Robust AC Optimal Power Flow with Convex Restriction

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***Abstract*—Power systems are increasingly being subjected to uncertain fluctuations from load demands and renewable generators. In order to avoid constraint violations and maintain steady-state stability, operators must account for uncertainties when determining the dispatch point for a power system. Accordingly, this paper considers the robust AC OPF problem, which requires that both the operational limits and the AC power flow equations are satisfied for all uncertainty realizations within a specified uncertainty set. Guaranteeing robustness is challenging due to the non-convex, nonlinear AC power flow equations, which may not always have a solution. Using a *convex restriction* technique, we first derive a provably robust feasibility condition for the OPF problem that is formulated as an explicit set of convex quadratic constraints. This condition is the first to rigorously guarantee robust feasibility with respect to both the operational limits and the AC power flow equations. We then use this condition in an algorithm that obtains robust solutions to AC OPF problems by solving a sequence of convex optimization problems. We demonstrate our algorithm and its ability to control robustness versus operating cost trade-offs using PGLib test cases.**

***Index Terms*—Robust optimal power flow, convex restriction**

I. Introduction

The optimal power flow (OPF) problem determines the minimum cost dispatch that satisfies both the power flow equations, which model the flow of power and balance supply and demand, and operational constraints such as voltage magnitude, line flow, and generator limits [1]. OPF problems rely on supply and demand forecasts, which enter as parameters in the problems. Increasing penetrations of stochastic renewable generation, often observed as larger variations in the demand due to behind-the-meter solar PV, lead to the actual operating conditions differing from the forecast considered in the OPF problem. Forecast errors can result in unacceptable violations of operational limits and the potential loss of steady-state stability (i.e., a condition where the system attempts to reach an operating point for which the AC power flow equations do not admit any solutions). An AC representation of the power flow equations is needed to accurately model these effects.

Common approaches for considering uncertainties in OPF problems include *stochastic* [2], *chance-constrained* [3]–[8] and *robust* [9]–[17] formulations. Robust OPF formulations, which are considered in this paper, prohibit constraint violations for all uncertainty realizations within a deterministic uncertainty set [18]. Fig. 1 illustrates the difference between a nominal OPF solution and a robust OPF solution for a 9-bus system. The feasible region (in blue) has two "holes" induced by two voltage magnitude constraints. We observe that the nominal solution (in purple) is at the voltage limit, making it vulnerable to changes in the parameters. The robust solution (red dot) is inside a *confidence ellipsoid*, and any parameter variation within that confidence ellipsoid remains feasible. One option for defining the size of the confidence ellipsoid (or other uncertainty set) is to require that it captures the uncertain variable with a certain probability (e.g., with 95% confidence) [18]. This establishes a natural link to chance-constrained formulations, which require that constraint violation probabilities are less than a specified threshold.

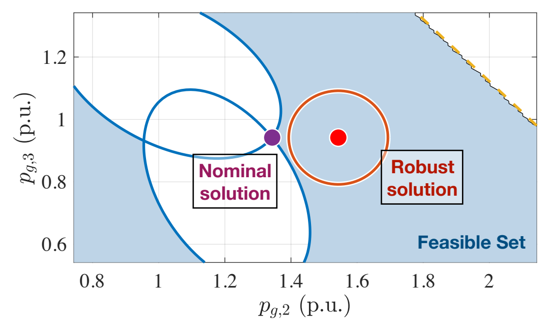

Fig. 1. Illustration of the nominal (non-robust) solution and a robust OPF solution for a 9-bus system. The blue region is the nonconvex feasible space, which is defined by the maximum voltage magnitude limits at buses 6 and 8 (two blue ovals) and minimum active power generation limit at the slack bus (yellow dashed line at the top right corner). Whereas generation uncertainty at buses 2 and 3 can result in the nominal solution (purple dot) easily violating the voltage limits, the robust solution (red dot) withstands all uncertainty realizations that are within the confidence ellipsoid (red circle).

Guaranteeing the robustness of AC OPF solutions requires two conditions: 1) the operational limits formulated as inequality constraints must be satisfied and 2) there must exist a solution to the AC power flow equality constraints. While several existing algorithms have targeted the former issue, there are no existing robust AC OPF algorithms which guarantee solvability of the AC power flow equations. Most of the algorithms developed for chance-constrained and robust AC OPF problems simplify the AC power flow equations [19], particularly through the use of linearizations [3], [4], [6], [9], [20] and convex relaxations [5]. Other approaches in [11]–[13] use convex relaxations to verify operational constraint satisfaction for a given uncertainty set, but do not guarantee AC power flow solvability for all uncertainty realizations. The approach in [16] relaxes the AC power flow equations by assuming that all buses have adjustable generators such that all nodal power balance constraints can be represented with inequalities. Reference [7] uses polynomial chaos expansion to approximate the impacts of the uncertainties, with increasing accuracy coming at the expense of computational tractability. Other related work in [14], [15] considers multi-period OPF problems in the presence of uncertainty by iteratively solving the AC OPF problems with linearizations or convex relaxations. Alternative methods in [2], [8], [10], [17] use scenario-based approaches that enforce feasibility for selected uncertainty realizations.

No existing algorithms provide rigorous feasibility guarantees for robust AC OPF problems. In this paper, we address this open problem using *convex restriction* techniques [21] that provide convex sufficient conditions which ensure feasibility of both the AC power flow equations and the operational constraints in OPF problems. We previously used convex restrictions to solve OPF problems and compute feasible paths connecting different operating points [22] via *sequential convex restriction*, i.e., solving a sequence of convex quadratic problems formulated using convex restriction techniques. In this paper, we utilize the convex restriction to enforce robust feasibility. The main contributions of this paper are:

1. We develop a *convex sufficient condition for robust feasibility* of a dispatch point against power injection uncertainty in an ellipsoidal uncertainty set. The condition ensures feasibility of all the operating constraints and the AC power flow equations, and hence provably guarantees robust feasibility. In addition, the convexity of the condition brings computational advantages.
2. We use the convex feasibility condition to (i) check robust feasibility for specified operating points and (ii) propose a tractable algorithm for AC OPF problems that provides solutions with provable robust feasibility guarantees. To the best of the authors' knowledge, these are the first solution methods to provide rigorous robust feasibility guarantees for nonlinear AC OPF problems.
3. We demonstrate the effectiveness and tractability of this algorithm using numerical experiments on PGLib test cases [23]. The results illustrate the ability of our algorithm to control the trade-off between the level of robustness and the operating cost.

This paper is organized as follows. Section II introduces the system model. Section III formalizes robust feasibility. Section IV derives the sufficient condition for robust feasibility. Section V leverages this condition to develop our proposed robust AC OPF algorithm. Section VI empirically demonstrates this algorithm's performance. Section VII concludes the paper.

II. System Model and Preliminaries

This section introduces the OPF problem with consideration of uncertainty in power injections. We use a distributed slack generator formulation, generalizing the model in [21], [22]. The distributed slack plays an important role in determining the generators' response to the uncertain power injections.

