Recovering Voltage Sensitivities to Active and Reactive Power Injections in Distribution Systems

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***Abstract*—We analyze the sensitivity matrices relating active and reactive power injections to the voltage magnitudes—which are block submatrices of the inverse AC power flow Jacobian. We develop a simple method to recover these sensitivity matrices under low-observability conditions from advanced metering infrastructure (AMI) data via matrix completion techniques. The method provides an efficient approach for computing linearizations of the node voltages in the network, which can be updated iteratively. We also provide an analysis to show that these two sensitivity matrices take distinct values.**

***Index Terms*—matrix completion, sensitivity analysis, distribution systems, physics-based structured learning**

I. Introduction

The operation of distribution systems relies on software models of the network for engineering analyses. These models can be difficult to maintain and may contain errors, causing the results of the analysis to be inaccurate [1].

Sensitivity coefficients describe small changes in an independent variable of interest, Δx, to small changes in a dependent variable Δy [2]. The growth in sensor deployment such as advanced metering infrastructure (AMI) in distribution systems has spurred research interest in the development of engineering methods to recover these sensitivity coefficients solely from data. This allows for the network behavior to be approximated even when the model is inaccurate, out of date, or unavailable [3]. The coefficients form a linear approximation of the non-linear AC power flow equations, which is useful for modeling distribution networks where knowledge of the topology and circuit parameters is limited.

The contribution of this paper is a method to update or recover the matrices describing the sensitivity of voltage magnitudes to active and reactive power injections—which are submatrices of the inverse of the AC power flow Jacobian—by using limited amounts of AMI data as an input to several matrix completion algorithms. We also discuss in Section III that the entries of the two matrices take distinct values, which may be useful for predicting the impact of both active and reactive power injections with only voltage magnitude data as an input. In Section IV, we develop structured matrix completion algorithms to recover these matrices jointly from the AMI measurements.

II. Background and Problem Description

Recent works have shown the effectiveness of applying matrix completion techniques [4] to power system estimation problems, such as estimating low-observability voltage phasors [5] and evaluating voltage stability [6]. In this paper, we extend these ideas to discovering the behavior of a radial network in terms of the voltage magnitude sensitivities to active and reactive power. We rely on the results of [7] that establish unique solutions for the voltage phasor sensitivities for radial networks. Our work builds upon previous work on adaptive linearizations of the power flow equations [8] and measurement-based estimation methods for sensitivity factors [9], [10]. These methods assume access to voltage phase angles. We also draw inspiration from [11], which establishes theoretical guarantees for completion of high-rank matrices with low-rank subspaces.

The methods presented in this paper assume a three-phase, unbalanced radial distribution network, where the set $\mathcal{N}$ comprises the PQ nodes of the network. We assume that there is a single slack bus, and thus, three slack nodes. The results of this paper are primarily useful for distribution network analysis, because an important application of this method is when the physical system model is of dubious quality or unknown.

A. Data Input Assumptions

An advanced metering infrastructure (AMI) dataset $\mathcal{D}_i$ for nodes $i = 1, \ldots, n$, where $n \triangleq |\mathcal{N}|$ is defined as:

$$\mathcal{D}_i = \{\mathbf{X}_i^{(t)}\}_{t=1}^{m} = \{(V_i^{(t)}, P_i^{(t)}, Q_i^{(t)})\}_{t=1}^{m}, \tag{1}$$

where $V_i^{(t)}, P_i^{(t)}$, and $Q_i^{(t)}$ are the nodal voltage magnitude measurements, net active, and net reactive power injections at node i at time steps $t = 1, \ldots, m$, respectively. We will assume the errors of the AMI sensors to be normally distributed with variance that is on the order of 0.5%. AMI sensors typically have errors between 0.07% and 4% depending on the power quality of the load [12]. Throughout the paper we operate under the assumption that voltage regulating devices are held fixed throughout the system. In the next section, we will drop the superscript t for brevity.

