Identifying and Characterizing Non-Convexities in the Feasible Spaces of Optimal Power Flow Problems

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***Abstract*—Optimal power flow (OPF) is an important problem in the operation of electric power systems. The solution to an OPF problem provides a minimum cost operating point that satisfies constraints imposed by both the non-linear power flow equations and engineering limits. These constraints can yield non-convex feasible spaces that result in significant computational challenges. This paper proposes an algorithm that identifies and characterizes non-convexities in OPF feasible spaces. This algorithm searches for a pair of feasible points whose connecting line segment contains an infeasible point. Such points certify the existence of a non-convexity in the OPF feasible space. Moreover, the constraint violations at the infeasible point along the connecting line segment physically characterize a cause of the non-convexity. Numerical demonstrations include a small illustrative example as well as applications to various test cases.**

I. Introduction

The optimal power flow (OPF) problem seeks a minimum cost operating point for an electric power system, subject to both the power flow equations modeling the network physics and engineering limits. The OPF problem is non-convex [1], may have local optima [2], and is generally NP-hard [3], [4]. A variety of local solution algorithms and convex relaxation techniques have been applied to OPF problems [5]–[7].

The difficulty inherent to solving an OPF problem depends on the non-convexities present in the problem's feasible space. Accordingly, many research efforts, e.g., [1], [2], [8]–[21], have studied OPF feasible space geometry, including analyses of convexity characteristics, local optima, exactness guarantees for certain convex relaxations, etc. (See [7] for a detailed survey.) Many of these papers consider special cases, such as very small systems, radial network topologies, a subset of the constraints, etc. Moreover, much of the literature focuses on conditions for which the feasible space is convex or convexifiable, as opposed to studying possible non-convexities.

This paper builds on previous literature by proposing an algorithm that identifies and characterizes non-convexities in OPF feasible spaces without focusing on very small systems or special cases. The proposed algorithm seeks a pair of feasible points whose connecting line segment contains an infeasible point. Specifically, the algorithm identifies non-convexities via a point on the connecting line segment for which 1) the power flow equations are solvable and 2) all power flow solutions violate at least one engineering limit. Non-convexities associated with power flow insolvability are not considered by this algorithm.

By the definition of convexity, such a combination of points certifies that the feasible space is non-convex, with the specific constraints violated by the infeasible point characterizing the cause of the non-convexity. Characterizing typical non-convexities is useful for informing the development of future OPF solution algorithms and creating challenging test cases for algorithmic benchmarking purposes.

This paper is organized as follows. Section II overviews the OPF problem. Section III describes the proposed algorithm for identifying non-convexities in OPF feasible spaces. Section IV presents numerical results. Section V concludes the paper.

Energy Systems Div., Argonne National Laboratory, dmolzahn@anl.gov. Funding from ARPA-E, U.S. Dept. of Energy, award DE-AC02-06CH11357 is gratefully acknowledged. Views and opinions expressed herein do not necessarily state or reflect those of the U.S. Government or any agency thereof.

II. Optimal Power Flow Overview

This section overviews the OPF problem. Consider an n-bus system, where $\mathcal{N} = \{1,\ldots,n\}$, $\mathcal{G}$, and $\mathcal{L}$ are the sets of buses, generators, and lines. Let $\mathcal{S}$ denote the slack bus. Let $P_{Di} + \mathbf{j}Q_{Di}$ and $P_{Gi} + \mathbf{j}Q_{Gi}$, where $\mathbf{j} = \sqrt{-1}$, represent the active and reactive load demand and generation, respectively, at bus $i \in \mathcal{N}$. Let $V_i\angle\theta_i$ represent the voltage phasor at bus $i \in \mathcal{N}$. Denate the shunt admittance at bus $i \in \mathcal{N}$ as $g_{sh,i} + \mathbf{j}b_{sh,i}$. Each generator $i \in \mathcal{G}$ has a quadratic cost function with coefficients $c_{2,i} \geq 0$, $c_{1,i}$, and $c_{0,i}$. Superscripts "max" and "min" denote upper and lower limits. Buses $i \in \mathcal{N} \setminus \mathcal{G}$ have generation limits set to zero.

