Moment-Based Relaxations of the Optimal Power Flow Problem

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Outline

- Optimal power flow overview
- Moment relaxations
- Investigation of feasible spaces
- Exploiting sparsity
- Conclusions

Classical OPF Problem

Cost

$$\min \sum_{k\in\mathcal{G}} f_k\left(P_{Gk}\right) = \sum_{k\in\mathcal{G}} c_{2k}P_{Gk}^2 + c_{1k}P_{Gk} + c_{0k}$$

Engineering Constraints

subject to

$$P_{Gk}^{\min} \leq P_{Gk} \leq P_{Gk}^{\max}$$

$$Q_{Gk}^{\min} \leq Q_{Gk} \leq Q_{Gk}^{\max}$$

$$\left(V_k^{\min}\right)^2 \leq V_{dk}^2 + V_{qk}^2 \leq \left(V_k^{\max}\right)^2$$

$$|S_{lm}| \leq S_{lm}^{\max}$$

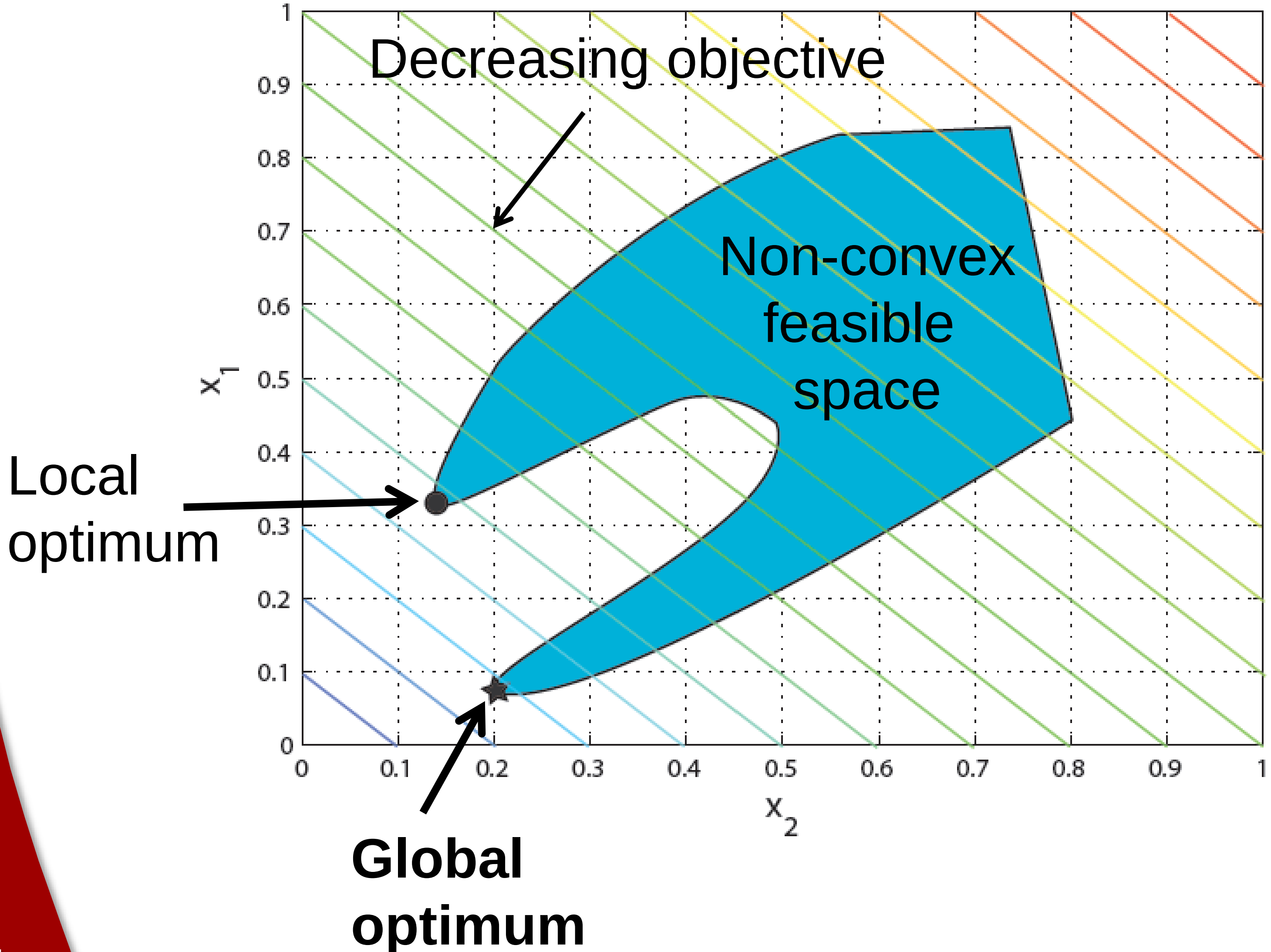
Physical Laws

$$P_{Gk} - P_{Dk} = V_{dk}\sum_{i=1}^{n}\left(\mathrm{G}_{ik}V_{di} - \mathrm{B}_{ik}V_{qi}\right) + V_{qk}\sum_{i=1}^{n}\left(\mathrm{B}_{ik}V_{di} + \mathrm{G}_{ik}V_{qi}\right)$$

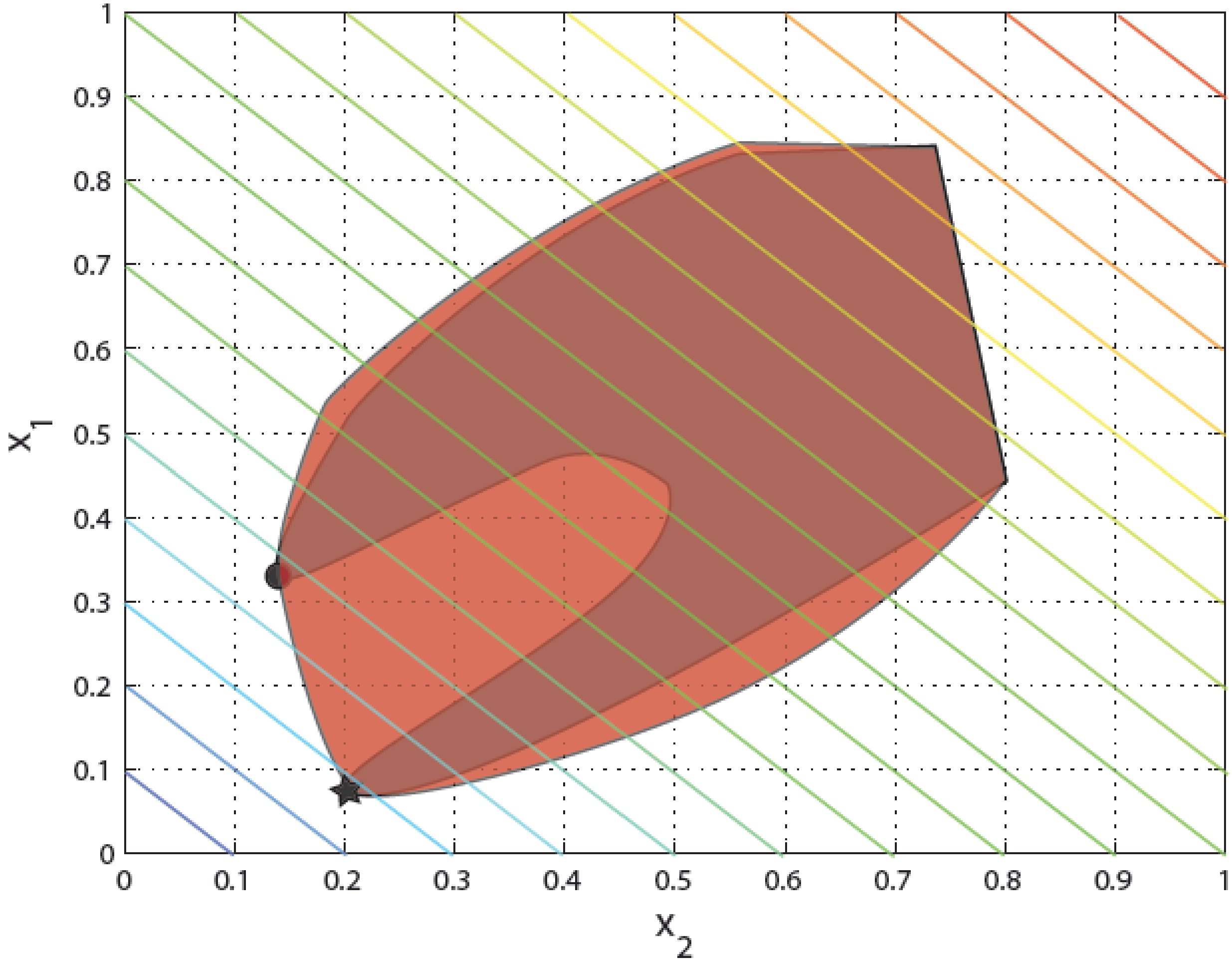
$$Q_{Gk} - Q_{Dk} = V_{dk}\sum_{i=1}^{n}\left(-\mathrm{B}_{ik}V_{di} - \mathrm{G}_{ik}V_{qi}\right) + V_{qk}\sum_{i=1}^{n}\left(\mathrm{G}_{ik}V_{di} - \mathrm{B}_{ik}V_{qi}\right)$$

Rectangular voltage coordinates: $\vec{V}_i = V_{di} + jV_{qi}$

Convex Relaxation

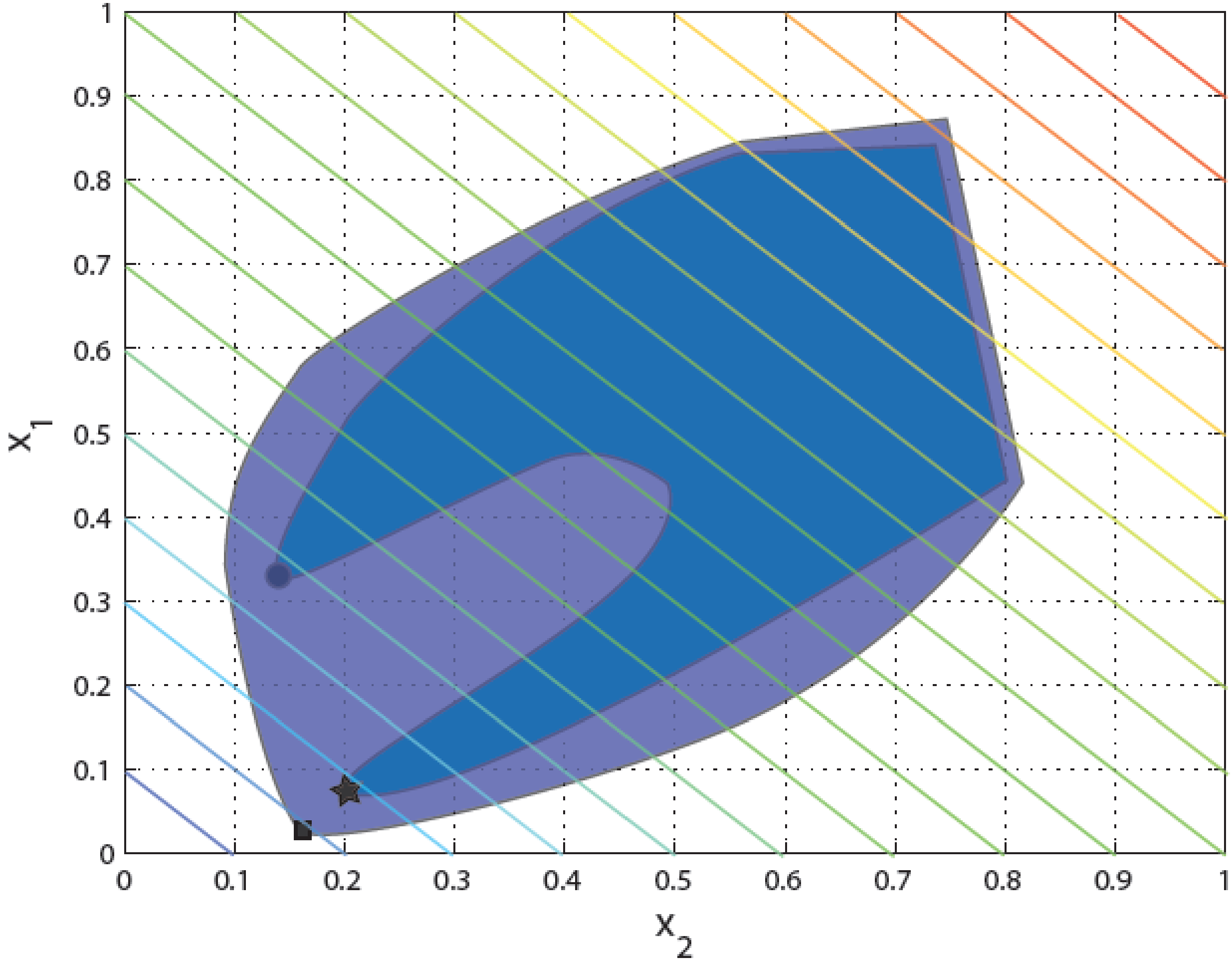


Convex Relaxation



Relaxation finds global optimum
(**zero relaxation gap**)