A. Notation

The scalars n_b, n_g, n_{pq}, n_l, and n_d denote the number of buses, generators, PQ buses, lines, and loads, respectively. The variables $p_\mathrm{g} \in \mathbb{R}^{n_g}$ and $q_\mathrm{g} \in \mathbb{R}^{n_g}$ represent the generators' active and reactive power outputs. Uncontrollable active and reactive power injections are denoted by $p_\mathrm{d} \in \mathbb{R}^{n_d}$ and $q_\mathrm{d} \in \mathbb{R}^{n_d}$ where positive values indicate stochastic loads and negative values indicate uncertain generation such as renewables. The voltage magnitudes and phase angles are $v \in \mathbb{R}^{n_b}$ and $\theta \in \mathbb{R}^{n_b}$. The *from* and *to* buses for the lines are denoted by "f" and "t". The non-reference, PV, and PQ elements of a vector are denoted with subscripts "ns", "pv", and "pq". Let $E \in \mathbb{R}^{n_b \times n_l}$ be the incidence matrix of the grid. The connection matrices for generator buses and load buses are denoted by $C_\mathrm{g} \in \mathbb{R}^{n_b \times n_g}$ and $C_\mathrm{d} \in \mathbb{R}^{n_b \times n_d}$, respectively. The matrices I and $\mathbf{0}$ denote identity and zero matrices of appropriate size. The vertical concatenation of vectors a and b is denoted by (a, b). The operations $\mathbf{diag}(\cdot)$ and $\mathbf{blkdiag}(\cdot)$ denote the diagonal and block-diagonal matrices with the argument vector being the diagonal elements.

B. AC Optimal Power Flow Problem Formulation

For notational convenience, we denote the angle differences between the terminals of the transmission lines as φ:

$$\varphi_l = \theta^\mathrm{f}_l - \theta^\mathrm{t}_l, \qquad l = 1, \ldots, n_l, \tag{1}$$

where θ^f_i and θ^t_i are the phase angles of the *from* bus and *to* bus of line l. The AC OPF problem is:

$$\underset{x,u,\overline{s}^\mathrm{f},\overline{s}^\mathrm{t}}{\text{minimize}} \quad c(p_\mathrm{g}) = \sum_{i=1}^{n_g} c_i(p_{\mathrm{g},i}) \tag{2}$$

subject to

$$p_k^\mathrm{inj} = \sum_{l=1}^{n_l} v_l^\mathrm{f} v_l^\mathrm{t} \left(G^c_{kl} \cos\varphi_l + B^s_{kl} \sin\varphi_l \right) + G^d_{kk} v_k^2, \tag{3a}$$

$$q_k^\mathrm{inj} = \sum_{l=1}^{n_l} v_l^\mathrm{f} v_l^\mathrm{t} \left(G^s_{kl} \sin\varphi_l - B^c_{kl} \cos\varphi_l \right) - B^d_{kk} v_k^2, \tag{3b}$$

$$p_{\mathrm{g},i}^\mathrm{min} \le p_{\mathrm{g},i} \le p_{\mathrm{g},i}^\mathrm{max}, \qquad i = 1, \ldots, n_g, \tag{4a}$$
$$q_{\mathrm{g},i}^\mathrm{min} \le q_{\mathrm{g},i} \le q_{\mathrm{g},i}^\mathrm{max}, \qquad i = 1, \ldots, n_g, \tag{4b}$$
$$v_i^\mathrm{min} \le v_i \le v_i^\mathrm{max}, \qquad i = 1, \ldots, n_b \tag{4c}$$
$$\varphi_l^\mathrm{min} \le \varphi_l \le \varphi_l^\mathrm{max}, \qquad l = 1, \ldots, n_l, \tag{4d}$$
$$(s^\mathrm{f}_{p,l})^2 + (s^\mathrm{f}_{q,l})^2 \le (s_l^\mathrm{max})^2, \qquad l = 1, \ldots, n_l, \tag{4e}$$
$$(s^\mathrm{t}_{p,l})^2 + (s^\mathrm{t}_{q,l})^2 \le (s_l^\mathrm{max})^2, \qquad l = 1, \ldots, n_l. \tag{4f}$$

where the matrices G^c, G^s, B^c, $B^\mathrm{s} \in \mathbb{R}^{n_b \times n_l}$ and G^d, $B^d \in \mathbb{R}^{n_b \times n_b}$ are transformed admittance matrices for the respective conductance and susceptance terms. The exact definitions of the transformed matrices are available in [22]. The objective $c : \mathbb{R}^{n_g} \to \mathbb{R}$ is a monotonically increasing function of the active power generation. The active and reactive power injections are $p^\mathrm{inj} = C_\mathrm{g} p_\mathrm{g} - C_\mathrm{d} p_\mathrm{d}$ and $q^\mathrm{inj} = C_\mathrm{g} q_\mathrm{g} - C_\mathrm{d} q_\mathrm{d}$. Superscripts max and min denote the maximum and minimum limits of the associated quantity. Constraints (4a) and (4b) impose the generators' active and reactive power output limits. Constraints (4c) and (4d) limit the voltage magnitudes and the angle differences. Constraints (4e) and (4f) impose line flow limits where $s^\mathrm{f/t}_{p,l}$ and $s^\mathrm{f/t}_{q,l}$ are the active and reactive power flowing into the line l at the *from* and *to* buses, respectively.

C. Power Injection Uncertainty Modelling

This paper focuses on obtaining a robustly feasible operating point with respect to uncertainty in the uncontrollable power injections, such as load variations and forecast errors from renewable energy sources. The variable $w = (p_d,\, q_d) \in \mathbb{R}^{2n_d}$ consists of uncertain active and reactive power injections $p_\mathrm{d} \in \mathbb{R}^{n_d}$ and $q_\mathrm{d} \in \mathbb{R}^{n_d}$. The nominal value of the uncertain variable is $w^{(0)}$. We consider a bounded uncertainty set $\mathcal{W}$ modeled with a *confidence ellipsoid* containing all power injections within a ball of radius γ centered on the nominal power injections:

$$\mathcal{W}(\gamma) = \{w \mid (w - w^{(0)})^T \Sigma^{-1} (w - w^{(0)}) \leq \gamma^2\}. \quad (5)$$

The power injection covariance matrix $\Sigma \in \mathbb{R}^{2n_d \times 2n_d}$ and the radius $\gamma \in \mathbb{R}$ determine the shape and size, respectively, of the uncertainty set. By choosing an appropriate value for γ, the confidence ellipsoid can be designed such that the probability of containing the uncertainty realization is greater than the desired threshold. For example, if the uncertainty is drawn from a univariate normal distribution, we can ensure that 95% of the uncertainty realizations are within the confidence ellipsoid by setting γ to be twice the variance.

