Invertibility Conditions for the Admittance Matrices of Balanced Power Systems

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***Abstract*—The admittance matrix encodes the network topology and electrical parameters of a power system in order to relate the current injection and voltage phasors. Since admittance matrices are central to many power engineering analyses, their characteristics are important subjects of theoretical studies. This paper focuses on the key characteristic of *invertibility*. Previous literature has presented an invertibility condition for admittance matrices. This paper first identifies and fixes a technical issue in the proof of this previously presented invertibility condition. This paper then extends this previous work by deriving new conditions that are applicable to a broader class of systems with lossless branches and transformers with off-nominal tap ratios.**

***Index Terms*—Admittance matrix, circuit analysis.**

NOTATION

j	The imaginary unit ($j^2+1=0$)
a, A	(No boldface letter) scalar
$\mathbf{a}$	(Boldface lowercase letter) column vector
$\mathbf{A}$	(Boldface uppercase letter) matrix
$\mathcal{A}$	(Calligraphic font uppercase letter) set
$\mathrm{Re}(\,\cdot\,)$	Element-wise real part operator
$\mathrm{Im}(\,\cdot\,)$	Element-wise imaginary part operator
$(\,\cdot\,)^*$	Element-wise conjugate operator
$(\,\cdot\,)^T$	Transpose operator
$(\,\cdot\,)^H$	Conjugate transpose operator
$\{\mathbf{a}\}_k$	k-th element of vector $\mathbf{a}$ (scalar)
$\{\mathbf{A}\}_k$	k-th row of matrix $\mathbf{A}$ (row vector)
$\{\mathbf{A}\}_{ij}$	Element of matrix $\mathbf{A}$ in row i, column j (scalar)
$\lvert a\rvert$	Absolute value of scalar a
$\lvert\mathcal{A}\rvert$	Cardinality of set $\mathcal{A}$
$\lVert\mathbf{a}\rVert_1$	1-norm of vector $\mathbf{a}$: $\lVert\mathbf{a}\rVert_1 = \sum_k \left\lvert\{\mathbf{a}\}_k\right\rvert$
$\lVert\mathbf{a}\rVert$	Euclidean norm of vector $\mathbf{a}$: $\lVert\mathbf{a}\rVert = (\sum_k \left\lvert\{\mathbf{a}\}_k\right\rvert^2)^{1/2}$
$\mathrm{diag}\,(\mathbf{a})$	Diagonal matrix such that $\{\mathrm{diag}\,(\mathbf{a})\}_{kk} = \{\mathbf{a}\}_k$. $\mathrm{diag}\,(\mathbf{a})$ has as rows and columns as the size of $\mathbf{a}$
$\mathrm{rank}\,(\mathbf{A})$	Rank of matrix $\mathbf{A}$ (scalar)
$\mathrm{Null}\,(\mathbf{A})$	Null space (kernel) of matrix $\mathbf{A}$ (vector space)
$\dim(\,\cdot\,)$	Dimension of a vector space (scalar)
$\mathrm{Sym}\,(\mathbf{B})$	Symmetric part of square matrix $\mathbf{B}$: $\mathrm{Sym}\,(\mathbf{B}) = \left(\mathbf{B}+\mathbf{B}^T\right)/2$
$\mathbf{B}\succeq\mathbf{0}$	Square matrix $\mathbf{B}$ is positive-semidefinite (for all $\mathbf{x}\neq\mathbf{0}$, $\mathrm{Re}\left(\mathbf{x}^H\mathbf{B}\mathbf{x}\right)\geq 0$), but not necessarily Hermitian
$\mathbf{B}\succ\mathbf{0}$	Square matrix $\mathbf{B}$ is positive-definite (for all $\mathbf{x}\neq\mathbf{0}$, $\mathrm{Re}\left(\mathbf{x}^H\mathbf{B}\mathbf{x}\right)> 0$), but not necessarily Hermitian

I. INTRODUCTION

THE admittance matrix, which relates the current injections to the bus voltages, is one of the most fundamental concepts in power engineering. In the phasor domain, admittance matrices are complex-valued square matrices. These matrices are used in many applications, including system modeling, power flow, optimal power flow, state estimation, stability analyses, etc. [1], [2]. This paper thoroughly characterizes the invertibility of admittance matrices, which is a fundamental property for many power system applications.