Convex Relaxation



Relaxation does not find global optimum

(**non-zero relaxation gap**)

Semidefinite Programming

- Convex optimization
- Interior point methods solve for the global optimum in polynomial time

$$\min_{\mathbf{W}} \operatorname{trace}(\mathbf{BW})$$

$$\text{subject to}$$

$$\operatorname{trace}(\mathbf{A}_i\mathbf{W}) = c_i$$

$$\mathbf{W} \succeq 0$$

where $\mathbf{B}$ and $\mathbf{A}_i$ are specified symmetric matrices

Recall: $\operatorname{trace}\left(\mathbf{A}^{\mathsf{T}}\mathbf{W}\right) = \mathbf{A}_{11}\mathbf{W}_{11} + \mathbf{A}_{12}\mathbf{W}_{12} + \ldots + \mathbf{A}_{nn}\mathbf{W}_{nn}$

$$\mathbf{W} \succeq 0 \ \text{ if and only if } \ \operatorname{eig}(\mathbf{W}) \geq 0$$

Moment-Based Relaxations

Preliminaries

- Exploit moment-based semidefinite relaxations for polynomial optimization problems [Lasserre '10]

$$\boxed{\begin{aligned} &\min_x f(x) \quad \text{subject to} \\ &g_i(x) \geq 0 \end{aligned}}$$

where $f(x)$ and $g_i(x)$ are polynomial functions of $x = \begin{bmatrix} V_{d1} & V_{d2} & \dots & V_{dn} & V_{q1} & V_{q2} & \dots & V_{qn} \end{bmatrix}^\intercal$

- Define linear functional L_y of polynomials $f(x)$ and $g_i(x)$

$$\boxed{g(x) = \sum_{\alpha \in \mathbb{N}^{2n}} g_\alpha x^\alpha} \qquad \boxed{L_y\{g(x)\} = \sum_{\alpha \in \mathbb{N}^{2n}} g_\alpha y_\alpha}$$

- Define vector x_d containing all monomials up to order d

$$x_d = \begin{bmatrix} 1 & V_{d1} & \dots & V_{qn} & V_{d1}^2 & V_{d1}V_{d2} & \dots & V_{qn}^2 & V_{d1}^3 & V_{d1}^2 V_{d2} & \dots & V_{qn}^d \end{bmatrix}^\intercal$$

Moment-Based Relaxation

- The order-d moment-based relaxation is

$$\min_{y} L_y\left\{f\left(x\right)\right\} \qquad \text{subject to}$$

$$L_y\left\{g_i\left(x\right) x_{d-1} x_{d-1}^{\intercal}\right\} \succeq 0 \qquad \text{[localizing matrices]}$$

$$L_y\left\{x_d x_d^{\intercal}\right\} \succeq 0 \qquad \text{[moment matrix]}$$

- Increasing d yields a tighter relaxation but has a computational cost
- Recover global optimum if $\operatorname{rank}\left(L_y\left\{x_1 x_1^{\intercal}\right\}\right) = 1$

Two-Bus Example (First Order)

$$x_1 = \begin{bmatrix} 1 & V_{d1} & V_{d2} & V_{q2} \end{bmatrix}^\intercal$$

(Eliminate V_{q1} *to enforce reference angle)*

$$L_y\left\{x_1 x_1^\intercal\right\} = \begin{bmatrix} y_{000} & y_{100} & y_{010} & y_{001} \\ y_{100} & y_{200} & y_{110} & y_{101} \\ y_{010} & y_{110} & y_{020} & y_{011} \\ y_{001} & y_{101} & y_{011} & y_{002} \end{bmatrix} \succeq 0$$

[moment constraint]

Lower limit of 0.9 per unit for voltage magnitude at bus 2:

$$g(x) = V_{d2}^2 + V_{q2}^2 - 0.9^2 \geq 0$$

$$\begin{aligned} L_y\left\{g(x)\, x_0 x_0^\intercal\right\} &= L_y\left\{\left(V_{d2}^2 + V_{q2}^2 - 0.9^2\right)\cdot 1\right\} \\ &= \boxed{y_{020} + y_{002} - 0.9^2\, y_{000}} \geq 0 \end{aligned}$$

[localizing constraint]

Two-Bus Example (Second Order)

$$x_2 = \begin{bmatrix} 1 & V_{d1} & V_{d2} & V_{q2} & V_{d1}^2 & V_{d1}V_{d2} & V_{d1}V_{q2} & V_{d2}^2 & V_{d2}V_{q2} & V_{q2}^2 \end{bmatrix}^\intercal$$

(Eliminate V_{q1} *to enforce reference angle)*

$$L_y\left(x_2 x_2^\intercal\right) = \begin{bmatrix}
y_{000} & y_{100} & y_{010} & y_{001} & y_{200} & y_{110} & y_{101} & y_{020} & y_{011} & y_{002} \\
y_{100} & y_{200} & y_{110} & y_{101} & y_{300} & y_{210} & y_{201} & y_{120} & y_{111} & y_{102} \\
y_{010} & y_{110} & y_{020} & y_{011} & y_{210} & y_{120} & y_{111} & y_{030} & y_{021} & y_{012} \\
y_{001} & y_{101} & y_{011} & y_{002} & y_{201} & y_{111} & y_{102} & y_{021} & y_{012} & y_{003} \\
y_{200} & y_{300} & y_{210} & y_{201} & y_{400} & y_{310} & y_{301} & y_{220} & y_{211} & y_{202} \\
y_{110} & y_{210} & y_{120} & y_{111} & y_{310} & y_{220} & y_{211} & y_{130} & y_{121} & y_{112} \\
y_{101} & y_{201} & y_{111} & y_{102} & y_{301} & y_{211} & y_{202} & y_{121} & y_{112} & y_{103} \\
y_{020} & y_{120} & y_{030} & y_{021} & y_{220} & y_{130} & y_{121} & y_{040} & y_{031} & y_{022} \\
y_{011} & y_{111} & y_{021} & y_{012} & y_{211} & y_{121} & y_{112} & y_{031} & y_{022} & y_{013} \\
y_{002} & y_{102} & y_{012} & y_{003} & y_{202} & y_{112} & y_{103} & y_{022} & y_{013} & y_{004}
\end{bmatrix} \succeq 0$$

[moment constraint]

Two-Bus Example (Second Order)

$$x_2 = \begin{bmatrix} 1 & V_{d1} & V_{d2} & V_{q2} & V_{d1}^2 & V_{d1}V_{d2} & V_{d1}V_{q2} & V_{d2}^2 & V_{d2}V_{q2} & V_{q2}^2 \end{bmatrix}^\intercal$$

(Eliminate V_{q1} to enforce reference angle)

$$L_y\left(x_2 x_2^\intercal\right) = \left[\begin{array}{cccc|cccccc} y_{000} & y_{100} & y_{010} & y_{001} & y_{200} & y_{110} & y_{101} & y_{020} & y_{011} & y_{002} \\ y_{100} & y_{200} & y_{110} & y_{101} & y_{300} & y_{210} & y_{201} & y_{120} & y_{111} & y_{102} \\ y_{010} & y_{110} & y_{020} & y_{011} & y_{210} & y_{120} & y_{111} & y_{030} & y_{021} & y_{012} \\ y_{001} & y_{101} & y_{011} & y_{002} & y_{201} & y_{111} & y_{102} & y_{021} & y_{012} & y_{003} \\ \hline y_{200} & y_{300} & y_{210} & y_{201} & y_{400} & y_{310} & y_{301} & y_{220} & y_{211} & y_{202} \\ y_{110} & y_{210} & y_{120} & y_{111} & y_{310} & y_{220} & y_{211} & y_{130} & y_{121} & y_{112} \\ y_{101} & y_{201} & y_{111} & y_{102} & y_{301} & y_{211} & y_{202} & y_{121} & y_{112} & y_{103} \\ y_{020} & y_{120} & y_{030} & y_{021} & y_{220} & y_{130} & y_{121} & y_{040} & y_{031} & y_{022} \\ y_{011} & y_{111} & y_{021} & y_{012} & y_{211} & y_{121} & y_{112} & y_{031} & y_{022} & y_{013} \\ y_{002} & y_{102} & y_{012} & y_{003} & y_{202} & y_{112} & y_{103} & y_{022} & y_{013} & y_{004} \end{array}\right] \succeq 0$$

$L_y\left(x_1 x_1^\intercal\right)$

[moment constraint]

Two-Bus Example (Second Order)

- Lower limit of 0.9 per unit for voltage magnitude at bus 2: $V_{d2}^2 + V_{q2}^2 - 0.9^2 \geq 0$

$$x_1 = \begin{bmatrix} 1 & V_{d1} & V_{d2} & V_{q2} \end{bmatrix} \qquad x_1 x_1^\intercal = \begin{bmatrix} 1 & V_{d1} & V_{d2} & V_{q2} \\ V_{d1} & V_{d1}^2 & V_{d1}V_{d2} & V_{d1}V_{q2} \\ V_{d2} & V_{d1}V_{d2} & V_{d2}^2 & V_{d2}V_{q2} \\ V_{q2} & V_{d1}V_{q2} & V_{d2}V_{q2} & V_{q2}^2 \end{bmatrix}$$

$$L_y\left\{\left(V_{d2}^2 + V_{q2}^2 - 0.9^2\right) x_1 x_1^\intercal\right\} =$$

$$\begin{bmatrix} y_{020} + y_{002} - (0.9)^2 y_{000} & y_{120} + y_{102} - (0.9)^2 y_{100} & y_{030} + y_{012} - (0.9)^2 y_{010} & y_{021} + y_{003} - (0.9)^2 y_{001} \\ y_{120} + y_{102} - (0.9)^2 y_{100} & y_{220} + y_{202} - (0.9)^2 y_{200} & y_{130} + y_{112} - (0.9)^2 y_{110} & y_{121} + y_{103} - (0.9)^2 y_{101} \\ y_{030} + y_{012} - (0.9)^2 y_{010} & y_{130} + y_{112} - (0.9)^2 y_{110} & y_{040} + y_{022} - (0.9)^2 y_{020} & y_{031} + y_{013} - (0.9)^2 y_{011} \\ y_{021} + y_{003} - (0.9)^2 y_{001} & y_{121} + y_{103} - (0.9)^2 y_{101} & y_{031} + y_{013} - (0.9)^2 y_{011} & y_{022} + y_{004} - (0.9)^2 y_{002} \end{bmatrix} \succeq 0$$

[localizing constraints]