Each line $(l,m) \in \mathcal{L}$ is modeled as a Π circuit with mutual admittance $g_{lm} + \mathbf{j}b_{lm}$ and shunt susceptance $\mathbf{j}b_{sh,lm}$. Let p_{lm} and q_{lm} represent the active and reactive power flows on the line $(l,m) \in \mathcal{L}$. Denote $\theta_{lm} = \theta_l - \theta_m$.

Using these definitions, the OPF problem is

$$\min \quad \sum_{i\in\mathcal{G}} c_{2i}\, P_{Gi}^2 + c_{1i}\, P_{Gi} + c_{0i} \tag{1a}$$

$$\text{s.t. } \left(\forall i \in \mathcal{N},\ \forall\, (l,m) \in \mathcal{L}\,\right)$$

$$p_{lm} = V_l^2\, g_{lm} - V_l V_m \left(g_{lm}\cos\left(\theta_{lm}\right) + b_{lm}\sin\left(\theta_{lm}\right)\right), \tag{1b}$$

$$q_{lm} = -V_l^2\left(b_{lm} + b_{sh,lm}/2\right) + V_l V_m\left(b_{lm}\cos(\theta_{lm}) - g_{lm}\sin(\theta_{lm})\right), \tag{1c}$$

$$P_{Gi} - P_{Di} = \sum_{\substack{(l,m)\in\mathcal{L}\\ \text{s.t. } l=i}} p_{lm} + \sum_{\substack{(l,m)\in\mathcal{L}\\ \text{s.t. } m=i}} p_{ml} + g_{sh,i}\, V_i^2, \tag{1d}$$

$$Q_{Gi} - Q_{Di} = \sum_{\substack{(l,m)\in\mathcal{L}\\ \text{s.t. } l=i}} q_{lm} + \sum_{\substack{(l,m)\in\mathcal{L}\\ \text{s.t. } m=i}} q_{ml} - b_{sh,i}\, V_i^2, \tag{1e}$$

$$\theta_s = 0, \quad s \in \mathcal{S}, \tag{1f}$$

$$P_{Gi}^{min} \leq P_{Gi} \leq P_{Gi}^{max}, \tag{1g}$$

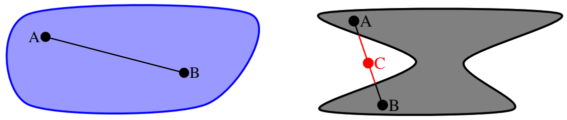
$$Q_{Gi}^{min} \leq Q_{Gi} \leq Q_{Gi}^{max}, \tag{1h}$$

$$V_i^{min} \leq V_i \leq V_i^{max}, \tag{1i}$$

$$\theta_{lm}^{min} \leq \theta_{lm} \leq \theta_{lm}^{max}, \tag{1j}$$

$$\left(p_{lm}\right)^2 + \left(q_{lm}\right)^2 \leq \left(s_{lm}^{max}\right)^2,\ \left(p_{ml}\right)^2 + \left(q_{ml}\right)^2 \leq \left(s_{lm}^{max}\right)^2. \tag{1k}$$

The objective (1a) minimizes the active power generation cost. The power flow equations (1b)–(1e) relate the voltage phasors and power injections. Constraint (1f) sets the angle reference. Constraints (1g)–(1i) limit the power generation and voltage magnitudes. Constraint (1k) limits the apparent power flows



(a) Convex space (b) Non-convex space

Fig. 1. Illustrative examples of convex and non-convex feasible spaces. In Fig. 1a, all points on the line connecting the feasible points A and B are in the feasible space. This is true for all pairs of feasible points. In Fig. 1b, the infeasible point C is on the line connecting the feasible points A and B. Points A, B, and C certify that the feasible space in Fig. 1b is non-convex.

Algorithm 1 Identify and Characterize OPF Non-Convexities

for each constraint $f \in \mathcal{C}$ **do**
  Solve (2) to obtain points A, B, and C.
  if problem (2) is feasible, solve a relaxation of (3).
    if problem (3) is infeasible, certify a non-convexity associated with f.

into both terminal buses of each line. Let $\mathcal{C}$ denote the set of all inequality constraints (1g)–(1k).

III. An Algorithm for Identifying and Characterizing OPF Non-Convexities

This section proposes an algorithm for finding and characterizing OPF non-convexities, including discussions of the mathematical details, computational tractability, and comparisons to alternative approaches.