Several applications directly rely on the invertibility of the admittance matrix. For instance, Kron reduction [3] is a popular technique for reducing the number of independent bus voltages modeled in a power system. The feasibility of Kron reduction is contingent on the invertibility of an appropriate sub-block of the admittance matrix. Additionally, various fault analysis techniques require the explicit computation of the inverse of the admittance matrix (the impedance matrix) [4]. The DC power flow [5] and its derivative applications [6], [7] also require the invertibility of admittance matrices for purely inductive systems. The invertibility of the admittance matrix is a requirement seen in both classical literature and recent research efforts (see, e.g., [8], [9]).

Checking invertibility of a matrix can be accomplished via rank-revealing factorizations [10], [11]. However, this approach is computationally costly for large matrices. Invertibility can also be checked approximately by computing the condition number via iterative algorithms that have lower complexity than matrix factorizations [12]. However, iterative estimation of the condition number can be inaccurate [13]. In some applications, such as transmission switching [14] and topology reconfiguration [15], [16], the admittance matrix changes as part of the problem and checking invertibility for every case is intractable. Recent research has studied the theoretical characteristics of the admittance matrix in order to guarantee invertibility without the need for computationally expensive explicit checks [17]–[19].

One of the most important results regarding theoretical invertibility guarantees comes from [17]. The authors of [17] show that the admittance matrix is invertible for connected networks consisting of reciprocal branches without mutual coupling and at least one shunt element. (See [20] for the definition and properties of reciprocal branches.) This result relies on additional modeling assumptions requiring that all admittances have positive conductances and prohibiting transformers with off-nominal tap ratios (including on-load tap changers which control the voltage magnitudes or phase shifters which control the voltage angles).

These requirements can be restrictive for practical power system models. While perfectly lossless branches do not exist in physical circuits, power system datasets often approximate certain branches as lossless. For instance, out of the 41 systems with more than 1000 buses in the PGLib test case repository [21], zero-conductance branches exist in 26 systems (63.4%). We further note that transformers with off-nominal tap ratios and non-zero phase shifts are also present in many practical datasets (e.g., 39 of the 41 PGLib systems (95.1%)).

In addition to these modeling restrictions, there is a technical issue with the proof presented in [17]. This paper demonstrates that the result of [17] can still be achieved and generalized to a broader class of power system models. We first detail the technical issue in the proof in [17]. We then prove invertibility of the admittance matrix under several conditions that generalize the requirements in [17]. These conditions hold for a broad class of realistic systems, including systems with lossless branches and transformers with off-nominal tap ratios. Finally, we discuss the implications of the new conditions and how they can be used to enforce invertibility of the admittance matrix by applying small modifications to the network model.

The rest of the paper is organized as follows. Section II describes the result of previous research and the technical issue in their proof. Section III states the modifications and additional lemmas required to amend and generalize the previous result to systems with purely reactive elements and more general transformer models. Section III also discusses practical implications of our results. Section IV concludes the paper.

II. Claims from Previous Literature and Limitations

Using the notation of [17], the admittance matrix is

$$\mathbf{Y}_{\mathcal{N}} = \mathbf{A}_{(\mathcal{N},\mathcal{L})}^{T}\mathbf{Y}_{\mathcal{L}}\mathbf{A}_{(\mathcal{N},\mathcal{L})} + \mathbf{Y}_{\mathcal{T}}, \tag{1}$$

where $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ is the oriented incidence matrix of the network graph (excluding ground), $\mathbf{Y}_{\mathcal{L}} = \operatorname{diag}(\mathbf{y}_{\mathcal{L}})$ is the diagonal matrix with the series admittances of each branch, and $\mathbf{Y}_{\mathcal{T}} = \operatorname{diag}(\mathbf{y}_{\mathcal{T}})$ is the diagonal matrix with the total shunt admittances at each node. $\mathcal{N}$ is the set of nodes (excluding ground) and $\mathcal{L}$ is the set of branches. Reference [17] states the following hypothesis and lemmas (presented here with some minor extensions as described below):

Hypothesis 1. *The branches are not electromagnetically coupled and have nonzero admittance, hence $\mathbf{Y}_{\mathcal{L}}$ is full-rank.*

Lemma 1. *The rank of the oriented incidence matrix of a connected graph with $|\mathcal{N}|$ nodes, $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$, is $|\mathcal{N}|-1$. The vector of ones $\mathbf{1}$ forms a basis of the null space of $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$.*

While the second statement regarding the basis of the null space is not included in Lemma 1 as presented in [17], it is a well-known characteristic of oriented incidence matrices[1] that we will use later in this paper.

