Comparison of Various Trilinear Monomial Envelopes for Convex Relaxations of Optimal Power Flow Problems

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***Abstract*—Solutions to optimal power flow (OPF) problems provide operating points for electric power systems that minimize operational costs while satisfying both engineering limits and the power flow equations. OPF problems are non-convex and may have multiple local optima. To search for global optima, recent research has developed a variety of convex relaxations to bound the optimal objective values of OPF problems. Certain relaxations, such as the quadratic convex (QC) relaxation, are derived from OPF representations that contain trilinear monomials. Previous work has considered three techniques for relaxing these trilinear monomials: *recursive McCormick* (RMC) envelopes, *Meyer and Floudas* (MF) envelopes, and *extreme-point* (EP) envelopes. This paper compares the tightness and computational times resulting from relaxations that employ each of these techniques. Forming the convex hulls of the trilinear monomials, MF and EP envelopes are equivalently tight and are both tighter than the RMC envelopes. Empirical results indicate that the EP envelopes have advantages in computational speed and numerical stability when used with state-of-the-art second-order cone programming solvers.**

I. Introduction

Optimal power flow (OPF) is a fundamental problem in power system operation and control. OPF problems seek operating points that optimize a specified objective function (often generation cost minimization) subject to engineering limits and power flow constraints that model the network physics [1]. OPF problems are non-convex, may have multiple local solutions [2], and are generally NP-hard [3], [4]. Since being introduced by Carpentier in 1962 [5], many solution techniques have been developed for OPF problems [6], [7].

Recently, a plethora of convex relaxation techniques have been applied to OPF problems in order to compute bounds on the objective values and, in some cases, obtain the globally optimal decision variables. Convex relaxations can also certify the infeasibility of OPF problems and provide initializations for local solution algorithms [8]. Convex relaxations have been formulated as semidefinite programs [9]–[11], second-order cone programs (SOCP) [12]–[18], and linear programs [19]–[21]. A detailed survey is provided in [22].

Some relaxations, such as the quadratic convex (QC) relaxation, are derived using polar representations of the complex voltage phasors. Polar representations result in trilinear products consisting of the voltage magnitudes and trigonometric functions of voltage angle differences for each pair of connected buses. The corresponding non-convex trilinear monomials are relaxed using convex envelope enclosures. The tightness of these envelopes and their particular mathematical formulations significantly impact a relaxation's solution quality and computational tractability.

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Three formulations for these envelopes have been proposed in previous OPF relaxation literature: recursively applied McCormick (RMC) envelopes [15], Meyer and Floudas (MF) envelopes [23], and extreme-point (EP) envelopes [24]. RMC envelopes first form lifted variables representing voltage magnitude products using the McCormick envelope for bilinear monomials [25], and then use another McCormick envelope to represent the products of these lifted variables with variables corresponding to the trigonometic functions. Even though the McCormick envelopes yield the convex hulls of bilinear monomials, recursive application of these envelopes does not necessarily yield the convex hulls of trilinear monomials.

Meyer and Floudas derived envelopes constructed via sets of hyperplanes which form the convex hulls of trilinear monomials [26], [27]. Using MF envelopes instead of RMC envelopes tightens the QC relaxation [23]. The convex hulls of trilinear envelopes can also be formulated via an EP characterization [28]–[30]. EP envelopes are applied to tighten the QC relaxation in [24]. Both the MF and the EP envelopes form the convex hulls of the trilinear monomials and therefore result in equivalently tight relaxations. However, their mathematical representations are quite different, which can result in differing numerical performance.

To characterize the performance of various envelopes, this paper compares the solution quality and computational tractability resulting from each of these three approaches for handling trilinear monomials in QC relaxations of OPF problems. Applying each approach to a wide variety of test cases using various solvers indicates that the MF and the EP envelopes outperform the RMC technique in providing tighter envelopes for trilinear monomials. Application of multiple solvers indicates that EP and RMC envelopes are numerically stable on all the test cases with comparable computational speeds. MF envelopes yield numerical issues for some solvers.

This paper is organized as follows. Section II overviews the OPF problem. Section III reviews the QC relaxation of the OPF problem and presents different approaches for handling trilinear monomials. Section IV empirically compares each approach for various test cases. Section V concludes the paper.

II. Optimal Power Flow Overview

This section reviews an OPF formulation using a polar representation of the voltage phasors. The power system network is modeled by a graph ($\mathcal{N}$, $\mathcal{L}$) with $\mathcal{N}$ and $\mathcal{L}$ representing the sets of buses and branches, respectively. Let "ref" denote the reference bus. Let $P_i^d + \mathbf{j}Q_i^d$ and $P_i^g + \mathbf{j}Q_i^g$ represent the complex power demand and generation at bus $i \in \mathcal{N}$, where $\mathbf{j} = \sqrt{-1}$. Let $g_{sh,i} + \mathbf{j}b_{sh,i}$ denote the shunt admittance at bus $i \in \mathcal{N}$. Let V_i and θ_i represent the voltage magnitude and angle at bus $i \in \mathcal{N}$. For each generator $i \in \mathcal{N}$, define a quadratic generation cost function with coefficients $c_{2,i} \geqslant 0$, $c_{1,i}$, and $c_{0,i}$. Denote $\theta_{lm} = \theta_l - \theta_m$. Specified upper and lower limits are denoted by $(\overline{\cdot})$ and $(\underline{\cdot})$, respectively. Buses without generators have generation limits set to zero.