D. Generation Recourse for System Balancing

To balance the system during variation in loads, we consider a "distributed slack" model where each generator adjusts its active power output to account for the system-wide power imbalance. These adjustments occur according to the generators' participation factors. The distributed slack model is formulated via an affine control policy:

$$p_{\mathrm{g},i}(w) = p_{\mathrm{g,ref},i} + \alpha_i \Delta, \quad (6)$$

where α_i is the participation factor for generator i, $p_{\mathrm{g,ref},i}$ is the nominal setpoint for the generator's active power output, and $\Delta \in \mathbb{R}$ is the system-side power imbalance. We consider constant participation factors α_i that are obtained from the generator specifications. The variable Δ is implicitly defined by the AC power flow equations (3a) such that the active power is balanced across the system. Fluctuations in power injections lead to changes in Δ and consequent adjustments to the active power generation according to (6). The distributed slack model generalizes a single slack bus formulation, which can be retrieved by setting all participation factors to 0 except for the participation factor of the slack bus, which is set to 1. The generators' reactive power outputs are also implicitly determined by the power flow equations (3b).

E. Variable Definitions

The buses are divided into two types according to the standard definitions in the distributed slack model:

- PV (generator) buses: $p_\mathrm{g,ref}$ and v_g are specified by the operators; q_pv and θ_pv are implicitly defined by the AC power flow equations.
- PQ (load) buses: p_d and q_d are either fixed or uncertain parameters; v_pq and θ_pq are implicitly defined by the AC power flow equations.

The angle at one arbitrarily chosen reference bus is set to zero.

Based on the bus types, we categorize all variables as *dispatch variables* or *internal states*, which are denoted by $u \in \mathbb{R}^{2n_g}$ and $x \in \mathbb{R}^{n_b+n_{pq}}$, respectively, with the following definitions:

- *Dispatch variables* refer to controllable quantities that can be set by the system operator, specifically, the active power generation setpoint and the voltage magnitude setpoint. A *dispatch point* is defined by a vector composed of dispatch variables and is denoted by $u = (p_\mathrm{g,ref},\, v_\mathrm{g})$.
- *Internal states* are physical quantities that are implicitly determined by the physics of the power grid and are computed through the AC power flow equations. These include the phase angles at the non-reference buses, the voltage magnitudes at PQ buses, and the distributed slack variable Δ. Internal states are denoted by $x = (\theta_\mathrm{ns},\, v_\mathrm{pq},\, \Delta)$.

Moreover, we introduce a transformed state variable $z = (\varphi,\, v_\mathrm{pq},\, \Delta)$ that converts the phase angles θ_ns to phase angle differences φ. The transformed state is defined as $z = \widetilde{C}x$ where $\widetilde{C} = \mathbf{blkdiag}(E_\mathrm{ns}^T, I_{n_\mathrm{pq} \times n_\mathrm{pq}}, 1)$. The transformed state z enables working directly with the angle differences φ that appear in the power flow equations' trigonometric functions.

Finally, we use the superscript (0) to indicate a nominal point of any variable when the uncertainty is equal to its nominal value. For example, the nominal state variable $x^{(0)}$ is the solution to the AC power flow equations with $w = w^{(0)}$.

III. Robust Feasibility: Definition and Problem Formulation

In this section, we introduce uncertainty into the nominal OPF problem. We first describe the unknown-but-bounded uncertainty set and then define the robust AC OPF problem.

A. Robust Feasibility Against Power Injection Uncertainty

Given the uncertainty model, the robust feasibility of a dispatch point is defined by the following statement:

Definition 1. A dispatch point u is *robustly feasible* if the OPF constraints are satisfied for all realizations of w within the given uncertainty set $\mathcal{W}(\gamma)$. That is, for all $w \in \mathcal{W}(\gamma)$, there exists $x = (\theta_\mathrm{ns},\, v_\mathrm{pq},\, \Delta)$ such that (3) and (4) are satisfied.

Robust feasibility is defined for a dispatch point u, which corresponds to the variables in the OPF problem that are directly controllable. Power injection fluctuations w change the internal states x (e.g., the voltage magnitudes and phase angles) according to the AC power flow equations. Hence, the internal states adapt to the uncertainty realizations.

B. Basis Function Formulation of the Power Flow Equations

In this section, we rewrite the AC power flow equations in terms of basis functions and include uncertain variables. The basis functions serve as building blocks for the power flow nonlinearities. The vector of nonlinear functions, $\psi : (\mathbb{R}^{n_l+n_{pq}+1}, \mathbb{R}^{2n_g}, \mathbb{R}^{n_d}) \to \mathbb{R}^{3n_b+2n_l}$, denotes the basis function, $\psi(z,u) = (p_\mathrm{g},\, q_\mathrm{g},\, \psi^{\cos},\, \psi^{\sin},\, \psi^{\mathrm{quad}})$, where

$$\begin{aligned} \psi_l^{\cos}(z,u) &= v_l^\mathrm{f} v_l^\mathrm{t} \cos(\varphi_l), \qquad && l = 1,\ldots,n_l, \\ \psi_l^{\sin}(z,u) &= v_l^\mathrm{f} v_l^\mathrm{t} \sin(\varphi_l), \qquad && l = 1,\ldots,n_l, \\ \psi_k^{\mathrm{quad}}(z,u) &= v_k^2, && k = 1,\ldots,n_b. \end{aligned} \quad (7)$$

The AC power flow equations (3) can be written in terms of the basis functions and the uncertain variables as follows:

$$\underbrace{\begin{bmatrix} C_{\mathrm{g}} & \mathbf{0} & -G^{\mathrm{c}} & -B^{\mathrm{s}} & -G^{\mathrm{d}} \\ \mathbf{0} & C_{\mathrm{g,pq}} & B^{\mathrm{c}}_{\mathrm{pq}} & -G^{\mathrm{s}}_{\mathrm{pq}} & B^{\mathrm{d}}_{\mathrm{pq}} \end{bmatrix}}_{M} \psi(z,u) - \underbrace{\begin{bmatrix} C_{\mathrm{d}} & \mathbf{0} \\ \mathbf{0} & C_{\mathrm{d,pq}} \end{bmatrix}}_{\widetilde{B}} w = 0, \tag{8}$$

where $M \in \mathbb{R}^{(n_b+n_{\mathrm{pq}})\times(3n_b+2n_l)}$ and $\tilde{B} \in \mathbb{R}^{(n_b+n_{\mathrm{pq}})\times 2n_d}$ are constant matrices. The matrix $B^{\mathrm{c}}_{\mathrm{pq}} \in \mathbb{R}^{n_{pq}\times n_l}$ is a submatrix of $B^{\mathrm{c}} \in \mathbb{R}^{n_b \times n_l}$ containing the rows corresponding to the PQ buses. Matrices $G^{\mathrm{s}}_{\mathrm{pq}}$, $B^{\mathrm{d}}_{\mathrm{pq}}$, $C_{\mathrm{g,pq}}$, and $C_{\mathrm{d,pq}}$ are defined similarly.