Lemma 2. *The sum of the columns of $\mathbf{Y}_{\mathcal{N}}$ equals the transpose of the sum of its rows, which also equals the vector of shunt elements $\mathbf{y}_{\mathcal{T}}$ (see [23]).*

Lemma 3. *For any matrix $\mathbf{M}$, $\operatorname{rank}\left(\mathbf{M}^{T}\mathbf{M}\right) = \operatorname{rank}(\mathbf{M})$.*

As we will discuss shortly, *Lemma 3 as stated above is incorrect*. This is the technical issue in [17] mentioned above.

Lemma 4. *For square matrices $\mathbf{N}_L$ and $\mathbf{N}_R$ with full rank and matching size, $\operatorname{rank}(\mathbf{N}_L\mathbf{M}) = \operatorname{rank}(\mathbf{M}) = \operatorname{rank}(\mathbf{M}\mathbf{N}_R)$. Furthermore, $\operatorname{Null}(\mathbf{N}_L\mathbf{M}) = \operatorname{Null}(\mathbf{M})$.*

While the second statement regarding the relationship between the null spaces is not included in Lemma 4 as presented in [17], it is a well-known result from matrix theory.[2]

One of the main results of [17] is the following theorem:

Theorem 1. *If the graph $(\mathcal{N},\mathcal{L})$ defines a connected network and Hypothesis 1 holds, then:*

$$\operatorname{rank}(\mathbf{Y}_{\mathcal{N}}) = \begin{cases} |\mathcal{N}|-1 & \text{if } \mathbf{y}_{\mathcal{T}} = \mathbf{0}, \\ |\mathcal{N}| & \text{otherwise.} \end{cases} \tag{2}$$

[1]The sum of the elements of each row of $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ is always zero since every row has exactly one entry of 1 and one entry of -1 with the rest of the entries equal to zero; see [22].

[2]Since the only solution of $\mathbf{N}_L\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, we make $\mathbf{x} = \mathbf{M}\mathbf{z}$ for some vector $\mathbf{z}$ and the result follows.

The authors of [17] prove Theorem 1 by cases. They first assume $\mathbf{y}_{\mathcal{T}} = \mathbf{0}$ and decompose $\mathbf{Y}_{\mathcal{L}}$ into square matrices:

$$\mathbf{Y}_{\mathcal{L}} = \mathbf{B}^{T}\mathbf{B}, \tag{3}$$

where $\mathbf{B}$ is clearly full-rank. Therefore:

$$\mathbf{Y}_{\mathcal{N}} = \mathbf{A}_{(\mathcal{N},\mathcal{L})}^{T}\mathbf{B}^{T}\mathbf{B}\mathbf{A}_{(\mathcal{N},\mathcal{L})}, \tag{4a}$$

$$\mathbf{Y}_{\mathcal{N}} = \left(\mathbf{B}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right)^{T}\mathbf{B}\mathbf{A}_{(\mathcal{N},\mathcal{L})}, \tag{4b}$$

$$\mathbf{Y}_{\mathcal{N}} = \mathbf{M}^{T}\mathbf{M}, \tag{4c}$$

where $\mathbf{M} = \mathbf{B}\mathbf{A}_{(\mathcal{N},\mathcal{L})}$. According to Lemma 1, $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ has rank $|\mathcal{N}|-1$. According to Lemma 4, $\operatorname{rank}\left(\mathbf{B}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) = \operatorname{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right)$, so $\operatorname{rank}(\mathbf{M}) = |\mathcal{N}|-1$. Finally, according to Lemma 3, $\operatorname{rank}(\mathbf{Y}_{\mathcal{N}}) = |\mathcal{N}|-1$.

There is a technical issue in the proof of Theorem 1 resulting from the fact that Lemma 3 only holds for real-valued matrices. A complex-valued counterexample is the following:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ j & 0 \end{bmatrix}, \qquad \operatorname{rank}(\mathbf{M}) = 1, \tag{5}$$

$$\mathbf{M}^{T}\mathbf{M} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \operatorname{rank}\left(\mathbf{M}^{T}\mathbf{M}\right) = 0. \tag{6}$$

However, there is a very similar property to Lemma 3 that holds for complex-valued matrices:

Lemma 3 (Corrected). *For any matrix $\mathbf{M}$ with complex entries, $\operatorname{rank}\left(\mathbf{M}^{H}\mathbf{M}\right) = \operatorname{rank}(\mathbf{M})$. Furthermore, $\operatorname{Null}\left(\mathbf{M}^{H}\mathbf{M}\right) = \operatorname{Null}(\mathbf{M})$.*