A. Description of the Algorithm

The proposed algorithm is based on the definition of convexity: a feasible space is convex if and only if it contains all points on the line segments connecting every pair of feasible points. As shown in Fig. 1, one approach for demonstrating non-convexity is to show that there exist points (A, B, and C in Fig. 1b) that violate this definition. The proposed algorithm uses local optimization techniques to seek such points for a specified OPF problem.

There are two relevant spaces within which one may seek non-convexities: the space of voltage phasors and the space of power injections, voltage magnitudes, and line flows. OPF non-convexities may exist in either or both spaces. The former space may be non-convex while the latter is convex (or vice-versa) since the power flow equations (1b)–(1f) are a non-linear map between these spaces [1]. With an objective and constraints specified directly for the power injections, voltage magnitudes, and line flows, characterizing non-convexities in the latter space is most relevant for OPF problems.

As summarized in Algorithm 1, the proposed approach consists of two steps, each of which solves families of optimization problems. The first step seeks triplets of points denoted A, B, and C. Points A and B are in the feasible space of the OPF problem. Point C must satisfy two conditions: 1) lie along the line segment connecting points A and B in the space of active power generation and voltage magnitudes, and 2) have a corresponding "high-voltage" power flow solution that is infeasible in the OPF problem. The second step guarantees that *all* power flow solutions corresponding to point C are infeasible in the OPF problem in order to rigorously certify that the OPF problem's feasible space is non-convex.

Formally, define a pair of points consisting of the active power generation at all non-slack generator buses $i \in \mathcal{G} \setminus \mathcal{S}$, denoted P_G^A and P_G^B, and voltage magnitudes at all generator buses $i \in \mathcal{G}$, denoted V^A and V^B. In the space of active power generation and voltage magnitudes, any point along the line segment connecting these points has active power generation at non-slack generator buses given by $\lambda\, P_G^A + (1-\lambda)\, P_G^B$ and generator bus voltage magnitudes given by $\lambda\, V^A + (1-\lambda)\, V^B$ for some value of the scalar $\lambda \in [0,1]$. A non-convexity is identified by values for P_G^A, P_G^B, V^A, V^B, and λ such that:

1) there exist power flow solutions corresponding to P_G^A, V^A and P_G^B, V^B that are feasible for the OPF problem;
2) all power flow solutions corresponding to the point $\lambda\, P_G^A + (1-\lambda)\, P_G^B$, $\lambda\, V^A + (1-\lambda)\, V^B$ (i.e., point C in Fig. 1b) are infeasible for the OPF problem.

This section presents the mathematical formulations used to address both of these conditions.

Step 1–Identify a triplet of points A, B, and C: The following optimization problem seeks to satisfy the first condition through violation of a specific inequality constraint. For each inequality constraint in the OPF problem (1g)–(1k), abstractly denoted as $f\left(V,\, \theta,\, P_G,\, Q_G,\, p,\, q\right) \geq 0$, $f \in \mathcal{C}$, define

$$\min \sum_{i \in \mathcal{G}} \left(c_{2i} \left(P_{Gi}^A \right)^2 + c_{1i} P_{Gi}^A + c_{0i} \right) + \epsilon_{obj}\, f\left(V^C,\, \theta^C,\, P_G^C,\, Q_G^C,\, p^C,\, q^C \right) \tag{2a}$$

$$\text{s.t.} \;\; \left(\forall i \in \mathcal{N},\; \forall (l,m) \in \mathcal{L},\; \forall \kappa \in \{A,B,C\},\; \forall \sigma \in \{A,B\} \right)$$

$$p_{lm}^\kappa = \left(V_l^\kappa \right)^2 g_{lm} - V_l^\kappa V_m^\kappa \left(g_{lm} \cos(\theta_{lm}) + b_{lm} \sin(\theta_{lm}^\kappa) \right), \tag{2b}$$

$$q_{lm}^\kappa = -\left(V_l^\kappa \right)^2 \left(b_{lm} + b_{sh,lm}/2 \right) + V_l^\kappa V_m^\kappa \left(b_{lm} \cos(\theta_{lm}^\kappa) - g_{lm} \sin(\theta_{lm}^\kappa) \right), \tag{2c}$$