Each line $(l,m) \in \mathcal{L}$ is modeled as a Π circuit with mutual admittance $g_{lm} + \mathbf{j}b_{lm}$ and shunt susceptance $\mathbf{j}b_{c,lm}$. Denote the complex power flow on the line $(l,m) \in \mathcal{L}$ as $P_{lm} + \mathbf{j}Q_{lm}$.

Using these definitions, the OPF problem is

$$\min \sum_{i\in\mathcal{N}} c_{2,i}\,(P_i^g)^2 + c_{1,i}\,P_i^g + c_{0,i} \tag{1a}$$

subject to $\quad (\forall i \in \mathcal{N},\ \forall\,(l,m)\in\mathcal{L})$

$$P_i^g - P_i^d = g_{sh,i}\,V_i^2 + \sum_{\substack{(l,m)\in\mathcal{L}\\ \text{s.t. } l=i}} P_{lm} + \sum_{\substack{(l,m)\in\mathcal{L}\\ \text{s.t. } m=i}} P_{ml}, \tag{1b}$$

$$Q_i^g - Q_i^d = -b_{sh,i}\,V_i^2 + \sum_{\substack{(l,m)\in\mathcal{L}\\ \text{s.t. } l=i}} Q_{lm} + \sum_{\substack{(l,m)\in\mathcal{L}\\ \text{s.t. } m=i}} Q_{ml}, \tag{1c}$$

$$\theta_{ref} = 0, \tag{1d}$$

$$\underline{P}_i^g \leqslant P_i^g \leqslant \overline{P}_i^g, \quad \underline{Q}_i^g \leqslant Q_i^g \leqslant \overline{Q}_i^g, \tag{1e}$$

$$\underline{V}_i \leqslant V_i \leqslant \overline{V}_i, \quad \underline{\theta}_{lm} \leqslant \theta_{lm} \leqslant \overline{\theta}_{lm}, \tag{1f}$$

$$P_{lm} = g_{lm}V_l^2 - g_{lm}V_lV_m\cos(\theta_{lm}) - b_{lm}V_lV_m\sin(\theta_{lm}), \tag{1g}$$

$$\begin{aligned} Q_{lm} = &-(b_{lm} + b_{c,lm}/2)\,V_l^2 + b_{lm}V_lV_m\cos(\theta_{lm}) \\ &- g_{lm}V_lV_m\sin(\theta_{lm}), \end{aligned} \tag{1h}$$

$$P_{lm}^2 + Q_{lm}^2 \leqslant (\overline{S}_{lm})^2, \quad P_{ml}^2 + Q_{ml}^2 \leqslant (\overline{S}_{lm})^2. \tag{1i}$$

The quadratic objective (1a) minimizes the total generation cost. Constraints (1b) and (1c) enforce power balance at each bus. Constraint (1d) sets the angle reference. Constraints (1e)–(1f) limit the active and reactive power generation, voltage magnitudes, and angle differences between connected buses. Constraints (1g)–(1h) model the power flows on each line, and (1i) limits the apparent power flows into each line terminal. Note that (1) can be extended to more detailed transformer models, such as off-nominal tap ratios and non-zero phase shifts, which are used in computing our numerical results.

III. The QC Relaxation

The relevant nonlinear expressions in (1) are V_i^2, $\forall i \in \mathcal{N}$, $V_l\,V_m\cos(\theta_{lm})$, and $V_l\,V_m\sin(\theta_{lm})$, $\forall (l,m) \in \mathcal{L}$.[1] The QC relaxation encloses these expressions in convex envelopes.

A. Squared voltage magnitude and trigonometric envelopes

The envelope $\langle x^2\rangle^T$ is the convex hull of the squared function:

$$\langle x^2\rangle^T = \left\{\breve{x} : \begin{cases}\breve{x} \geqslant x^2, & \breve{x} \leqslant (\overline{x} + \underline{x})\,x - \overline{x}\underline{x},\end{cases}\right. \tag{2}$$

where $\breve{x}$ is a "lifted" variable representing the set. Squared voltage magnitudes are relaxed as $w_{ii} \in \langle V_i^2\rangle^T$.

Envelopes for the sine and cosine functions are

$$\langle \sin(x)\rangle^S = \left\{\breve{S} : \begin{cases}\breve{S} \leqslant \cos\left(\frac{x^m}{2}\right)\left(x - \frac{x^m}{2}\right) + \sin\left(\frac{x^m}{2}\right), \\ \breve{S} \geqslant \cos\left(\frac{x^m}{2}\right)\left(x + \frac{x^m}{2}\right) - \sin\left(\frac{x^m}{2}\right),\end{cases}\right. \tag{3a}$$

$$\langle \cos(x)\rangle^C = \left\{\breve{C} : \begin{cases}\breve{C} \leqslant 1 - \frac{1-\cos(x^m)}{(x^m)^2}x^2, \\ \breve{C} \geqslant \frac{\cos(\underline{x}) - \cos(\overline{x})}{\underline{x} - \overline{x}}\,(x - \underline{x}) + \cos(\underline{x}).\end{cases}\right. \tag{3b}$$

[1]The objective (1d) and constraint (1i) are representable as SOCPs.