B. The Newton-Raphson Power Flow

Consider the power balance equations for a node $i \in \mathcal{N}$:

$$P_i = V_i \sum_{k=1}^{n} V_k (G_{ik} \cos(\theta_i - \theta_k) + B_{ik} \sin(\theta_i - \theta_k)), \quad (2)$$

$$Q_i = V_i \sum_{k=1}^{n} V_k (G_{ik} \sin(\theta_i - \theta_k) + B_{ik} \cos(\theta_i - \theta_k)), \quad (3)$$

where V_i, V_k are the voltage magnitudes at node i and node k and G_{ik}, B_{ik} are the real and imaginary parts of the ik-th entry of the nodal admittance matrix $Y_{ik} \triangleq G_{ik} + jB_{ik}$, respectively. In order to solve the systems (2) and (3), a classical approach is the Newton-Raphson (NR) algorithm, which iteratively solves the system of equations (4):

$$\underbrace{\begin{bmatrix} \Delta\mathbf{P} \\ \Delta\mathbf{Q} \end{bmatrix}}_{(2n\times1)} = \underbrace{\left[\begin{array}{c|c} \frac{\partial\mathbf{P}}{\partial\boldsymbol{\theta}} & \frac{\partial\mathbf{P}}{\partial\mathbf{V}} \\ \hline \frac{\partial\mathbf{Q}}{\partial\boldsymbol{\theta}} & \frac{\partial\mathbf{Q}}{\partial\mathbf{V}} \end{array}\right]}_{(2n\times2n)} \underbrace{\begin{bmatrix} \Delta\boldsymbol{\theta} \\ \Delta\mathbf{V} \end{bmatrix}}_{(2n\times1)} = \boldsymbol{J} \begin{bmatrix} \Delta\boldsymbol{\theta} \\ \Delta\mathbf{V} \end{bmatrix}, \quad (4)$$

where $\Delta\mathbf{P}, \Delta\mathbf{Q} \in \mathbb{R}^n$ are vectors of small deviations in active and reactive power, respectively. The power flow Jacobian is known to be relatively constant with respect to small changes in power injections [9], [10], [13], [14]. Let us consider the block submatrices of the inverse power flow Jacobian. We refer to blocks of the inverse Jacobian as sensitivity matrices and their elements as sensitivity coefficients. Denote these blocks as $\left(\boldsymbol{S}_y^x\right)$. The inverse problem of (4) can be written as:

$$\underbrace{\begin{bmatrix} \Delta\boldsymbol{\theta} \\ \Delta\mathbf{V} \end{bmatrix}}_{(2n\times1)} = \underbrace{\left[\begin{array}{c|c} \left(\boldsymbol{S}_p^\theta\right) & \left(\boldsymbol{S}_q^\theta\right) \\ \hline \left(\boldsymbol{S}_p^v\right) & \left(\boldsymbol{S}_q^v\right) \end{array}\right]}_{(2n\times2n)} \underbrace{\begin{bmatrix} \Delta\mathbf{P} \\ \Delta\mathbf{Q} \end{bmatrix}}_{(2n\times1)} = \boldsymbol{J}^{-1} \begin{bmatrix} \Delta\mathbf{P} \\ \Delta\mathbf{Q} \end{bmatrix}. \quad (5)$$

C. Phaseless Approximation of the Power Flow Equations

Particularly in distribution systems, access to phase angle information $\Delta\boldsymbol{\theta}$ may be unavailable due to low penetrations of phasor measurement units (PMUs), making it difficult to analyze (4) and (5) in real time. However, the sensitivity matrices are relatively constant inter-temporally, allowing for model behavior to be approximated via linear methods [9], [10], [13], [14]. The voltage magnitude of node i, V_i, can be written as a first-order linear approximation around a given operating condition:

$$V_i \approx V_i^0 + \frac{\partial V_i}{\partial \mathbf{P}} \Delta\mathbf{P} + \frac{\partial V_i}{\partial \mathbf{Q}} \Delta\mathbf{Q}, \quad (6)$$

where V_i^0 is the voltage magnitude of node i in the given operating condition and $\frac{\partial V_i}{\partial \mathbf{P}}, \frac{\partial V_i}{\partial \mathbf{Q}}$ are the i-th rows of the matrices describing voltage magnitude sensitivities with respect to the power injections. From (5) we can write a rectangular linearized system which relates voltage magnitude variations to active and reactive power variations:

$$\underbrace{\Delta\mathbf{V}}_{(n\times1)} \approx \underbrace{\begin{bmatrix} \left(\boldsymbol{S}_p^v\right) & \left(\boldsymbol{S}_q^v\right) \end{bmatrix}}_{(n\times2n)} \underbrace{\begin{bmatrix} \Delta\mathbf{P} \\ \Delta\mathbf{Q} \end{bmatrix}}_{(2n\times1)} = \tilde{\boldsymbol{S}}\boldsymbol{x}. \quad (7)$$