$$P_{Gi}^\kappa - P_{Di}^\kappa = \sum_{\substack{(l,m) \in \mathcal{L} \\ \text{s.t. } l=i}} p_{lm}^\kappa + \sum_{\substack{(l,m) \in \mathcal{L} \\ \text{s.t. } m=i}} p_{ml}^\kappa + g_{sh,i} \left(V_i^\kappa \right)^2, \tag{2d}$$

$$Q_{Gi}^\kappa - Q_{Di}^\kappa = \sum_{\substack{(l,m) \in \mathcal{L} \\ \text{s.t. } l=i}} q_{lm}^\kappa + \sum_{\substack{(l,m) \in \mathcal{L} \\ \text{s.t. } m=i}} q_{ml}^\kappa - b_{sh,i} \left(V_i^\kappa \right)^2, \tag{2e}$$

$$\theta_s^\kappa = 0, \quad s \in \mathcal{S}, \tag{2f}$$

$$P_{Gi}^{min} \leq P_{Gi}^\sigma \leq P_{Gi}^{max}, \tag{2g}$$

$$Q_{Gi}^{min} \leq Q_{Gi}^\sigma \leq Q_{Gi}^{max}, \tag{2h}$$

$$V_i^{min} \leq V_i^\sigma \leq V_i^{max}, \tag{2i}$$

$$\theta_{lm}^{min} \leq \theta_{lm}^\sigma \leq \theta_{lm}^{max}, \tag{2j}$$

$$\left(p_{lm}^\sigma \right)^2 + \left(q_{lm}^\sigma \right)^2 \leq \left(s_{lm}^{max} \right)^2,\; \left(p_{ml}^\sigma \right)^2 + \left(q_{ml}^\sigma \right)^2 \leq \left(s_{lm}^{max} \right)^2, \tag{2k}$$

$$P_{Gh}^C = \lambda\, P_{Gh}^A + (1-\lambda)\, P_{Gh}^B, \quad \forall h \in \mathcal{G} \setminus \mathcal{S}, \tag{2l}$$

$$V_k^C = \lambda\, V_k^A + (1-\lambda)\, V_k^B, \quad \forall k \in \mathcal{G}, \tag{2m}$$

$$0 \leq \lambda \leq 1, \tag{2n}$$

$$f\left(V^C, \theta^C, P_G^C, Q_G^C, p^C, q^C \right) \leq -\epsilon_{vio}, \tag{2o}$$

$$V_i^C \geq V_{lv}, \tag{2p}$$

where ϵ_{obj}, ϵ_{vio}, and V_{lv} are specified positive scalars and superscripts denote a quantity at the corresponding point A, B, or C. The decision variables in (2) consist of the scalar λ and the voltage magnitudes, voltage angles, power injections, power generation, and power flows at points A, B, and C.

Constraints (2b)–(2e) relate the power injections, line flows, and voltage phasors for each point A, B, and C. Constraint (2f) sets the reference angle. Constraints (2g)–(2k) ensure that points A and B are in the OPF problem's feasible space.

Constraints (2l)–(2n) constrain point C to be on the line between points A and B. Constraint (2o) forces the inequality constraint $f \geq 0$ associated with this problem to be violated at point C by at least $\epsilon_{vio} > 0$, ensuring that the solution obtained for point C is infeasible in the OPF problem (1).

The objective (2a) selects among multiple valid choices for points A, B, and C based on two criteria, weighted by ϵ_{obj}:

1) A maximum violation of the constraint $f \geq 0$ associated with this optimization problem. Larger violations are likely associated with more extreme non-convexities.
2) A minimum generation cost at one of the points (arbitrarily selected to be point A). This criteria attempts to find a non-convexity associated with an optimum of the OPF problem.