Proof. Suppose a vector $\mathbf{z}$ is in the null space of $\mathbf{M}$, then:

$$\mathbf{0} = \mathbf{M}\mathbf{z}, \quad \Longrightarrow \quad \mathbf{0} = \mathbf{M}^{H}\mathbf{M}\mathbf{z}, \tag{7}$$

so $\mathbf{z}$ is also in the null space of $\mathbf{M}^{H}\mathbf{M}$. Moreover, suppose a vector $\mathbf{z}$ is in the null space of $\mathbf{M}^{H}\mathbf{M}$. Then, we have

$$\mathbf{0} = \mathbf{M}^{H}\mathbf{M}\mathbf{z}, \quad \Longrightarrow \quad 0 = \mathbf{z}^{H}\mathbf{M}^{H}\mathbf{M}\mathbf{z} = \|\mathbf{M}\mathbf{z}\|^{2} \tag{8a}$$

$$\Longrightarrow \quad \mathbf{0} = \mathbf{M}\mathbf{z}, \tag{8b}$$

so $\mathbf{z}$ is also in the null space of $\mathbf{M}$. In conclusion, $\mathbf{z}$ is in the null space of $\mathbf{M}$ if and only if it is in the null space $\mathbf{M}^{H}\mathbf{M}$; this means that $\operatorname{Null}\left(\mathbf{M}^{H}\mathbf{M}\right) = \operatorname{Null}(\mathbf{M})$. Now we apply the rank-nullity theorem (see [24]) to complete the proof. □

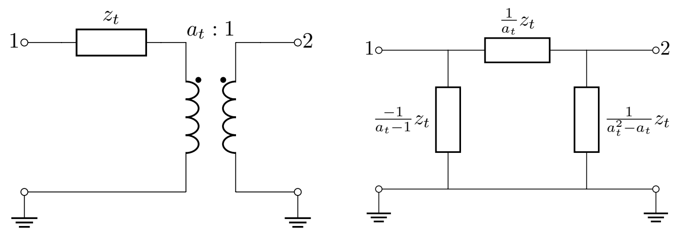
With the corrected version of Lemma 3 and a modeling restriction to systems where all branches are strictly lossy (have positive conductances), we can fix the proof of Theorem 1 as stated above. More specifically, the assumptions of [17] imply Condition 2 and Condition 3 of the generalized version of Theorem 1 stated in the next section.

We now turn our attention to the modeling restrictions of [17]. Before generalizing Theorem 1, we need to understand why a system that violates the modeling restrictions may not satisfy the theorem. Consider the circuit modeling a transformer with an off-nominal tap ratio shown in Fig. 1a. Let $y_t = 1/z_t$. The transformer's turns ratio a_t is an arbitrary complex number. The transformer's admittance matrix is:

$$\mathbf{Y}_t = \begin{bmatrix} y_t & -a_t y_t \\ -a_t^{*} y_t & |a_t|^{2} y_t \end{bmatrix} = y_t\, \mathbf{a}_t\, \mathbf{a}_t^{H}, \tag{9}$$

where $\mathbf{a}_t^{H} = [1, -a_t]$. If a_t is purely real, then $a_t^{*} = a_t$ and we can model the transformer with the π circuit in Fig. 1b [1].

The transformer's π circuit is a two-port network with $\operatorname{rank}(\mathbf{Y}_t) = 1$. This π circuit violates the requirement of



(a) Transformer circuit providing a counterexample to Theorem 1 in [17].

(b) π-equivalent circuit for the transformer in Fig. 1a for a real-valued turns ratio a_t.

Fig. 1: Transformer circuits.

strictly lossy branches, as one one of the shunts will always have non-positive conductance. Notice that the impedances around the loop in the π circuit have the sum $\frac{-1}{a_t-1} z_t + \frac{1}{a_t} z_t + \frac{1}{a_t^2 - a_t} z_t = 0$. With a zero-impedance loop (i.e., a closed path through the circuit where the sum of the impedances along the path equals zero), it is mathematically possible to have non-null voltages even in the case of null current injections. This means that the admittance matrix is singular. More generally, admittance matrix singularity can result from other power system models with zero-impedance loops besides those associated with transformers.