III. Analytical Results

Prior numerical results have empirically indicated that the voltage magnitude sensitivities for active and reactive power injections are distinct [15]–[17]. In this section, we provide an analytical discussion to show that the voltage magnitude sensitivities to active and reactive power injections take distinct non-zero values in radial networks. The complex power injection S_i at node i can be related to the network's phasor voltages as:

$$S_i = \bar{V}_i \left(\sum_{j \in \mathcal{N} \cup \mathcal{S}} Y_{ij} \bar{V}_j \right)^* \forall i \in \mathcal{N}, \quad (8)$$

where $\bar{V}_i$ is the voltage phasor of node i and $(\cdot)^*$ denotes the complex conjugate operator. Following [7], differentiating (8) with respect to active and reactive power individually yields the linear differential equations (9) and (10) respectively, whose solutions are the sensitivities of *phasor* voltages to active and reactive power injections:

$$\delta_{il} = \frac{\partial \bar{V}_i^*}{\partial P_l} \sum_{j \in \mathcal{S} \cup \mathcal{N}} Y_{i,j} \bar{V}_j + \bar{V}_i^* \sum_{j \in \mathcal{N}} Y_{i,j} \frac{\partial \bar{V}_j}{\partial P_l}, \quad (9)$$

$$-j\delta_{il} = \frac{\partial \bar{V}_i^*}{\partial Q_l} \sum_{j \in \mathcal{S} \cup \mathcal{N}} Y_{i,j} \bar{V}_j + \bar{V}_i^* \sum_{j \in \mathcal{N}} Y_{i,j} \frac{\partial \bar{V}_j}{\partial Q_l}, \quad (10)$$

where δ_{il} is the Kronecker delta, defined as:

$$\delta_{il} = \begin{cases} 1 & \text{if } i = l, \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

The voltage *phasor* sensitivities to active and reactive power injections, $\frac{\partial \bar{V}_i}{\partial P_l}, \frac{\partial \bar{V}_i}{\partial Q_l}$ are of particular interest in distribution systems, because they are known to have a unique solution [7] for radial networks.

Remark 1. *For the nontrivial solutions of the equations in the systems* (9) *and* (10)*, i.e., where* $\delta_{il} \neq 0$*, the unknowns* $\frac{\partial \bar{V}_i}{\partial Q_l}$ *and* $\frac{\partial \bar{V}_i}{\partial P_l}$ *achieve distinct complex values.*

For the next propositions we denote by $\Re\{\cdot\}$ and $\Im\{\cdot\}$ the real and imaginary part operators, respectively.

Proposition 1. *The voltage magnitude sensitivity coefficients of a network can be written as* (12) *and* (13)*.*

$$\frac{\partial V_i}{\partial Q_l} = \frac{1}{V_i} \Re\left\{ \bar{V}_i^* \frac{\partial \bar{V}_i}{\partial Q_l} \right\}, \quad (12)$$

$$\frac{\partial V_i}{\partial P_l} = \frac{1}{V_i} \Re\left\{ \bar{V}_i^* \frac{\partial \bar{V}_i}{\partial P_l} \right\}. \quad (13)$$

Proof. Let V_i and θ_i be the real-valued magnitude and angle of the voltage phasor. Write the voltage phasor sensitivity at node i to a quantity X as:

$$\frac{\partial \bar{V}_i}{\partial X} = \frac{\partial}{\partial X}\{V_i e^{j\theta_i}\} = \frac{\partial V_i}{\partial X} e^{j\theta_i} + jV_i \frac{\partial \theta_i}{\partial X} e^{j\theta_i}, \tag{14}$$

so we have that:

$$e^{-j\theta_i}\frac{\partial \bar{V}_i}{\partial X} = \frac{\partial V_i}{\partial X} + jV_i \frac{\partial \theta_i}{\partial X} \implies \Re\left\{e^{-j\theta_i}\frac{\partial \bar{V}_i}{\partial X}\right\} = \frac{\partial V_i}{\partial X}. \tag{15}$$

Next, observe that $\bar{V}_i^*/V_i = e^{-j\theta}$. Therefore,

$$\Re\left\{e^{-j\theta_i}\frac{\partial \bar{V}_i}{\partial X}\right\} = \frac{1}{V_i}\Re\left\{\bar{V}_i^*\frac{\partial \bar{V}_i}{\partial X}\right\}, \tag{16}$$

$$\frac{\partial V_i}{\partial X} = \frac{1}{V_i}\Re\left\{\bar{V}_i^*\frac{\partial \bar{V}_i}{\partial X}\right\}. \tag{17}$$

Make $X = P_l$ or $X = Q_l$ to get the desired result. □

Next, we will show that if $\frac{\partial \bar{V}_i}{\partial Q_l}$ and $\frac{\partial \bar{V}_i}{\partial P_l}$ have unique solutions then we can say the same for the voltage magnitudes.

