Transient Stability Analysis of Power Systems via Occupation Measures

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***Abstract*— This paper proposes the application of occupation measure theory to the classical problem of transient stability analyses of electric power systems. This enables the computation of certified inner and outer approximations of the region of attraction for a nominal operating point. System operators can use these inner and outer approximations to quickly determine whether a post-disturbance point requires corrective actions to ensure stability. In order to determine whether a point is stable, one would simply need to check the sign of a polynomial evaluated at that point. Thus, computationally expensive dynamical simulations are only required for post-disturbance points in the region between the inner and outer approximations. We focus on the nonlinear swing equations but voltage dynamics could also be included. The proposed approach is formulated as a hierarchy of convex relaxations which take the form of semidefinite programs stemming from an infinite dimensional linear program in a measure space, with a natural dual sum-of-squares perspective. On the theoretical side, this paper lays the groundwork for exploiting the oscillatory structure of power system physics by using Hermitian sums-of-squares instead of real sums-of-squares and connects the proposed approach to recent results from algebraic geometry.**

I. Introduction

The application of sum-of-squares techniques to electric power systems dates back to 2000 in Parrilo's PhD thesis [1, Chapter 7.4], where they are used for robust bifurcation analysis. More recently, there has been a growing interest in the power systems community regarding applications of sum-of-squares techniques and, in their dual form, moment relaxation hierarchies. In particular, these techniques are used to find global solutions to alternating current optimal power flow (ACOPF) problems, which seek minimum cost operating points that satisfy both the power flow equations and engineering constraints on power injections, voltage magnitudes, line flows, etc. [2]–[5]. We also note the application of sum-of-squares techniques to ensuring satisfaction of voltage constraints for systems subject to load dynamics [6].

The use of these techniques is justified when weaker relaxations, such as the Shor relaxation [7], do not provide a global solution, but rather a strict lower bound [8]. References [2]–[5] show that the Lasserre hierarchy of moment relaxations [9], [10] can solve ACOPF problems for small power systems (with up to 10 buses) to global optimality using low orders of the hierarchy (typically the first or second orders). This is crucial since the Lasserre hierarchy becomes computationally expensive with increasing relaxation order. By exploiting sparsity [5], [11], [12], the Lasserre hierarchy can solve practical instances of ACOPF problems [13] with thousands of variables and constraints. This is achieved by applying the second-order relaxation to only a subset of the constraints with the first-order relaxation applied to the remainder, i.e., a *multi-ordered* Lasserre hierarchy [12]. Other techniques to improve the tractability of hierarchies for polynomial optimization include the use of chordal sparsity [14], the BSOS hierarchy [15] and Sparse-BSOS hierarchy [16], a mixed semidefinite/second-order cone hierarchy [17], the DSOS and SDSOS hierarchies [18]–[21], the exploitation of complex variable structures via Hermitian sums-of-squares [12], and ADMM for sums-of-squares [22].

In this paper, we demonstrate that the problem of transient stability analysis in power systems can be addressed using similar techniques. Transient stability analysis considers the behavior of a power system following a major disturbance. A power system must return to a stable condition and preserve synchronous operation after the switching of various devices and after faults. Electric power systems are growing in complexity due to increasing shares of renewable generation, increasing peak loads, and the expected wide-scale uses of demand response and energy storage. New computational tools are needed to benefit from high-performance computing and advances in sensing and communication equipment, such as phasor measurement units. Moreover, the control of power systems is complicated by phase-shifting transformers, high-voltage direct current lines, special protection schemes, etc. In this paper, we focus on the case of uncontrolled dynamics as a first step towards certified computations regarding the region of attraction around a nominal operating point.

Similar to ACOPF problems, we find that transient stability analysis problems can be solved by convexifying the problem using measure theory, following the work of [23] which admits a dual sum-of-squares perspective. To the best of our knowledge, sum-of-squares were first used to obtain estimates of the region of attraction of dynamical systems in [1, Chapter 7.3]. In the context of power systems, they were pioneered by the work [24], which uses a Lyapunov approach. (See [25]–[27] for related works.) This general approach either leads to bilinear matrix linear inequalities or conservative LMI conditions. The authors of [24] devise a method for coping with the conservativeness using an expanding interior algorithm. One could say that their approach

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is dual while the approach to be proposed in this paper is primal. The key distinction is that the primal approach results in a single semidefinite program with no additional data required besides the problem description and a hierarchy order parameter. We thus believe that it bears great potential for transmission systems operators, provided that sparsity may be exploited as in ACOPF problems.

We next summarize some recent work on transient stability analysis for power systems. Wang et al. [28] propose stability analyses using a hybrid direct-time-domain method and a partial energy function. The analysis of the power system is reduced to several pairs of "coupled" machines with large rotor speed differences. Owusu-Mireku and Chiang [29] propose an energy-based method for the stability analysis of a power system transmission switching event. Their method determines a relevant controlling unstable equilibrium point for a switching event and then uses an energy margin to assess stability. Dasgupta and Vaidya [30] develop a methodology for finite-time rotor stability analysis. The authors draw on the theory of normal hyperbolic surfaces in order to bring new insights to existing techniques. In particular, they connect repulsion rates of normally hyperbolic surfaces and finite-time stability. All these contributions are confirmed numerically on relevant test cases, such as those in [13].