The strict-lossiness restriction in [17] forbids the existence of zero-impedance loops, as the sum over any loop will have positive real part. However, this also restricts the presence of transformers with off-nominal tap ratios and branches modeled as purely reactive elements, both of which appear in practical power system datasets as discussed in Section I. To circumvent this issue, we will treat transformers as general series elements while modeling the shunt elements of the transformer π circuit by employing an appropriate representation of the admittance matrix. With this approach, the conditions we derive in this paper only forbid the existence of *non-transformer* zero-impedance loops. Further, the new representation allows us to generalize Theorem 1 to systems with purely reactive elements and transformers with off-nominal tap ratios.

III. Main Results

This section describes our process for fixing and generalizing the main theorem. We first state and prove all necessary lemmas that will be used to prove the main result. We also declare several additional reasonable hypotheses that allow us to extend the result to systems with general transformer models. We then state the generalized version of the main theorem, which requires some relaxed conditions in order to hold. We next state and prove the validity of the generalized Theorem 1 under each of these conditions, individually. We end this section by discussing the implications of our results.

A. Preliminaries

We start by introducing the following hypothesis:

Hypothesis 2. *For any series branch $l \in \mathcal{L}$ from node i to node k, the admittance matrix associated with just this element can be written as $\mathbf{Y}_l = y_l \mathbf{a}_l \mathbf{a}_l^H \in \mathbb{C}^{|\mathcal{N}| \times |\mathcal{N}|}$, where $\{\mathbf{a}_l\}_i = 1$, $\{\mathbf{a}_l\}_k = -a_l^*$ (a_l is a non-zero complex number) and all other entries of $\mathbf{a}_l$ are zero.*

Transmission lines and transformers (including transformers with off-nominal tap ratios) satisfy Hypothesis 2. Transmission lines can be modeled as transformers with $a_l = 1$ along with some shunt elements. Using Hypothesis 2, the admittance matrix of the full system is:

$$\mathbf{Y}_{\mathcal{N}} = \sum_{l \in \mathcal{L}} \mathbf{Y}_l + \mathbf{Y}_{\mathcal{T}}. \tag{10}$$

In (10), note that $\mathbf{Y}_{\mathcal{T}}$ does not include the shunt elements in the transformers' π circuits as these elements are instead included in $\mathbf{Y}_l$. The sum of the matrices can be rewritten as:

$$\mathbf{Y}_{\mathcal{N}} = \mathbf{A}_{(\mathcal{N},\mathcal{L})}^H \mathbf{Y}_{\mathcal{L}} \mathbf{A}_{(\mathcal{N},\mathcal{L})} + \mathbf{Y}_{\mathcal{T}}, \tag{11}$$

where, in a slight abuse of notation relative to Section II, $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ is the *generalized incidence matrix*, whose l-th row is $\left\{\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right\}_l = \mathbf{a}_l^H$; $\mathbf{Y}_{\mathcal{L}} = \operatorname{diag}(\mathbf{y}_{\mathcal{L}})$ is the diagonal matrix containing the series admittances for each branch; and $\mathbf{Y}_{\mathcal{T}} = \operatorname{diag}(\mathbf{y}_{\mathcal{T}})$ is the diagonal matrix containing the total shunt admittances at each node. This is the default representation for the rest of the paper.

Normally, there is no need to have parallel branches, as in most cases they can be reduced to a single branch by adding the admittances. However, it is possible that the parallel admittances could sum to zero, in which case the network could fail to satisfy the connectedness condition. To prevent this, we introduce an additional hypothesis:

Hypothesis 3. *There are no parallel branches.*

Parallel transformers with different tap ratios cannot be represented as single branch in the form required by Hypothesis 2. Hence, Hypothesis 3 restricts us to cases where all parallel transformers have equal tap ratios. In practice, this restriction normally holds, so the main results are still widely applicable.

We next state the rank-nullity theorem as we will use it several times throughout the paper:

Rank-nullity theorem ([Theorem 2.3] in [24]). *Let $\mathbf{M} \in \mathbb{C}^{m \times n}$ be an arbitrary matrix, then:*

$$\operatorname{rank}(\mathbf{M}) + \dim(\operatorname{Null}(\mathbf{M})) = n. \tag{12}$$

Now we extend Lemma 1 to generalized incidence matrices:

Lemma 1 (Extended). *The rank of the generalized incidence matrix of a connected network with $|\mathcal{N}|$ nodes, $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$, is at least $|\mathcal{N}| - 1$. If $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ is not full column rank, then none of the basis vectors of its null space have null entries.*