where $x^m = \max(|\underline{x}|, |\overline{x}|)$ and the lifted variables $\breve{S}$ and $\breve{C}$ represent the corresponding set. For each line $(l,m) \in \mathcal{L}$, the QC relaxation is strengthened via constraints proposed in [31] that relate the squared magnitudes of current flows, ℓ_{lm}, the squared voltage magnitudes, and the power flows on the lines:

$$P_{lm} + P_{ml} = \frac{g_{lm}}{g_{lm}^2 + b_{lm}^2}\left(\ell_{lm} + \frac{b_{c,lm}^2}{4}V_l^2 + b_{c,lm}\,Q_{lm}\right), \tag{4a}$$

$$\begin{aligned} Q_{lm} + Q_{ml} = &\frac{-b_{lm}}{g_{lm}^2 + b_{lm}^2}\left(\ell_{lm} + \frac{b_{c,lm}^2}{4}V_l^2 + b_{c,lm}\,Q_{lm}\right) \\ &- (b_{c,lm}/2)\left(V_l^2 + V_m^2\right), \end{aligned} \tag{4b}$$

$$P_{lm}^2 + Q_{lm}^2 \leqslant V_l^2\,\ell_{lm}. \tag{4c}$$

Relaxing $\sin(\theta_{lm})$ and $\cos(\theta_{lm})$ via $s_{lm} \in \langle\sin(\theta_{lm})\rangle^S$ and $c_{lm} \in \langle\cos(\theta_{lm})\rangle^C$ yields the trilinear monomials $V_l\,V_m\,s_{lm}$ and $V_l\,V_m\,c_{lm}$, $\forall (l,m) \in \mathcal{L}$. This section next presents various relaxations of these monomials.

B. Recursive McCormick envelopes for trilinear monomials

The McCormick envelope $\langle x\,y\rangle^M$ forms the convex hull of the bilinear monomial xy. A McCormick envelope is formulated using four linear inequality constraints:

$$\langle x\,y\rangle^M = \left\{\widetilde{xy} : \begin{cases}\widetilde{xy} \geqslant \underline{x}y + \underline{y}x - \underline{x}\underline{y}, & \widetilde{xy} \geqslant \overline{x}y + \overline{y}x - \overline{x}\overline{y}, \\ \widetilde{xy} \leqslant \underline{x}y + \overline{y}x - \underline{x}\overline{y}, & \widetilde{xy} \leqslant \overline{x}y + \underline{y}x - \overline{x}\underline{y},\end{cases}\right. \tag{5}$$

where $\widetilde{xy}$ is a lifted variable. To address trilinear monomials, the QC relaxation in [15] recursively applies McCormick envelopes by first constructing a lifted variable w_{lm} that relaxes the product of the voltage magnitudes, $V_l\,V_m$, i.e., $w_{lm} \in \langle V_l\,V_m\rangle^M$ for all $(l,m) \in \mathcal{L}$. McCormick envelopes are then again applied to represent the trilinear monomials $V_l\,V_m\,s_{lm}$ and $V_l\,V_m\,c_{lm}$ as $w_{s,lm} \in \langle w_{lm}\,s_{lm}\rangle^M$ and $w_{c,lm} \in \langle w_{lm}\,c_{lm}\rangle^M$, respectively, for all $(l,m) \in \mathcal{L}$.

Recursive McCormick envelopes do not generally yield the convex hulls of the trilinear monomials [30], [32]. The following sections describe two tighter envelopes that yield the convex hulls of the trilinear monomials.

C. Meyer and Floudas envelopes for trilinear monomials

MF envelopes [26], [27] are hyperplane representations of the convex hulls of trilinear monomials. Accordingly, MF envelopes are generally tighter than recursive McCormick envelopes. MF envelopes are formed using linear inequalities that are applied based on the signs of the bounds on the variables that make up the trilinear monomials. We denote the envelopes for $V_l\,V_m\,s_{lm}$ and $V_l\,V_m\,c_{lm}$ as $w_{s,lm} \in \langle V_l\,V_m\,s_{lm}\rangle^{MF}$ and $w_{c,lm} \in \langle V_l\,V_m\,c_{lm}\rangle^{MF}$, respectively, $\forall (l,m) \in \mathcal{L}$.

The cases that are relevant to the monomials $V_l\,V_m\,s_{lm}$, $\forall (l,m) \in \mathcal{L}$, are presented in the boxes denoted "Cases I–VII", where the subscripts on the s_{lm} variable bounds are dropped for notational brevity. The upper portion of each box gives the conditions which must all be satisfied for the constraints in the lower portion to apply. Note that multiple cases apply simultaneously (e.g., Case IV implies Case I).

The same procedure is applied for the monomials $V_l\,V_m\,c_{lm}$, $\forall (l,m) \in \mathcal{L}$, with c_{lm} replacing s_{lm}. Since the cosine function is non-negative in the first and fourth quadrants, only Cases II and III are applicable for these monomials.