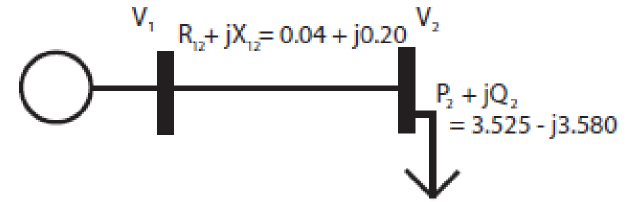
Feasible Space Investigation

Test System Results

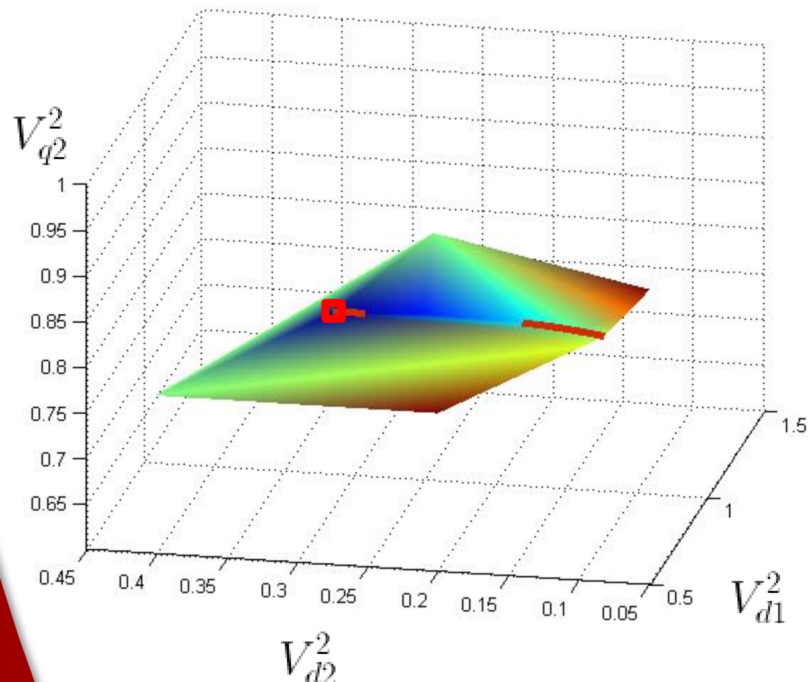
- First-order relaxation is exact for many problems
  [Lavaei & Low ’12, Molzahn et al. ‘13]
  - IEEE 14, 30, 57, and 118-bus test systems
  - Polish 2736, 2737, and 2746-bus systems in MATPOWER distribution
- Both small and large example systems where first-order relaxation fails to be exact
  - Second and third-order relaxations globally solve many problems where first-order relaxation fails

Disconnected Feasible Space

- Two-bus example OPF problem
  [Bukhsh et al. ‘11]

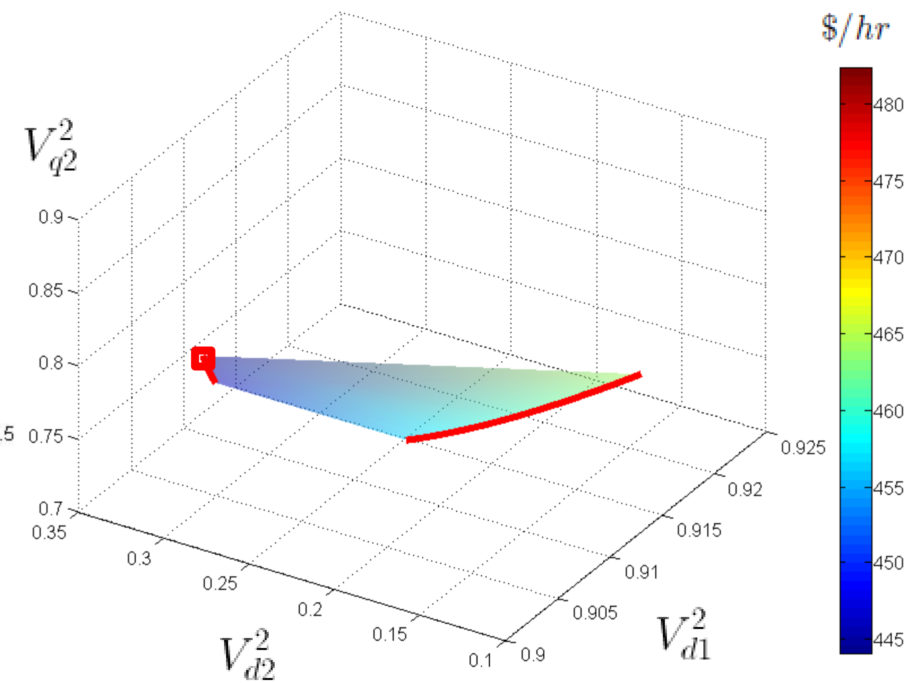


Feasible Space of
First-Order Relaxation



$$V_2^{max} = 1.05 \text{ per unit}$$

Feasible Space of
Second-Order Relaxation



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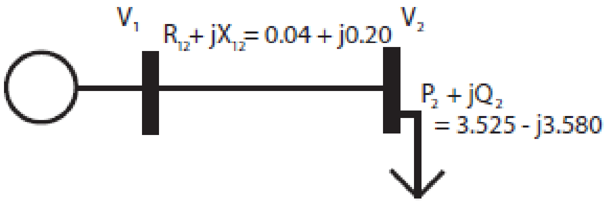

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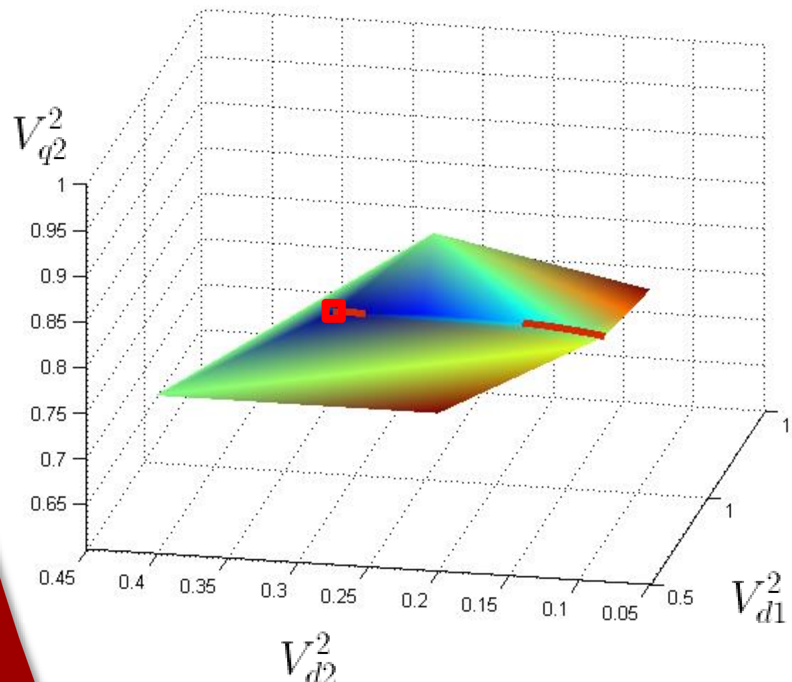

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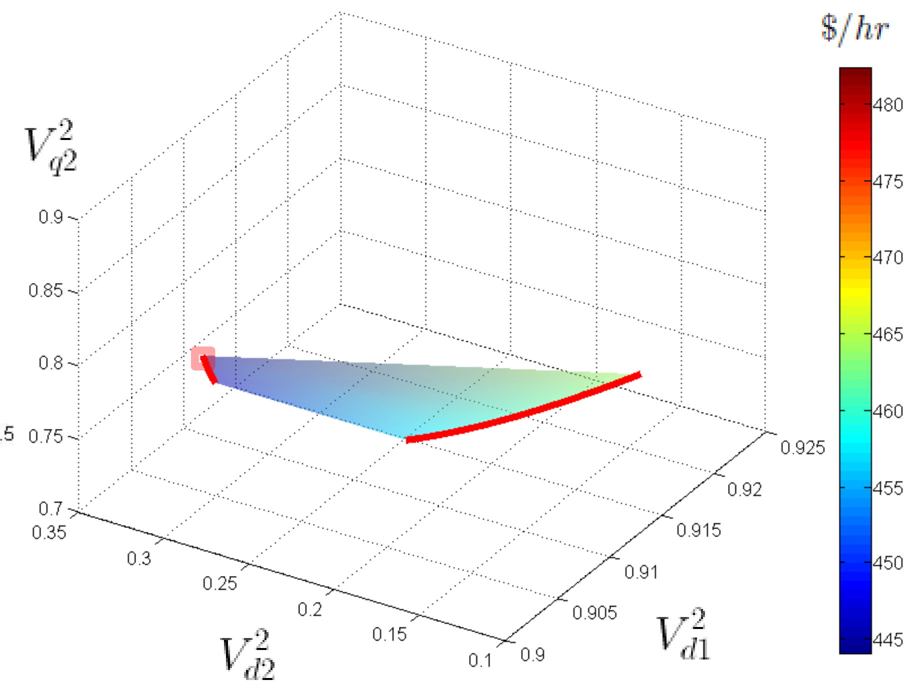
Feasible Space of
Second-Order Relaxation



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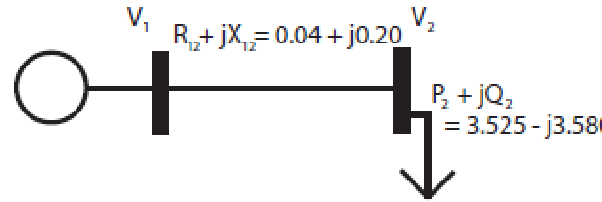

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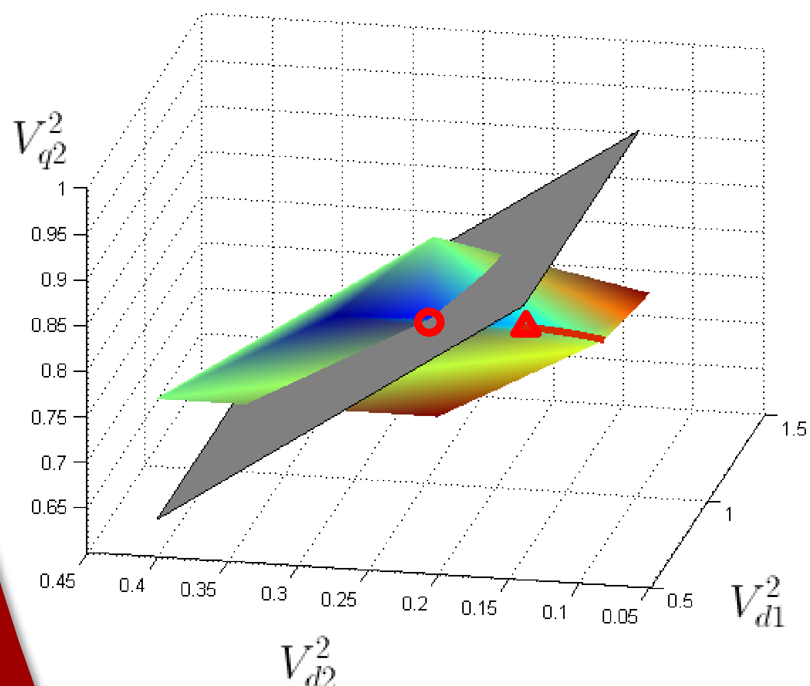

$$V_2^{max} = 1.02 \text{ per unit}$$

Feasible Space of
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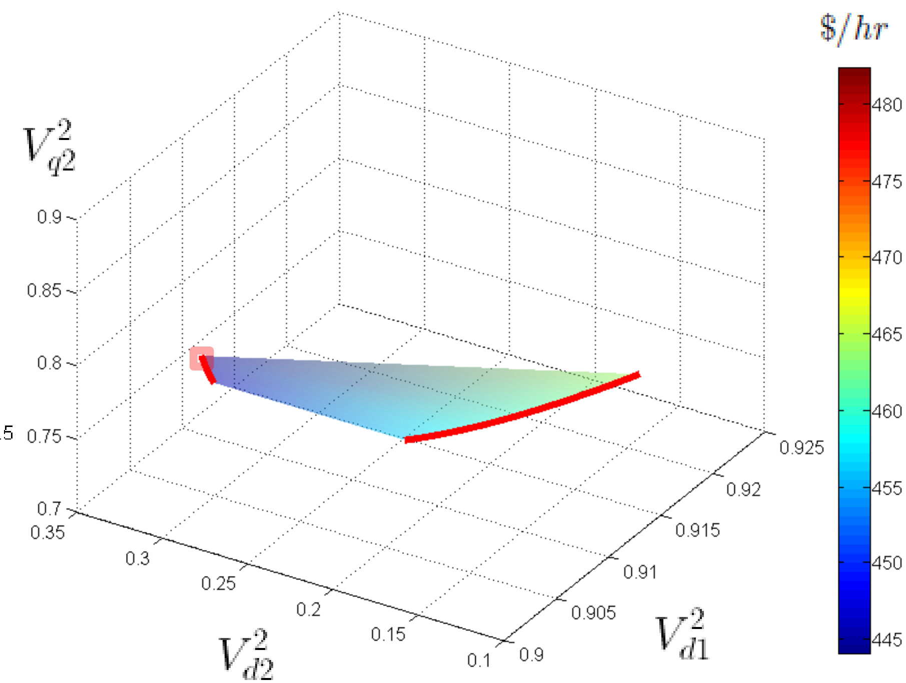