To explain constraint (2p), observe that there may exist multiple solutions for voltage phasors $V^C\angle\theta^C$ which satisfy (2b)–(2f) for any specified choice of V_k^C, $\forall k \in \mathcal{G}$, and P_{Gh}^C, $\forall h \in \mathcal{G} \setminus \mathcal{S}$, in (2l) and (2m) (i.e., the *power flow equations* defined by (2b)–(2f) may have multiple solutions). There is typically a "high-voltage" solution that corresponds to a normal operating condition and "low-voltage" solutions that generally correspond to undesirable operating conditions [22]. OPF non-convexity is certified by a point C for which *all* corresponding power flow solutions violate one or more OPF inequality constraints (1g)–(1k). In contrast, the optimization problem (2) seeks *any* power flow solution corresponding to point C that is infeasible in the OPF problem (1). In the absence of (2p), the optimization problem (2) could choose voltage phasors $V^C\angle\theta^C$ corresponding to a low-voltage solution at point C for which there exists a high-voltage solution that is feasible in the OPF problem (1). Constraint (2p) forces the voltage magnitudes at point C to be greater than a scalar parameter V_{lv} in an attempt to preclude a low-voltage power flow solution at point C. Selecting $V_{lv} = 0.7$ per unit (p.u.) will typically exclude low-voltage solutions while enabling discovery of non-convexities associated with high-voltage solutions.

In addition to the engineering limits (1g)–(1k), note that OPF non-convexities can also be caused by insolvability of the power flow equations (2b)–(2f) for some choices of generator outputs within the valid generation range (i.e., there may exist choices of P_G and Q_G satisfying (1g) and (1h) that lack corresponding voltage phasors which solve (1b)–(1f)) [1].[1] Since (2) requires that the power flow equations (2b)–(2f) at point C have a solution, the proposed algorithm cannot detect non-convexities associated with power flow insolvability.

To search for non-convexities in the OPF problem, Algorithm 1 applies a local solver to the optimization problems (2) corresponding to the inequality constraints in the OPF problem. If the local solver indicates infeasibility for all of these problems, no non-convexities are identified and the algorithm terminates. Due to the non-convexity of (2), note that failure of Algorithm 1 to identify a non-convexity does not necessarily imply that the OPF problem is convex.

For any feasible solutions obtained from (2), Algorithm 1 applies the following step to certify OPF non-convexity.

Step 2–Certify that all power flow solutions corresponding to point C are infeasible in the OPF problem: Solving (2) indicates that *at least one* power flow solution corresponding to point C fails to satisfy the OPF problem's constraints. Rigorously certifying OPF non-convexity requires proving that *all* power flow solutions corresponding to point C are infeasible in the OPF problem (1). Algorithm 1 accomplishes this by using convex relaxation techniques to compute a sufficient condition for infeasibility of all power flow solutions corresponding to point C. Specifically, consider the feasibility problem

$$\text{Find a point satisfying the constraints (1b)–(1k),} \tag{3a}$$
$$P_{Gi} = P_{Gi}^C, \quad \forall i \in \mathcal{G} \setminus S, \tag{3b}$$
$$V_i = V_i^C, \quad \forall i \in \mathcal{G}, \tag{3c}$$

where P_{Gi}^C and V_i^C are the values obtained from solving (2). The feasibility problem (3) can be formulated as an optimization problem with a constant objective function (i.e., $\min\ 1$ subject to (3)). If (3) is infeasible, then there does not exist a power flow solution corresponding to point C that is feasible in the OPF problem. In this case, points A, B, and C from (2) certify non-convexity of the OPF problem's feasible space.

As a non-convex problem, proving infeasibility of (3) can be challenging. Algorithm 1 therefore leverages techniques based on, e.g., semidefinite programming (SDP) [23]–[26] or second-order cone programming [12], [27]–[31] to solve a convex relaxation of (3). Infeasibility of a convex relaxation is sufficient to certify infeasibility of (3) and therefore prove that the OPF problem's feasible space is non-convex.

B. Discussion of Computational Tractability

Algorithm 1 is tractable for reasonably large systems. The computational burden of (2) is determined by the number of OPF inequality constraints, with each requiring one call to a local solver. The inequality constraints consist of $2\,|\mathcal{N}|$ limits on voltage magnitudes, $2\,|\mathcal{L}|$ limits on line flows (one for each line terminal), $2\,|\mathcal{L}|$ limits on angle differences, and $4\,|\mathcal{G}|$ limits on active and reactive outputs of generators, where $|\cdot|$ denotes the cardinality of a set.[2] While large problems can have many inequalities, practical applications of Algorithm 1 will likely focus on those in a specific subregion of interest.

Each solution from (2) requires one call to a convex programming solver in order to evaluate the feasibility of (3), which has a computational complexity that depends on the chosen relaxation. More computationally intensive relaxations are typically capable of certifying infeasibility of a broader class of problems, so there is a trade-off in computation time versus generality of the algorithm. Recently developed convex relaxations often provide tight bounds on OPF objective values and are tractable for large problems [7]. The second step of Algorithm 1 is therefore not overly burdensome.