Case I: $\overline{s} \leqslant 0$.

$$
\begin{aligned}
\check{s}_{lm} &\geqslant \overline{V}_m\underline{s}V_l + \underline{V}_l\underline{s}V_m + \underline{V}_l\underline{V}_m\check{S} - \underline{V}_l\overline{V}_m\underline{s} - \underline{V}_l\underline{V}_m\underline{s},\\
\check{s}_{lm} &\geqslant \overline{V}_m\underline{s}V_l + \underline{V}_l\overline{s}V_m + \underline{V}_l\overline{V}_m\check{S} - \underline{V}_l\overline{V}_m\underline{s} - \underline{V}_l\overline{V}_m\overline{s},\\
\check{s}_{lm} &\geqslant \underline{V}_m\overline{s}V_l + \overline{V}_l\underline{s}V_m + \overline{V}_l\underline{V}_m\check{S} - \overline{V}_l\underline{V}_m\overline{s} - \overline{V}_l\underline{V}_m\underline{s},\\
\check{s}_{lm} &\geqslant \underline{V}_m\overline{s}V_l + \overline{V}_l\overline{s}V_m + \overline{V}_l\overline{V}_m\check{S} - \overline{V}_l\underline{V}_m\overline{s} - \overline{V}_l\overline{V}_m\overline{s},\\
\check{s}_{lm} &\geqslant \underline{V}_m\underline{s}V_l + \overline{V}_l\underline{s}V_m + \underline{V}_l\underline{V}_m\check{S} - \overline{V}_l\underline{V}_m\underline{s} - \underline{V}_l\underline{V}_m\underline{s},\\
\check{s}_{lm} &\geqslant \overline{V}_m\overline{s}V_l + \underline{V}_l\overline{s}V_m + \overline{V}_l\overline{V}_m\check{S} - \overline{V}_l\overline{V}_m\overline{s} - \underline{V}_l\overline{V}_m\overline{s}.
\end{aligned}
$$

Case II: $\underline{s} \geqslant 0$.

$$
\begin{aligned}
\check{s}_{lm} &\leqslant \underline{V}_m\underline{s}V_l + \overline{V}_l\underline{s}V_m + \overline{V}_l\overline{V}_m\check{S} - \overline{V}_l\overline{V}_m\underline{s} - \overline{V}_l\underline{V}_m\underline{s},\\
\check{s}_{lm} &\leqslant \overline{V}_m\underline{s}V_l + \underline{V}_l\underline{s}V_m + \overline{V}_l\overline{V}_m\check{S} - \overline{V}_l\overline{V}_m\underline{s} - \underline{V}_l\overline{V}_m\underline{s},\\
\check{s}_{lm} &\leqslant \underline{V}_m\underline{s}V_l + \overline{V}_l\overline{s}V_m + \overline{V}_l\underline{V}_m\check{S} - \overline{V}_l\underline{V}_m\overline{s} - \overline{V}_l\underline{V}_m\underline{s},\\
\check{s}_{lm} &\leqslant \overline{V}_m\overline{s}V_l + \underline{V}_l\underline{s}V_m + \underline{V}_l\overline{V}_m\check{S} - \underline{V}_l\overline{V}_m\overline{s} - \underline{V}_l\overline{V}_m\underline{s},\\
\check{s}_{lm} &\leqslant \underline{V}_m\overline{s}V_l + \overline{V}_l\overline{s}V_m + \underline{V}_l\underline{V}_m\check{S} - \overline{V}_l\underline{V}_m\overline{s} - \underline{V}_l\underline{V}_m\overline{s},\\
\check{s}_{lm} &\leqslant \overline{V}_m\overline{s}V_l + \underline{V}_l\overline{s}V_m + \underline{V}_l\underline{V}_m\check{S} - \underline{V}_l\overline{V}_m\overline{s} - \underline{V}_l\underline{V}_m\overline{s}.
\end{aligned}
$$

Case III: $\underline{s} \geqslant 0$. Map $\{V_l, V_m, s\}$ to $\{x, y, z\}$ such that $\overline{x}\underline{y}\underline{z} + \underline{x}\overline{y}\overline{z} \leqslant \underline{x}\overline{y}\underline{z} + \overline{x}\underline{y}\overline{z}$ and $\overline{x}\underline{y}\underline{z} + \underline{x}\overline{y}\overline{z} \leqslant \overline{x}\overline{y}\underline{z} + \underline{x}\underline{y}\overline{z}$.

$$
\begin{aligned}
\check{s}_{lm} &\geqslant \underline{y}\underline{z}x + \underline{x}\underline{z}y + \underline{x}\underline{y}z - 2\underline{x}\underline{y}\underline{z},\\
\check{s}_{lm} &\geqslant \overline{y}\overline{z}x + \overline{x}\overline{z}y + \overline{x}\overline{y}z - 2\overline{x}\overline{y}\overline{z},\\
\check{s}_{lm} &\geqslant \underline{y}\overline{z}x + \underline{x}\overline{z}y + \overline{x}\underline{y}z - \underline{x}\underline{y}\overline{z} - \overline{x}\underline{y}\overline{z},\\
\check{s}_{lm} &\geqslant \overline{y}\underline{z}x + \overline{x}\underline{z}y + \underline{x}\overline{y}z - \overline{x}\overline{y}\underline{z} - \underline{x}\overline{y}\underline{z},\\
\check{s}_{lm} &\geqslant \frac{\Lambda_3}{\overline{x}-\underline{x}}x + \overline{x}\underline{z}y + \overline{x}\underline{y}z - \frac{\Lambda_3\underline{x}}{\overline{x}-\underline{x}} - \overline{x}\overline{y}\underline{z} - \overline{x}\underline{y}\overline{z} + \underline{x}\overline{y}\overline{z},\\
&\quad \text{where } \Lambda_3 = \overline{x}\overline{y}\underline{z} - \underline{x}\overline{y}\overline{z} - \overline{x}\underline{y}\underline{z} + \overline{x}\underline{y}\overline{z},\\
\check{s}_{lm} &\geqslant \frac{\Gamma_3}{\underline{x}-\overline{x}}x + \underline{x}\overline{z}y + \underline{x}\overline{y}z - \frac{\Gamma_3\overline{x}}{\underline{x}-\overline{x}} - \underline{x}\underline{y}\overline{z} - \underline{x}\overline{y}\underline{z} + \overline{x}\underline{y}\underline{z},\\
&\quad \text{where } \Gamma_3 = \underline{x}\underline{y}\overline{z} - \overline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\overline{z} + \underline{x}\overline{y}\underline{z}.
\end{aligned}
$$