Finally, we note that power system dynamics can be studied via the theory of differential algebraic equations [31], for which there have been several recent developments. Taha et al. [32] design centralized or decentralized state-feedback controllers for generators while considering worst-case uncertainty. In particular, they introduce $\mathscr{L}_\infty$ robust control and stability for uncertain power networks. The model they consider accounts for unknown disturbances from renewables and loads, making it relevant for current issues faced by transmission system operators. For some progress related to the theory of power system stability, see [33] which presents a new energy-based formulation of the Kuramoto model as port-Hamiltonian system of differential-algebraic equations. This facilitates the development of a robust representation of the physical system with respect to disturbances.

This paper is organized as follows. Section II formulates the transient stability analysis problem. Section III explains the Lyapunov approach to the problem and discusses its advantages and disadvantages. Section IV presents the proposed occupation-measure-based method as well as some foundational theoretical results. Section V describes numerical experiments conducted to show the practical relevance of the proposed method. Section VI gives future research directions regarding computational tractability. Finally, Section VII concludes the paper.

II. Problem Formulation

A. Transient stability of power systems

Consider a power system composed of n synchronous generators with respective complex voltages $v_1,\dots,v_n$. We assume, as it is common in the literature, that the voltage magnitudes $|v_1|,\dots,|v_n|$ are fixed during the transient period, while the phase angles $\theta_1,\dots,\theta_n$ are variable (compared to the rotating frame) with respective angular speeds $\omega_1,\dots,\omega_n$. In addition, the loads in the network are considered to be constant and passive impedances. After a fault occurs, the phases will satisfy the following set of differential equations:

$$\left\{\begin{array}{rcl} \dot\theta_k &=& \omega_k, \\ \dot\omega_k &=& -\lambda_k\omega_k + \frac{1}{M_k}\left(P_k^{\text{mec}} - P_k^{\text{elec}}(\theta_1,\dots,\theta_n)\right), \end{array}\right. \tag{1}$$

where P_k^{mec} is the (fixed) mechanical power input at bus k and $P_k^{\text{elec}}(\theta_1,\dots,\theta_n)$ is the electrical power output of each generator k with value given by

$$G_{kk}|V_k|^2 + \sum_{l\neq k}|v_k||v_l|\left\{B_{kl}\sin(\theta_k-\theta_l)+G_{kl}\cos(\theta_k-\theta_l)\right\}. \tag{2}$$

The quantities B_{kl} and G_{kl} denote the line susceptances and conductances, and M_k denotes the generator inertia constant. The constant λ_k is related to the damping coefficient of each generator.

We assume that there exists an equilibrium to these equations, i.e., values of θ^{eq} that satisfy

$$P_k^{\text{mec}} = P_k^{\text{elec}}(\theta_1^{\text{eq}},\dots,\theta_n^{\text{eq}}), \quad k=1,\dots,n. \tag{3}$$

In other words, θ^{eq} corresponds to a steady-state operating point of an AC transmission system. As usual, we choose one bus, denoted by subscript "ref", to serve as the reference bus, with $\theta_{\text{ref}}^{\text{eq}}=0$ (often referred to as slack bus). Indeed, the equations are invariant up to a phase shift. Although the focus of the paper is on frequency analysis, the results apply to a more comprehensive model coupled with voltage dynamics. The details are omitted for brevity.

B. Illustrative example

The transient stability analyses described in this paper rely on polynomial reformulations of the dynamical system model (1)–(3). To illustrate these reformulations, we use the three-bus example from Chiang et al. [34], which is composed of three synchronous machines connected in a cycle. Since the third bus sets the reference angle (i.e., $\theta_3=0$), we only need two phase angle variables, θ_1, θ_2, and two rotor speed variables, ω_1, ω_2, to describe the dynamics:

$$\left\{\begin{array}{rcl} \dot\theta_1 &=& \omega_1, \\ \dot\theta_2 &=& \omega_2, \\ \dot\omega_1 &=& -\sin(\theta_1)-0.5\sin(\theta_1-\theta_2)-0.4\omega_2, \\ \dot\omega_2 &=& -0.5\sin(\theta_2)-0.5\sin(\theta_2-\theta_1)-0.5\omega_2+0.05. \end{array}\right.$$

A stable equilibrium is given by $(\theta_1^{\text{eq}},\theta_2^{\text{eq}})=(0.02,0.06)$. Following [24], the coordinates can be shifted so that $(\theta_1^{\text{eq}},\theta_2^{\text{eq}})=(0.00,0.00)$ is a stable equilibrium. Applying this shift and trigonometric angle difference equations yields

$$\begin{cases} \dot{\theta}_1 &= \omega_1 \\ \dot{\theta}_2 &= \omega_2 \\ \dot{\omega}_1 &= 0.0200\cos(\theta_1)\cos(\theta_2) - 0.0200\cos(\theta_1) \\ & \quad -0.9998\sin(\theta_1) - 0.4000\omega_1 \\ & \quad +0.4996\cos(\theta_1)\sin(\theta_2) - 0.4996\cos(\theta_2)\sin(\theta_1) \\ & \quad +0.0200\sin(\theta_1)\sin(\theta_2) \\ \dot{\omega}_2 &= 0.4996\cos(\theta_2)\sin(\theta_1) - 0.0299\cos(\theta_2) \\ & \quad -0.4991\sin(\theta_2) - 0.0200\cos(\theta_1)\cos(\theta_2) \\ & \quad -0.4996\cos(\theta_1)\sin(\theta_2) - 0.5000\omega_2 \\ & \quad -0.0200\sin(\theta_1)\sin(\theta_2) + 0.0500. \end{cases}$$

