbf{M}\right)$, and $\operatorname{rank}\left(\operatorname{Re}\left(\mathbf{M}\right)\right) = \operatorname{rank}\left(\mathbf{M}\right)$.*

**Proof.** As $\mathbf{M}$ is Hermitian and positive-semidefinite, it can be factored as $\mathbf{M} = \mathbf{A}^H\mathbf{A}$. Now we expand $\operatorname{Re}\left(\mathbf{M}\right)$:

$$\operatorname{Re}\left(\mathbf{M}\right) = \operatorname{Re}\left(\mathbf{A}\right)^T\operatorname{Re}\left(\mathbf{A}\right) + \operatorname{Im}\left(\mathbf{A}\right)^T\operatorname{Im}\left(\mathbf{A}\right) \succeq \mathbf{0}. \quad (30)$$

Note that $\operatorname{Re}\left(\mathbf{M}\right)$ is symmetric and each term in the right hand side is positive semidefinite. Applying Lemmas 6 and 3:

$$\operatorname{Null}\left(\operatorname{Re}\left(\mathbf{M}\right)\right) = \operatorname{Null}\left(\operatorname{Re}\left(\mathbf{A}\right)\right) \cap \operatorname{Null}\left(\operatorname{Im}\left(\mathbf{A}\right)\right), \quad (31a)$$

$$\operatorname{Null}\left(\operatorname{Re}\left(\mathbf{M}\right)\right) \subseteq \operatorname{Null}\left(\mathbf{A}\right) = \operatorname{Null}\left(\mathbf{M}\right). \quad (31b)$$

Recall that $\operatorname{Re}\left(\mathbf{M}\right) \succeq \mathbf{0}$. Applying Lemma 5 yields:

$$\operatorname{Null}\left(\mathbf{M}\right) \subseteq \operatorname{Null}\left(\operatorname{Re}\left(\mathbf{M}\right)\right). \quad (32)$$

Since $\operatorname{Null}\left(\operatorname{Re}\left(\mathbf{M}\right)\right) \subseteq \operatorname{Null}\left(\mathbf{M}\right)$ and $\operatorname{Null}\left(\mathbf{M}\right) \subseteq \operatorname{Null}\left(\operatorname{Re}\left(\mathbf{M}\right)\right)$, we have that $\operatorname{Null}\left(\operatorname{Re}\left(\mathbf{M}\right)\right) = \operatorname{Null}\left(\mathbf{M}\right)$. The claim follows after applying the rank-nullity theorem. □

### B. Admittance Matrix Invertibility Theorem

We now have the tools to present the amended version of Theorem 1 and prove its validity under various conditions.

**Theorem 1 (Generalized).** *Let the graph $(\mathcal{N},\mathcal{L})$ define a connected network and let $\mathcal{T}$ define the shunts of the network. If Assumptions 1 and 2 hold and $\operatorname{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) \subseteq \operatorname{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right)$, then:*

$$\operatorname{rank}\left(\mathbf{Y}_{\mathcal{N}}\right) = \begin{cases} \operatorname{rank}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right) & \text{if } \mathcal{T} = \emptyset, \\ |\mathcal{N}| & \text{otherwise.} \end{cases} \quad (33)$$

**Proof.** First assume that $\mathcal{T} = \emptyset$, then

$$\mathbf{Y}_{\mathcal{N}} = \mathbf{A}^H_{\mathcal{L},\mathcal{N}}\mathbf{Y}_{\mathcal{L}}\mathbf{A}_{\mathcal{L},\mathcal{N}}. \quad (34)$$

Clearly, any vector $\mathbf{w}$ such that $\mathbf{A}_{\mathcal{L},\mathcal{N}}\mathbf{w} = \mathbf{0}$ also satisfies $\mathbf{Y}_{\mathcal{N}}\mathbf{w} = \mathbf{0}$. This means that

$$\operatorname{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) \supseteq \operatorname{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right), \quad (35)$$

so

$$\operatorname{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) = \operatorname{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right). \quad (36)$$

Applying the rank-nullity theorem, we conclude that (33) holds for this case. Next assume that $\mathcal{T} \neq \emptyset$, then

$$\mathbf{Y}_{\mathcal{N}} = \mathbf{A}^H_{\mathcal{L},\mathcal{N}}\mathbf{Y}_{\mathcal{L}}\mathbf{A}_{\mathcal{L},\mathcal{N}} + \mathbf{Y}_{\mathcal{T}}. \quad (37)$$

If $\mathbf{A}_{\mathcal{L},\mathcal{N}}$ is full rank, then the fact that $\operatorname{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) \subseteq \operatorname{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right)$ and the rank-nullity theorem imply that $\mathbf{Y}_{\mathcal{N}}$ is invertible, meaning that (33) holds. If $\mathbf{A}_{\mathcal{L},\mathcal{N}}$ is not full rank, then we take an arbitrary vector $\mathbf{x} \in \operatorname{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right)$. From Lemma 1 (extended) we have that $\mathbf{x} = \alpha\mathbf{u}$ where $\mathbf{u}$ is a vector with no null entries. We now calculate $\mathbf{Y}_{\mathcal{N}}\mathbf{x}$:

$$\mathbf{Y}_{\mathcal{N}}\mathbf{x} = \alpha\mathbf{Y}_{\mathcal{N}}\mathbf{u}, \quad (38a)$$

$$\mathbf{Y}_{\mathcal{N}}\mathbf{x} = \alpha\left(\mathbf{A}^T_{\mathcal{L},\mathcal{N}}\mathbf{Y}_{\mathcal{L}}\mathbf{A}_{\mathcal{L},\mathcal{N}} + \mathbf{Y}_{\mathcal{T}}\right)\mathbf{u}, \quad (38b)$$

$$\mathbf{Y}_{\mathcal{N}}\mathbf{x} = \alpha\left(\mathbf{A}^T_{\mathcal{L},\mathcal{N}}\mathbf{Y}_{\mathcal{L}}\mathbf{A}_{\mathcal{L},\mathcal{N}}\mathbf{u} + \mathbf{Y}_{\mathcal{T}}\mathbf{u}\right), \quad (38c)$$

$$\mathbf{Y}_{\mathcal{N}}\mathbf{x} = \alpha\mathbf{Y}_{\mathcal{T}}\mathbf{u}. \quad (38d)$$

Since $\mathbf{Y}_{\mathcal{T}} = \operatorname{diag}\left(\mathbf{y}_{\mathcal{T}}\right)$, $\mathbf{y}_{\mathcal{T}} \neq \mathbf{0}$, and $\mathbf{u}$ has no null entries, we observe that $\mathbf{Y}_{\mathcal{N}}\mathbf{u}$ cannot be $\mathbf{0}$ unless $\alpha = 0$. This means the only vector in the null space of $\mathbf{A}_{\mathcal{L},\mathcal{N}}$ that is also in the null space of $\mathbf{Y}_{\mathcal{N}}$ is $\mathbf{0}$. This implies that $\mathbf{Y}_{\mathcal{N}}$ is full-rank, so (33) holds. □

We have recovered the results of [20], at the cost of requiring that the condition $\operatorname{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) \subseteq \operatorname{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right)$ holds. With the next theorem, we will show that the problem of verifying the condition for the whole network can be reduced to multiple smaller problems of the same nature.