D. Extreme point envelopes for trilinear monomials

EP envelopes capture the convex hulls of trilinear monomials, or multilinears in general, in a vertex representation [28]. Given a set X, a point $p \in X$ is *extreme* if it cannot be expressed as a convex combination of two distinct points from X, i.e., there *do not* exist two other distinct points $p_1, p_2 \in X$ and a non-negative multiplier $\lambda \in (0, 1)$ such that $p = \lambda p_1 + (1-\lambda)p_2$. Based on this definition of an extreme point, we now describe the convex envelope.

Let $\phi(x, y, z) = xyz$ be any trilinear term with respective variable bounds $[\underline{x}, \overline{x}], [\underline{y}, \overline{y}], [\underline{z}, \overline{z}]$. The extreme points of $\phi(\cdot)$ are given by the Cartesian product $(\underline{x}, \overline{x}) \times (\underline{y}, \overline{y}) \times (\underline{z}, \overline{z}) = \langle \xi_1, \xi_2, \ldots, \xi_8 \rangle$. We use ξ_k^i to denote the coordinate of x_i in ξ_k. The convex hull of the extreme points of $\phi(\cdot)$ is given by

$$
\check{x} = \sum_{k=1,\ldots,8} \lambda_k \, \phi(\xi_k), \quad x_i = \sum_{k=1,\ldots,8} \lambda_k \, \xi_k^i, \tag{6a}
$$

$$
\sum_{k=1,\ldots,8} \lambda_k = 1, \quad \lambda_k \geqslant 0, \quad k = 1, \ldots, 8. \tag{6b}
$$

Given a lifted variable $\check{x}$, the notation $\check{x} \in \langle xyz \rangle^{EP}$ represents the λ-based convex hull envelope of a trilinear term as in (6).

E. Formulation of the QC relaxation

Using the envelopes described above, the QC relaxation replaces the relevant nonlinearities in the OPF problem (1) to construct an SOCP:

$$
\min \quad \sum_{i \in \mathcal{G}} c_{2i} \, (P_i^g)^2 + c_{1i} \, P_i^g + c_{0i} \tag{7a}
$$

subject to $\quad (\forall i \in \mathcal{N}, \ \forall \, (l, m) \in \mathcal{L})$

Case IV: $\overline{s} \leqslant 0$,

$$
\begin{aligned}
&\underline{V}_l\underline{V}_m\underline{s} + \overline{V}_l\overline{V}_m\overline{s} \geqslant \overline{V}_l\underline{V}_m\underline{s} + \underline{V}_l\overline{V}_m\overline{s},\\
&\underline{V}_l\underline{V}_m\underline{s} + \overline{V}_l\overline{V}_m\overline{s} \geqslant \underline{V}_l\overline{V}_m\underline{s} + \overline{V}_l\underline{V}_m\overline{s}.
\end{aligned}
$$

$$
\begin{aligned}
\check{s}_{lm} &\leqslant \underline{V}_m\overline{s}V_l + \underline{V}_l\overline{s}V_m + \underline{V}_l\underline{V}_m\check{S} - 2\underline{V}_l\underline{V}_m\overline{s},\\
\check{s}_{lm} &\leqslant \overline{V}_m\underline{s}V_l + \overline{V}_l\underline{s}V_m + \overline{V}_l\overline{V}_m\check{S} - 2\overline{V}_l\overline{V}_m\underline{s},\\
\check{s}_{lm} &\leqslant \underline{V}_m\underline{s}V_l + \overline{V}_l\overline{s}V_m + \overline{V}_l\underline{V}_m\check{S} - \overline{V}_l\underline{V}_m\overline{s} - \overline{V}_l\underline{V}_m\underline{s},\\
\check{s}_{lm} &\leqslant \overline{V}_m\overline{s}V_l + \underline{V}_l\underline{s}V_m + \underline{V}_l\overline{V}_m\check{S} - \underline{V}_l\overline{V}_m\overline{s} - \underline{V}_l\overline{V}_m\underline{s},\\
\check{s}_{lm} &\leqslant \underline{V}_m\underline{s}V_l + \underline{V}_l\underline{s}V_m + \frac{\Lambda_4}{\underline{s}-\overline{s}}\check{S} - \frac{\Lambda_4\overline{s}}{\underline{s}-\overline{s}} - \overline{V}_l\underline{V}_m\underline{s}\\
&\quad - \underline{V}_l\overline{V}_m\underline{s} + \overline{V}_l\overline{V}_m\overline{s},\\
&\quad \text{where } \Lambda_4 = \overline{V}_l\underline{V}_m\underline{s} - \overline{V}_l\overline{V}_m\overline{s} - \underline{V}_l\underline{V}_m\underline{s} + \underline{V}_l\overline{V}_m\underline{s},\\
\check{s}_{lm} &\leqslant \overline{V}_m\overline{s}V_l + \overline{V}_l\overline{s}V_m - \frac{\Gamma_4}{\overline{s}-\underline{s}}\check{S} - \frac{\Gamma_4\underline{s}}{\overline{s}-\underline{s}} - \overline{V}_l\underline{V}_m\overline{s}\\
&\quad - \underline{V}_l\overline{V}_m\overline{s} + \underline{V}_l\underline{V}_m\underline{s},\\
&\quad \text{where } \Gamma_4 = \overline{V}_l\underline{V}_m\overline{s} - \underline{V}_l\underline{V}_m\underline{s} - \overline{V}_l\overline{V}_m\overline{s} + \underline{V}_l\overline{V}_m\overline{s}.
\end{aligned}
$$