This dynamical system can in turn be formulated as a differential algebraic system of equations in terms of polynomials, as suggested by Anghel et al. [24]. To that end, we introduce auxiliary variables

$$s_1 := \sin(\theta_1) \quad \text{and} \quad s_2 := \sin(\theta_2) \tag{4}$$

and

$$c_1 := 1 - \cos(\theta_1) \quad \text{and} \quad c_2 := 1 - \cos(\theta_2). \tag{5}$$

The reformulated dynamical system is

$$\begin{cases} \dot{\omega}_1 &= 0.4996 s_2 - 0.4\omega_1 - 1.4994 s_1 - 0.02 c_2 \\ & \quad +0.02 s_1 s_2 + 0.4996 s_1 c_2 - 0.4996 c_1 s_2 + 0.02 c_1 c_2 \\ \dot{\omega}_2 &= 0.4996 s_1 + 0.02 c_1 - 09986 s_2 + 0.05 c_2 \\ & \quad -0.5\omega_2 - 0.02 s_1 s_2 - 0.4996 s_1 c_2 \\ & \quad +0.4996 c_1 s_2 - 0.02 c_1 c_2 \\ \dot{s}_1 &= \omega_1 - c_1\omega_1 \\ \dot{s}_2 &= s_1\omega_1 \\ \dot{c}_1 &= \omega_2 - c_2\omega_2 \\ \dot{c}_2 &= s_2\omega_2 \\ 0 &= s_1^2 + c_1^2 - 2.0 c_1 \\ 0 &= s_2^2 + c_2^2 - 2.0 c_2 \end{cases}$$

Section VI will show that one can actually avoid increasing the number of variables and immediately obtain an algebraic differential system of equations in *complex-valued quantities*. For this example, it suffices to define

$$v_1 := \exp(i\theta_1) \quad \text{and} \quad v_2 := \exp(i\theta_2) \tag{6}$$

where i denotes the imaginary number. This yields the dynamical system

$$\begin{cases} \dot{v}_1 &= i\omega_1 v_1 \\ \dot{v}_2 &= i\omega_2 v_2 \\ \dot{\omega}_1 &= i0.5 v_1 - i0.5\overline{v}_1 + i0.25 v_1\overline{v}_2 - i0.25\overline{v}_1 v_2 \\ & \quad -0.4\omega_2 \\ \dot{\omega}_2 &= i0.25 v_2 - i0.25\overline{v}_2 + i0.25 v_2\overline{v}_1 - i0.25\overline{v}_2 v_1 \\ & \quad -0.5\omega_2 + 0.05 \\ 0 &= v_1\overline{v}_1 - 1 \\ 0 &= v_2\overline{v}_2 - 1 \end{cases}$$

where $\overline{(\cdot)}$ denotes the complex conjugate of the argument.

C. Region of attraction

Consider the dynamical system

$$\dot{x}(t) = f(x(t)), \quad \forall t \in [0, T], \tag{7}$$

where $x(\cdot) : [0, T] \longrightarrow \mathbb{C}^n$ and $f := (f_i)_{1 \leqslant i \leqslant n}$ is a given vector field with polynomial entries $f_i \in \mathbb{R}[x]$ and final time $T > 0$ (possibly $T = +\infty$). The state trajectory $x(\cdot)$ belongs to a basic algebraic set

$$X := \{\ x \in \mathbb{C}^n \mid g_i(x) = 0,\ i = 1, \ldots, n_X \} \tag{8}$$

where $g_1, \ldots, g_{n_X}$ are polynomials such that X is compact. In addition, the final state $x(T)$ must belong to a semi-algebraic set X_T. In this paper, we consider X_T to be a Euclidian ball with a small radius $\varepsilon > 0$ centered at 0 (the equilibrium).

The set of admissible trajectories starting from an initial condition x_0 is defined by

$$\mathscr{X}(x_0) := \{x(\cdot) \mid \dot{x}(t) = f(x(t)),\ x(0) = x_0, \\ x(t) \in X, \forall t \in [0, T],\ x(T) \in X_T\}. \tag{9}$$

The region of attraction X_0 is the set of initial conditions for which there exists an admissible trajectory:

$$X_0 := \{\ x_0 \in X \mid \mathscr{X}(x_0) \neq \emptyset\ \}. \tag{10}$$

Note that X_0 is a bounded set due to the compactness of X. The remainder of this paper describes approaches for computing inner and outer approximations to the region of attraction X_0.

III. Approximation of the Region of Attraction using Lyapunov Theory

In this section, we explain the Lyapunov approach for power systems proposed by Anghel et al. [24]. The idea is to construct a real-valued function $V(\cdot)$ defined on an open set $D \subset \mathbb{R}^n$ containing the equilibrium $x = 0$ that is continuously differentiable and satisfies

$$V(0) = 0, \tag{11a}$$

$$V(x) > 0, \qquad \forall x \in D \setminus \{0\}, \tag{11b}$$

$$-\dot{V}(x) = \left(\frac{dV}{dx}\right)^{\mathsf{T}} f(x) \geqslant 0, \qquad \forall x \in D, \tag{11c}$$

where $(\cdot)^{\mathsf{T}}$ denotes the transpose. Lyapunov theory then guarantees that all trajectories initialized at points $x_0 \in D$ asymptotically converge to the stable equilibrium point $x = 0$. In [24], an algorithm is designed to compute both a domain D and a Lyapunov function $V(\cdot)$. In order to address the system of differential algebraic equations, [24] defines the domain

$$\tilde{D} := \{x \in \mathbb{R}^n \mid \beta - p(x) \geqslant 0,\ g(x) = 0\}, \tag{12}$$

where $\beta > 0$, p is a positive definite polynomial, and $g(x) := (g_1(x), \ldots, g_{n_X}(x))$. The Lyapunov conditions (11) are met if the following inclusions hold:

$$\tilde{D} \setminus \{0\} \subset \{x \in \mathbb{R}^n \mid V(x) > 0\}, \tag{13a}$$

$$\tilde{D} \setminus \{0\} \subset \{x \in \mathbb{R}^n \mid \dot{V}(x) < 0\}. \tag{13b}$$