**Theorem 2.** *Let the graph $(\mathcal{N},\mathcal{L})$ define a connected network and let $\mathcal{T}$ define the shunts of the network. Assumptions 1 and 2 hold, $\operatorname{Re}\left(y_l\right) \geq 0$ for all $y_l$ of $\mathbf{y}_{\mathcal{L}}$ and $\operatorname{Re}\left(y_t\right) \geq 0$ for all $y_t$ of $\mathbf{y}_{\mathcal{T}}$. Let $\mathcal{G}$ be the set containing the ground node, let $\mathcal{L}' \subseteq \mathcal{L}$ be the set of purely reactive branches, let $\mathcal{N}' \subseteq \mathcal{N}$ be the set of non-isolated nodes with respect to $\mathcal{L}'$, and let $\mathcal{T}' \subseteq \mathcal{T}$ be the set of purely reactive shunts that are connected to some node in $\mathcal{N}'$. Let the reactive network $(\mathcal{N}',\mathcal{L}')$ have $K$ connected components ($K$ may be 0), indexed as $(\mathcal{N}'(k)\cup\mathcal{G},\mathcal{L}'(k)\cup\mathcal{T}'(k))$ for $k = 1,\ldots,K$, and let $\mathcal{T}'(k) \subseteq \mathcal{T}'$ be the set of shunts of component $k$. If the admittance matrices of all components, $\mathbf{Y}_{\mathcal{N}'(k)}$, satisfy that $\operatorname{Null}\left(\mathbf{Y}_{\mathcal{N}'(k)}\right) \subseteq \operatorname{Null}\left(\mathbf{A}_{\mathcal{L}'(k),\mathcal{N}'(k)}\right)$ (for $K = 0$ this is vacuously true), then:*

$$\operatorname{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) \subseteq \operatorname{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right).$$

**Proof.** For convenience we establish the following conventions:

- The incidence matrix of an empty branch set is $\mathbf{A}_{\emptyset,\mathcal{N}} = \mathbf{0} \in \mathbb{C}^{1\times|\mathcal{N}|}$.
- The series branch admittance matrix of an empty branch set is $\mathbf{Y}_{\emptyset} = 0$.
- The shunt admittance matrix of an empty shunt set is $\mathbf{Y}_{\emptyset} = \mathbf{0} \in \mathbb{C}^{|\mathcal{N}|\times|\mathcal{N}|}$.

We rely in context to discern between branch and shunt admittance matrices. Assume that both $\mathcal{L}'$ and $\mathcal{L}\setminus\mathcal{L}'$ are non-empty. We can write $\mathbf{A}_{\mathcal{L},\mathcal{N}}$ in block form as follows:

$$\mathbf{A}_{\mathcal{L},\mathcal{N}} = \begin{bmatrix} \mathbf{A}_{\mathcal{L}',\mathcal{N}} \\ \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}} \end{bmatrix}. \tag{39}$$

Writing $\mathbf{Y}_{\mathcal{N}}$ in terms of the block matrices yields:

$$\begin{aligned} \mathbf{Y}_{\mathcal{N}} &= \mathbf{A}_{\mathcal{L}',\mathcal{N}}^H \mathbf{Y}_{\mathcal{L}'} \mathbf{A}_{\mathcal{L}',\mathcal{N}} + \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}^H \mathbf{Y}_{\mathcal{L}\setminus\mathcal{L}'} \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}} \\ &\quad + \mathbf{Y}_{\mathcal{T}'} + \mathbf{Y}_{\mathcal{T}\setminus\mathcal{T}'}, \qquad (40a) \\ \mathbf{Y}_{\mathcal{N}} &= \mathbf{A}_{\mathcal{L}',\mathcal{N}}^H \mathbf{Y}_{\mathcal{L}'} \mathbf{A}_{\mathcal{L}',\mathcal{N}} + \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}^H \mathbf{Y}_{\mathcal{L}\setminus\mathcal{L}'} \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}} \\ &\quad + \mathbf{Y}_{\mathcal{T}'} + \mathbf{G}_{\mathcal{T}\setminus\mathcal{T}'} + j\mathbf{B}_{\mathcal{T}\setminus\mathcal{T}'}. \qquad (40b) \end{aligned}$$

Next we compute $\mathrm{Sym}\left(\mathbf{G}_{\mathcal{N}}\right)$ as follows:

$$\begin{aligned} \mathbf{Y}_{\mathcal{N}} &= \mathbf{G}_{\mathcal{N}} + j\mathbf{B}_{\mathcal{N}}, \qquad (41a) \\ \mathbf{G}_{\mathcal{N}} &= \mathrm{Re}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{Y}_{\mathcal{L}} \mathbf{A}_{\mathcal{L},\mathcal{N}} + \mathbf{Y}_{\mathcal{T}}\right), \qquad (41b) \end{aligned}$$

let $\mathbf{A}_R$ and $\mathbf{A}_I$ denote the real and imaginary parts of $\mathbf{A}_{\mathcal{L},\mathcal{N}}$, then

$$\begin{aligned} \mathbf{G}_{\mathcal{N}} &= \mathbf{A}_R^T \mathbf{G}_{\mathcal{L}} \mathbf{A}_R + \mathbf{A}_I^T \mathbf{G}_{\mathcal{L}} \mathbf{A}_I + \mathbf{A}_I^T \mathbf{B}_{\mathcal{L}} \mathbf{A}_R \\ &\quad - \mathbf{A}_R^T \mathbf{B}_{\mathcal{L}} \mathbf{A}_I + \mathbf{G}_{\mathcal{T}}, \qquad (42a) \\ \mathrm{Sym}\left(\mathbf{G}_{\mathcal{N}}\right) &= \mathbf{A}_R^T \mathbf{G}_{\mathcal{L}} \mathbf{A}_R + \mathbf{A}_I^T \mathbf{G}_{\mathcal{L}} \mathbf{A}_I + \mathbf{G}_{\mathcal{T}}. \qquad (42b) \end{aligned}$$

Notice that:

$$\mathrm{Re}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{G}_{\mathcal{L}} \mathbf{A}_{\mathcal{L},\mathcal{N}}\right) = \mathrm{Re}\left(\left(\mathbf{A}_R + j\mathbf{A}_I\right)^H \cdot \mathbf{G}_{\mathcal{L}} \cdot \left(\mathbf{A}_R + j\mathbf{A}_I\right)\right), \tag{43a}$$

$$\mathrm{Re}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{G}_{\mathcal{L}} \mathbf{A}_{\mathcal{L},\mathcal{N}}\right) = \mathrm{Re}\left(\left(\mathbf{A}_R^T - j\mathbf{A}_I^T\right) \cdot \mathbf{G}_{\mathcal{L}} \cdot \left(\mathbf{A}_R + j\mathbf{A}_I\right)\right), \tag{43b}$$

$$\mathrm{Re}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{G}_{\mathcal{L}} \mathbf{A}_{\mathcal{L},\mathcal{N}}\right) = \mathbf{A}_R^T \mathbf{G}_{\mathcal{L}} \mathbf{A}_R + \mathbf{A}_I^T \mathbf{G}_{\mathcal{L}} \mathbf{A}_I, \tag{43c}$$

where $\mathbf{A}_R^T \mathbf{G}_{\mathcal{L}} \mathbf{A}_R \succeq \mathbf{0}$ and $\mathbf{A}_R^T \mathbf{G}_{\mathcal{L}} \mathbf{A}_R \succeq \mathbf{0}$ (because all conductances are non-negative), so from Lemma 6 we have that $\mathrm{Re}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{G}_{\mathcal{L}} \mathbf{A}_{\mathcal{L},\mathcal{N}}\right) \succeq \mathbf{0}$. Replacing in (42b):

$$\mathrm{Sym}\left(\mathbf{G}_{\mathcal{N}}\right) = \mathrm{Re}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{G}_{\mathcal{L}} \mathbf{A}_{\mathcal{L},\mathcal{N}}\right) + \mathbf{G}_{\mathcal{T}}. \tag{44}$$