Proof. Let $\mathcal{S} \subseteq \mathcal{L}$ be a set of edges forming a spanning tree of the network graph. We can permute the edges in $\mathcal{L}$ in order to write $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ in blocks as follows:

$$\mathbf{A}_{(\mathcal{N},\mathcal{L})} = \begin{bmatrix} \mathbf{A}_{(\mathcal{N},\mathcal{S})} \\ \mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{S})} \end{bmatrix}, \tag{13}$$

where $\mathbf{A}_{(\mathcal{N},\mathcal{S})}$ is the generalized incidence matrix of the edges in $\mathcal{S}$ and $\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{S})}$ is the generalized incidence matrix of the remaining edges. For any vector $\mathbf{x}$ in the null space of $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$, $\mathbf{x}$ must be orthogonal to all rows of $\mathbf{A}_{(\mathcal{N},\mathcal{S})}$:

$$\mathbf{a}_s^H \mathbf{x} = 0, \qquad \forall s \in \mathcal{S}. \tag{14}$$

Take an arbitrary branch s that goes from node i to node k, then from (14) we have:

$$\{\mathbf{x}\}_i - a_s \{\mathbf{x}\}_k = 0, \tag{15}$$

where a_s is the tap ratio of branch s. We can write:

$$\{\mathbf{x}\}_i = a_s \{\mathbf{x}\}_k, \tag{16a}$$

$$\{\mathbf{x}\}_k = a_s^{-1}\{\mathbf{x}\}_i. \tag{16b}$$

We generalize this result and say that if nodes i and k are connected through a branch $b \in \mathcal{S}$ we can write:

$$\{\mathbf{x}\}_k = a_b^{d(b,\mathcal{S}(i,k))}\{\mathbf{x}\}_i, \tag{17}$$

where a_b is the tap ratio of branch b, $\mathcal{S}(i,k) \subseteq \mathcal{S}$ is the (unique) set of edges in $\mathcal{S}$ forming a path from node i to node k (in this case the only member of $\mathcal{S}(i,k)$ is b), and $d(b,\mathcal{S}(i,k))$ is a function that returns either 1 or -1 depending on the direction of branch b relative to the path defined by $\mathcal{S}(i,k)$ (if branch b goes from node i to node k then $d(b,\mathcal{S}(i,k)) = -1$, otherwise $d(b,\mathcal{S}(i,k)) = 1$). As $\mathcal{S}$ is a spanning tree, there exists a unique path from node 1 to every other node $k \neq 1$. Define $p(i,\mathcal{S}(i,k))$ as a function returning the node in the i-th position along the path from node 1 to node k ($p(1,\mathcal{S}(i,k)) = i$ and $p(1+|\mathcal{S}(i,k)|,\mathcal{S}(i,k)) = k$), and let $b(i,\mathcal{S}(i,k)) \in \mathcal{S}(i,k)$ be the branch connecting nodes $p(i,\mathcal{S}(i,k))$ and $p(i+1,\mathcal{S}(i,k))$. Let $D(k) = |\mathcal{S}(1,k)|$. We write $\{\mathbf{x}\}_k$ in terms of $\{\mathbf{x}\}_1$ by chaining (17) for each pair of consecutive nodes in the path between nodes 1 and k:

$$1 \xrightarrow{b(1,\mathcal{S}(1,k))} p(2,\mathcal{S}(1,k)) \xrightarrow{b(2,\mathcal{S}(1,k))} \cdots$$
$$\cdots p(D(k),\mathcal{S}(1,k)) \xrightarrow{b(D(k),\mathcal{S}(1,k))} k.$$

We backtrack the chain of equations starting from node k until we reach node 1:

$$\{\mathbf{x}\}_k = a_{b(D(k),\mathcal{S}(1,k))}^{d(b(D(k),\mathcal{S}(1,k)),\mathcal{S}(1,k))}\{\mathbf{x}\}_{p(D(k),\mathcal{S}(1,k))}, \tag{18a}$$

$$\begin{aligned}\{\mathbf{x}\}_k = {} & a_{b(D(k),\mathcal{S}(1,k))}^{d(b(D(k),\mathcal{S}(1,k)),\mathcal{S}(1,k))} \cdot a_{b(D(k)-1,\mathcal{S}(1,k))}^{d(b(D(k)-1,\mathcal{S}(1,k)),\mathcal{S}(1,k))} \\ & \cdot \{\mathbf{x}\}_{p(D(k)-1,\mathcal{S}(1,k))},\end{aligned} \tag{18b}$$

$$\vdots$$

$$\{\mathbf{x}\}_k = \{\mathbf{x}\}_1 \prod_{i=1}^{D(k)} a_{b(i,\mathcal{S}(1,k))}^{d(b(i,\mathcal{S}(1,k)),\mathcal{S}(1,k))}, \tag{18c}$$

or written more succinctly (as the product is commutative):

$$\{\mathbf{x}\}_k = \{\mathbf{x}\}_1 \prod_{s \in \mathcal{S}(1,k)} a_s^{d(s,\mathcal{S}(1,k))}. \tag{19}$$