C. *Robust AC OPF Problem Formulation*

The robust AC OPF problem seeks the dispatch point with the minimum worst-case generation cost:

$$\underset{x,z,u,\overline{s}^{\mathrm{f}},\overline{s}^{\mathrm{t}}}{\text{minimize}} \quad c(p_{\mathrm{g}}) = \sum_{i=1}^{n_g} c_i(p_{\mathrm{g},i}) \tag{9}$$

subject to: for all w in $\mathcal{W}(\gamma)$, there exists x such that

$$M\psi(z,u) - \widetilde{B}w = 0, \quad z = \widetilde{C}x, \tag{10}$$

$$\begin{bmatrix} \varphi^{\min} \\ v^{\min}_{\mathrm{pq}} \end{bmatrix} \le z \le \begin{bmatrix} \varphi^{\max} \\ v^{\max}_{\mathrm{pq}} \end{bmatrix}, \quad \begin{bmatrix} p^{\min}_{\mathrm{g}} \\ v^{\min}_{\mathrm{g}} \end{bmatrix} \le \begin{bmatrix} p_{\mathrm{g}} \\ v_{\mathrm{g}} \end{bmatrix} \le \begin{bmatrix} p^{\max}_{\mathrm{g}} \\ v^{\max}_{\mathrm{g}} \end{bmatrix}, \tag{11}$$

$$q^{\mathrm{inj}}(q^{\min}_{\mathrm{g}}, w) \le L_{\mathrm{q}}\psi(z,u) \le q^{\mathrm{inj}}(q^{\max}_{\mathrm{g}}, w), \tag{12}$$

$$\begin{gathered} |L^{\mathrm{f}}_{\mathrm{line}}\psi(z,u)| \le \overline{s}^{\mathrm{f}}, \quad |L^{\mathrm{t}}_{\mathrm{line}}\psi(z,u)| \le \overline{s}^{\mathrm{t}}, \\ \left(\overline{s}^{\mathrm{f}}_p\right)^2 + \left(\overline{s}^{\mathrm{f}}_q\right)^2 \le (s^{\max})^2, \quad \left(\overline{s}^{\mathrm{t}}_p\right)^2 + \left(\overline{s}^{\mathrm{t}}_q\right)^2 \le (s^{\max})^2. \end{gathered} \tag{13}$$

The constraints in (10) are the AC power flow equations. The constraints in (11) impose the phase angle difference, voltage magnitude, and active power generation limits. The constraints in (12) and (13) limit reactive power generation, $q^{\mathrm{inj}}(q,w)$, and line flows, where L_{q} and $L^{\mathrm{f/t}}_{\mathrm{line}}$ are defined in Appendix A. In the absence of uncertainty (i.e., $\mathcal{W}(0) = \{w^{(0)}\}$), (9)–(13) is an equivalent formulation of the AC OPF problem in (2)–(4).

IV. A Sufficient Condition for Robust Feasibility Against Power Injection Uncertainty

This section derives a convex sufficient condition for robust feasibility under power injection uncertainty. Robust feasibility requires that the uncertainty set is contained in the nonconvex feasible set of the OPF problem. We show that the dispatch point is robustly feasible by ensuring that the uncertainty set is contained in a convex restriction of the feasible set.

A. *Preliminaries for Convex Restriction*

This section reviews the concept of convex restriction, which is described in more detail in [21], [22]. A convex restriction is a convex inner approximation of the feasible set in the space of dispatch variables. The derivation of this inner approximation relies on the use of Brouwer's fixed-point theorem, and requires the definition of (i) upper convex and lower concave envelopes for the non-linear functions $\psi(z,u)$, and (ii) an outer approximation of the possible values for the internal state variables x and z.

1) Fixed-Point Representation of the AC Power Flow Equations: The power flow equations (8) can be converted to a fixed-point formulation inspired by Newton's iteration:

$$x = -J^{-1}_{f,(0)}Mg(z,u) + J^{-1}_{f,(0)}\widetilde{B}w, \tag{14}$$

where $g : (\mathbb{R}^{n_l+n_{pq}+1}, \mathbb{R}^{2n_g}) \to \mathbb{R}^{3n_b+2n_l}$ is the residual of the basis functions,

$$g(z,u) = \psi(z,u) - J_{\psi,(0)}z. \tag{15}$$

The matrices $J_{f,(0)}$ and $J_{\psi,(0)}$ are Jacobians evaluated at the nominal operating point, defined by

$$J_{f,(0)} = \nabla_x f \mid_{(x^{(0)},u^{(0)})}, \quad J_{\psi,(0)} = \nabla_z \psi \mid_{(z^{(0)},u^{(0)})}. \tag{16}$$

These Jacobians are related by $J_{f,(0)} = MJ_{\psi,(0)}\widetilde{C}$. Equations (10) and (14) are equivalent given that the power flow Jacobian $J_{f,(0)}$ is non-singular.

2) Upper Convex and Lower Concave Envelopes: Denote the k-th element of the residual $g(z,u)$ as $g_k(z,u)$. Upper convex and lower concave envelopes of the function g_k are defined using lower and upper bounding functions $g^{\ell}_k(z,u)$ and $g^u_k(z,u)$, respectively, such that

$$g^{\ell}_k(z,u) \le g_k(z,u) \le g^u_k(z,u), \tag{17}$$

where $g^u_k(z,u)$ is a convex function and $g^{\ell}_k(z,u)$ is a concave function with respect to z and u. Similarly, let ψ^{ℓ} and ψ^u define upper convex and lower concave envelopes for ψ. Expressions for the concave envelopes used in AC OPF problems are given in [21].

3) Outer Approximations of the State Variables: The internal states adapt to both the uncertainty realizations and the choice of dispatch variables. To bound the values taken by the internal states, we define an outer-approximation of the set of internal states as

$$\mathcal{P}(\tilde{z}) = \left\{ x \;\middle|\; \begin{bmatrix} \widetilde{C} \\ -\widetilde{C} \end{bmatrix} x \le \tilde{z}, \ \tilde{z} = \begin{bmatrix} z^u \\ -z^{\ell} \end{bmatrix} \right\}, \tag{18}$$

which is described by the parameter $\tilde{z} \in \mathbb{R}^{2n_z}$. This set can be equivalently written as $\mathcal{P}(\tilde{z}) = \{x \mid \varphi^{\ell} \le E_{\mathrm{ns}}\theta_{\mathrm{ns}} \le \varphi^u, \ v^{\ell}_{\mathrm{pq}} \le v_{\mathrm{pq}} \le v^u_{\mathrm{pq}}, \ \Delta^{\ell} \le \Delta \le \Delta^u\}$. The outer approximations in $\tilde{z}$ are *not* the same as the operational limits in (4), but rather are decision variables in the optimization problem which must satisfy the operational limits (e.g., $\varphi^u \le \varphi^{\max}$.)