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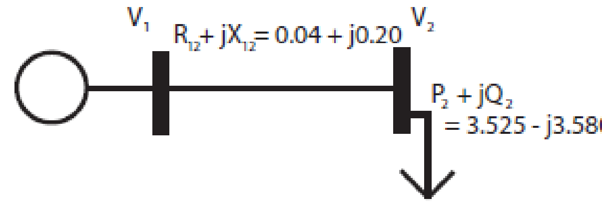

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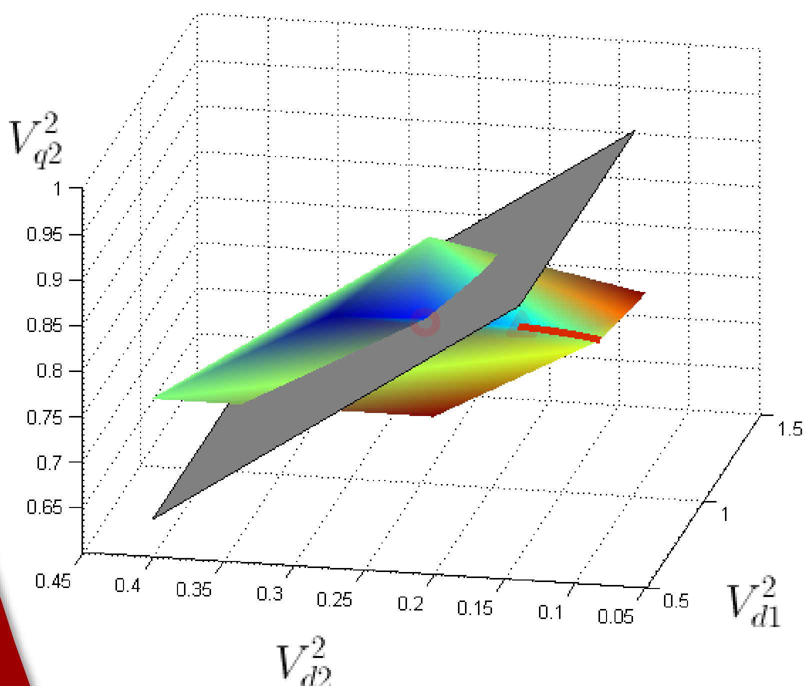
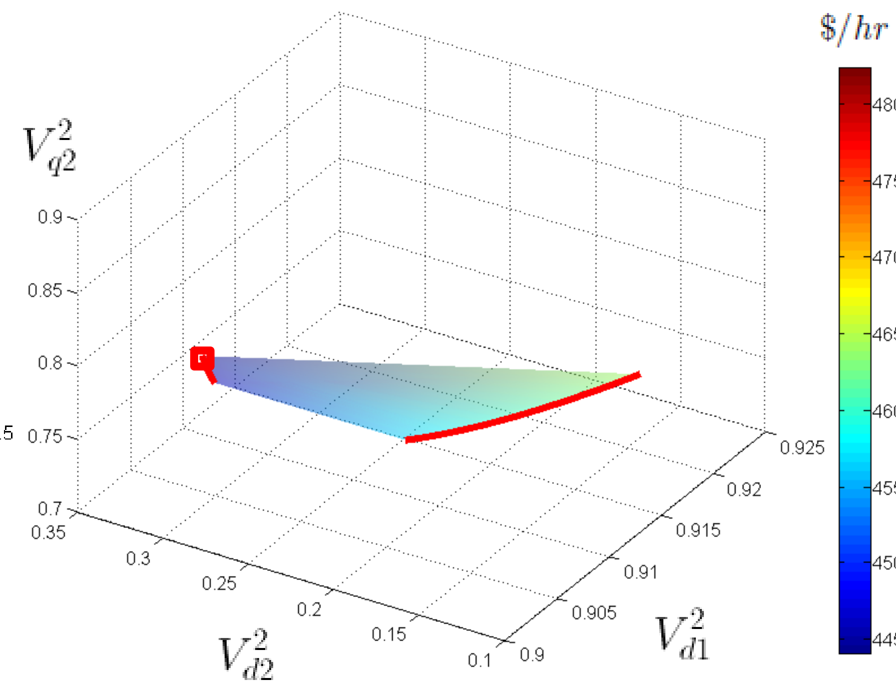


$V_2^{max} = 1.02 \text{ per unit}$

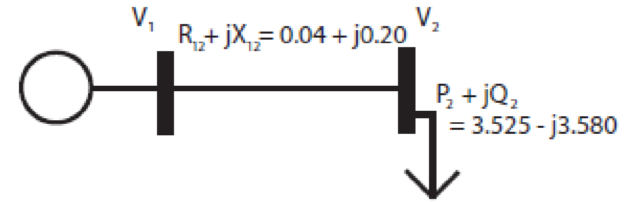
Feasible Space of
Second-Order Relaxation



$\boxed{V_2^{max} = 1.05 \text{ per unit}}$

Disconnected Feasible Space

- Two-bus example OPF problem
  [Bukhsh et al. '11]



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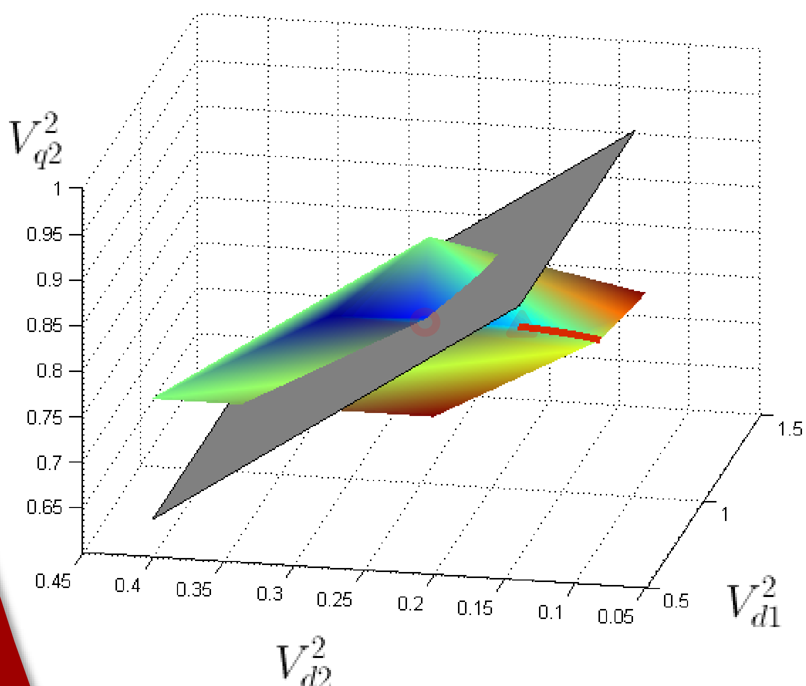

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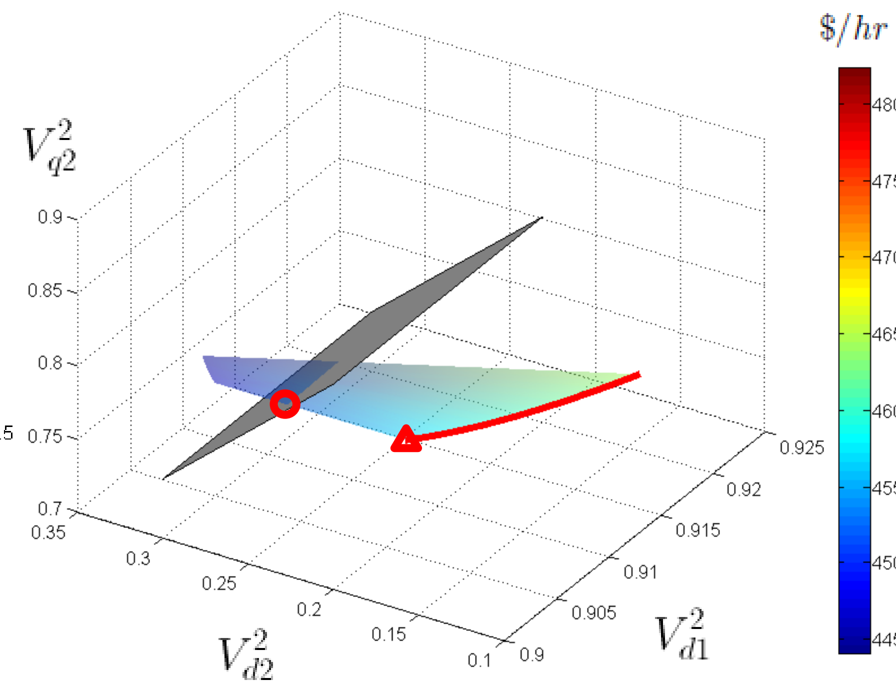

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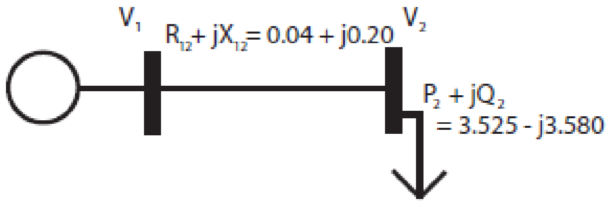

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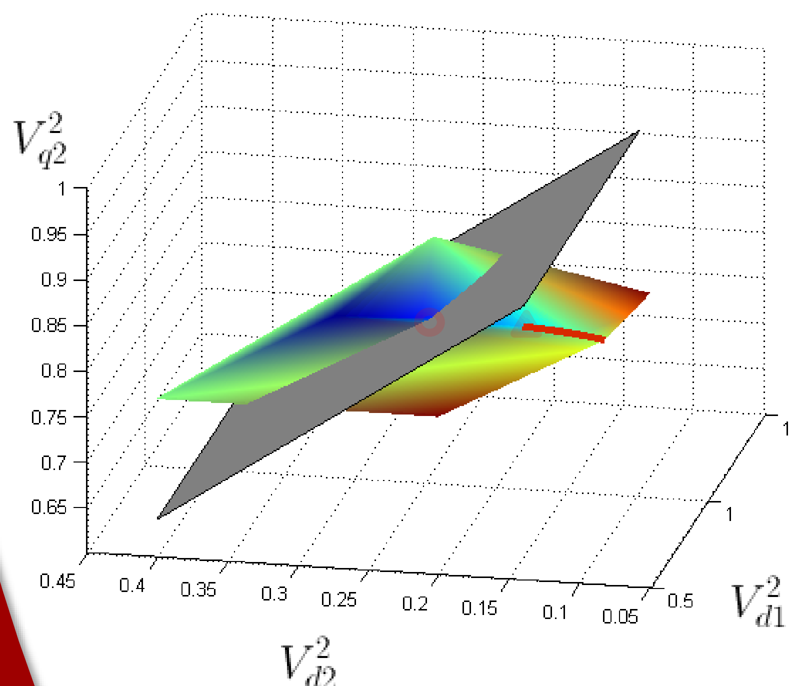