In particular, these inclusions are true if both of the following conditions hold:

$$\{x \in \mathbb{R}^n \mid \beta - p(x) \geqslant 0,\ g(x) = 0,\ l_1(x) \neq 0, V(x) \leqslant 0\} = \emptyset, \tag{14a}$$

$$\{x \in \mathbb{R}^n \mid \beta - p(x) \geqslant 0,\ g(x) = 0,\ l_2(x) \neq 0, \dot{V}(x) \geqslant 0\} = \emptyset, \tag{14b}$$

where $l_1(x)$ and $l_2(x)$ are length-n_X vectors of sum-of-squares polynomials. Using Stengle's Positivstellensatz (see the appendix or [35, Theorem 2.1] for further details), the above conditions hold if there exists $\beta > 0$, polynomials $V(x)$, $\lambda_1(x)$, $\lambda_2(x)$, and sums-of-squares polynomials $s_1(x), \ldots, s_6(x)$ such that $V(0) = 0$ and the polynomials

$$\begin{gathered} -s_1(x)(\beta - p(x)) + s_2(x)V(x) + s_3(x)(\beta - p(x))V(x) \\ -\lambda_1(x)^\intercal g(x) - l_1(x) \\[1em] -s_4(x)(\beta - p(x)) + s_5(x)V(x) - s_6(x)(\beta - p(x))\dot{V}(x) \\ -\lambda_2(x)^\intercal g(x) - l_2(x) \end{gathered} \tag{15}$$

are sums-of-squares. For a given polynomial $p(x)$ and a constant β (and thus a domain $\tilde{D}$), one can use semidefinite programming to compute a function $V(\cdot)$ that ensures asymptotic stability over the domain $\tilde{D}$. The Lyapunov function $V(\cdot)$ provides an inner approximation to the region of attraction in the form of level sets $\{x \in \mathbb{R}^n \mid V(x) \leqslant c,\ g(x) = 0\}$, where c is a constant. Indeed, these level sets describe positively invariant regions contained in the region of attraction of the equilibrium point. In order to find the best inner approximation using this approach, Anghel et al. [24] propose to maximize the constant c once $V(\cdot)$ and β are computed and $p(x)$ is given. The corresponding optimization problem is

$$\begin{array}{rl} \max & c \\ \text{s.t.} & \{x \in \mathbb{R}^n \mid c - V(x) \geqslant 0,\ g(x) = 0\} \\ & \subset \{x \in \mathbb{R}^n \mid \beta - p(x) \geqslant 0,\ g(x) = 0\} \end{array} \tag{16}$$

In order to convert this into a semidefinite program, the problem is reformulated as

$$\begin{array}{rl} \max & c \\ \text{s.t.} & \{x \in \mathbb{R}^n \mid c - V(x) \geqslant 0,\ g(x) = 0, \\ & \qquad \beta - p(x) \geqslant 0,\ \beta - p(x) \neq 0\} = \emptyset, \end{array} \tag{17}$$

ultimately leading to the following semidefinite program:

$$\begin{array}{rl} \max & c \\ \text{s.t.} & -s_1(x)(c - V(x)) - s_2(x)(p - \beta) - s_3(x)(p(x) - \beta) \\ & -s_4(x)(c - V(x))(p(x) - \beta) - \lambda(x)^\intercal g(x) \\ & -(p(x) - \beta)^2 \text{ is a sum-of-squares.} \end{array} \tag{18}$$

Note the bilinearity in (18), with the variable c that appears in the objective multiplying other variables. Thus, (18) is a non-convex problem that is solved using a bisection search on the variable c. Another issue that arises with this approach is that it tends to provide conservative inner approximations that may be too restrictive for some power systems applications (see, e.g., [24, Figure 3]). Based on an algorithm from [36], the authors of [24] propose an expanding interior algorithm that has better performance (see [24, Figure 5]). Nevertheless, in order to begin the expanding procedure, one must chose an appropriate initial positive polynomial $p(x)$ and initial constant β, which may be difficult to ascertain. Hence, we propose a different approach that does not require any information besides the problem data and a relaxation order parameter. We detail this approach in the next section.