Proposition 2. *Let the rectangular form of the complex sensitivities be* $\frac{\partial \bar{V}_i}{\partial Q_l} = \alpha + j\beta$ *and* $\frac{\partial \bar{V}_i}{\partial P_l} = \gamma + j\delta$ *respectively. If we have:*

$$(\alpha, \beta) \notin \{(\alpha, \beta) : \Re\{\bar{V}_i\}\alpha + \Im\{\bar{V}_i\}\beta = 0\}, \tag{18}$$

$$(\gamma, \delta) \notin \{(\gamma, \delta) : \Re\{\bar{V}_i\}\gamma + \Im\{\bar{V}_i\}\delta = 0\}, \tag{19}$$

then $\frac{\partial V_i}{\partial Q_l} \neq \frac{\partial V_i}{\partial P_l}\ \forall i, l$.

Proof. Using (12), the voltage magnitude sensitivity coefficients for node i to reactive power injections at node l is:

$$\frac{\partial V_i}{\partial Q_l} = \frac{1}{V_i}\Re\left\{(\Re\{\bar{V}_i\} - j\Im\{\bar{V}_i\})\frac{\partial \bar{V}_i}{\partial Q_l}\right\}, \tag{20}$$

and in the same way, for active power, we have:

$$\frac{\partial V_i}{\partial P_l} = \frac{1}{V_i}\Re\left\{(\Re\{\bar{V}_i\} - j\Im\{\bar{V}_i\})\frac{\partial \bar{V}_i}{\partial P_l}\right\}. \tag{21}$$

Simplifying the above results in:

$$\frac{\partial V_i}{\partial Q_l} = \frac{1}{V_i}(\Re\{\bar{V}_i\}\alpha + \Im\{\bar{V}_i\}\beta), \tag{22}$$

$$\frac{\partial V_i}{\partial P_l} = \frac{1}{V_i}(\Re\{\bar{V}_i\}\gamma + \Im\{\bar{V}_i\}\delta). \tag{23}$$

From Remark 1, if $\frac{\partial \bar{V}_i}{\partial Q_l} \neq \frac{\partial \bar{V}_i}{\partial P_l}$ for nonzero solutions this implies that either $\alpha \neq \gamma$ or $\beta \neq \delta$. So, provided that the sensitivities are not zero, i.e., (18) and (19) are satisfied, then it must also be true that $\frac{\partial V_i}{\partial Q_l} \neq \frac{\partial V_i}{\partial P_l}\ \forall i, l$. □

Propositions 1 and 2 imply that the matrices $\boldsymbol{S}_p^v$, $\boldsymbol{S}_q^v$ in (5) are full rank for a radial network, and the matrix $\tilde{\boldsymbol{S}}$ has full column rank for any subset of the columns whose cardinality is less than $\frac{n}{2}$. Essentially, the voltage magnitude sensitivities to active and reactive power injections in a radial network take distinct values. Practically, what this means is that it is possible to quantify both active and reactive power impacts on distribution systems using only voltage magnitudes.

IV. Recovering Voltage Sensitivity Models

In this section, we develop methods for recovering the voltage magnitude sensitivity matrices in low-observability distribution system modeling settings. Specifically, we compute the best-fit estimate $\tilde{\boldsymbol{S}}^{\#}$ of the sensitivity matrix $\tilde{\boldsymbol{S}}$ via matrix completion techniques.