We next define variables for over- and under-estimators $g^u_{\mathcal{P},k}$ and $g^{\ell}_{\mathcal{P},k}$ of $g_k(z,u)$ over the set $\mathcal{P}(\tilde{z})$. These over- and under-estimators are defined such that, for a given $z = \tilde{C}x$,

$$g^{\ell}_{\mathcal{P},k}(u,\tilde{z}) \le g_k(z,u) \le g^u_{\mathcal{P},k}(u,\tilde{z}), \quad \forall\, x \in \mathcal{P}(\tilde{z}). \tag{19}$$

Using the envelopes defined in (17), this constraint is imposed by ensuring that $g^u_{\mathcal{P},k}(u,\tilde{z}) \ge g^u_k(z,u)$ and $g^{\ell}_{\mathcal{P},k}(u,\tilde{z}) \le g^{\ell}_k(z,u)$ for all $z \in \{z \mid z_i \in \{z^{\ell}_i, z^u_i\}, \tilde{z} = (z^u, -z^{\ell})\}$. This is accomplished by considering the maximum and minimum values of the function $g(u,\tilde{C}x)$ over the domain $x \in \mathcal{P}(\tilde{z})$. Details regarding this aspect of the formulation and its computational tractability are given in [21], [22]. With this definition of $\mathcal{P}(\tilde{z})$, we know that the internal states determined by the AC power flow equations, for any uncertainty realization, are guaranteed to be bounded by $\mathcal{P}(\tilde{z})$.

B. Sufficient Condition for Robust Feasibility

In this section, we provide a convex sufficient condition for robust feasibility of a dispatch point u. The condition allows us to solve a deterministic convex optimization problem to obtain a feasible solution for the robust optimization problem in (9)–(13). Our approach combines the convex restriction with well-known results used in robust optimization [18], which give an analytical solution for the following supremum problem:

$$\sup\{R_i\, w \mid w \in \mathcal{W}(\gamma)\} = R_i\, w^{(0)} + \gamma \| R_i\, \Sigma^{1/2}\|_2, \quad (20)$$

where R_i is an arbitrary row vector and $\mathcal{W}(\gamma)$ is the confidence ellipsoid in (5). For the definition of robust feasibility in Section III-A, the following theorem provides a convex sufficient condition for robustness of a dispatch point:

Theorem 1. (Robust Feasibility Condition for AC OPF) The dispatch point $u = (p_{g,\text{ref}},\, v_{\text{g}})$ is robustly feasible with respect to the uncertainty set $\mathcal{W}(\gamma)$ if there exist bounds on the internal states $\tilde{z} = (\varphi^u,\, v^u_{\text{pq}},\, \Delta^u,\, -\varphi^\ell,\, -v^\ell_{\text{pq}}, -\Delta^\ell)$ and line flows $(\overline{s}^{\text{f}}, \overline{s}^{\text{t}})$ such that

$$K^+ g^u_{\mathcal{P}}(u,\tilde{z}) + K^- g^\ell_{\mathcal{P}}(u,\tilde{z}) + \xi(\gamma) \le \tilde{z} \quad (21)$$

$$p_{\text{pv,ref}} + \alpha\Delta^u \le p^{\max}_{\text{pv}}, \qquad v_{\text{g}} \le v^{\max}_{\text{g}} \quad (22a)$$

$$p_{\text{pv,ref}} + \alpha\Delta^\ell \ge p^{\min}_{\text{pv}}, \qquad v_{\text{g}} \ge v^{\min}_{\text{g}} \quad (22b)$$

$$\begin{bmatrix} \varphi^u \\ v^u_{\text{pq}} \end{bmatrix} \le \begin{bmatrix} \varphi^{\max} \\ v^{\max}_{\text{pq}} \end{bmatrix}, \quad \begin{bmatrix} \varphi^\ell \\ v^\ell_{\text{pq}} \end{bmatrix} \ge \begin{bmatrix} \varphi^{\min} \\ v^{\min}_{\text{pq}} \end{bmatrix} \quad (22c)$$

$$L^+_{\text{q}}\psi^u_{\mathcal{P}}(u,\tilde{z}) + L^-_{\text{q}}\psi^\ell_{\mathcal{P}}(u,\tilde{z}) + \zeta(\gamma) \le q^{\text{inj}}(q^{\max}_g, w^{(0)}) \quad (23a)$$

$$L^-_{\text{q}}\psi^u_{\mathcal{P}}(u,\tilde{z}) + L^+_{\text{q}}\psi^\ell_{\mathcal{P}}(u,\tilde{z}) - \zeta(\gamma) \ge q^{\text{inj}}(q^{\min}_g, w^{(0)}) \quad (23b)$$

$$L^{\text{k},+}_{\text{line}}\psi^u_{\mathcal{P}}(u,\tilde{z}) + L^{\text{k},-}_{\text{line}}\psi^\ell_{\mathcal{P}}(u,\tilde{z}) \le \overline{s}^{\text{k}}, \quad \text{k} \in \{\text{f},\text{t}\} \quad (24a)$$

$$-L^{\text{k},-}_{\text{line}}\psi^u_{\mathcal{P}}(u,\tilde{z}) - L^{\text{k},+}_{\text{line}}\psi^\ell_{\mathcal{P}}(u,\tilde{z}) \le \overline{s}^{\text{k}}, \quad \text{k} \in \{\text{f},\text{t}\} \quad (24b)$$

$$\left(\overline{s}^{\text{k}}_p\right)^2 + \left(\overline{s}^{\text{k}}_q\right)^2 \le (s^{\max})^2, \quad \text{k} \in \{\text{f},\text{t}\} \quad (24c)$$

where the margins $\xi_i(\gamma)$ and $\zeta_i(\gamma)$ are linear functions of γ:

$$\xi_i(\gamma) = R_i w^{(0)} + \gamma \left\| R_i\, \Sigma^{1/2} \right\|_2, \quad (25)$$

$$\zeta_i(\gamma) = \gamma \left\| \left[\mathbf{0}_{n_b \times n_d} \quad C_{\text{d},i}\right] \Sigma^{1/2} \right\|_2. \quad (26)$$

The constant matrices K and R are defined by

$$K = \begin{bmatrix} -I \\ I \end{bmatrix} \widetilde{C} J^{-1}_{f,0} M, \quad R = \begin{bmatrix} I \\ -I \end{bmatrix} \widetilde{C} J^{-1}_{f,0} \widetilde{B}, \quad (27)$$

and K^+, $K^- \in \mathbb{R}^{n_1 \times n_2}$ are defined as $K^+_{ij} = \max\{K_{ij}, 0\}$ and $K^-_{ij} = \min\{K_{ij}, 0\}$ for each element of K. The matrices L^+ and L^- are defined similarly.