From the definition of $\mathcal{L}'$, we have

$$\begin{aligned} \mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{G}_{\mathcal{L}} \mathbf{A}_{\mathcal{L},\mathcal{N}} &= \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}^H \mathbf{G}_{\mathcal{L}\setminus\mathcal{L}'} \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}, \qquad (45a) \\ \mathrm{Re}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{G}_{\mathcal{L}} \mathbf{A}_{\mathcal{L},\mathcal{N}}\right) &= \mathrm{Re}\left(\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}^H \mathbf{G}_{\mathcal{L}\setminus\mathcal{L}'} \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\right), \qquad (45b) \\ \mathbf{G}_{\mathcal{T}} &= \mathbf{G}_{\mathcal{T}\setminus\mathcal{T}'}. \qquad (45c) \end{aligned}$$

Replacing in (44) yields:

$$\mathrm{Sym}\left(\mathbf{G}_{\mathcal{N}}\right) = \mathrm{Re}\left(\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}^H \mathbf{G}_{\mathcal{L}\setminus\mathcal{L}'} \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\right) + \mathbf{G}_{\mathcal{T}\setminus\mathcal{T}'}. \tag{46}$$

As all conductances are non-negative, we know that $\mathbf{G}_{\mathcal{T}\setminus\mathcal{T}'} \succeq \mathbf{0}$. Applying Lemma 6, we conclude that $\mathrm{Sym}\left(\mathbf{G}_{\mathcal{N}}\right) \succeq \mathbf{0}$ and its null space is the intersection of the null spaces of $\mathrm{Re}\left(\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}^H \mathbf{G}_{\mathcal{L}\setminus\mathcal{L}'} \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\right)$ and $\mathbf{G}_{\mathcal{T}\setminus\mathcal{T}'}$. As $\mathrm{Sym}\left(\mathbf{G}_{\mathcal{N}}\right) \succeq \mathbf{0}$, then $\mathbf{G}_{\mathcal{N}} \succeq \mathbf{0}$ as well, so we can apply Lemma 5:

$$\begin{aligned} \mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) &\subseteq \mathrm{Null}\left(\mathrm{Sym}\left(\mathbf{G}_{\mathcal{N}}\right)\right), \qquad (47a) \\ \mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) &\subseteq \mathrm{Null}\left(\mathrm{Re}\left(\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}^H \mathbf{G}_{\mathcal{L}\setminus\mathcal{L}'} \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\right)\right) \\ &\quad \cap \mathrm{Null}\left(\mathbf{G}_{\mathcal{T}\setminus\mathcal{T}'}\right). \qquad (47b) \end{aligned}$$

Applying Lemma 8 yields:

$$\begin{aligned} \mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) \subseteq \mathrm{Null}&\left(\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}^H \mathbf{G}_{\mathcal{L}\setminus\mathcal{L}'} \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\right) \\ &\cap \mathrm{Null}\left(\mathbf{G}_{\mathcal{T}\setminus\mathcal{T}'}\right). \qquad (48) \end{aligned}$$

From the way $\mathcal{L}'$ is defined, we know that $\mathbf{G}_{\mathcal{L}\setminus\mathcal{L}'} \succ \mathbf{0}$ so we can factor $\mathbf{G}_{\mathcal{L}\setminus\mathcal{L}'}$ as $\mathbf{G}_{\mathcal{L}\setminus\mathcal{L}'} = \mathbf{D}^H\mathbf{D}$, $\mathbf{D} \succ \mathbf{0}$. Next, we apply Lemma 3 and Lemma 4 to show that

$$\begin{aligned} &\mathrm{Null}\left(\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}^H \mathbf{G}_{\mathcal{L}\setminus\mathcal{L}'} \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\right) = \\ &\qquad \mathrm{Null}\left(\left(\mathbf{D}\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\right)^H \left(\mathbf{D}\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\right)\right), \qquad (49a) \\ &\mathrm{Null}\left(\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}^H \mathbf{G}_{\mathcal{L}\setminus\mathcal{L}'} \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\right) = \mathrm{Null}\left(\mathbf{D}\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\right), \qquad (49b) \\ &\mathrm{Null}\left(\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}^H \mathbf{G}_{\mathcal{L}\setminus\mathcal{L}'} \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\right) = \mathrm{Null}\left(\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\right). \qquad (49c) \end{aligned}$$

Substituting into (47) yields:

$$\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) \subseteq \mathrm{Null}\left(\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\right) \cap \mathrm{Null}\left(\mathbf{G}_{\mathcal{T}\setminus\mathcal{T}'}\right). \tag{50}$$

With our established conventions, we note that (50) holds even if $\mathcal{L}\setminus\mathcal{L}'$ is empty, so from now on we drop such assumption and only assume that $\mathcal{L}' \neq \emptyset$. Let $\mathbf{v} \in \mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right)$, then $\mathbf{0} = \mathbf{Y}_{\mathcal{N}}\mathbf{v}$. From (40b) we have that

$$\begin{aligned} \mathbf{0} &= \mathbf{A}_{\mathcal{L}',\mathcal{N}}^H \mathbf{Y}_{\mathcal{L}'} \mathbf{A}_{\mathcal{L}',\mathcal{N}}\mathbf{v} + \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}^H \mathbf{Y}_{\mathcal{L}\setminus\mathcal{L}'} \mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\mathbf{v} + \mathbf{Y}_{\mathcal{T}'}\mathbf{v} \\ &\quad + \mathbf{G}_{\mathcal{T}\setminus\mathcal{T}'}\mathbf{v} + j\mathbf{B}_{\mathcal{T}\setminus\mathcal{T}'}\mathbf{v}. \qquad (51) \end{aligned}$$