Let $\{\mathbf{x}\}_1 = \alpha$, for an arbitrary α. We can then write $\mathbf{x}$ as:

$$\mathbf{x} = \alpha\mathbf{v}, \tag{20a}$$

$$\{\mathbf{v}\}_1 = 1, \tag{20b}$$

$$\{\mathbf{v}\}_k = \prod_{s \in \mathcal{S}(1,k)} a_s^{d(s,\mathcal{S}(1,k))}, \qquad k = 2,\ldots,|\mathcal{N}|. \tag{20c}$$

Since $\mathbf{x}$ has only one free parameter (α), the rank-nullity theorem implies that $\operatorname{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{S})}\right) = |\mathcal{N}| - 1$. Furthermore, as $a_l \neq 0$ for all $l \in \mathcal{L}$, then all entries of $\mathbf{v}$ are non-zero.

Since $\mathbf{x}$ must also be orthogonal to all rows of $\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{S})}$, we have, for each row of $\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{S})}$, the following equation:

$$\alpha\left(\prod_{s \in \mathcal{S}(1,i)} a_s^{d(s,\mathcal{S}(1,i))} - a_l \prod_{s \in \mathcal{S}(1,k)} a_s^{d(s,\mathcal{S}(1,k))}\right) = 0, \tag{21}$$

for any branch $l \in \mathcal{L}/\mathcal{S}$ going from node i to k. If the term inside the parentheses is null for all rows, then the (directed) product of tap ratios a_l across edges in a cycle is 1, for all cycles. In that case, α is a free parameter and $\operatorname{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) = |\mathcal{N}| - 1$. Otherwise $\alpha = 0$, and so $\operatorname{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) = |\mathcal{N}|$ (i.e., $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ is full column rank). $\square$

We also require some new lemmas. We start with Lemma 5, which is a simple extension of Lemma 1 from [19]:

Lemma 5. *Consider a matrix* $\mathbf{Y} = \mathbf{G} + j\mathbf{B} \in \mathbb{C}^{n\times n}$ *with* $\mathbf{G},\mathbf{B} \in \mathbb{R}^{n\times n}$. *Suppose* $\mathbf{G} \succeq \mathbf{0}$, *then* $\operatorname{Null}(\mathbf{Y}) \subseteq \operatorname{Null}(\operatorname{Sym}(\mathbf{G}))$ *and* $\operatorname{rank}(\operatorname{Sym}(\mathbf{G})) \leq \operatorname{rank}(\mathbf{Y})$.

Proof. Consider a vector $\mathbf{x} \in \mathbb{C}^n$ in the null space of $\mathbf{Y}$. We can write $\mathbf{x}$ in rectangular form as $\mathbf{x} = \mathbf{x}_R + j\mathbf{x}_I$ with $\mathbf{x}_R, \mathbf{x}_I \in \mathbb{R}^n$. Using the definition of the null space, we have:

$$0 = \operatorname{Re}\left(\mathbf{x}^H\mathbf{Y}\mathbf{x}\right), \tag{22a}$$

$$0 = \mathbf{x}_R^T\mathbf{G}\mathbf{x}_R + \mathbf{x}_I^T\mathbf{G}\mathbf{x}_I + \mathbf{x}_I^T\mathbf{B}\mathbf{x}_R - \mathbf{x}_R^T\mathbf{B}\mathbf{x}_I. \tag{22b}$$