$V_2^{max} = 1.02$ per unit

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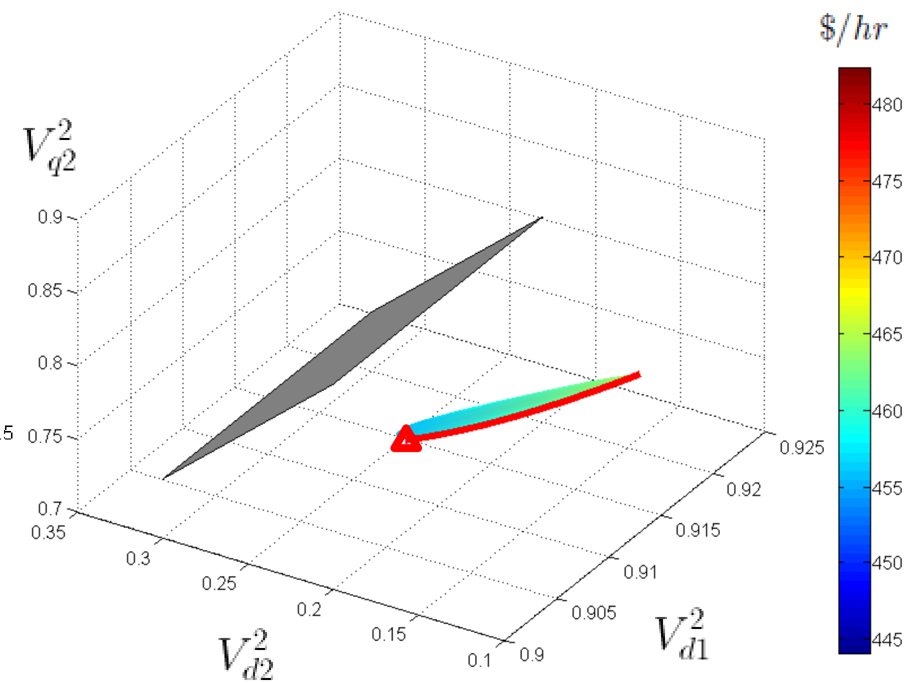

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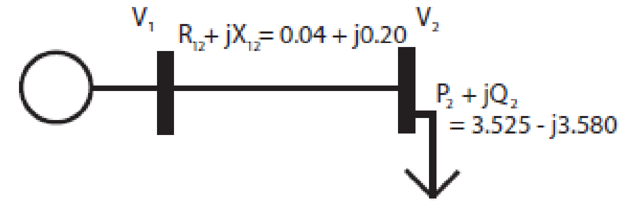

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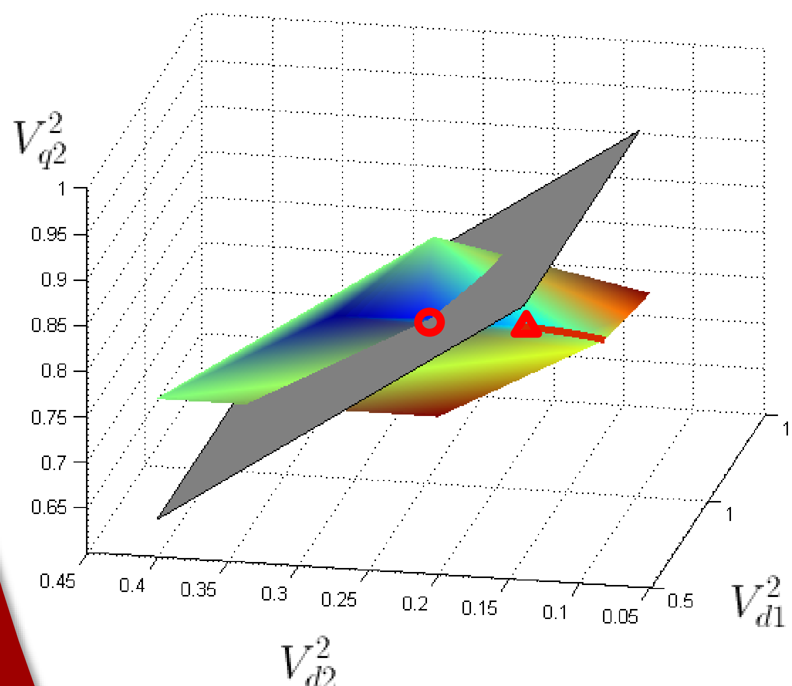

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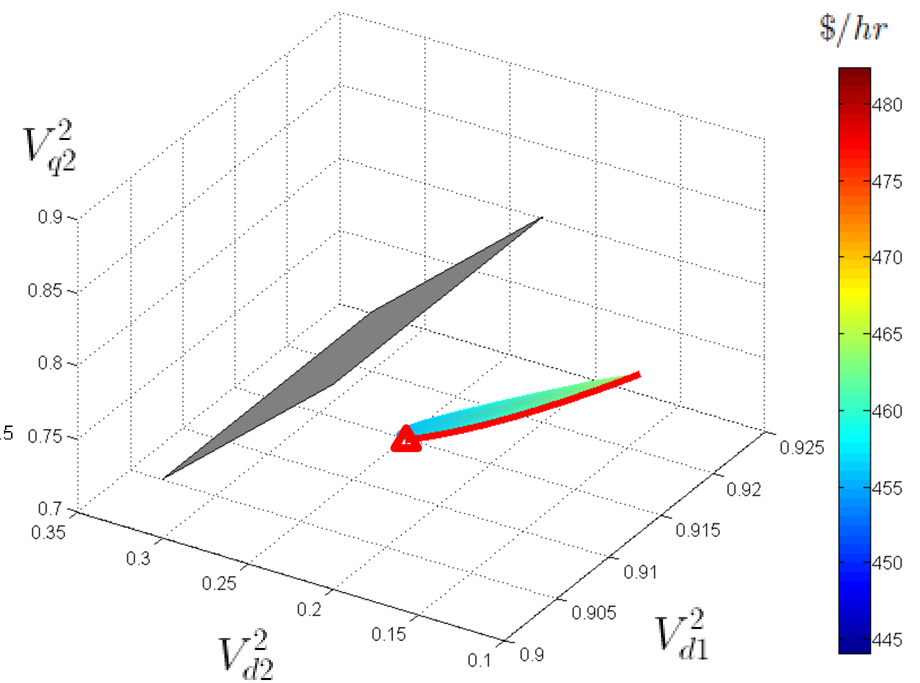
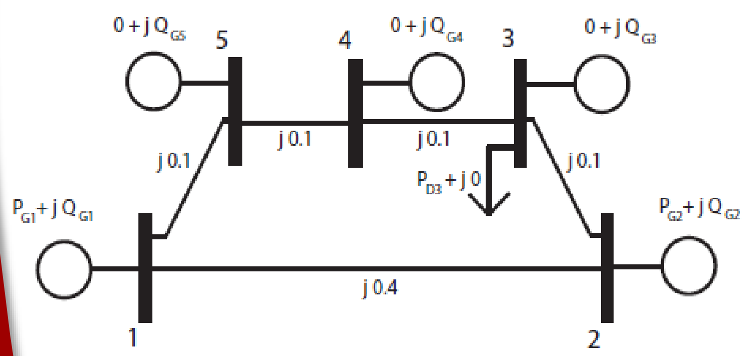
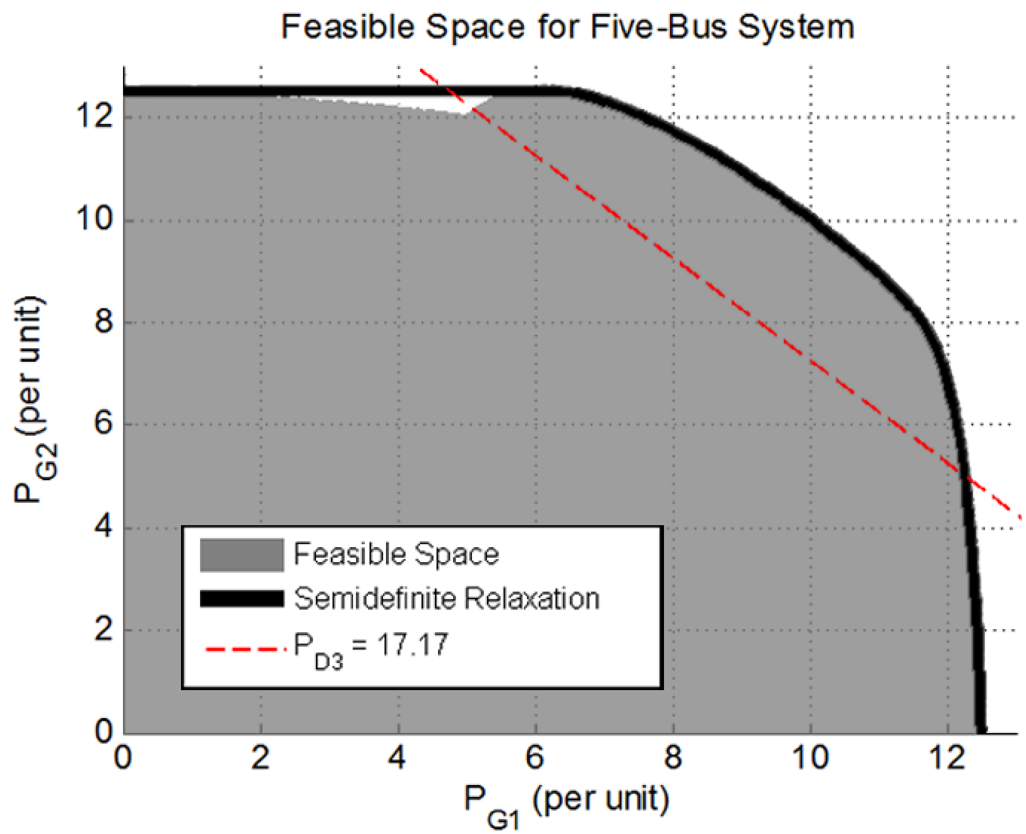


$V_2^{max} = 1.02$ per unit

Connected But Non-Convex Space

- Five-bus example OPF problem [Lesieutre & Hiskens '05]





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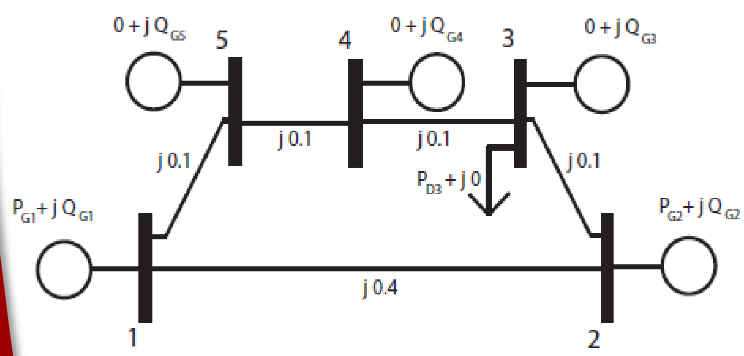