We assume that the sensitivity matrix $\tilde{\boldsymbol{S}}$ is "approximately low-rank", i.e., that the spectral content is highly skewed towards a few singular values, and there are many small singular values, as shown in Fig. 1. Therefore, we can approximate the matrix in terms of a truncated singular value decomposition (SVD) as:

$$\tilde{\boldsymbol{S}} \approx \sum_{k=1}^{R} \sigma_k \mathbf{u}_k \mathbf{v}_k^T, \tag{24}$$

where $\sigma_k, \mathbf{u}_k$, and $\mathbf{v}_k$, $k = 1, \ldots, R$ are the R largest singular values and corresponding singular vectors. The low-rank structure inherent in this matrix can be observed empirically, as shown in Fig. 1, which visualizes a spectral analysis of the voltage sensitivities for the IEEE 13-bus test feeder.

A. Low-Observability Modeling

Suppose that we have an incomplete sensitivity matrix $\tilde{\boldsymbol{S}}_0 := [\boldsymbol{S}_{p,0}^v, \boldsymbol{S}_{q,0}^v]$ where the set $\Omega = \{i, j : [\tilde{\boldsymbol{S}}_0]_{i,j} = 0\}$ represents $|\Omega|$ entries of $\tilde{\boldsymbol{S}}_0$ where we do not have access to voltage sensitivity relationships, as shown in Fig. 2. At observable nodes, the coefficients can be estimated from AMI data using linear regression if voltage regulation equipment is held fixed.

Remark 2. *The approximate low-rank structure of* $\tilde{\boldsymbol{S}}$ *results from the columns belonging to a union of multiple low-rank subspaces. Empirically, we have found these are related to groupings of the injection type (P/Q) and phase (A/B/C).*

The matrix describing the sensitivities of voltage magnitudes to active and reactive power injections can be recovered as the solution to the following program, subject to the constraint that the estimated matrix has rank R:

$$\tilde{\boldsymbol{S}}^{\#} = \underset{\boldsymbol{S}\in\mathbb{R}^{n\times 2n}}{\arg\min}\ ||\tilde{\boldsymbol{S}}_0 - \boldsymbol{S}||_F^2 \quad \text{s.t.} \quad \text{rank}(\boldsymbol{S}) = R, \tag{25}$$

where $||\cdot||_F^2$ is the squared Frobenius norm, which is defined for a matrix $\boldsymbol{X} \in \mathbb{R}^{d_1\times d_2}$ as $||\boldsymbol{X}||_F^2 = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2}|X_{i,j}|^2$.

The program (25) is non-convex, but a closed form solution can be tractably found by truncating the SVD as in (24). Choosing R is equivalent to tuning a real-valued hyperparameter $\lambda \geq 0$ in the Lagrangian of this program,

$$\tilde{\boldsymbol{S}}^{\#} = \underset{\boldsymbol{S}}{\arg\min}\left(||\tilde{\boldsymbol{S}}_0 - \boldsymbol{S}||_F^2 + \lambda\,(\text{rank}(\boldsymbol{S}))\right). \tag{26}$$

The rank constraint on the optimization variable $\tilde{\mathbf{S}}$ is also non-convex, and the solution requires *hard-thresholding*, i.e., selecting an integer R in (24). Additionally, we cannot solve (26) in this way, as we cannot take the truncated SVD of a matrix with unknown values. Following [4], this leads us to

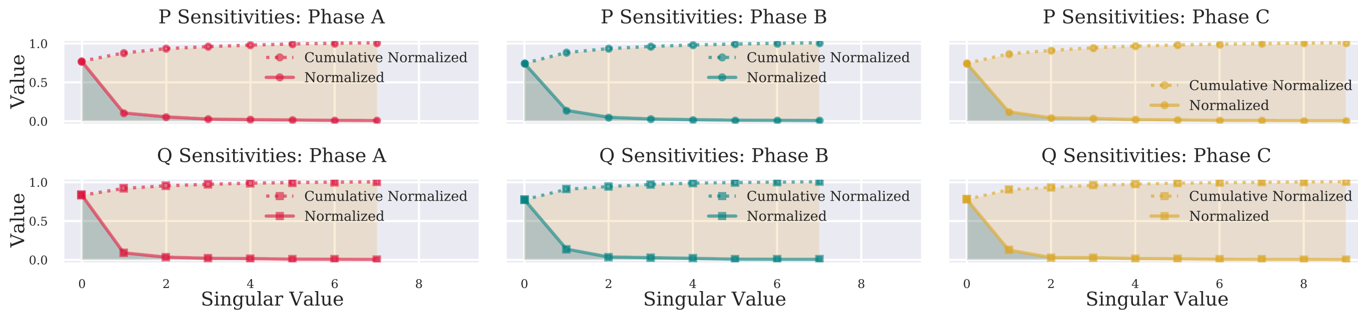

Figure 1. Spectral analysis of sensitivity matrix in (7) by phase and injection type for the IEEE 13-bus test feeder showing approximate low-rank structure.