From (50), we conclude that $\mathbf{G}_{\mathcal{T}\setminus\mathcal{T}'}\mathbf{v} = \mathbf{0}$. Both $\mathbf{G}_{\mathcal{T}\setminus\mathcal{T}'}$ and $\mathbf{B}_{\mathcal{T}\setminus\mathcal{T}'}$ are diagonal, and the position of the null columns of $\mathbf{G}_{\mathcal{T}\setminus\mathcal{T}'}$ also correspond to null columns of $\mathbf{B}_{\mathcal{T}\setminus\mathcal{T}'}$. We conclude that $\mathrm{Null}\left(\mathbf{G}_{\mathcal{T}\setminus\mathcal{T}'}\right) \subseteq \mathrm{Null}\left(\mathbf{B}_{\mathcal{T}\setminus\mathcal{T}'}\right)$ and so $\mathbf{B}_{\mathcal{T}\setminus\mathcal{T}'}\mathbf{v} = \mathbf{0}$. We also have from (50) that $\mathbf{A}_{\mathcal{L}\setminus\mathcal{L}',\mathcal{N}}\mathbf{v} = \mathbf{0}$. Removing these terms, the equation becomes:

$$\mathbf{0} = \mathbf{A}_{\mathcal{L}',\mathcal{N}}^H \mathbf{Y}_{\mathcal{L}'} \mathbf{A}_{\mathcal{L}',\mathcal{N}}\mathbf{v} + \mathbf{Y}_{\mathcal{T}'}\mathbf{v}. \tag{52}$$

As we assumed that $\mathcal{L}'$ is non-empty then $K \geq 1$. We assume, without loss of generality, that the nodes of $\mathcal{N}$ and $\mathcal{N}'$ are sorted such that we can write:

$$\mathbf{A}_{\mathcal{L}',\mathcal{N}} = \begin{bmatrix} \mathbf{A}_{\mathcal{L}'(1),\mathcal{N}} \\ \vdots \\ \mathbf{A}_{\mathcal{L}'(K),\mathcal{N}} \end{bmatrix}, \tag{53a}$$

$$\begin{aligned} \mathbf{A}_{\mathcal{L}'(k),\mathcal{N}} = \big[&\mathbf{0}_{|\mathcal{L}'(k)|\times|\mathcal{N}'(1)|}, \ldots, \mathbf{A}_{\mathcal{L}'(k),\mathcal{N}'(k)}, \ldots, \\ &\mathbf{0}_{|\mathcal{L}'(k)|\times|\mathcal{N}'(K)|}, \mathbf{0}_{|\mathcal{L}'(k)|\times|\mathcal{N}\setminus\mathcal{N}'|}\big], \qquad (53b) \end{aligned}$$

$$\mathbf{A}_{\mathcal{L}',\mathcal{N}} = \begin{bmatrix} \mathbf{A}_{\mathcal{L}'(1),\mathcal{N}'(1)} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{A}_{\mathcal{L}'(K),\mathcal{N}'(K)} & \mathbf{0} \end{bmatrix}. \tag{53c}$$

Similarly:

$$\mathbf{Y}_{\mathcal{T}'} = \begin{bmatrix} \mathbf{Y}'_{\mathcal{T}'(1)} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{Y}'_{\mathcal{T}'(K)} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{54a}$$

$$\mathbf{Y}_{\mathcal{L}'} = \begin{bmatrix} \mathbf{Y}_{\mathcal{L}'(1)} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{Y}_{\mathcal{L}'(K)} \end{bmatrix}, \tag{54b}$$

where $\mathbf{Y}_{\mathcal{L}'(k)}$ has size $|\mathcal{L}'(k)| \times |\mathcal{L}'(k)|$ and $\mathbf{Y}'_{\mathcal{T}'(k)}$ has size $|\mathcal{N}'(k)| \times |\mathcal{N}'(k)|$. Notice that the admittance matrix of the network $(\mathcal{N}'(k) \cup \mathcal{G}, \mathcal{L}'(k) \cup \mathcal{T}'(k))$, using the node in $\mathcal{G}$ as ground, is

$$\mathbf{Y}_{\mathcal{N}'(k)} = \mathbf{A}^H_{\mathcal{L}'(k),\mathcal{N}'(k)} \mathbf{Y}_{\mathcal{L}'(k)} \mathbf{A}_{\mathcal{L}'(k),\mathcal{N}'(k)} + \mathbf{Y}'_{\mathcal{T}'(k)}. \quad (55)$$

Replacing (53), (54), and (55) in (52), we have

$$\mathbf{0} = \begin{bmatrix} \mathbf{Y}_{\mathcal{N}'(1)} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{Y}_{\mathcal{N}'(K)} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{v}. \quad (56)$$

Taking only the entries associated with nodes of $\mathcal{N}'(k)$ yields

$$\mathbf{0} = \mathbf{R}_k \mathbf{v}, \quad (57)$$

where

$$\mathbf{R}_k = \big[\mathbf{0}_{|\mathcal{N}'(k)| \times |\mathcal{N}'(1)|}, \ldots, \mathbf{Y}_{\mathcal{N}'(k)}, \ldots, \mathbf{0}_{|\mathcal{N}'(k)| \times |\mathcal{N}'(K)|}, \mathbf{0}_{|\mathcal{L}'(k)| \times |\mathcal{N} \setminus \mathcal{N}'|}\big]. \quad (58)$$

Then, by definition:

$$\mathbf{v} \in \mathrm{Null}(\mathbf{R}_k), \quad \forall k = 1, \ldots, K, \quad (59a)$$
$$\mathbf{v} \in \cap_{k=1}^K \mathrm{Null}(\mathbf{R}_k). \quad (59b)$$

The null space of $\mathbf{R}_k$ can be computed directly as the following Cartesian product:

$$\mathrm{Null}(\mathbf{R}_k) = \prod_{i=1}^{k-1} \mathbb{R}^{|\mathcal{N}'(i)|} \times \mathrm{Null}(\mathbf{Y}_{\mathcal{N}'(k)}) \times \prod_{i=k+1}^{K} \mathbb{R}^{|\mathcal{N}'(i)|} \times \mathbb{R}^{|\mathcal{N} \setminus \mathcal{N}'|} \quad (60)$$

From the statement of Theorem 2, we have

$$\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}'(k)}\right) \subseteq \mathrm{Null}\left(\mathbf{A}_{\mathcal{L}'(k),\mathcal{N}'(k)}\right). \quad (61)$$

Replacing this in (60) yields

$$\mathrm{Null}(\mathbf{R}_k) \subseteq \prod_{i=1}^{k-1} \mathbb{R}^{|\mathcal{N}'(i)|} \times \mathrm{Null}\left(\mathbf{A}_{\mathcal{L}'(k),\mathcal{N}'(k)}\right) \times \prod_{i=k+1}^{K} \mathbb{R}^{|\mathcal{N}'(i)|} \times \mathbb{R}^{|\mathcal{N} \setminus \mathcal{N}'|}. \quad (62)$$

From (53b), we have

$$\mathrm{Null}\left(\mathbf{A}_{\mathcal{L}'(k),\mathcal{N}}\right) = \mathrm{Null}\left(\big[\mathbf{0}_{|\mathcal{L}'(k)| \times |\mathcal{N}'(1)|}, \ldots, \mathbf{A}_{\mathcal{L}'(k),\mathcal{N}'(k)}, \ldots, \mathbf{0}_{|\mathcal{L}'(k)| \times |\mathcal{N}'(K)|}, \mathbf{0}_{|\mathcal{L}'(k)| \times |\mathcal{N} \setminus \mathcal{N}'|}\big]\right) \quad (63a)$$

$$\mathrm{Null}\left(\mathbf{A}_{\mathcal{L}'(k),\mathcal{N}}\right) = \prod_{i=1}^{k-1} \mathbb{R}^{|\mathcal{N}'(i)|} \times \mathrm{Null}\left(\mathbf{A}_{\mathcal{L}'(k),\mathcal{N}'(k)}\right) \times \prod_{i=k+1}^{K} \mathbb{R}^{|\mathcal{N}'(i)|} \times \mathbb{R}^{|\mathcal{N} \setminus \mathcal{N}'|}. \quad (63b)$$