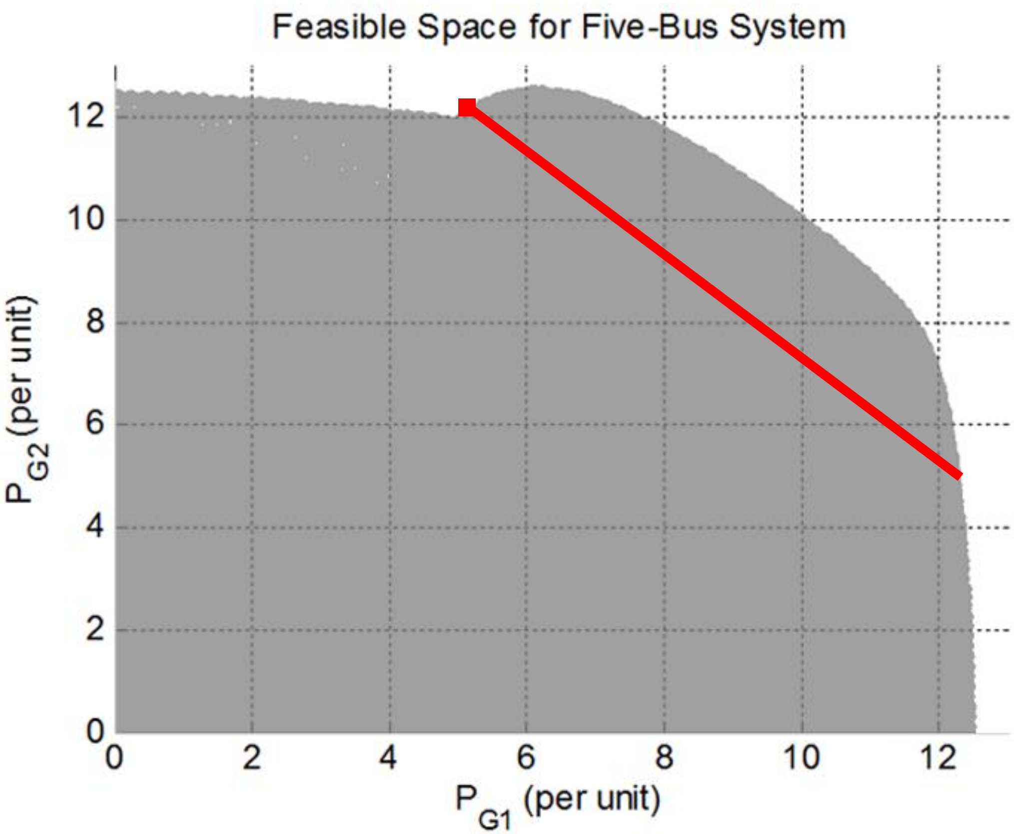

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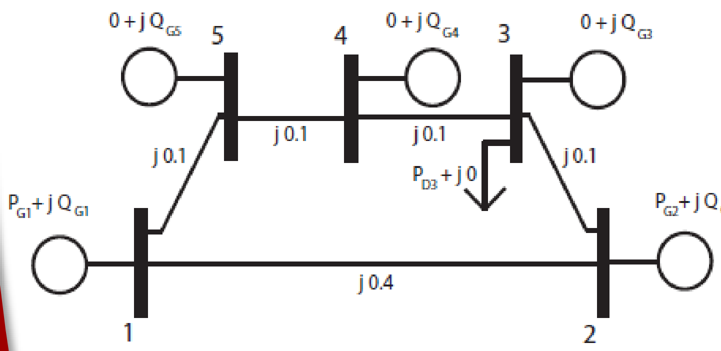

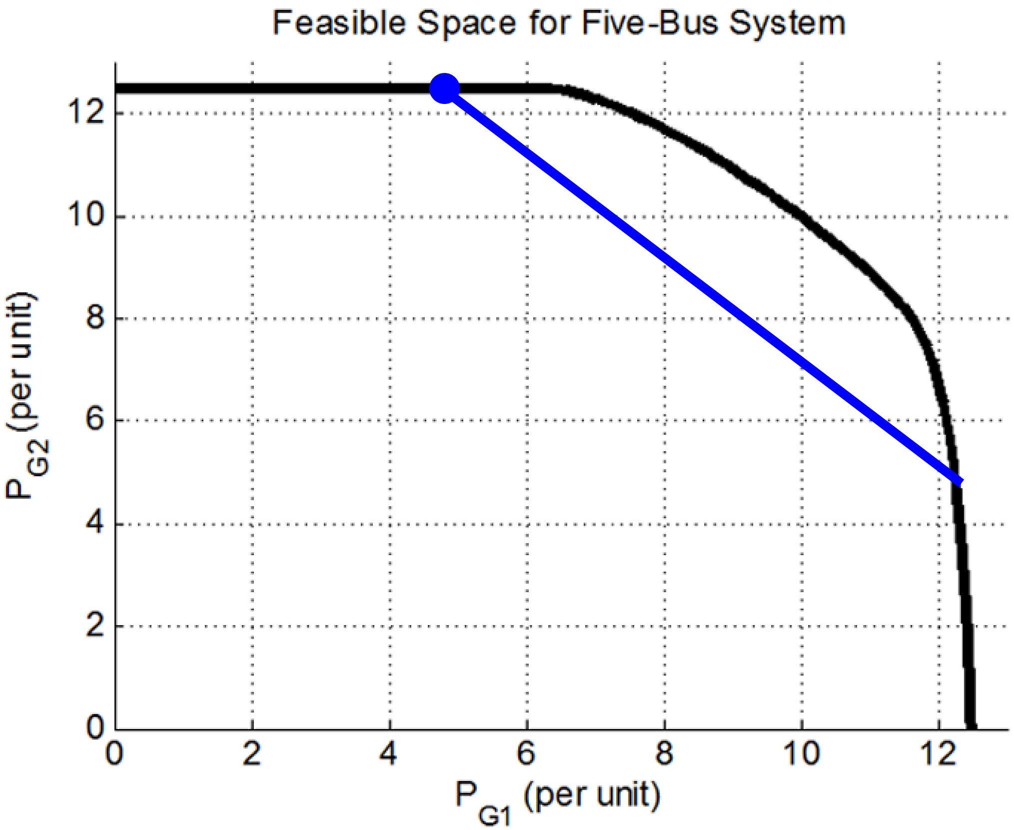

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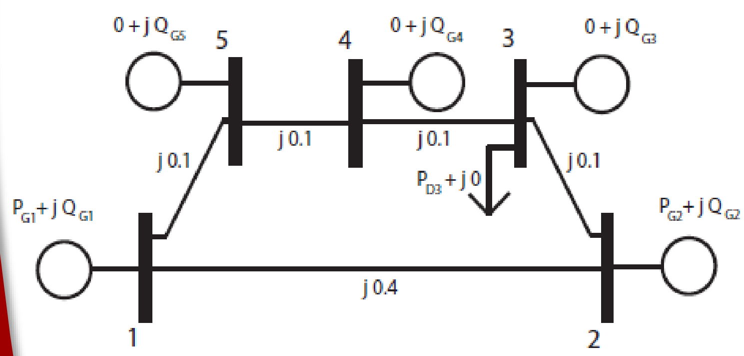

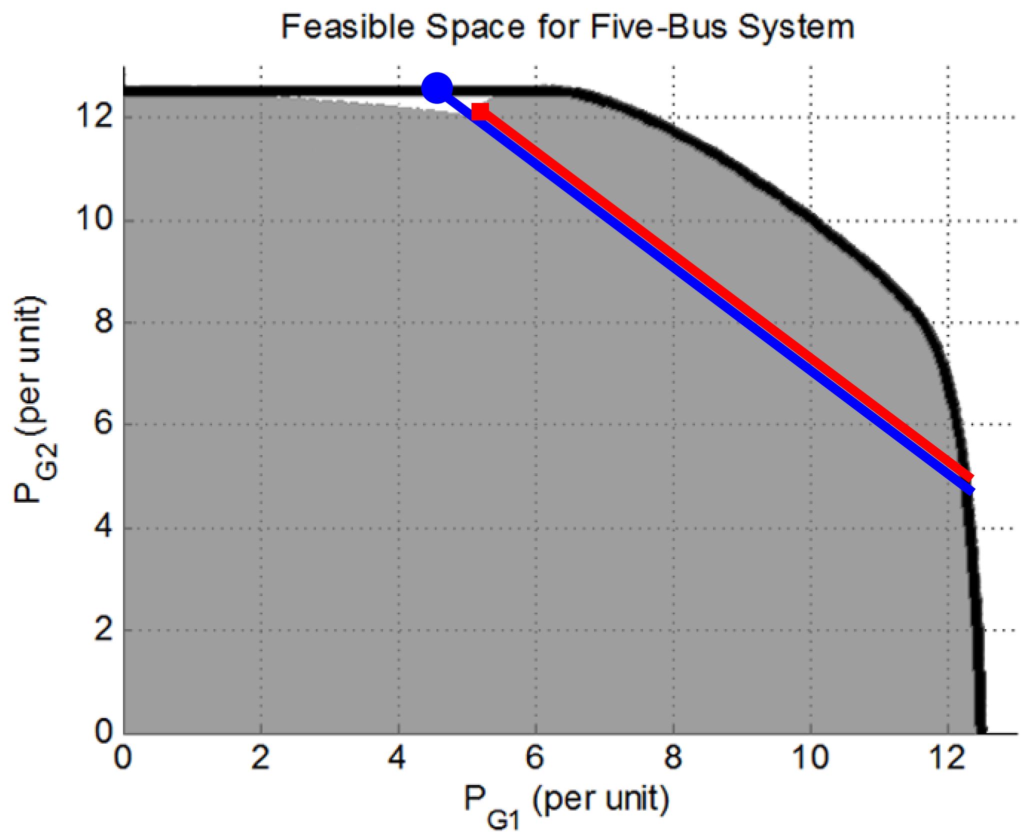

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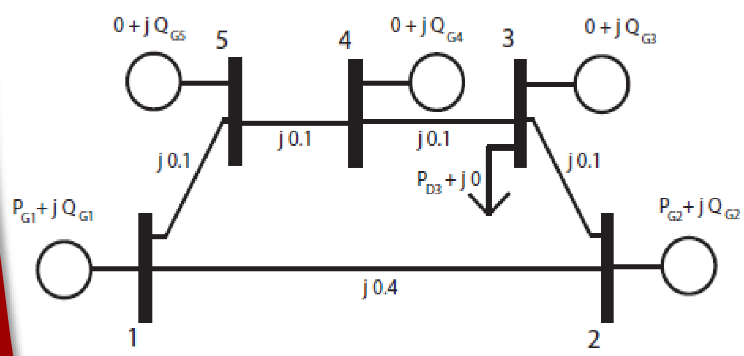
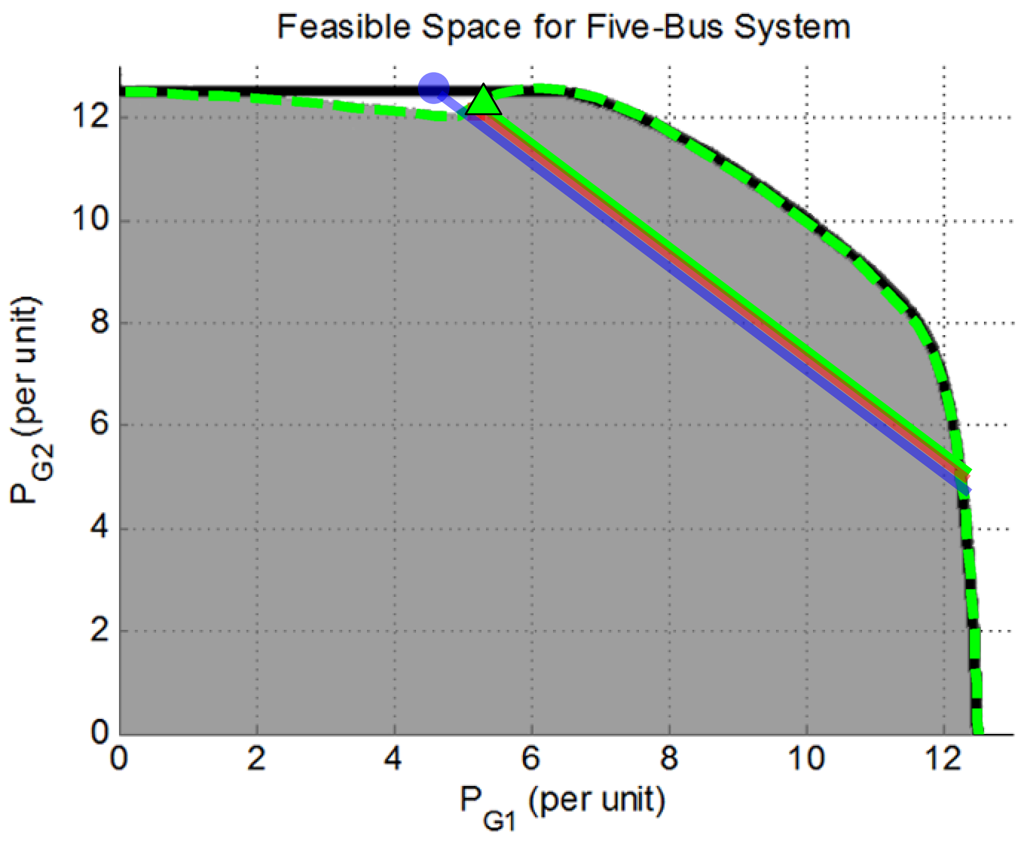
- Five-bus example OPF problem [Lesieutre & Hiskens '05]