Case V: $\overline{s} \leqslant 0$,

$$
\begin{aligned}
&\overline{V}_l\underline{V}_m\underline{s} + \underline{V}_l\overline{V}_m\overline{s} \geqslant \underline{V}_l\overline{V}_m\underline{s} + \overline{V}_l\underline{V}_m\overline{s},\\
&\underline{V}_l\underline{V}_m\underline{s} + \overline{V}_l\overline{V}_m\overline{s} < \overline{V}_l\underline{V}_m\underline{s} + \underline{V}_l\overline{V}_m\overline{s},\\
&\underline{V}_l\underline{V}_m\underline{s} + \overline{V}_l\overline{V}_m\overline{s} < \underline{V}_l\overline{V}_m\underline{s} + \overline{V}_l\underline{V}_m\overline{s}.
\end{aligned}
$$

$$
\begin{aligned}
\check{s}_{lm} &\leqslant \underline{V}_m\overline{s}V_l + \underline{V}_l\overline{s}V_m + \underline{V}_l\underline{V}_m\check{S} - 2\underline{V}_l\underline{V}_m\overline{s},\\
\check{s}_{lm} &\leqslant \overline{V}_m\underline{s}V_l + \overline{V}_l\underline{s}V_m + \overline{V}_l\overline{V}_m\check{S} - 2\overline{V}_l\overline{V}_m\underline{s},\\
\check{s}_{lm} &\leqslant \underline{V}_m\underline{s}V_l + \underline{V}_l\underline{s}V_m + \underline{V}_l\overline{V}_m\check{S} - \underline{V}_l\underline{V}_m\underline{s} - \underline{V}_l\overline{V}_m\underline{s},\\
\check{s}_{lm} &\leqslant \overline{V}_m\overline{s}V_l + \overline{V}_l\overline{s}V_m + \overline{V}_l\underline{V}_m\check{S} - \overline{V}_l\underline{V}_m\overline{s} - \overline{V}_l\overline{V}_m\overline{s},\\
\check{s}_{lm} &\leqslant \underline{V}_m\underline{s}V_l + \frac{\Lambda_5}{\underline{V}_m-\overline{V}_m}V_m + \overline{V}_l\underline{V}_m\check{S} - \frac{\Lambda_5\overline{V}_m}{\underline{V}_m-\overline{V}_m}\\
&\quad - \underline{V}_l\underline{V}_m\underline{s} - \overline{V}_l\underline{V}_m\overline{s} + \underline{V}_l\overline{V}_m\overline{s},\\
&\quad \text{where } \Lambda_5 = \underline{V}_l\underline{V}_m\underline{s} - \underline{V}_l\overline{V}_m\overline{s} - \overline{V}_l\underline{V}_m\underline{s} + \overline{V}_l\underline{V}_m\overline{s},\\
\check{s}_{lm} &\leqslant \overline{V}_m\overline{s}V_l + \frac{\Gamma_5}{\overline{V}_m-\underline{V}_m}V_m + \underline{V}_l\overline{V}_m\check{S} - \frac{\Gamma_5\underline{V}_m}{\overline{V}_m-\underline{V}_m}\\
&\quad - \underline{V}_l\overline{V}_m\underline{s} - \overline{V}_l\overline{V}_m\overline{s} + \overline{V}_l\underline{V}_m\underline{s},\\
&\quad \text{where } \Gamma_5 = \underline{V}_l\overline{V}_m\underline{s} - \overline{V}_l\underline{V}_m\underline{s} - \underline{V}_l\overline{V}_m\overline{s} + \overline{V}_l\overline{V}_m\overline{s}.
\end{aligned}
$$

Case VI: $\underline{s} \leqslant 0, \overline{s} \geqslant 0$.