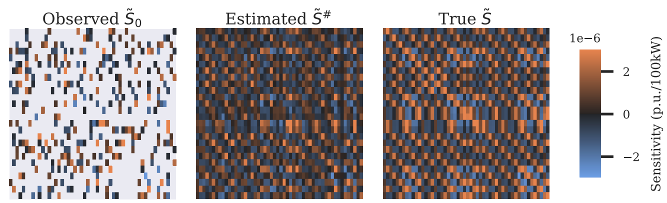

Figure 2. Example sensitivity matrix completion for the IEEE 13-bus feeder. Left: $\tilde{\boldsymbol{S}}_0$ with 25% observability. Center: Estimated matrix found with (27) with $\lambda = 2 \times 10^{-4}$. Right: "true" matrix computed with the perturb-and-observe technique in OpenDSS.

the convex relaxation (27), which replaces the rank penalty term with the *nuclear norm* of the decision matrix:

$$\begin{aligned} &\arg\min_{\boldsymbol{S}} \left(||\tilde{\boldsymbol{S}}_0 - \boldsymbol{S}||_F^2 + \lambda ||\boldsymbol{S}||_* \right), \\ &\text{s.t.} \quad ||\tilde{\boldsymbol{S}}_0 - \boldsymbol{S}_\Omega||_F \leq \delta, \end{aligned} \tag{27}$$

where $[\boldsymbol{S}_\Omega]_{i,j} = 0 \ \forall (i,j) \in \Omega$. The operator $|| \cdot ||_*$ is the nuclear norm, which is the sum of the singular values of $\boldsymbol{S}$. The hyperparameter δ reflects how accurately we wish to match the coefficients that are known beforehand in $\tilde{\boldsymbol{S}}_0$. The program (26) promotes solutions with skewed singular values, which are "approximately low-rank". This is also known as "soft-thresholding".

B. Online Model Update

We can use the real-time AMI datastreams in (1) to form an adaptive approximation of the sensitivity matrix at time t, $\tilde{\boldsymbol{S}}_t^{\#}$, by solving the online convex optimization problem (28):

$$\begin{aligned} \tilde{\boldsymbol{S}}_t^{\#} &= \arg\min_{\boldsymbol{S}} ||\Delta \mathbf{V}_t - \boldsymbol{S}\boldsymbol{x}_t||_2^2 + \lambda ||\boldsymbol{S}||_* + c \sum_{s=1}^{t-1} \gamma^s ||\tilde{\boldsymbol{S}}_{t-s}^{\#} - \boldsymbol{S}||_F^2, \\ &\text{s.t.} \quad ||\tilde{\boldsymbol{S}}_0 - \boldsymbol{S}_\Omega||_F \leq \delta. \end{aligned} \tag{28}$$

The summation term in the optimization is a penalty term: if we consider $\tilde{\boldsymbol{S}}_t^{\#}$ for all t as a time series, then the summation is equivalent to an exponential smoother. The time constant of the smoother is $\gamma \in (0,1)$ and the strength of this penalty term is given by hyperparameter c. The purpose of this term is to smooth out any sharp difference between the various $\tilde{\boldsymbol{S}}_t^{\#}$ at contiguous time steps. The voltage and power perturbations at time t are the vectors $\Delta \mathbf{V}_t \in \mathbb{R}^n, \ \boldsymbol{x}_t \in \mathbb{R}^{2n}$.

C. Other Applications

If some nodes of a network do not have AMI sensors, or if some sensors are deemed faulty or compromised, the set Ω can be used to represent these $|\Omega|$ suspect sensitivities, and a linearization of the node voltages at these locations can be recovered or updated via the methods of this paper to model to determine if the sensors are compromised.

The method does not require phase angle measurements from devices such as phasor measurement units (PMUs). While PMUs enhance the observability and accuracy of network modeling techniques, the cost of installing PMUs is high and their deployment is low in distribution systems. The future installation of such sensors may be intractable in rural or underserved areas, as they require very fast broadband communications and strong GPS signals. By adaptively linearizing equations for the node voltages using AMI data, the methods proposed in this paper allow engineers to approximate model dynamics without access to PMUs.