Results for Other Systems

- Second and third-order moment-based relaxations globally solve small OPF problems

Case	Number of Buses	Parameters	Minimum Order
Lesieutre, Molzahn, Borden, & DeMarco '11	3	50 MVA line limit	2
Molzahn, Lesieutre, & DeMarco '14	3	100 MVA line limit	2
Bukhsh, Grothey, McKinnon & Trodden '13	3	$P_{D3} = 17.17$ per unit	2
Lesieutre & Hiskens '05	5		2
Bukhsh, Grothey, McKinnon & Trodden '13	5	$-50 \leq Q_5^{\min} \leq -27.36$ MVAR $-27.35 \leq Q_5^{\min} \leq -27.04$ MVAR $-27.03 \leq Q_2^{\min} \leq 0$ MVAR	2 3 2
Bukhsh, Grothey, McKinnon & Trodden '13	9		2
Madani, Sojoudi & Lavaei '13 (ex. 1)	10		> 2
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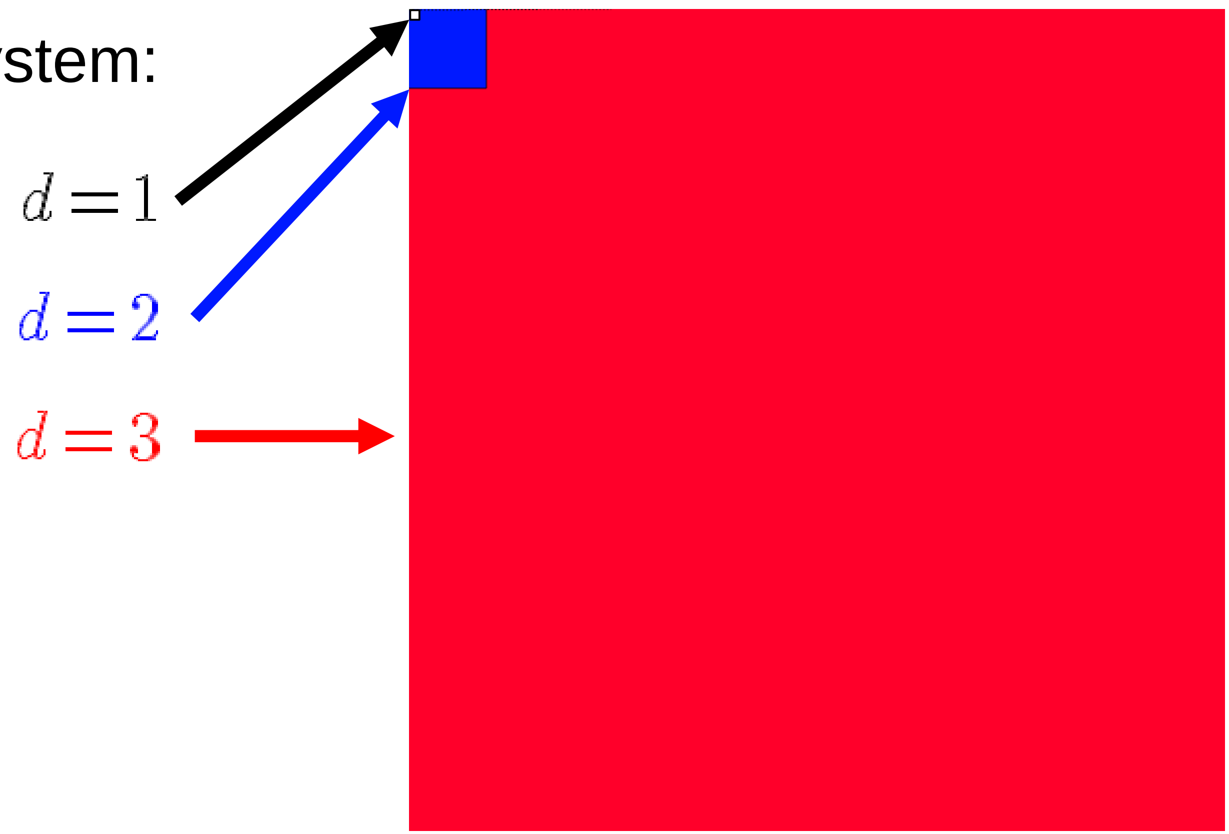
Exploiting Sparsity

Computational Challenges

- Moment matrix size for order-d relaxation of n-bus system: $\dfrac{(2n+d)!}{(2n)!d!}$

14-Bus System:

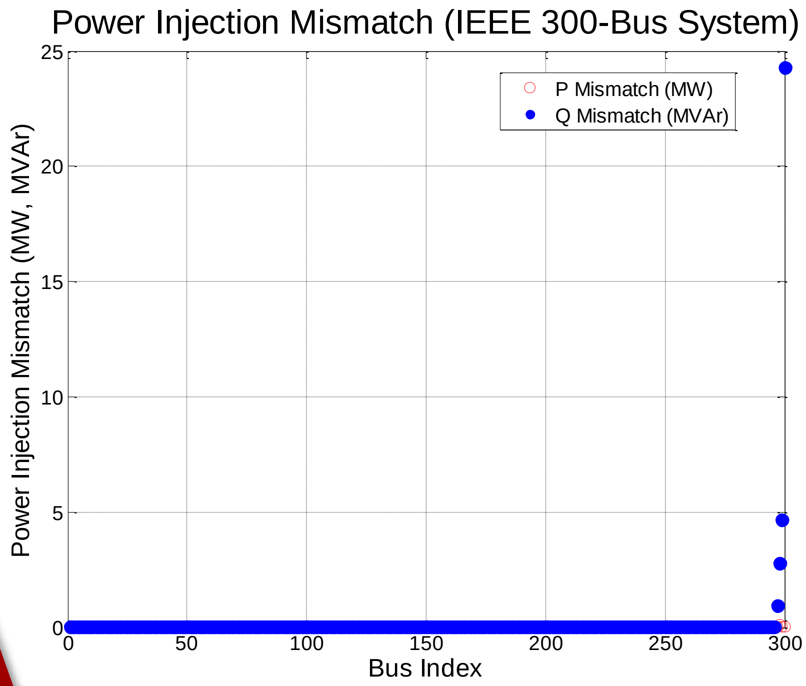


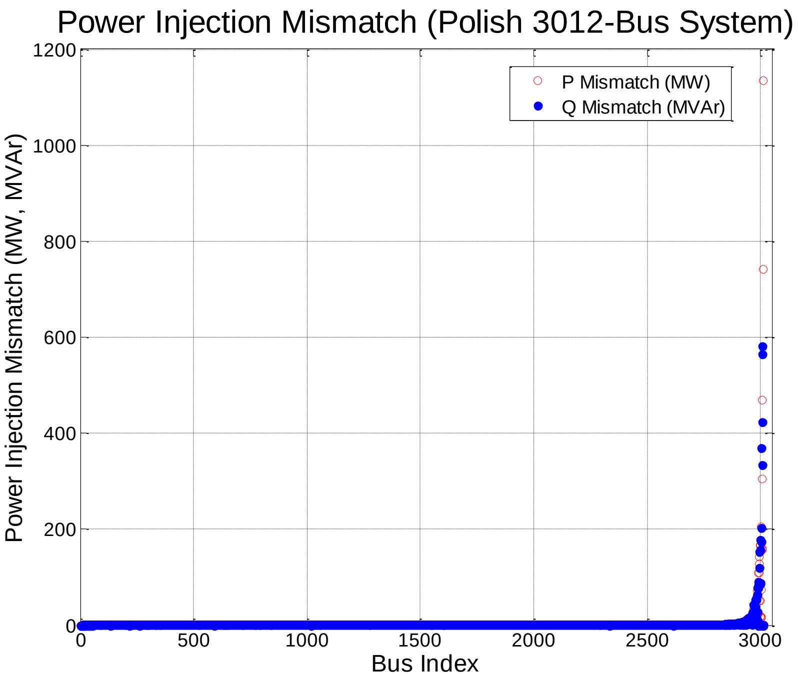
Extension to Large Problems

- Exploit sparsity using chordal extension and matrix completion decomposition [Waki et al. ’06]
  - Similar to existing approaches [Jabr ‘11, Molzahn et al. ’13]
- Naïvely exploiting sparsity allows for solving second-order relaxation with up to approximately 40 buses
- To extend to larger systems, only apply higher-order relaxation to problematic regions of large networks
  - Heuristic using power injection mismatches

Large-Scale First-Order Relaxations

- Global solution to some OPF problems
- Fails for other problems, but mismatch (using closest rank one matrix) at only a few buses





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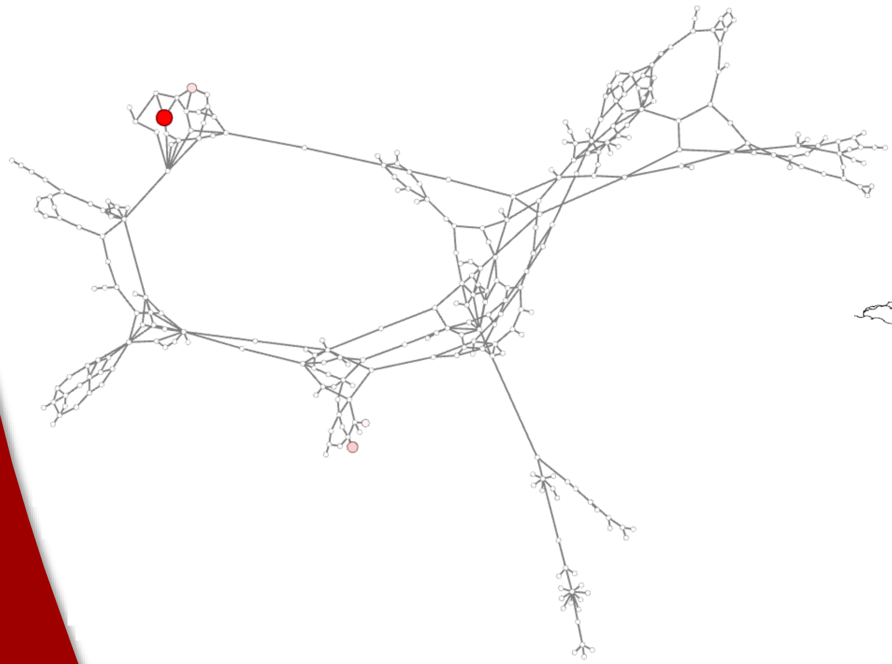
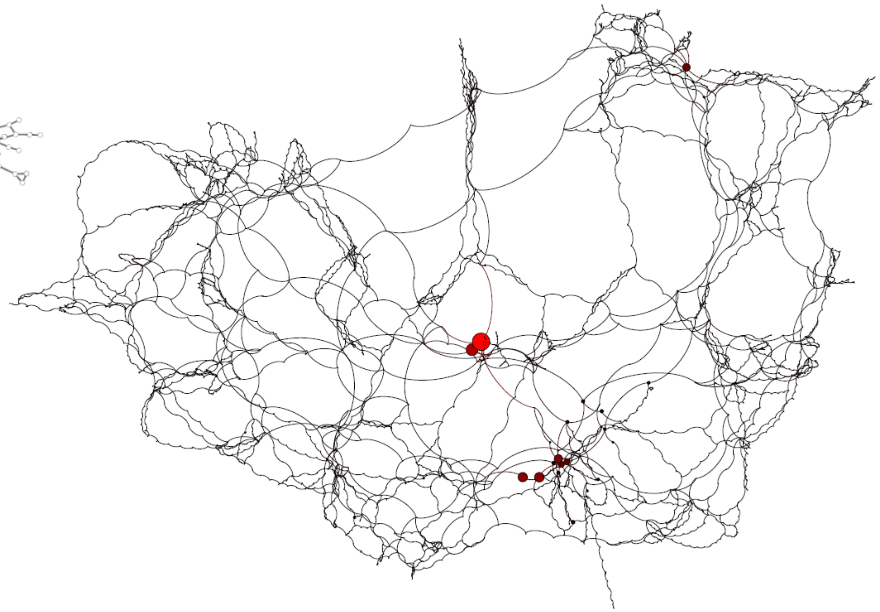
Power Injection Mismatch (IEEE 300-Bus System)