Replacing this in (62) yields

$$\mathrm{Null}(\mathbf{R}_k) \subseteq \mathrm{Null}\left(\mathbf{A}_{\mathcal{L}'(k),\mathcal{N}}\right), \quad (64)$$

therefore:

$$\mathbf{v} \in \cap_{k=1}^K \mathrm{Null}\left(\mathbf{A}_{\mathcal{L}'(k),\mathcal{N}}\right). \quad (65)$$

From (53a), we know that the matrices $\mathbf{A}_{\mathcal{L}'(k),\mathcal{N}}$ are the row blocks of $\mathbf{A}_{\mathcal{L}',\mathcal{N}}$, and hence

$$\mathrm{Null}\left(\mathbf{A}_{\mathcal{L}',\mathcal{N}}\right) = \cap_{k=1}^K \mathrm{Null}\left(\mathbf{A}_{\mathcal{L}'(k),\mathcal{N}}\right). \quad (66)$$

Replacing:

$$\mathbf{v} \in \mathrm{Null}\left(\mathbf{A}_{\mathcal{L}',\mathcal{N}}\right), \quad (67a)$$
$$\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) \subseteq \mathrm{Null}\left(\mathbf{A}_{\mathcal{L}',\mathcal{N}}\right). \quad (67b)$$

Combining (50) and (67b) yields

$$\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) \subseteq \mathrm{Null}\left(\mathbf{A}_{\mathcal{L}',\mathcal{N}}\right) \cap \mathrm{Null}\left(\mathbf{G}_{\mathcal{T} \setminus \mathcal{T}'}\right) \cap \mathrm{Null}\left(\mathbf{A}_{\mathcal{L} \setminus \mathcal{L}',\mathcal{N}}\right), \quad (68a)$$

$$\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) \subseteq \mathrm{Null}\left(\mathbf{A}_{\mathcal{L}',\mathcal{N}}\right) \cap \mathrm{Null}\left(\mathbf{A}_{\mathcal{L} \setminus \mathcal{L}',\mathcal{N}}\right). \quad (68b)$$

From (39), we know that $\mathbf{A}_{\mathcal{L}',\mathcal{N}}$ and $\mathbf{A}_{\mathcal{L} \setminus \mathcal{L}',\mathcal{N}}$ are the row blocks of $\mathbf{A}_{\mathcal{L},\mathcal{N}}$, and thus

$$\mathrm{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right) = \mathrm{Null}\left(\mathbf{A}_{\mathcal{L}',\mathcal{N}}\right) \cap \mathrm{Null}\left(\mathbf{A}_{\mathcal{L} \setminus \mathcal{L}',\mathcal{N}}\right). \quad (69)$$

We note that both (68b) and (69) hold even if $\mathcal{L}'$ is empty, so from now on we drop such assumption. Finally, from (68b) and (69) we conclude that in general:

$$\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) \subseteq \mathrm{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right). \qquad \square$$

Qualitatively speaking, Theorem 2 is a recursive reduction: we can apply Theorem 1 to the network admittance matrix if we can also apply Theorem 1 to the reactive components of the network (defined by the subgraphs $(\mathcal{N}'(k), \mathcal{L}'(k))$). If there are no such reactive components, then we only require the standard condition of non-negative conductances in order to apply Theorem 1. We still need to prove that the conditions of Theorem 1 hold over the reactive components of the network. In the general case such proof may be too complex or even unattainable. However, we will prove that the conditions hold for common cases of reactive components with simple structures. Moreover, as we will see in the experiments section, the reactive components of practical power systems often have such structures, making the theory practically applicable. We next show the validity of Theorem 1 over several cases.

**Theorem 3.** *Let the graph $(\mathcal{N}, \mathcal{L})$ define a connected network and let $\mathcal{T}$ define the shunts of the network. Moreover, Assumptions 1 and 2 hold. If the network satisfies at least one of the following conditions:*

1) *$(\mathcal{N}, \mathcal{L})$ is a tree and there exists a root node $r \in \mathcal{N}$ such that equivalent admittance of the network, measured between node $r$ and ground, is non-zero. Moreover, the equivalent admittance of any node to ground, under the condition of the parent node being grounded, is finite.*
2) *$(\mathcal{N}, \mathcal{L})$ is a tree and $\mathcal{T} = \emptyset$.*
3) *There are only inductors or there are only capacitors.*

*then* $\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right) \subseteq \mathrm{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right)$.

**Proof, Condition *1)*.** Let $\mathbf{v} \in \mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right)$ and define the vectors $\mathbf{i}_{\mathcal{L}} = \mathbf{Y}_{\mathcal{L}} \mathbf{A}_{\mathcal{L},\mathcal{N}} \mathbf{v}$ and $\mathbf{i}_{\mathcal{T}} = \mathbf{Y}_{\mathcal{T}} \mathbf{v}$. We have that

$$\mathbf{0} = \mathbf{Y}_{\mathcal{N}} \mathbf{v} \quad (70a)$$
$$\mathbf{0} = \mathbf{A}^H_{\mathcal{L},\mathcal{N}} \mathbf{i}_{\mathcal{L}} + \mathbf{i}_{\mathcal{T}}. \quad (70b)$$

We define $\mathcal{V}(0) \subseteq \mathcal{N}$ as the leaves of tree $(\mathcal{N}, \mathcal{L})$ (not including the root node $r$, see [29]). For $l > 0$, we define $\mathcal{V}(l) \subseteq \mathcal{N}$ as the set of nodes having all their children in $\cup_{k=0}^{l-1} \mathcal{V}(k)$, but do not belong to $\cup_{k=0}^{l-1} \mathcal{V}(k)$ themselves (i.e.