V. Case Study

We compute the voltage sensitivities to active and reactive power injections for the IEEE 13-bus test case using OpenDSS and the *perturb-and-observe* technique as a baseline for comparison. Note that this method is itself an approximation of the true sensitivity matrix. We use CVXPY [18] to design the matrix completion algorithms. We vary the number of observed sensitivity coefficients, $|\Omega|$, between 20-90% of the total entries. This represents a varying degree of AMI sensor penetration. We vary the value of the nuclear norm penalty λ between 1×10^{-6} and 8×10^{-6}. We fix $\delta = 6 \times 10^{-3}$. We reconstruct the active and reactive power sensitivities with a mean absolute error below 1.25×10^{-6} for all levels of data observability, as shown in Fig. 3.

As seen in Fig. 2, the IEEE 13-bus sensitivities range between -3×10^{-6} and 3×10^{-6} in our experiments. Therefore, the reconstruction error in Fig. 3 indicates that the matrix completion method performs reasonably well, even for a large number of unknown entries.

Using the online method (28), we choose a smoothing factor of $\gamma = 0.9$, a nuclear norm penalty of $\lambda = 1.25 \times 10^{-4}$, and a weighting factor of $c = 1 \times 10^{-8}$ for the smoothing term. We run the online optimization problem at 15-minute time steps

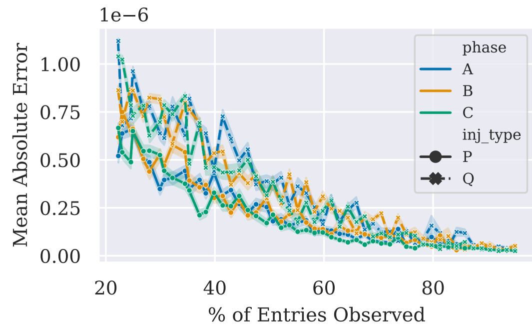

Figure 3. Mean absolute error for the entries of the IEEE 13-bus feeder sensitivity matrices by phase and injection type vs. percentage of entries known a priori. Shaded region represents a 95% bootstrap confidence interval.

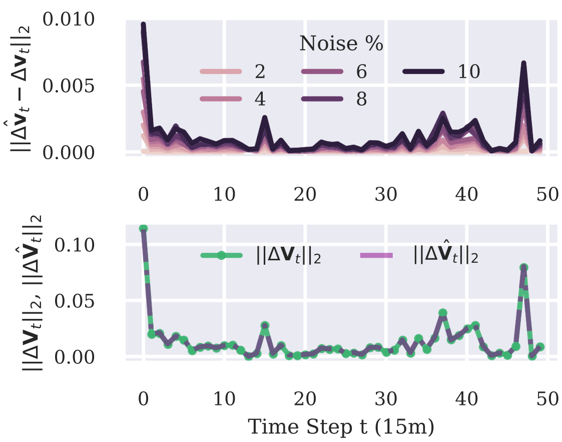

Figure 4. Online sensitivity estimation results. The top figure shows the Euclidean distance vs. time between the observed and predicted voltage deviations at all nodes for varying noise levels. The bottom figure shows the Euclidean norm vs. time of the predicted (purple) and the observed (green) voltage deviations for all nodes given the lowest noise level.

at 10 different noise levels, as shown in Fig. 4. A default OpenDSS load shape is used to generate the time-series.

The presented methods approximate the response of the voltage magnitude fluctuations of a radial network as a function of power perturbation states via a linear system of equations. This allows for an iterative linearization of the node voltages in the network using AMI data, including at nodes that may not have direct measurements. This is helpful for improving today's electric distribution network models, which are often error-prone [1]. These errors can cause inaccuracies in the model's simulation of underlying system physics, which the proposed method can help correct.

VI. Conclusion

This paper presented an analysis of the voltage magnitude sensitivity matrices for radial electrical networks. We showed that these matrices achieve distinct values and developed a method for recovering them in low-observability modeling scenarios. The methods introduced allow networks to be analyzed with significantly reduced data input requirements. Problems that typically require energy measurements can be solved using voltage magnitude data by exploiting the difference in the voltage sensitivities created by active and reactive power injections. The presented results are useful for updating a linearization of the node voltages in low-observability settings, or if the circuit model is of questionable quality.

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