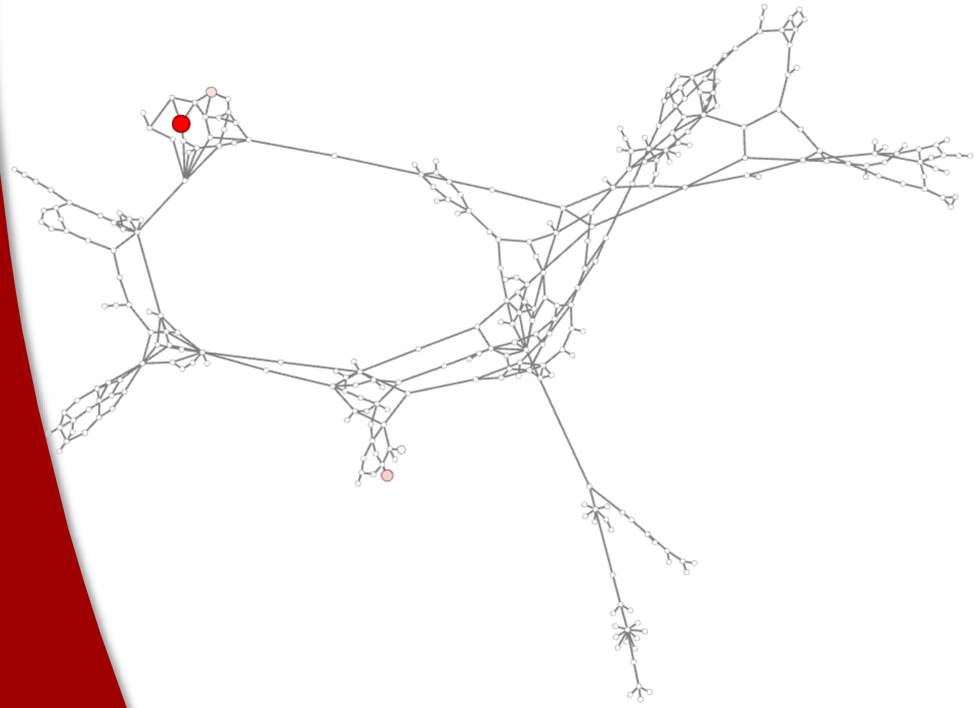
Power Injection Mismatch (Polish 3012-Bus System)

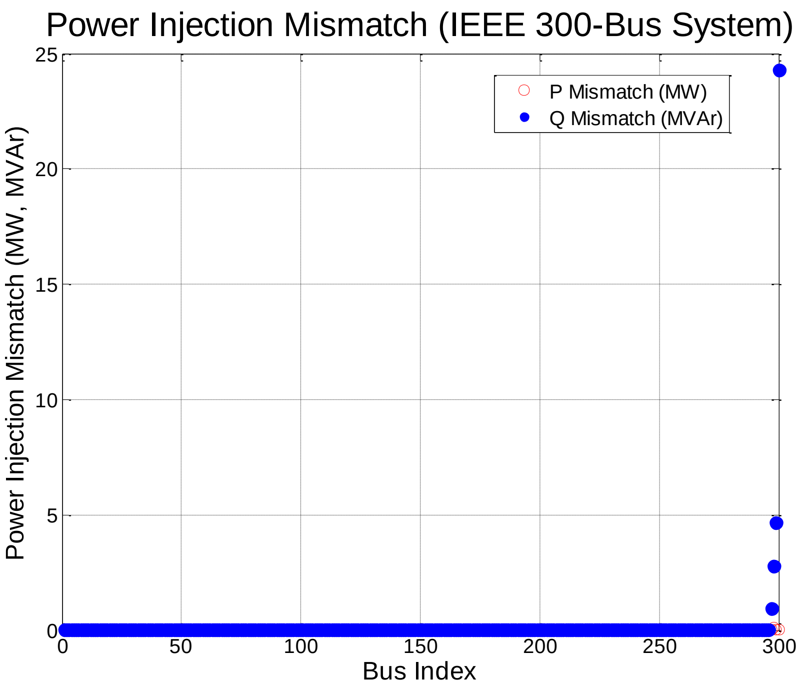


Proof-of-Concept Example

- Global solution from application of second-order constraints to two buses in the IEEE 300-bus system
  - 22% increase in computation time

First-order relaxation





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- Global solution from application of second-order constraints to two buses in the IEEE 300-bus system
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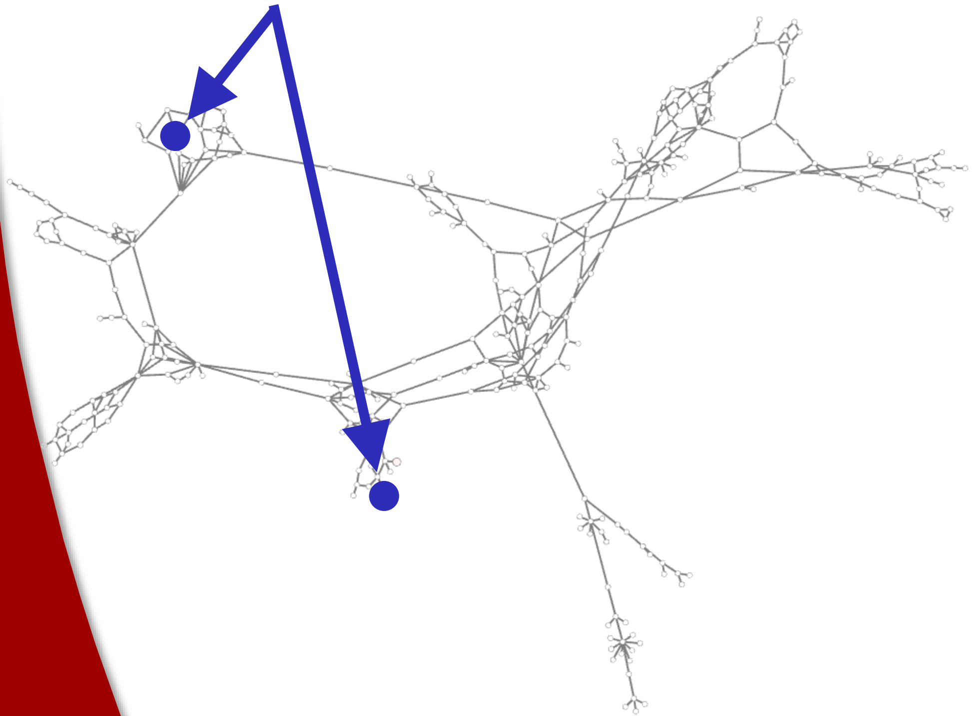
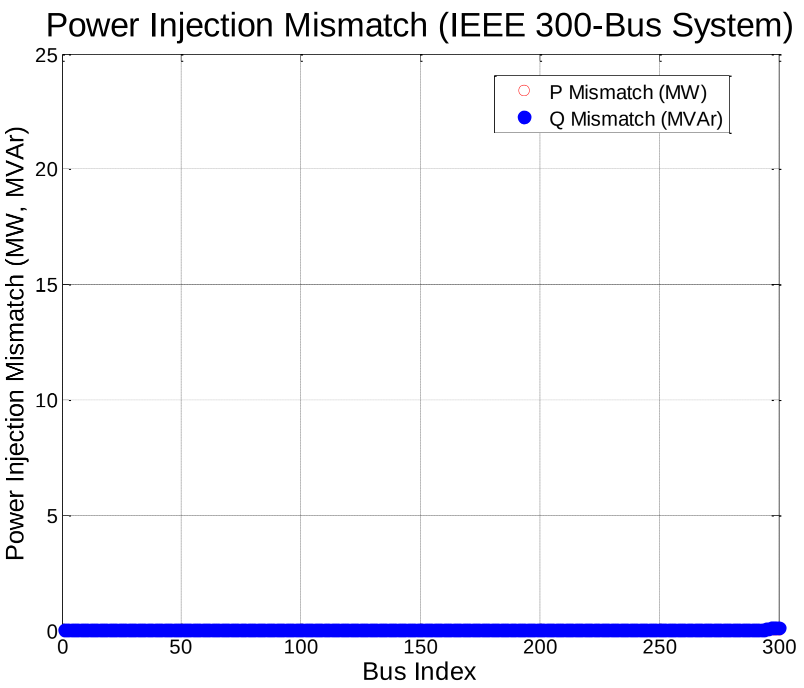
Second-order relaxation here





First-order relaxation elsewhere

Conclusion

Conclusion

- Moment-based relaxations find global solutions to OPF problems
- Investigation of the feasible spaces for first and second-order moment relaxations
- Exploiting sparsity and selective application of higher-order relaxations to globally solve larger problems
  - Proof-of-concept example with moderate-size problem

Ongoing Work

- Improve computational tractability for large problems
  - Challenges: memory, numerical precision, computational speed
- Implement distributed solution algorithms
  - Proven successful for existing OPF relaxations
    [Lam, Zhang, & Tse ’11, Kraning, Chu, Lavaei, & Boyd ‘14]
- Find and explore cases where low-order relaxations fail
- Extend to other problems
  - State estimation, voltage stability margins, etc.

Related Publications

[1] D.K. Molzahn, "Application of Semidefinite Optimization Techniques to Problems in Electric Power Systems," *Ph.D. Dissertation, University of Wisconsin–Madison Department of Electrical and Computer Engineering*, August 2013.

[2] B.C. Lesieutre, D.K. Molzahn, A.R. Borden, and C.L. DeMarco, "Examining the Limits of the Application of Semidefinite Programming to Power Flow Problems," *49th Annual Allerton Conference on Communication, Control, and Computing (Allerton), 2011*, pp. 1492-1499, 28-30 September 2011.

[3] D.K. Molzahn, J.T. Holzer, and B.C. Lesieutre, and C.L. DeMarco, "Implementation of a Large-Scale Optimal Power Flow Solver Based on Semidefinite Programming," *IEEE Transactions on Power Systems,* vol. 28, no. 4, pp. 3987-3998, November 2013.

[4] D.K. Molzahn, B.C. Lesieutre, and C.L. DeMarco, "Investigation of Non-Zero Duality Gap Solutions to a Semidefinite Relaxation of the Optimal Power Flow Problem," *47th Hawaii International Conference on System Sciences (HICSS), 2014*, 6-9 January 2014.

[5] D.K. Molzahn and I.A. Hiskens, "Moment-Based Relaxation of the Optimal Power Flow Problem," To appear in *18th Power Systems Computation Conference, 2014*, 18-22 August 2014. Preprint available: http://arxiv.org/abs/1312.1992

[6] D.K. Molzahn and I.A. Hiskens, "Sparsity-Exploiting Moment-Based Relaxations of the Optimal Power Flow Problem," Submitted. Preprint available: http://arxiv.org/abs/1404.5071

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W.A. Bukhsh, A. Grothey, K.I. McKinnon, and P.A. Trodden, “Local Solutions of Optimal Power Flow,” University of Edinburgh School of Mathematics, Tech. Rep. ERGO 11-017, 2011.

W.A. Bukhsh, A. Grothey, K.I. McKinnon, and P.A. Trodden, “Local Solutions of the Optimal Power Flow Problem,” *IEEE Transactions on Power Systems*, vol. 28, no. 4, pp. 4780-4788, 2013.

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M. Kraning, E. Chu, J. Lavaei, and S. Boyd, “Dynamic Network Energy Management via Proximal Message Passing,” *Foundations and Trends in Optimization,* vol. 1, no. 2, pp. 70-122, January 2014.

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R. Madani, S. Sojoudi, and J. Lavaei, “Convex Relaxation for Optimal Power Flow Problem: Mesh Networks,” in *Asilomar Conference on Signals, Systems, and Computers*, 3-6 November 2013.

H. Waki, S. Kim, M. Kojima, and M. Mauramatsu, “Sums of Squares and Semidefinite Program Relaxation for Polynomial Optimization Problems with Structured Sparsity,” *SIAM Journal on Optimization*, vol. 17, no. 1, pp. 218-242, 2006.

Questions?


relaxation
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non-zero
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problems
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