The proof of Theorem 1 is provided in Appendix B. The basic idea is to ensure that there exists a fixed point for (14) that is within the set $\mathcal{P}(\tilde{z})$ for all realizations ω in the uncertainty set $\mathcal{W}(\gamma)$. Equation (21) ensures that the AC power flow equations have a solution with a corresponding internal state x within $\mathcal{P}(\tilde{z})$ for every uncertainty realization. Equations (22)–(24) ensure that all points in the set $\mathcal{P}(\tilde{z})$ satisfy the operational constraints.

Theorem 1 generalizes the convex restriction we proposed in [21], which is retrieved by setting $\gamma = 0$ (i.e., the case where the uncertainty set only contains the nominal power injections). Equation (21) shows that robust feasibility is guaranteed by introducing an extra margin, $\xi(\gamma)$, into the solvability condition from previous convex restriction formulations.

The robust feasibility condition is convex with respect to all decision variables x, z, u, $\overline{s}^{\text{f}}$, and $\overline{s}^{\text{t}}$ as well as the uncertainty set size parameter γ. This allows us to use well-developed theory and algorithms in convex optimization via commercial solvers such as Mosek, CPLEX, and Gurobi. Moreover, the number of associated constraints scales linearly with the number of buses and the number of lines in the system, which makes the approach tractable for large systems.

V. Algorithms for Robust OPF Problems

OPF formulations seek the dispatch point with minimum generation cost while considering the load demands and renewable generation. We consider two important questions:

- How robust is a dispatch point against uncertainty from load demands and renewable generator outputs?
- How can we compute a low-cost dispatch point that is robust with respect to a given uncertainty set?

We next develop algorithms that use the previously presented robust feasibility condition in order to rigorously answer both of these questions.

A. Robustness Margin for a Specified Dispatch Point

We first consider a setting where the operating point $u^{(0)}$ has been determined and the operator wants to compute the robustness margin. The problem can be formulated as

$$\underset{\gamma,\, \tilde{z},\, g^u_{\mathcal{P}},\, g^\ell_{\mathcal{P}}}{\text{maximize}} \quad \gamma \quad \text{subject to (21)–(26) and } u = u^{(0)}. \quad (28)$$

Since the constraints in (28) include the sufficient condition for robustness, the solution $u^{(0)}$ is robust against any uncertainty realization ω in $\mathcal{W}(\gamma)$. The margin γ computed by solving (28) is a guaranteed lower bound on the robustness margin. Problem (28) can be solved using convex optimization.

B. Robust AC OPF Algorithm

We next consider a setting where the system operator wants to solve the robust AC OPF problem (9)–(13) with a robustness margin of at least γ_{req}. Robust optimization is a "worst-case" approach where the constraints need to be satisfied for all uncertainty realizations and the objective function minimizes the maximum cost among all uncertainty realizations. We therefore introduce the variable c^u and the following constraint in order to upper bound the system-wide generation cost function (9) for all uncertainty realizations w in $\mathcal{W}(\gamma)$:

$$c^u \ge \sum_{i=1}^{n_g} c_i(p_{\text{g,ref},i} + \alpha_i \Delta^u). \quad (29)$$

where Δ^u is an over-estimator of the distributed slack imbalance variable Δ. The over-estimator Δ^u is an element of the vector $\tilde{z}$ in Theorem 1 and is therefore a decision variable.

Replacing the constraints (10)–(13) with the robust feasibility condition (21)–(26) from Theorem 1 and the objective function (9) with (29) yields a tractable optimization problem whose solution is certifiably robustly feasible for (9)–(13). Formally, consider the following optimization problem:

$$\underset{u,\tilde{z},g^{u}_{\mathcal{P}},g^{\ell}_{\mathcal{P}},c^u}{\text{minimize}} \quad c^u \ \text{ subject to (21)–(26), (29), } \gamma = \gamma_{\text{req}}, \tag{30}$$

where the conditions in (21)–(24) are constructed around a nominal dispatch point denoted as $u^{(k)}$. The dependence on $u^{(k)}$ appears in (27) where the Jacobian $J_{f,0}$ is evaluated at the given nominal point.

Due to this dependence on the nominal operating point, we use an iterative solution approach. We start by solving the nominal AC OPF problem with any existing algorithm [1] to determine the initial dispatch point $u^{(0)}$, which is not necessarily robustly feasible. We improve this dispatch point via a *sequential convex restriction* algorithm that iterates between (i) constructing (30) by updating the convex restriction based on the dispatch point $u^{(k)}$ and (ii) solving (30) to obtain a new dispatch point $u^{(k+1)}$.

The following steps describe our robust AC OPF solution algorithm:

Step 1: *Initialization*: Solve the nominal AC OPF problem, i.e., (9)–(13) with $\mathcal{W}(0) = \{w^{(0)}\}$. Set $u^{(0)}$ to the resulting dispatch point. Set $\gamma = \gamma_{\text{req}}$ and $k = 0$.

Step 2: Use Newton's method to solve the AC power flow equations for the dispatch point $u^{(k)}$ with the nominal uncertain variable $w^{(0)}$ in order to obtain the corresponding nominal internal states $x^{(k)}$.

Step 3: Set $x^{(k)}$ as the nominal operating point and evaluate (16) to update the convex restriction.

Step 4: Solve the robust AC OPF (30) and update $u^{(k+1)}$ to the associated dispatch solution.

Step 5: If the worst-case generation cost decreases from the previous iteration (i.e., $c^{u,(k+1)} < c^{u,(k)}$), repeat from Step 2 with $k = k + 1$. Otherwise, return the dispatch point $u^{(k)}$ and terminate the algorithm.

The dispatch point resulting from the proposed algorithm is guaranteed to be robustly feasible. To the best of our knowledge, this is the first algorithm that is able to rigorously certify robust feasibility for general AC OPF problems.

However, due to the conservativeness of the convex restrictions, our algorithm may not find the optimal solution to the robust AC OPF. In other words, there may exist lower cost dispatch points that are robustly feasible. Unfortunately, we cannot directly check if this is the case since there currently does not exist an algorithm that globally solves robust AC OPF problems. Nevertheless, we can obtain a bound on the optimality gap of our solution by comparing it to the solution of the nominal (non-robust) problem, which provides a lower bound on the cost of the robust solution. The empirical studies in the next section show that the robustly feasible dispatch points obtained from our algorithm often have generation costs that are close ($\lesssim 1\%$) to those of the nominal AC OPF solutions. The small difference between the nominal and robust objective values in our case studies hence indicates that our robustly feasible dispatch points are at least close to optimal.

VI. Numerical Studies

This section provides numerical studies to demonstrate our algorithms. These numerical studies were performed on a computer with a 3.3 GHz Intel Core i7 processor and 16 GB of RAM. Our implementations used JuMP/Julia [25]. The convex quadratic problems were solved with MOSEK. The power flow equations and the initial OPF problems were solved using PowerModels.jl [26] and IPOPT [27]. The numerical studies examine dispatch points subject to uncertainty from uniform and Gaussian distributions to establish a link between stochastic uncertainty and the robustness margin.