$$
\begin{aligned}
\check{s}_{lm} &\geqslant \overline{V}_m\overline{s}V_l + \overline{V}_l\overline{s}V_m + \overline{V}_l\overline{V}_m\check{S} - 2\overline{V}_l\overline{V}_m\overline{s},\\
\check{s}_{lm} &\geqslant \overline{V}_m\underline{s}V_l + \underline{V}_l\overline{s}V_m + \underline{V}_l\overline{V}_m\check{S} - \underline{V}_l\overline{V}_m\underline{s} - \underline{V}_l\overline{V}_m\overline{s},\\
\check{s}_{lm} &\geqslant \overline{V}_m\underline{s}V_l + \underline{V}_l\underline{s}V_m + \underline{V}_l\underline{V}_m\check{S} - \underline{V}_l\overline{V}_m\underline{s} - \underline{V}_l\underline{V}_m\underline{s},\\
\check{s}_{lm} &\geqslant \underline{V}_m\overline{s}V_l + \overline{V}_l\underline{s}V_m + \overline{V}_l\underline{V}_m\check{S} - \overline{V}_l\underline{V}_m\overline{s} - \overline{V}_l\underline{V}_m\underline{s},\\
\check{s}_{lm} &\geqslant \underline{V}_m\underline{s}V_l + \overline{V}_l\underline{s}V_m + \underline{V}_l\underline{V}_m\check{S} - \overline{V}_l\underline{V}_m\underline{s} - \underline{V}_l\underline{V}_m\underline{s},\\
\check{s}_{lm} &\geqslant \underline{V}_m\overline{s}V_l + \underline{V}_l\overline{s}V_m + \frac{\Lambda_6}{\overline{s}-\underline{s}}\check{S} - \frac{\Lambda_6\underline{s}}{\overline{s}-\underline{s}} - \underline{V}_l\overline{V}_m\overline{s}\\
&\quad - \overline{V}_l\underline{V}_m\overline{s} + \overline{V}_l\overline{V}_m\underline{s},\\
&\quad \text{where } \Lambda_6 = \underline{V}_l\overline{V}_m\overline{s} - \overline{V}_l\overline{V}_m\underline{s} - \underline{V}_l\underline{V}_m\overline{s} + \overline{V}_l\underline{V}_m\overline{s}.
\end{aligned}
$$

Case VII: $\underline{s} \leqslant 0, \overline{s} \geqslant 0$.

$$
\begin{aligned}
\check{s}_{lm} &\leqslant \overline{V}_m\underline{s}V_l + \overline{V}_l\underline{s}V_m + \overline{V}_l\overline{V}_m\check{S} - 2\overline{V}_l\overline{V}_m\underline{s},\\
\check{s}_{lm} &\leqslant \underline{V}_m\underline{s}V_l + \overline{V}_l\overline{s}V_m + \overline{V}_l\underline{V}_m\check{S} - \overline{V}_l\underline{V}_m\overline{s} - \overline{V}_l\underline{V}_m\underline{s},\\
\check{s}_{lm} &\leqslant \overline{V}_m\overline{s}V_l + \underline{V}_l\overline{s}V_m + \underline{V}_l\underline{V}_m\check{S} - \underline{V}_l\overline{V}_m\overline{s} - \underline{V}_l\underline{V}_m\overline{s},\\
\check{s}_{lm} &\leqslant \overline{V}_m\overline{s}V_l + \underline{V}_l\underline{s}V_m + \underline{V}_l\overline{V}_m\check{S} - \underline{V}_l\overline{V}_m\overline{s} - \underline{V}_l\overline{V}_m\underline{s},\\
\check{s}_{lm} &\leqslant \underline{V}_m\overline{s}V_l + \overline{V}_l\overline{s}V_m + \underline{V}_l\underline{V}_m\check{S} - \overline{V}_l\underline{V}_m\overline{s} - \underline{V}_l\underline{V}_m\overline{s},\\
\check{s}_{lm} &\leqslant \underline{V}_m\underline{s}V_l + \underline{V}_l\underline{s}V_m + \frac{\Lambda_7}{\underline{s}-\overline{s}}\check{S} - \frac{\Lambda_7\overline{s}}{\underline{s}-\overline{s}} - \overline{V}_l\underline{V}_m\underline{s}\\
&\quad - \underline{V}_l\overline{V}_m\underline{s} + \overline{V}_l\overline{V}_m\overline{s},\\
&\quad \text{where } \Lambda_7 = \overline{V}_l\underline{V}_m\underline{s} - \overline{V}_l\overline{V}_m\overline{s} - \underline{V}_l\underline{V}_m\underline{s} + \underline{V}_l\overline{V}_m\underline{s}.
\end{aligned}
$$

Equations (1b), (1c), (1g), (1h), (4) with substitutions $V_i^2 \to w_{ii}$,

$$
V_l \, V_m \sin(\theta_{lm}) \to w_{s,lm}, V_l \, V_m \cos(\theta_{lm}) \to w_{c,lm}, \tag{7b}
$$

Equations (1d)–(1f), (1i) (7c)

$$
w_{lm} \in \langle V_l \, V_m \rangle^M, \ w_{s,lm} \in \langle w_{lm} \, s_{lm} \rangle^M, w_{c,lm} \in \langle w_{lm} \, c_{lm} \rangle^M, \text{or}
$$

$$
w_{s,lm} \in \langle V_l \, V_m \, s_{lm} \rangle^{MF \vee EP}, \ w_{c,lm} \in \langle V_l \, V_m \, c_{lm} \rangle^{MF \vee EP}. \tag{7d}
$$

IV. Comparison of the Trilinear Envelopes

There are trade-offs inherent to the choices of trilinear envelopes in the QC relaxation (7d). The MF and EP envelopes both yield the convex hulls of the trilinear monomials and therefore are equivalently tight. Moreover, both are generally tighter than the recursive McCormick envelopes.

The number of variables and constraints necessary to describe the trilinear envelopes for a given trilinear monomial is tabulated in Table I, where $(\cdot)^{\leqslant}$ and $(\cdot)^{=}$ represent the number of inequality and equality constraints, respectively.

Table I
Variables and constraints per trilinear monomial envelope.