$\mathcal{V}(l) \cap (\cup_{k=0}^{l-1} \mathcal{V}(k)) = \emptyset$). The *height* of the tree is the unique integer $L$ such that $\{r\} = \mathcal{V}(L)$. The sets $\mathcal{V}(0), \ldots, \mathcal{V}(L)$ form a partition of $\mathcal{N}$. We also define

$$\mathbf{Y}^{i,k} = \begin{bmatrix} y_{11}^{i,k} & y_{12}^{i,k} \\ y_{21}^{i,k} & y_{22}^{i,k} \end{bmatrix}, \tag{71}$$

as the $2 \times 2$ admittance matrix formed by considering only nodes $i$ and $k$ (in that order), and all branches connecting them (shunts excluded). Lastly, we define $i^{i,k}$ as

$$i^{i,k} = y_{11}^{i,k} \{\mathbf{v}\}_i + y_{12}^{i,k} \{\mathbf{v}\}_k . \tag{72}$$

Consider a node $n \in \mathcal{V}(l)$ for some $l < L$ (so $n \neq r$). Let $\mathcal{C}(n)$ be index set of all branches connecting $n$ to some child node, let $p$ be the parent node of $n$, and let $k$ be the index of the branch connecting $n$ and $p$. The scalar equation of (70b) associated with node $n$ is

$$0 = y_{21}^{p,n} \{\mathbf{v}\}_p + y_{22}^{p,n} \{\mathbf{v}\}_n + \{\mathbf{i}_\mathcal{T}\}_n + \sum\nolimits_{i \in \mathcal{C}(n)} i^{n,i}. \tag{73}$$

We assume for induction that for any $i^{n,i} \in \mathcal{C}(n)$ we can write

$$i^{n,i} = y_i^b \{\mathbf{v}\}_n , \qquad y_i^b \in \mathbb{C}, \tag{74}$$

for some finite $y_i^b$. We recall that if $l = 0$ then $n$ is a leaf node, hence $\mathcal{C}(n) = \emptyset$ and the induction hypothesis holds vacuously. Let the shunt of node $n$ be $y_n^s = \{\mathbf{Y}_\mathcal{T}\}_{nn}$, then from (74) and the definition of $\mathbf{i}_\mathcal{T}$ we get that

$$0 = y_{21}^{p,n} \{\mathbf{v}\}_p + y_{22}^{p,n} \{\mathbf{v}\}_n + y_n^s \{\mathbf{v}\}_n + \sum\nolimits_{i \in \mathcal{C}(n)} y_i^b \{\mathbf{v}\}_n , \tag{75a}$$

$$0 = y_{21}^{p,n} \{\mathbf{v}\}_p + \left(y_{22}^{p,n} + y_n^{sb}\right) \{\mathbf{v}\}_n , \tag{75b}$$

where

$$y_n^{sb} = y_n^s + \sum\nolimits_{i \in \mathcal{C}(n)} y_i^b. \tag{76}$$

Multiplying by $y_{12}^{p,n}$ on both sides of (75b) we get that

$$0 = y_{21}^{p,n} y_{12}^{p,n} \{\mathbf{v}\}_p + \left(y_{22}^{p,n} + y_n^{sb}\right) y_{12}^{p,n} \{\mathbf{v}\}_n . \tag{77}$$

Notice that:

$$i^{p,n} = y_{11}^{p,n} \{\mathbf{v}\}_p + y_{12}^{p,n} \{\mathbf{v}\}_n , \tag{78a}$$

$$y_{12}^{p,n} \{\mathbf{v}\}_n = i^{p,n} - y_{11}^{p,n} \{\mathbf{v}\}_p , \tag{78b}$$

hence

$$0 = y_{21}^{p,n} y_{12}^{p,n} \{\mathbf{v}\}_p + \left(y_{22}^{p,n} + y_n^{sb}\right) \left(i^{p,n} - y_{11}^{p,n} \{\mathbf{v}\}_p\right) . \tag{79}$$

The term $y_{22}^{p,n} + y_n^{sb}$ is the equivalent admittance between node $n$ and ground, under the condition of node $p$ being ground. Hence $y_{22}^{p,n} + y_n^{sb} \neq 0$ according to Condition *1)*, and

$$i^{p,n} = \left(y_{11}^{p,n} - \frac{y_{12}^{p,n} y_{21}^{p,n}}{y_{22}^{p,n} + y_n^{sb}}\right) \{\mathbf{v}\}_p , \tag{80a}$$

$$i^{p,n} = y_n^b \{\mathbf{v}\}_p , \tag{80b}$$

where

$$y_n^b = y_{11}^{p,n} - \frac{y_{12}^{p,n} y_{21}^{p,n}}{y_{22}^{p,n} + y_n^{sb}}. \tag{81}$$

We conclude that the induction hypothesis holds for any node $n \neq r$. We remark that $y_n^b$ is finite because $y_{22}^{p,n} + y_n^{sb} \neq 0$. Now we write the scalar equation of (70b) associated with the root node $r$:

$$0 = \{\mathbf{i}_\mathcal{T}\}_r + \sum\nolimits_{i \in \mathcal{C}(r)} i^{r,i}, \tag{82a}$$

$$0 = y_r^s \{\mathbf{v}\}_r + \sum\nolimits_{i \in \mathcal{C}(r)} y_i^b \{\mathbf{v}\}_r , \tag{82b}$$

$$0 = y_r^{sb} \{\mathbf{v}\}_r , \tag{82c}$$

so $y_r^{sb}$ is the equivalent admittance of the network. Condition *1)* states that $y_r^{sb}$ is non-zero, so we conclude that $\{\mathbf{v}\}_r = 0$. Now we propose a backward induction hypothesis: for every node $m \in \cup_{k=l}^{L} \mathcal{V}(k), l > 0$ we have that $\{\mathbf{v}\}_m = 0$ (which trivially holds for $l = L$). We take any node $n \in \mathcal{V}(l-1)$, let $p$ be the parent node of $n$, then $p \in \cup_{k=l}^{L} \mathcal{V}(k)$ and so $\{\mathbf{v}\}_p = 0$. As $y_{21}^{p,n}$ is finite, we get from (75b) that

$$0 = \left(y_{22}^{p,n} + y_n^{sb}\right) \{\mathbf{v}\}_n , \tag{83}$$

and as $y_{22}^{p,n} + y_n^{sb} \neq 0$ we conclude that $\{\mathbf{v}\}_n = 0$, proving the induction hypothesis. This means that $\mathbf{v} = \mathbf{0}$, so

$$\mathrm{Null}\left(\mathbf{Y}_\mathcal{N}\right) = \{\mathbf{0}\} \subseteq \mathrm{Null}\left(\mathbf{A}_{\mathcal{N},\mathcal{L}}\right) . \qquad \square$$

**Proof, Condition *2)*.** In this case we have that $\mathbf{Y}_\mathcal{N} = \mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{Y}_\mathcal{L} \mathbf{A}_{\mathcal{L},\mathcal{N}}$ (see (34)), and thus $\mathrm{Null}\left(\mathbf{Y}_\mathcal{N}\right) \supseteq \mathrm{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right)$. We also know, since $(\mathcal{N}, \mathcal{L})$ is a tree, that the network has exactly $|\mathcal{N}| - 1$ branches. This means that $\mathbf{A}_{\mathcal{L},\mathcal{N}}$ has size $|\mathcal{N}| - 1 \times |\mathcal{N}|$, so from Lemma 1 we have that $\mathrm{rank}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right) = |\mathcal{N}| - 1$. Applying the Frobenius inequality (see exercise 4.5.17 in [28]) to (34), we have

$$\mathrm{rank}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{Y}_\mathcal{L}\right) + \mathrm{rank}\left(\mathbf{Y}_\mathcal{L} \mathbf{A}_{\mathcal{L},\mathcal{N}}\right) \leq \mathrm{rank}\left(\mathbf{Y}_\mathcal{L}\right) + \mathrm{rank}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{Y}_\mathcal{L} \mathbf{A}_{\mathcal{L},\mathcal{N}}\right) , \tag{84a}$$

$$\mathrm{rank}\left(\mathbf{Y}_\mathcal{N}\right) \geq \mathrm{rank}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{Y}_\mathcal{L}\right) + \mathrm{rank}\left(\mathbf{Y}_\mathcal{L} \mathbf{A}_{\mathcal{L},\mathcal{N}}\right) - \mathrm{rank}\left(\mathbf{Y}_\mathcal{L}\right) . \tag{84b}$$