A. Illustrative Example using a 9-Bus System

We begin by considering the 9-bus system from [24] with uncertain loads at buses 5 and 7. The participation factors are set to 1 for the generator at bus 1 and 0 for the other generators.

1) Comparison of nominal and robust solution: We first consider normally distributed loads with mean equal to the nominal load $w^{(0)}$ and standard deviation equal to 10% of $w^{(0)}$ without correlation between loads. For our robust AC OPF algorithm, we use an uncertainty set $\mathcal{W}(\gamma_{\text{req}})$ that encloses two standard deviations of the considered load uncertainty by setting $\Sigma = \mathbf{diag}(p_{\text{d}}^2)$ and $\gamma_{\text{req}} = 0.2$. To assess the quality of our solutions, we compute the probability of constraint violations using 10,000 samples of the random loads.

The solutions to the nominal and robust OPF problems are shown in Figs. 2(a) and (b), respectively. For the nominal (non-robust) OPF solution in Fig. 2(a), the operational cost is 5296.69 \$/hr and constraint violations occur in 55.18% of the samples. The most common violations are the maximum voltage magnitude limits at buses 6 and 8. For the robust OPF solution in Fig. 2(b), we observe that all uncertainty realizations within the considered uncertainty set (red circle) are feasible, as guaranteed by our algorithm. The generation cost for the robust dispatch point (5342.99 \$/hr) is 0.87% greater than the cost for the nominal dispatch point, but only 0.13% of the samples result in the constraint violations.

2) Robustness Margin of a Given Dispatch Point: We now examine the robustness margin for correlated loads. Given the robust dispatch point from the previous experiment (the red dot in Fig. 2(b)), robustness margins are computed by solving (28). The legend of Fig. 2(c) gives the robustness margins γ obtained for different correlation matrices Σ, with the corresponding uncertainty sets shown by the ellipses.

3) Computation Time: Since the robustness condition is convex, these problems can be solved efficiently. For the 9-bus system, the robust AC OPF algorithm converged in two iterations, each taking an average of 0.0497 seconds to compute. The average computation time for obtaining the robustness margins is 0.0585 seconds.

B. Robustness vs. Cost Trade-Off for the IEEE 118-Bus System

We next study the trade-off between operating cost and the probability of constraint violations by solving the robust AC OPF problem for the IEEE 118-bus system [23]. Load uncertainty is modeled via a Gaussian distribution with variance

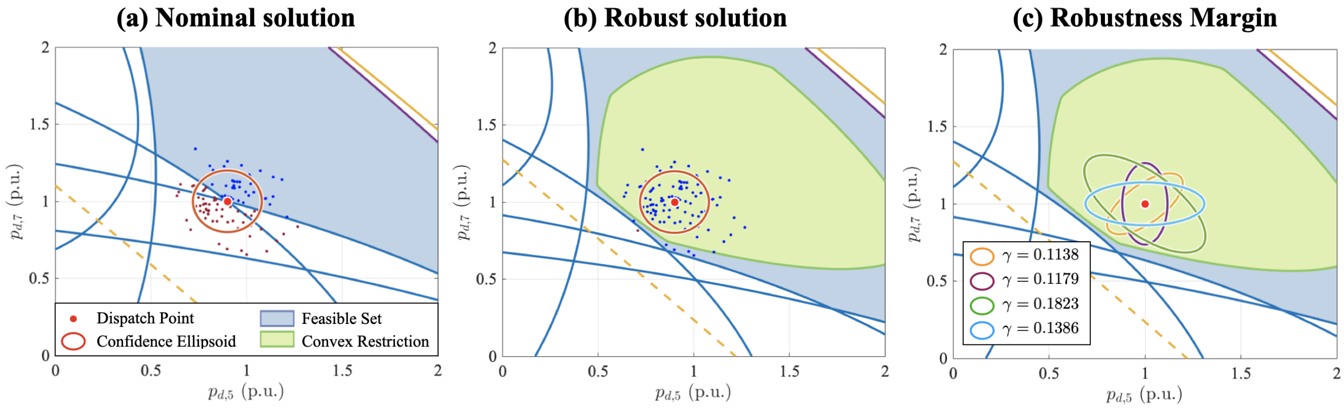

Fig. 2. Fig. 2(a) shows the nominal (non-robust) AC OPF solution for the 9-bus system from [24] with the feasible region (in blue) defined by uncertain load variation at buses 5 and 7. Scattered dots show 100 load samples drawn from Gaussian distribution where blue and brown dots represent safe and unsafe realizations, respectively. Fig. 2(b) shows the robust AC OPF solution obtained using the algorithm in Section V-B with a robustness margin equal to 20% of the nominal load. The feasible space is shown in blue and the convex restriction is shown in green. Sampled uncertainty realizations that satisfy the operational constraints are shown by the blue points. Samples resulting in constraint violations are plotted in brown. Observe that all samples within the considered uncertainty set (red circle) are feasible. Fig. 2(c) shows the robustness margins γ obtained by solving (30) for various correlation matrices Σ.

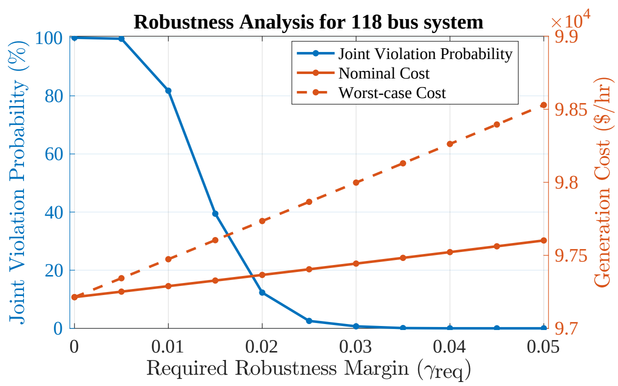

Fig. 3. The trade-off between generation cost and robustness for the IEEE 118-bus system. The x-axis shows the required robustness margin, γ_{req}. The solid and dashed red lines show the nominal and worst-case generation costs, $c^{(0)}$ and c^{u}, respectively. The blue lines show the joint probability of constraint violations with stochastic uncertainty.

equal to 1% of the nominal load. The results are plotted in Fig. 3 with the enforced robustness margin γ_{req} on the x-axis. The left y-axis (in blue) shows the empirically determined join violation probability, i.e., the probability that a sample from the considered uncertainty distribution violates one or more constraints. The right y-axis (in orange) shows both the nominal ($c^{(0)}$) and worst-case (c^{u}) generation costs.

We observe that increasing the robustness margin results in lower violation probability and higher generation cost. By examining this trade-off, operators can balance generation cost and robustness based on the assessed level of power injection uncertainty. For example, we can avoid constraint violations with 99% probability against random samples from the Gaussian distribution by setting $\gamma_{\text{req}} = 0.03$, which increases the cost relative to the nominal solution by 0.24%.