IV. Approximation of the Region of Attraction via Occupation Measures

In this section, we explain the general approach proposed by Henrion and Korda [23], [37]. Their idea is to provide a convex formulation of controlled polynomial dynamical systems using the notion of *occupation measures* [38], which quantify the time spent by the trajectory of the state in a subset of X:

$$\mu(A \times B | x_0) := \int_0^T I_{A \times B}(t, x(t|x_0))\, dt \tag{19}$$

where $A \subset [0, T]$, $B \subset X$, and I is the indicator function. An occupation measure satisfies the important property that for all measurable functions φ, it holds that

$$\int_0^T \varphi(t, x(t|x_0))\, dt = \int_{[0,T]\times X} \varphi(t, x)\, d\mu(t, x|x_0). \tag{20}$$

Next, define the linear operator $\mathscr{L} : C^1([0, T] \times X) \to C([0, T] \times X)$ by

$$v \longmapsto \mathscr{L}v := \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i(t, x) = \frac{\partial v}{\partial t} + \text{grad } v \cdot f. \tag{21}$$

and the adjoint operator $\mathscr{L}' : C([0, T] \times X)' \to C^1([0, T] \times X)'$ by

$$\langle \mathscr{L}'\xi, v\rangle := \langle \xi, \mathscr{L}v\rangle = \int_{[0,T]\times X} \mathscr{L}v(t, x)\, d\xi(t, x). \tag{22}$$

We have that

$$\begin{aligned} v(T, x(T|x_0)) &= v(0, x_0) + \int_0^T \tfrac{d}{dt} v(t, x(t|x_0))\, dt, \\ &= v(0, x_0) + \int_0^T \mathscr{L}v(t, x(t|x_0))\, dt, \\ &= v(0, x_0) + \int_{[0,T]\times X} \mathscr{L}v(t, x)\, d\mu(t, x|x_0), \\ &= v(0, x_0) + \langle \mathscr{L}'\mu(\cdot, x_0), v\rangle. \end{aligned} \tag{23}$$

If instead of an initial point x_0, we consider a probability distribution μ_0 supported on the feasible set X, one may define the average occupation measure

$$\mu(A \times B) := \int_X \mu(A \times B|x_0)\, d\mu_0(x_0), \tag{24}$$

$$\mu_T(B) := \int_X I_B(x(T|x_0))\, d\mu_0(x_0). \tag{25}$$

Integrating (23) with respect to μ_0, we obtain that

$$\begin{aligned} \int_{X_T} v(T, x)\, d\mu_T(x) = \int_X & v(0, x)\, d\mu_0(x) \\ &+ \int_{[0,T]\times X} \mathscr{L}v(t, x)\, d\mu(t, x). \end{aligned} \tag{26}$$

We will refer to the above equation as Louiville's equation. Indeed, using distributional derivatives, one can interpret the above equation as Louiville's partial differential equation. The problem of finding the region of attraction is then formulated as the following optimization problem:

$$\begin{array}{rl} \sup & \lambda(\text{supp } \mu_0) \\ \text{s.t.} & \text{Louiville equation (26)} \\ & \mu \geqslant 0,\ \mu_0 \geqslant 0,\ \mu_T \geqslant 0, \\ & \text{supp}(\mu) \subset [0,T] \times X, \\ & \text{supp}(\mu_0) \subset X,\ \text{supp}(\mu_T) \subset X_T, \end{array} \tag{27}$$

where λ denotes the Lebesgue measure on $\mathbb{R}^n$ and supp denotes the support of a measure. Henrion and Korda's approach [23], [37] replaces the above problem with

$$\begin{array}{rl} \sup & \mu_0(X) \\ \text{s.t.} & \text{Louiville equation (26),} \\ & \mu \geqslant 0,\ \lambda \geqslant \mu_0 \geqslant 0,\ \mu_T \geqslant 0, \\ & \text{supp}(\mu) \subset [0,T] \times X, \\ & \text{supp}(\mu_0) \subset X,\ \text{supp}(\mu_T) \subset X_T. \end{array} \tag{28}$$

The optimal value of the above optimization problem is equal to the volume of the region of attraction [23, Theorem 1]. Importantly, the supremum is attained and the optimal solution is such that μ_0 is equal to the restriction of the Lebesgue measure to the region of attraction. The problem can be more conveniently written as

$$\begin{array}{rl} \sup & \mu_0(X) \\ \text{s.t.} & \text{Louiville equation (26),} \\ & \mu_0 + \hat{\mu}_0 = \lambda, \\ & \mu \geqslant 0,\ \mu_0 \geqslant 0,\ \mu_T \geqslant 0,\ \hat{\mu}_0 \geqslant 0, \\ & \text{supp}(\mu) \subset [0,T] \times X, \\ & \text{supp}(\mu_0) \subset X,\ \text{supp}(\mu_T) \subset X_T, \\ & \text{supp}(\hat{\mu}_0) \subset X. \end{array} \tag{29}$$

Louiville's equation and $\mu_0 + \hat{\mu}_0 = \lambda$ induce a linear relationship between the moments of the four measures. In Louisville's equation, we can take the test functions $v(t,x) := t^\alpha x^\beta$, such that

$$\begin{array}{rcl} \int_{X_T} T^\alpha x^\beta \, d\mu_T(x) & = & \int_X 0^\alpha x^\beta \, d\mu_0(x) \\ & & + \int_{[0,T]\times X} \mathscr{L} t^\alpha x^\beta \, d\mu(t,x). \end{array} \tag{30}$$

In the other equation, one may take the test functions $w(x) := x^\alpha$, such that

$$\int_X x^\alpha \, d\mu_0(x) + \int_X x^\alpha \, d\hat{\mu}_0(x) = \int_X x^\alpha d\lambda(x). \tag{31}$$