Applying Lemma 4 and the fact that $\mathbf{Y}_\mathcal{L}$ is square and full rank, we get that

$$\mathrm{rank}\left(\mathbf{Y}_\mathcal{N}\right) \geq \mathrm{rank}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}^H\right) + \mathrm{rank}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right) - \mathrm{rank}\left(\mathbf{Y}_\mathcal{L}\right) , \tag{85a}$$

$$\mathrm{rank}\left(\mathbf{Y}_\mathcal{N}\right) \geq |\mathcal{N}| - 1. \tag{85b}$$

Applying the rank-nullity theorem:

$$\dim\left(\mathrm{Null}\left(\mathbf{Y}_\mathcal{N}\right)\right) \leq 1 = \dim\left(\mathrm{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right)\right) , \tag{86}$$

but $\mathrm{Null}\left(\mathbf{Y}_\mathcal{N}\right) \supseteq \mathrm{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right)$, and as they have equal dimension then $\mathrm{Null}\left(\mathbf{Y}_\mathcal{N}\right) = \mathrm{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right)$. This trivially implies that $\mathrm{Null}\left(\mathbf{Y}_\mathcal{N}\right) \subseteq \mathrm{Null}\left(\mathbf{A}_{\mathcal{L},\mathcal{N}}\right)$. $\square$

**Proof, Condition *3)*.** As the network is purely inductive (or purely capacitive) we can write

$$\mathbf{Y}_\mathcal{N} = \mathbf{A}_{\mathcal{L},\mathcal{N}}^H \left(jk\mathbf{B}_\mathcal{L}\right) \mathbf{A}_{\mathcal{L},\mathcal{N}} + \left(jk\mathbf{B}_\mathcal{T}\right) , \tag{87a}$$

$$\mathbf{Y}_\mathcal{N} = jk \left(\mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{B}_\mathcal{L} \mathbf{A}_{\mathcal{L},\mathcal{N}} + \mathbf{B}_\mathcal{T}\right) , \tag{87b}$$

where $\mathbf{B}_\mathcal{L}$ and $\mathbf{B}_\mathcal{T}$ are diagonal matrices with non-negative real entries, and $k = 1$ if the network is purely capacitive or $k = -1$ if the network is purely inductive. Now we consider an alternative network with a set of nodes $\mathcal{N}'$ identical to $\mathcal{N}$, a set of branches $\mathcal{L}'$ such that $\mathbf{A}_{\mathcal{L}',\mathcal{N}'} = \mathbf{A}_{\mathcal{L},\mathcal{N}}$ and $\mathbf{Y}_{\mathcal{L}'} = \mathbf{B}_\mathcal{L}$, and a set of shunts $\mathcal{T}'$ such that $\mathbf{Y}_{\mathcal{T}'} = \mathbf{B}_\mathcal{T}$. The admittance matrix of the alternative network is:

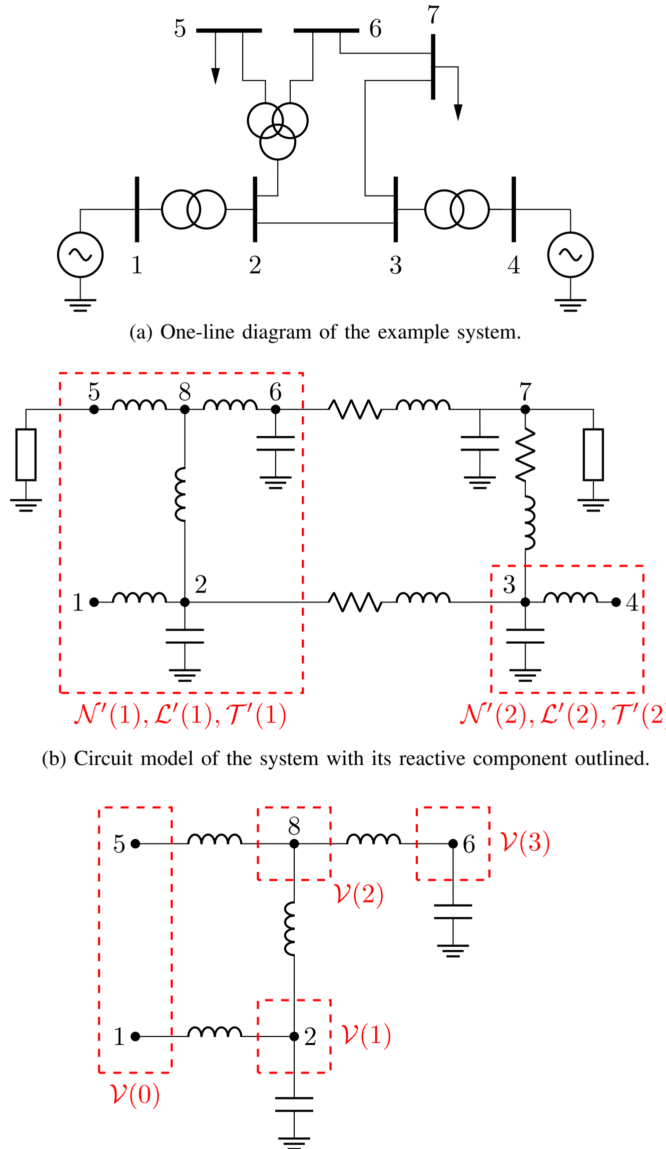
$$\mathbf{Y}_{\mathcal{N}'} = \mathbf{A}_{\mathcal{L},\mathcal{N}}^H \mathbf{B}_\mathcal{L} \mathbf{A}_{\mathcal{L},\mathcal{N}} + \mathbf{B}_\mathcal{T}, \tag{88}$$

therefore

$$\mathbf{Y}_{\mathcal{N}} = jk\mathbf{Y}_{\mathcal{N}'} \tag{89}$$

Notice that the alternative network satisfies Assumptions 1 and 2 and is purely resistive with no negative conductances. Hence $\mathbf{Y}_{\mathcal{N}'}$ satisfies Theorem 2. As $jk \neq 0$ we have that $\mathrm{Null}(\mathbf{Y}_{\mathcal{N}}) = \mathrm{Null}(\mathbf{Y}_{\mathcal{N}'})$. Moreover, we know that $\mathbf{A}_{\mathcal{L}',\mathcal{N}'} = \mathbf{A}_{\mathcal{L},\mathcal{N}}$, so $\mathrm{Null}(\mathbf{Y}_{\mathcal{N}}) \subseteq \mathrm{Null}(\mathbf{A}_{\mathcal{L},\mathcal{N}})$. $\square$

We now have enough tools to check the invertibility of the admittance matrix. First we apply Theorem 2 to obtain the reactive components of the network. Then we apply Theorem 3 to each component. If we succeed then from Theorem 1 we get the rank of the admittance matrix. To illustrate this idea consider the example system of Fig. 2. The one-line diagram of the system is shown in Fig. 2a. In Fig. 2b, we have the circuit model of the system, where the loads are modelled as constant admittances and each transmission line is modeled using a $\pi$ circuit. Applying Theorem 2, we obtain two reactive components, outlined in the figure (shunt loads are not included in the components, as they have a resistive part). If main condition of Theorem 2 can be proved for each component (by means of Theorem 3, for example), then Theorem 1 will hold for the system. The branches of the first component, $(\mathcal{N}'(1), \mathcal{T}'(1))$, form a tree. Choosing node 6 as root, we obtain the node partition shown in Fig. 2c. This partition can be used to check Condition *1)* of Theorem 3.