Convex envelope	No. of Variables	No. of Constraints
RMC	3 (original) + 2 (lifted)	$8^{\leqslant}$
EP	3 (original) + 9 (lifted)	$5^{=}$
MF	3 (original) + 1 (lifted)	$12^{\leqslant}$

Using the algebraic modeling language JuMP [33] in Julia, we formulate each version of the QC relaxation (7) by modifying the relaxation implementations in PowerModels.jl [34]. For each version of the QC relaxation, we apply the solvers *CPLEX* 12.8, *GUROBI* 8.0, and *IPOPT* 3.12.9 (with "ma27" HSL solver) to each OPF problem in the NESTA v0.7 archive [35]. Optimality gaps for the QC relaxation are given by $\frac{UB-LB}{UB}\cdot 100$, where UB is the local feasible solution obtained from solving (1) with *IPOPT* and LB is the lower bound obtained by applying the QC relaxation.

The results in Table II for a selected set of instances show that replacing RMC envelopes with MF or EP envelopes tighten the QC relaxation and can reduce the optimality gaps substantially. For instance, the optimality gap for case24_ieee_rts__api is reduced by 3.10%. We expect further gap reductions when the convex hull envelopes are used in combination with bound tightening procedures [16]–[18], [36].

The box-and-whisker plot shown in Figure 1 compares the run times of various SOCP solvers. The lower and upper ends of the boxes in Figure 1 reflect the first and third quartiles, the lines inside the boxes denote the median, and the plus marks are outliers. "Medium" and "Large" categories correspond to instances including "TYP", "API", and "SAD" with numbers of buses $1354 \leqslant |\mathcal{N}| \leqslant 3375$ and $|\mathcal{N}| \geqslant 6468$, respectively.

For each solver, the RMC envelopes yield the fastest or nearly the fastest results, but have larger optimality gaps than the MF and EP envelopes, particularly for the small problems in Table II. *GUROBI* and *CPLEX* are faster than *IPOPT* for the RMC and EP envelopes. While slightly faster than the EP envelopes when using *IPOPT*, the MF envelopes are substantially slower than the RMC and EP envelopes when using *GUROBI* and *CPLEX*. Moreover, the solvers *CPLEX* and *GUROBI* encounter numerical issues for approximately 19.5% of the instances when using the MF envelopes and hence do not converge to optimal values. We speculate that the MF envelopes yield dense columns, which is a known issue for the convergence of barrier-based algorithms for solving SOCPs. In summary, *IPOPT* is numerically stable on all the formulations and instances but is slower than *GUROBI* and *CPLEX*, and, while equivalently tight, the EP envelopes are significantly faster than the MF envelopes when using *CPLEX* and *GUROBI*.

Table II
QC relaxation gaps using recursive McCormick (RMC) vs. convex-hull envelopes (MF, EP).

Instances	RMC (%)	MF, EP (%)	Improvement (%)
case3_lmbd	1.21	0.96	0.25
case30_ieee	15.64	15.20	0.44
case2224_edin	6.03	6.01	0.02
case3_lmbd__api	1.79	1.59	0.20
case24_ieee_rts__api	11.88	8.78	3.10
case73_ieee_rts__api	10.97	9.64	1.33
case1397sp_eir_api	1.07	1.00	0.07
case3_lmbd__sad	1.42	1.37	0.05
case4_gs__sad	1.53	0.96	0.57
case5_pjm__sad	0.99	0.77	0.22
case24_ieee_rts__sad	2.93	2.77	0.16
case73_ieee_rts__sad	2.53	2.38	0.15
case118_ieee__sad	4.61	4.14	0.47

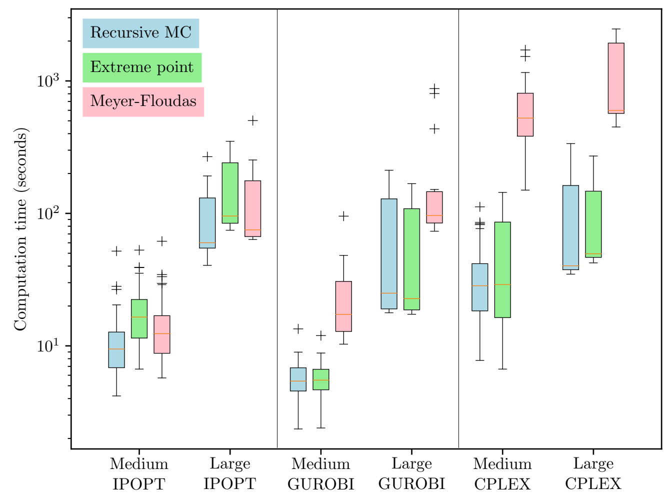

Figure 1. Run time comparisons of various formulations using three solvers.

V. Conclusions

Convex relaxations of OPF problems derived using polar voltage coordinates give rise to trilinear monomials. Using extensive empirical tests, this paper has compared three previously proposed techniques for addressing the trilinear monomials: recursive McCormick envelopes, Meyer and Floudas envelopes, and extreme point envelopes. The latter two envelopes yield the convex hulls of the trilinear monomials and are therefore generally tighter than recursive McCormick envelopes. Thus, the MF and EP envelopes improve the QC relaxation gaps, particularly on instances with less than 300 buses. Despite being equivalently tight, the differing mathematical formulations of the MF and EP envelopes yield differing computational performance with various solvers. Given its advantages in ease of implementation and numerical stability with state-of-the-art solvers like CPLEX and GUROBI, we recommend using EP envelopes for OPF relaxations.

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