The linear relationships (30)-(31) between the moments up to order $2k$ will be written succintly as $A_k(y, y_0, y_T, y_0) = b_k$ below. The primal perspective leads to the following truncated moment problem akin to the Lasserre hierarchy for polynomial optimization:

$$\begin{array}{rl} \max & (y_0)_0 \\ \text{s.t.} & A_k(y, y_0, y_T, y_0) = b_k, \\ & M_k(y) \succcurlyeq 0,\ M_{k-d_{X_i}}(g_i, y) = 0,\ i = 1, \ldots, n_X \\ & M_{k-1}[t(T-t), y] \succcurlyeq 0, \\ & M_k(y_0) \succcurlyeq 0,\ M_{k-d_{X_i}}(g_i, y_0) = 0,\ i = 1, \ldots, n_X \\ & M_k(y_T) \succcurlyeq 0,\ M_{k-d_{T_i}}(\varepsilon^2 - (x_1^2 + \ldots + x_n^2), y_T) \succcurlyeq 0, \\ & M_k(\hat{y}_0) \succcurlyeq 0,\ M_{k-d_{T_i}}(g_i, \hat{y}_0) = 0,\ i = 1, \ldots, n_X \end{array} \tag{32}$$

where the notation $\succcurlyeq 0$ indicates positive semidefiniteness of the corresponding matrix and $M_d(p,z) := (\sum_\gamma p_\gamma z_{\alpha+\beta+\gamma})_{|\alpha|,|\beta|\leqslant d}$ for a polynomial $p(x)$ and a pseudo-moment sequence z. In the above problem, the sequence of pseudo-moments $(y, y_0, y_T, \hat{y}_0)$ is truncated at order $2k$.

There exists a dual perspective to the approach:

$$\begin{array}{rll} \inf & \int_X w(x) \, d\lambda(x) & \\ \text{s.t.} & \mathscr{L} v(t,x) \leqslant 0, & \forall (t,x) \in [0,T] \times X \\ & w(x) \geqslant v(0,x) + 1, & \forall x \in X \\ & v(T,x) \geqslant 0, & \forall x \in X_T \\ & w(x) \geqslant 0, & \forall x \in X \end{array} \tag{33}$$

The constraint $\mathscr{L} v(t,x) \leqslant 0$ implies that v is non-increasing along the trajectories, and thus $v(0,x) \geqslant 0$ on X_0 due to the constraint $v(T,x) \geqslant 0$ on X_T. As a byproduct, we also have that $w(x) \geqslant 0$ on X_0. A nice property about the previous optimization problems is that there is no duality gap [23, Theorem 2].

This dual perspective naturally admits a sum-of-squares reformulation:

$$\begin{array}{rl} \inf & \mathbf{w}^\intercal h \\ \text{s.t.} & -\mathscr{L} v_k(t,x) = p(t,x) + q_0(t,x) t(T-t) \\ & \qquad + \sum_{i=1}^{n_X} q_i(t,x) g_i^X(x), \\ & w_k(x) - v_k(0,x) - 1 = p_0(x) + \sum_{i=1}^{n_X} q_{0_i}(x) g_i^X(x), \\ & v_k(T,x) = p_T(x) + \sum_{i=1}^{n_T} q_{T_i}(x) g_i^{X_T}(x), \\ & w_k(x) = s_0(x) + \sum_{i=1}^{n_X} s_{0_i}(x) g_i^X(x). \end{array} \tag{34}$$

where h is the vector of Lebesgue moments over X with indices in the canonical basis, and $\mathbf{w}$ is the vector of coefficient of $w_k(x)$ in that basis. The optimization variables include polynomials $v_k(t,x)$ and $w_k(x)$ of degree at most $2k$ as well as the sum-of-squares polynomials $p(x)$, $q_i(x)$, $p_0(x)$, $p_T(x)$, $q_{0_i}(x)$, $q_{T_i}(x)$, $s_0(x)$, and $s_{0_i}(x)$ with appropriate degrees that can be deduced from the constraints in the optimization problem. Again, there is no duality gap between the truncated problems at every order of the hierarchy [23, Theorem 4].

An outer approximation to the region of attraction is then given by

$$\boxed{\{\ x \in \mathbb{R}^n \mid v_k(0,x) \geqslant 0\ \}} \tag{35}$$

which converges towards to the region of attraction as the order k increases towards infinity [23, Theorem 6]. As with

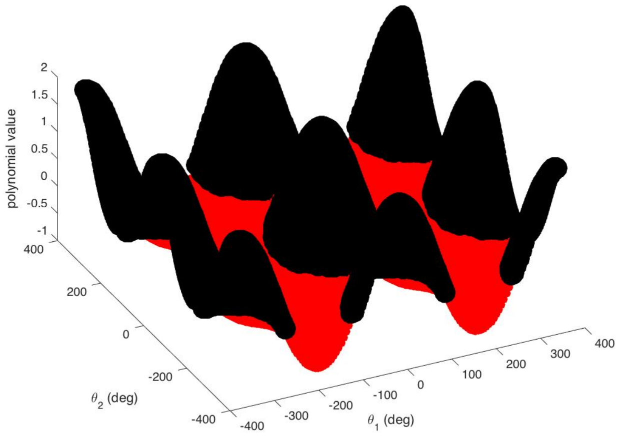

Fig. 1. The polynomial for the three-bus system whose zero level set, which is indicated by the back region, provides an outer approximation to the region of attraction (36). The projection shown is for $(\omega_1, \omega_2) = (0, 0)$.