(a) One-line diagram of the example system.

(b) Circuit model of the system with its reactive component outlined.

(c) Node partition for the tree $(\mathcal{N}'(1), \mathcal{L}'(1))$, using node 6 as root.

Fig. 2. Example system to illustrate how to apply the main theorems.

## IV. Implementation and Test Cases

We developed MATLAB R2012b code that numerically demonstrates the theory in Section III. The code is publicly available at the following page:

https://github.com/djturizo/ybus-inv-check

This code is not optimized for performance, but rather serves as a proof-of-concept for the complexity of checking the theorems. The interested reader can read the code and its comments to see that the program has a time complexity of $\mathcal{O}(|\mathcal{N}| + |\mathcal{L}|)$ (linear in the system size). We remark that comparisons cannot be exact due to the finite-precision computations, so the program uses a user-defined tolerance for all comparisons.

For the numerical experiments, we selected the test cases from the Power Grid Library PGLib [24] (from the OPF benchmarks, more specifically). Some of the PGLib test cases have a small number of negative resistance elements, precluding the application of Theorem 1. This is the result of modeling choices associated with equivalenced networks [30]. Since these non-passive branches are of an artificial (non-physical) nature, we focus on the other 44 PGLib test cases without negative resistance elements for our numerical experiments. With a tolerance of $10^{-12}$, we obtained the results shown in Table I. Of the 44 test cases, we found that 6 of them did not satisfy the conditions of the theorems. Thus, the program could not certify the invertibility of the admittance matrix for those 6 cases. These cases are identified with a dash in the last column of Table I to indicate that the theorems cannot certify whether or not the admittance matrix is invertible. The reason why the invertibility could not be certified for each of the 6 cases is because they have reactive components with topologies not covered by Theorem 3. Such components have inductors, capacitors and loops formed by branches. However, those complex topologies are uncommon, as for the other 38 cases (86% of all cases) the conditions of the theorems hold, so the program can certify the whether the admittance matrix is invertible or not for each case. The admittance matrix is known to be invertible for the test cases, so we get positive results whenever the theorems were applicable. The results show that the theorems can be used to certify the invertibility of the admittance matrices for a wide range of practical and realistic power systems. Moreover, the cases where the theorems cannot be applied to a realistic power system are uncommon.

## V. Conclusions

This paper studied the invertibility of the admittance matrix for balanced networks. First, we analyzed a theorem from the literature regarding conditions guaranteeing invertibility of the admittance matrix, and we found a technical issue in the proof of that theorem. Next, we developed a framework of

TABLE I
TEST CASES USED FOR CHECKING THE THEOREMS

| Test case | $\lvert\mathcal{N}\rvert$ | $\lvert\mathcal{L}\rvert$ | Satisfy thm. conditions? | $\mathbf{Y}_{\mathcal{N}}$ non-singular? |
|---|---|---|---|---|
| case3_lmbd | 3 | 3 | Yes | Yes |
| case5_pjm | 5 | 6 | Yes | Yes |
| case14_ieee | 14 | 20 | No | - |
| case24_ieee_rts | 24 | 38 | Yes | Yes |
| case30_as | 30 | 41 | No | - |
| case30_ieee | 30 | 41 | No | - |
| case39_epri | 39 | 46 | Yes | Yes |
| case57_ieee | 57 | 80 | No | - |
| case60_c | 60 | 88 | Yes | Yes |
| case73_ieee_rts | 73 | 120 | Yes | Yes |
| case89_pegase | 89 | 210 | Yes | Yes |
| case118_ieee | 118 | 186 | Yes | Yes |
| case162_ieee_dtc | 162 | 284 | Yes | Yes |
| case179_goc | 179 | 263 | Yes | Yes |
| case200_activ | 200 | 245 | Yes | Yes |
| case240_pserc | 240 | 448 | Yes | Yes |
| case300_ieee | 300 | 411 | No | - |
| case500_goc | 500 | 733 | Yes | Yes |
| case1354_pegase | 1354 | 1991 | Yes | Yes |
| case1888_rte | 1888 | 2531 | Yes | Yes |
| case1951_rte | 1951 | 2596 | Yes | Yes |
| case2000_goc | 2000 | 3639 | Yes | Yes |
| case2383wp_k | 2383 | 2896 | Yes | Yes |
| case2736sp_k | 2736 | 3504 | Yes | Yes |
| case2742_goc | 2742 | 4673 | Yes | Yes |
| case2746wp_k | 2746 | 3514 | Yes | Yes |
| case2848_rte | 2848 | 3776 | Yes | Yes |
| case2868_rte | 2868 | 3808 | Yes | Yes |
| case2869_pegase | 2869 | 4582 | No | - |
| case3970_goc | 3970 | 6641 | Yes | Yes |
| case4020_goc | 4020 | 6988 | Yes | Yes |
| case4601_goc | 4601 | 7199 | Yes | Yes |
| case4619_goc | 4619 | 8150 | Yes | Yes |
| case4837_goc | 4837 | 7765 | Yes | Yes |
| case6468_rte | 6468 | 9000 | Yes | Yes |
| case6470_rte | 6470 | 9005 | Yes | Yes |
| case6495_rte | 6495 | 9019 | Yes | Yes |
| case6515_rte | 6515 | 9037 | Yes | Yes |
| case9591_goc | 9591 | 15915 | Yes | Yes |
| case10000_goc | 10000 | 13193 | Yes | Yes |
| case10480_goc | 10480 | 18559 | Yes | Yes |
| case19402_goc | 19402 | 34704 | Yes | Yes |
| case24464_goc | 24464 | 37816 | Yes | Yes |
| case30000_goc | 30000 | 35393 | Yes | Yes |

lemmas and assumptions that allowed us to amend the proof of previous claims, developing relaxed conditions that guarantee the invertibility of the admittance matrix and generalizing the results to systems with branches modeled as purely reactive elements and transformers with off-nominal tap ratios. Finally, we implemented and publicly released a proof-of-concept program that uses the theorems to certify the invertibility of the admittance matrix. Numerical tests showed that the theorems are applicable in a large number of realistic power systems.

The theory developed in this paper has solely considered admittance matrices for balanced single-phase equivalent network representations. With rapidly increasing penetration of distributed energy resources in unbalanced distribution systems, extending the theory developed here to address the admittance matrices associated with polyphase networks is an important direction for future work. The authors of [20] considered this topic in [21], where they generalize Theorem 1 to polyphase networks. However, the theory in [21] also relies on the incorrectly stated Lemma 3 and hence may also benefit from amendments and extensions similar to those in this paper.

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