Note that each evaluation of (2) and (3) can be parallelized, which is an attractive option for improving performance.

C. Comparisons to Alternative Approaches

This section concludes with qualitative comparisons of the proposed algorithm to alternative approaches. Reference [18] describes a probabilistic approach for identifying

[1] Large enough increases in load demands result in saddle node bifurcations that decrease the number of power flow solutions. The power flow equations (2b)–(2f) are insolvable for sufficiently high load demands.

[2] Constraints (2l) and (2m) ensure that OPF constraints (1g) for $i \in \mathcal{G} \setminus \mathcal{S}$ and (1i) for $i \in \mathcal{G}$ are always satisfied at point C. It is therefore not necessary to consider these constraints in Algorithm 1, thus reducing the number of relevant inequality constraints by $4\,|\mathcal{G}| - 2$.

non-convexities using linear matrix inequalities. This approach is computationally challenging, and thus has only been demonstrated for small problems. Other related approaches in [19]–[21] compute discretized representations of OPF feasible spaces. Projections constructed from these representations can be used to visually identify and characterize non-convexities. This requires appropriate two- or three-dimensional projections from the high-dimensional feasible spaces, which can be difficult to find and challenging to interpret physically. Moreover, computing high-fidelity representations of the feasible spaces can be computationally burdensome. Some OPF problems have analytic representations for their feasible spaces or have provable convexity characteristics, but these are restricted to very particular special cases [1], [9], [13]. Other approaches, such as [8], [10], focus on the feasible spaces of power flow problems, neglecting the engineering limits in OPF problems.

In contrast to previous approaches, the algorithm proposed in this paper directly identifies and characterizes non-convexities and is not computationally limited to small problems. However, as discussed in Section III-A, note that the proposed algorithm can identify non-convexities associated with the engineering constraints (1g)–(1k) but not those associated with power flow insolvability (i.e., insolvability of (1b)–(1f)). Conversely, alternative approaches such as those in [18]–[20] can identify non-convexities associated with both the engineering constraints and power flow insolvability.

IV. Numerical Test Cases

This section describes the application of Algorithm 1 to several test cases. The optimization problems (2) are implemented in Julia [32] using PowerModels.jl [33] and the JuMP modeling language [34], with Ipopt [35] as the local solver. A convex relaxation of (3) is formulated using the combination of an SDP relaxation [23], [24] and the QC relaxation [27]–[29] along with a bound tightening technique [28]. This relaxation is implemented in MATLAB using YALMIP [36] and solved using MOSEK. Parameters for all the test cases are $V_{lv} = 0.7$ p.u. and $\epsilon_{obj} = 1$. The parameter ϵ_{vio} is 0.05 p.u. for power injections and line flows, 0.02 p.u. for voltage magnitudes, and 1° for angle differences.

As an illustrative example, Fig. 2 shows the feasible space for the nine-bus system "case9mod" from [2], computed using the algorithm in [19], along with points A, B, and C for various solutions to (2). Solving a relaxation of (3) indicates that all power flow solutions corresponding to point C are infeasible in the OPF problem for each solution to (2), thus assuring that these points certify OPF non-convexity.

Note that the solution to (2) is dependent on the initialization. For instance, the light and dark blue lines in Fig. 1 correspond to solutions of (2) for the same inequality constraint $f \geq 0$ (a lower reactive generation limit at bus 2) obtained with different initializations. A "flat start" initialization (voltages of $1\angle 0$ p.u.) works well in practice.

Table I summarizes selected results obtained from applying Algorithm 1 to various test cases, initialized using both a flat start and the OPF solution obtained from MATPOWER [37]. The "nesta" cases are from [38]. The "case39mod" cases are from [2], augmented with angle difference limits of $\pm 60^\circ$. Note that the values in Table I are *lower bounds* on the number of constraints associated with non-convexities since Algorithm 1 is not guaranteed to find all non-convexities.