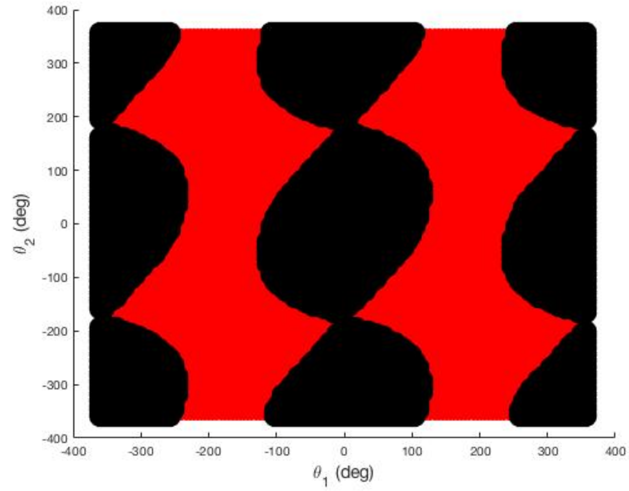

Fig. 2. An outer approximation of the region of attraction is indicated by the back region. The projection shown is for $(\omega_1, \omega_2) = (0, 0)$.

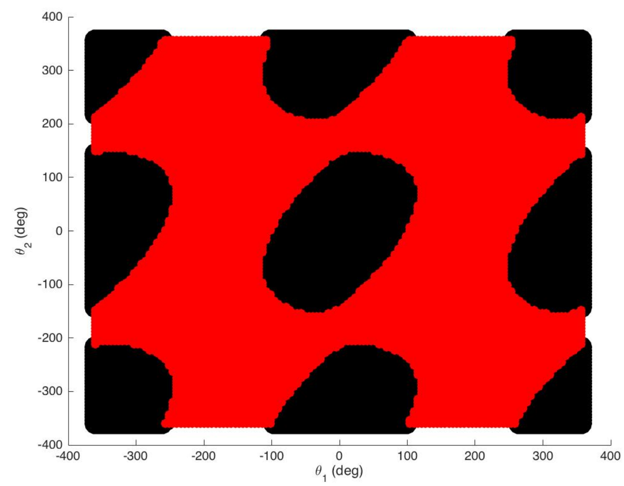

Fig. 3. An inner approximation of the region of attraction is indicated by the back region. The projection shown is for $(\omega_1, \omega_2) = (0,0)$.

the Lasserre hierarchy or the Lyapunov approach via sum-of-squares discussed in the previous section, the computational burden increases sharply as the order k increases.

We conclude this section by briefly discussing the approach for computing inner approximations. The machinery for inner approximations is very similar to the outer approximation approach discussed above. The key distinction is that the inner approximations consider an outer approximation to the complement of the region of attraction, $X_0^c := X \setminus X_0$. See [39] for further details.

V. Case study

To conduct our numerical experiments, we use MATLAB R2015b, YALMIP [40], SeDuMi 1.3 [41], and the "ROA" code of Henrion and Korda [23] to apply occupation measure theory to the three-bus example from [34] that is described in Section II-B.

We note that practical power system analyses require the ability to address significantly larger problems than the three-bus test case considered in this paper. However, the difficulty of constructing certified inner and outer approximations for the region of attraction leads to significant computational challenges for related algorithms. Similar to the test cases first used to demonstrate previous algorithms, e.g., [24], the numerical demonstrations in this paper use a small system as an initial step towards approximating the regions of attraction for larger systems. In our future work, we plan to scale our approach to larger systems by exploiting network sparsity and other problem structures. Decomposition approaches may also prove valuable [25]–[27]. Section VI discusses our ongoing efforts at improving computational tractability.

A. Outer approximation

With final time $T = 8$ and radius $\varepsilon = 0.1$, we find the following polynomial, $v_5(0,x)$, at the fifth-order relaxation ($k = 5$):

$$\begin{aligned}
v_5(0,x) = \quad & 1.8707 - 4.9538x_1 + 0.0017x_2 \\
& - 4.7856x_3 + 0.0018x_4 - 0.0037x_5 \\
& - 4.8546x_6 - 0.0131x_1^2 + 10.9412x_1x_2 \\
& - 0.0356x_1x_3 + 13.9529x_1x_4 + 0.0208x_1x_5 \\
& + 0.0142x_1x_6 + 16.4121x_2^2 + 0.0609x_2x_3 \\
& \qquad \vdots \\
& - 0.2755x_5^5x_6^5 + 0.0017x_5^4x_6^6 + 0.0170x_5^3x_6^7 \\
& - 0.0002x_5^2x_6^8 - 0.0021x_5x_6^9 - 0.0003x_6^{10}
\end{aligned}$$

whose zero level set, i.e.,

$$\{\ x \in \mathbb{R}^6 \mid v_5(0,x) \geqslant 0\ \}, \tag{36}$$

provides an outer approximation to the region of attraction. We illustrate the polynomial $v_5(0,\cdot)$ in Fig. 1 as a function of the original state variables (θ_1,θ_2). (We consider $(\omega_1,\omega_2) = (0,0)$ at the initial point in order to visualize the region of attraction but this is not a necessary restriction.) We illustrate the outer approximation to the region of attraction in Fig. 2.

B. Inner approximation

Likewise, with the final time $T = 8$ and radius $\varepsilon = 0.1$, we find at the third-order relaxation ($k = 3$) the inner approximation to the region of attraction presented in Fig. 3. We again consider $(\omega_1,\omega_2) = (0,0)$ at the initial point, but emphasize that this is not a necessary restriction.