C. Numerical Studies using the PGLib Test Cases

Finally, we show the effectiveness of our robust AC OPF algorithm using the PGLib test cases [23] with sizes up to 179

TABLE I
NUMERICAL EXPERIMENTS ON PGLIB TEST CASES

Test Case	Nominal OPF cost ($/h)	Robust OPF cost ($/h)	Cost diff (%)	Solver time (sec)
case3	5812.64	5830.66	0.31	0.02
case5	17551.89	17630.34	0.45	0.04
case14	2178.08	2187.52	0.43	0.11
case24	63352.20	63599.15	0.39	0.31
case30_as	803.13	803.14	0.00	0.36
case30_fsr	575.77	577.74	0.34	0.45
case30_ieee	8208.52	8223.18	0.18	0.32
case39	138415.56	138638.33	0.16	0.56
case57	37589.34	37613.36	0.06	1.02
case73	189764.08	190194.12	0.23	2.30
case118	97213.61	97289.82	0.08	5.23
case162	108075.64	109236.54	1.07	17.83
case179	754266.41	761952.87	1.02	14.21

buses. Each generator's participation factor is proportional to the generator's capacity, i.e., $\alpha_i = (p^{\max}_{\text{g},i} - p^{\min}_{\text{g},i}) / \sum_{i=1}^{n_g} (p^{\max}_{\text{g},i} - p^{\min}_{\text{g},i})$. Table I compares the generation cost of the nominal (non-robust) solution obtained using PowerModels.jl [26] with our robust dispatch point considering the uncertainty set containing 1% demand fluctuations at every load bus (i.e., $\Sigma = \textbf{diag}(p_{\text{d}}^2)$ and $\gamma_{\text{req}} = 0.01$). For each case, the algorithm terminates after one iteration. Experiments show that the single iteration is sufficient given the nominal AC OPF solution is set as the initial point. The cost difference is approximately 1% or less, indicating that only a marginal trade-off in generation cost is necessary to achieve this level of robustness.

VII. CONCLUSION

Power system operating conditions are uncertain and variable. Accordingly, this paper has considered the robust AC optimal power flow problem to address inherent uncertainties in the net power injections. We presented a robust feasibility condition for the AC OPF problem along with associated algorithms for computing robustness margins and a robust dispatch point. These algorithms are the first to provide rigorous guarantees with respect to both power flow solvability and operational constraint satisfaction for robust AC OPF problems. Since the key steps consist of solving convex optimization problems, these algorithms are computationally tractable. We demonstrated the algorithms on a range of test cases and studied the trade-off between cost and robustness.

Building on the uncertain power injections considered in this paper, our ongoing work is studying more general uncertainties, including uncertainty in the transmission line parameters and network topology (e.g., $N-1$ contingencies).

Appendix

A. AC OPF Constraints with Basis Functions

The reactive power injections at PV buses are

$$\underbrace{C_{\mathrm{g,pv}}q_{\mathrm{g}} - C_{\mathrm{d,pv}}q_{\mathrm{d}}}_{q^{\mathrm{inj}}(q_g,w)} = \underbrace{\begin{bmatrix}\mathbf{0} & \mathbf{0} & -B^{\mathrm{c}}_{\mathrm{pv}} & G^{\mathrm{s}}_{\mathrm{pv}} & -B^{\mathrm{d}}_{\mathrm{pv}}\end{bmatrix}}_{L_{\mathrm{q}}}\psi(z,u). \tag{31}$$

Similarly, the transmission line flows are

$$\underbrace{\begin{bmatrix} s^{\mathrm{f}}_p \\ s^{\mathrm{f}}_q \end{bmatrix}}_{s^{\mathrm{f}}} = \underbrace{\begin{bmatrix}\mathbf{0} & \mathbf{0} & G_{\mathrm{ft}} & B_{\mathrm{ft}} & G_{\mathrm{ff}}E^T_{\mathrm{f}} \\ \mathbf{0} & \mathbf{0} & -B_{\mathrm{ft}} & G_{\mathrm{ft}} & -B_{\mathrm{ff}}E^T_{\mathrm{f}}\end{bmatrix}}_{L^{\mathrm{f}}_{\mathrm{line}}}\psi(z,u), \tag{32a}$$

$$\underbrace{\begin{bmatrix} s^{\mathrm{t}}_p \\ s^{\mathrm{t}}_q \end{bmatrix}}_{s^{\mathrm{t}}} = \underbrace{\begin{bmatrix}\mathbf{0} & \mathbf{0} & G_{\mathrm{tf}} & -B_{\mathrm{tf}} & G_{\mathrm{tt}}E^T_{\mathrm{t}} \\ \mathbf{0} & \mathbf{0} & -B_{\mathrm{tf}} & -G_{\mathrm{tf}} & -B_{\mathrm{tt}}E^T_{\mathrm{t}}\end{bmatrix}}_{L^{\mathrm{t}}_{\mathrm{line}}}\psi(z,u). \tag{32b}$$

We refer to [22] for the definitions of G_{ff}, G_{tt}, B_{ff}, etc.

B. Proof of Theorem 1

The condition in inequality (21) ensures that, for $i = 1,\ldots,2(n_l+n_{pq}+1)$,

$$\begin{aligned}
&\max_{w\in\mathcal{W}}\max_{x\in\mathcal{P}(\tilde{z})} K_i g(Cx,u) + R_i w \\
&\leq \max_{x\in\mathcal{P}(\tilde{z})}\left\{K_i^+ g^u(Cx,u) + K_i^- g^\ell(Cx,u)\right\} + \max_{w\in\mathcal{W}} R_i w \\
&\leq K_i^+ \max_{x\in\mathcal{P}(\tilde{z})} g^u(Cx,u) + K_i^- \min_{x\in\mathcal{P}(\tilde{z})} g^\ell(Cx,u) + \xi(\gamma) \\
&= K_i^+ g^u_{\mathcal{P}}(u,\tilde{x}) + K_i^- g^\ell_{\mathcal{P}}(u,\tilde{z}) + \xi(\gamma) \leq \tilde{z}_i.
\end{aligned}$$

Then the nonlinear map $G_{u,w} : x \rightarrow -J^{-1}_{F,(0)}Mg(x,u) + J^{-1}_{F,(0)}\widetilde{B}w$ in (14) is self-mapping with the set $\mathcal{P}(\tilde{z})$ for any realizations of $w\in\mathcal{W}(\gamma)$ (i.e., $\forall w\in\mathcal{W}(\gamma)$ and $\forall x\in\mathcal{P}(\tilde{z})$, $G_{u,w}(x)\in\mathcal{P}(\tilde{z})$). By Brouwer's fixed point theorem, there exists a fixed point for (14), which is equivalent to satisfying the AC power flow equations in (8). The inequalities in (22)–(24) ensure that x satisfies the operational constraints in (12)–(13) for all $x\in\mathcal{P}(\tilde{z})$. Thus, for each uncertainty realization $w\in\mathcal{W}(\gamma)$, there exists associated internal states x that satisfy the power flow equations and the operational constraints.

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