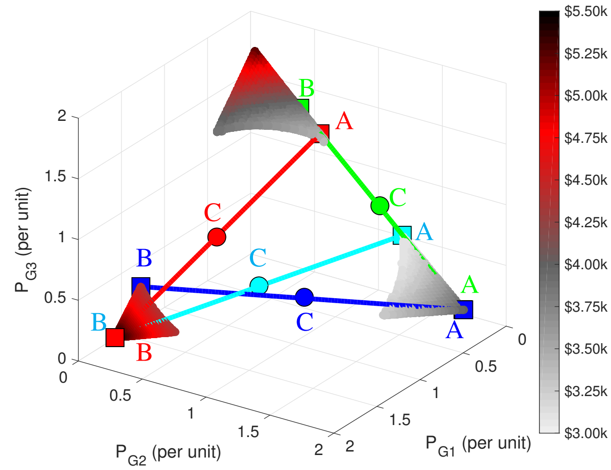


Fig. 2. A projection of the disconnected feasible space for the nine-bus system "case9mod" from [2]. The colored region represents the feasible space with the colors indicating the generation cost. For various solutions of (2), the squares at the endpoints of the colored lines represent points A and B and the circles represent point C. Each of the lines results from choosing $f \geq 0$ in (2) as the lower reactive generation limit at one of the three generator buses. The red line corresponds to bus 1, the light and dark blue lines correspond to bus 2 (using different initializations for the local solver), and the green line corresponds to bus 3. Note that the endpoints for the red and light blue lines as well as the green and dark blue lines overlap with each other.

TABLE I
NUMBER OF CONSTRAINTS (BY TYPE) ASSOCIATED WITH A NON-CONVEXITY AS IDENTIFIED BY ALGORITHM 1

Test Case	P_G^{min}	P_G^{max}	Q_G^{min}	Q_G^{max}	S^{max}
nesta_case5_pjm	1	0	2	3	0
nesta_case9_wscc	1	0	0	0	0
nesta_case14_ieee	0	0	0	0	0
nesta_case30_ieee	0	0	0	0	0
nesta_case39_epri	0	0	10	0	3
case39mod1	1	0	10	0	8
case39mod2	2	0	17	0	12
case39mod3	1	0	8	0	1
case39mod4	1	0	17	0	9
nesta_case57_ieee	0	0	0	0	0

Algorithm 1 does not identify any non-convexities associated with limits on voltage magnitudes (1i) or angle differences (1j).

Algorithm 1 does not identify any non-convexities for some of these test cases (the IEEE 14-, 30-, and 57-bus systems). This aligns with previous research suggesting that these cases are not difficult (e.g., relatively simple relaxations yield small optimality gaps [38]). Algorithm 1 identifies non-convexities in several other cases. While previous research suggests that some of these test cases (e.g., the WSCC 9-bus and the EPRI 39-bus cases) are relatively "simple," others are more challenging (e.g., the PJM 5-bus case and the "case39mod" systems from [2]), at least in terms of the non-negligible optimality gaps resulting from various relaxations and, for the "case39mod" systems, the existence of multiple local optima.

Although the presence of non-convexities does not necessarily mean that a problem is challenging, Algorithm 1 often finds multiple non-convexities for the challenging test cases. Many of these non-convexities are associated with lower reactive generation and apparent power flow limits, which corroborates other evidence regarding the importance of these limits with respect to OPF non-convexities [2], [11], [21], [39].

As an estimate of computational complexity, approximately 30 minutes is required to evaluate Algorithm 1 for all constraints in the 39-bus systems, of which over 90% is spent in locally solving (2). Ongoing efforts to speed computation include parallelizing the implementation and modifying the formulation to more quickly identify infeasibility of (2).

V. Conclusions and Future Work

This paper proposes an algorithm that identifies and characterizes non-convexities in OPF feasible spaces. The proposed algorithm is based on the definition of convexity: a feasible space is convex if and only if it contains all points on the line segments connecting any pair of feasible points. Using a local solver, the proposed algorithm searches for a triplet of points: two feasible points and a third point on the connecting line segment that is infeasible. The proposed algorithm then certifies infeasibility of the third point using convex relaxation techniques. Numerical results demonstrate the algorithm's ability to find non-convexities for a variety of OPF test cases.

Along with improving computational speed, ongoing work is leveraging the identified non-convexities to create test cases that challenge various OPF algorithms for benchmarking purposes. Other future work aims to use the identified non-convexities to both develop a better physical understanding of OPF problems and improve convex relaxation algorithms.

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