VI. Future Research Directions

In this section, we present some future research directions to improve the scalability of the proposed method.

A. Hermitian sum-of-squares

We next show how to use Hermitian sums-of-squares in the context of transient stability analyses of power systems. For ACOPF problems, applying Hermitian sums-of-squares yields computational advantages while preserving convergence guarantees [12]. The idea is to exploit the structure that comes from alternating current physics in order to reduce the computational burden. A common assumption in transient dynamics is that the magnitudes of the complex voltages, $|v|$, are fixed such that only the phase angles θ are variables. This allows us to define $v_k := \exp(i\theta_k)$ where 1 per unit denotes the nominal voltage magnitude, in order to obtain the relationship $\dot{v}_k = i\dot{\theta}_k \exp(i\theta_k)$. The dynamics can thus immediately be written as a differential algebraic system of equations:

$$\begin{cases} \dot{v}_k = & i\omega_k v_k, \\ \dot{\omega}_k = & -\lambda_k\omega_k + \frac{1}{M_k}\left(P_k - \frac{1}{2}\sum_{l\neq k} -G_{kl}|v_k|^2 - \overline{Y}_{kl}v_k\overline{v}_l - Y_{kl}v_l\overline{v}_k\right), \\ 0 = & |v_1|^2 - 1, \\ & \vdots \\ 0 = & |v_n|^2 - 1, \end{cases} \tag{37}$$

where Y_{kl} denotes the mutual admittance of the line connecting buses k and l.

It is straightforward to adapt the theory of occupation measures for polynomial control systems to the case of complex numbers by leveraging recent results in algebraic geometry regarding complex polynomials (see Theorem 2 and Theorem 3 in the appendix). Our ongoing research is implementing a complex version of the hierarchy of convex relaxations proposed by Henrion and Korda [23] in order to reduce computational complexity at a given relaxation order with the possible trade off in the tightness of the relaxations (and thus the tightness of the inner and outer approximations to the region of attraction) obtained at each order in the hierarchy.

B. Iterative algorithm between inner and outer approximations

In this paper, we considered the final time set X_T to be a Euclidian ball. In order to improve computational tractability by obtaining outer approximations with a polynomial $v_k(t,x)$ of lower degree, it may be advantageous to replace the final set by an inner approximation to the region of attraction. This approach is also the subject of our ongoing research.

VII. Conclusion

In the context of the transient stability analysis of power systems, this paper demonstrates the potential for using the theory of occupation measures (along with convex optimization techniques) to compute inner and outer approximations to the region of attraction for a stable equilibrium point. To the best of our knowledge, this is the first time that occupation measure theory has been applied to analyze transient stability problems for electric power systems. The resulting inner and outer approximations have the potential to provide analytically rigorous guarantees that can preclude the need for computationally expensive transient simulations. With computational tractability remaining an important challenge, future research will investigate how to exploit sparsity when using occupation measures.

Appendix

Each approach presented in this paper relies in some way on a result from algebraic geometry regarding positive polynomials. We state these results below for the reader's convenience.

Theorem 1 (Stengle's Positivstellenstatz [42]–[44]): Let $F := (f_i)_{i\in I_1}$, $G := (g_i)_{i\in I_2}$ and $H := (h_i)_{i\in I_3}$ denote finite sets of multivariate polynomials. Let $P(F)$ denote the preordering generated by $(f_i)_{i\in I_1}$ (see [35, Definition A.3]), let $M(G)$ denote the set of all finite products of g_i's, and let $I(H)$ denote the ideal generated by $(h_i)_{i\in I_3}$ (see [35, Definition A.7]). Then

$$\{x\in\mathbb{R}^n \mid f_i(x)\geqslant 0,\ \forall i\in I_1,\ g_i(x)\neq 0,\ \forall i\in I_2, \\ h_i(x) = 0,\ \forall i\in I_3\} = \emptyset \tag{38}$$

if and only if there exist $f\in P(F)$, $g\in M(G)$, and $h\in I(H)$ such that

$$f + g^2 + h = 0. \tag{39}$$

Given some real-valued complex polynomials $g_1,\ldots,g_m$, define the semialgebraic set $K := \{z\in\mathbb{C}^n \mid g_i(z,\bar{z})\geqslant 0,\ i = 1,\ldots,m\}$.

Theorem 2 (D'Angelo's and Putinar's Positivstellenstatz [45]): Assume that one of the constraints of K is a sphere $|z_1|^2+\cdots+|z_n|^2 = R^2$ for some radius $R > 0$. If $f > 0$ on K, then there exists Hermitian sums-of-squares $\sigma_0,\ldots,\sigma_m$ such that

$$f = \sigma_0 + \sum_{i=1}^{m}\sigma_i g_i. \tag{40}$$

This theorem naturally admits a dual perspective. Let $\mathscr{M}_+(K)$ denote the set of positive finite Borel measures on K.

Theorem 3 (Putinar and Scheiderer [46]): If one of the constraints of K is a sphere, then the following properties are equivalent:

1) $\exists \mu \in \mathscr{M}_+(K): \ \forall \alpha, \beta \in \mathbb{N}^n, \quad y_{\alpha,\beta} = \int_K z^\alpha \bar{z}^\beta \, d\mu$;
2) $\forall d \geqslant d^{\min}, \quad M_d(y) \succcurlyeq 0, \quad M_{d-k_i}(g_i y) \succcurlyeq 0.$

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