The quadratic terms are real, so they only depend on the symmetric part of the matrices:[3]

$$\begin{aligned}0 = {} & \mathbf{x}_R^T\operatorname{Sym}(\mathbf{G})\,\mathbf{x}_R + \mathbf{x}_I^T\operatorname{Sym}(\mathbf{G})\,\mathbf{x}_I \\ & + \mathbf{x}_I^T\operatorname{Sym}(\mathbf{B})\,\mathbf{x}_R - \mathbf{x}_R^T\operatorname{Sym}(\mathbf{B})\,\mathbf{x}_I,\end{aligned} \tag{23a}$$

$$\begin{aligned}0 = {} & \mathbf{x}_R^T\operatorname{Sym}(\mathbf{G})\,\mathbf{x}_R + \mathbf{x}_I^T\operatorname{Sym}(\mathbf{G})\,\mathbf{x}_I \\ & + \mathbf{x}_I^T\operatorname{Sym}(\mathbf{B})\,\mathbf{x}_R - \mathbf{x}_I^T\operatorname{Sym}(\mathbf{B})\,\mathbf{x}_R,\end{aligned} \tag{23b}$$

$$0 = \mathbf{x}_R^T\operatorname{Sym}(\mathbf{G})\,\mathbf{x}_R + \mathbf{x}_I^T\operatorname{Sym}(\mathbf{G})\,\mathbf{x}_I. \tag{23c}$$

As $\operatorname{Sym}(\mathbf{G}) \succeq \mathbf{0}$, both terms must be non-negative. Equality only holds if both terms are zero, and hence both $\mathbf{x}_R$ and $\mathbf{x}_I$ belong to the null space of $\operatorname{Sym}(\mathbf{G})$. Therefore if $\mathbf{Y}\mathbf{x} = \mathbf{0}$ then $\operatorname{Sym}(\mathbf{G})\,\mathbf{x} = \mathbf{0}$, so $\operatorname{Null}(\mathbf{Y}) \subseteq \operatorname{Null}(\operatorname{Sym}(\mathbf{G}))$. We apply the rank-nullity theorem to conclude the proof. $\square$

Lemma 6. *Let* $\mathbf{A} \succeq \mathbf{0}$ *and* $\mathbf{B} \succeq \mathbf{0}$ *be square matrices in* $\mathbb{R}^{n\times n}$. *Then the following equations hold:*

$$\mathbf{A} + \mathbf{B} \succeq \mathbf{0}, \tag{24}$$

$$\operatorname{Null}(\operatorname{Sym}(\mathbf{A}+\mathbf{B})) = \operatorname{Null}(\operatorname{Sym}(\mathbf{A})) \cap \operatorname{Null}(\operatorname{Sym}(\mathbf{B})), \tag{25}$$

$$\max(\operatorname{rank}(\operatorname{Sym}(\mathbf{A})), \operatorname{rank}(\operatorname{Sym}(\mathbf{B}))) \leq \operatorname{rank}(\operatorname{Sym}(\mathbf{A}+\mathbf{B})). \tag{26}$$

Proof. Let us calculate the quadratic form of $\mathbf{A} + \mathbf{B}$:

$$\mathbf{x}^T(\mathbf{A}+\mathbf{B})\,\mathbf{x} = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{B}\mathbf{x}. \tag{27}$$

As both $\mathbf{A}$ and $\mathbf{B}$ are positive semi-definite, then $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ and $\mathbf{x}^T\mathbf{B}\mathbf{x} \geq 0$, thus $\mathbf{x}^T(\mathbf{A}+\mathbf{B})\,\mathbf{x} \geq 0$. Then by definition $\mathbf{A} + \mathbf{B} \succeq \mathbf{0}$. Now let $\mathbf{z}$ be a vector in the null space of $\operatorname{Sym}(\mathbf{A}+\mathbf{B})$. This implies that:

$$\mathbf{z}^T\operatorname{Sym}(\mathbf{A}+\mathbf{B})\,\mathbf{z} = 0, \tag{28a}$$

$$\mathbf{z}^T\operatorname{Sym}(\mathbf{A})\,\mathbf{z} + \mathbf{z}^T\operatorname{Sym}(\mathbf{B})\,\mathbf{z} = 0. \tag{28b}$$

As both terms are non-negative:

$$\mathbf{z}^T\operatorname{Sym}(\mathbf{A})\,\mathbf{z} = 0, \qquad \mathbf{z}^T\operatorname{Sym}(\mathbf{B})\,\mathbf{z} = 0, \tag{29}$$

and hence $\mathbf{z}$ belongs to the null spaces of both $\operatorname{Sym}(\mathbf{A})$ and $\operatorname{Sym}(\mathbf{B})$. The converse can be proved trivially by reversing the steps, so $\operatorname{Null}(\operatorname{Sym}(\mathbf{A}+\mathbf{B})) = \operatorname{Null}(\operatorname{Sym}(\mathbf{A})) \cap \operatorname{Null}(\operatorname{Sym}(\mathbf{B}))$. We then apply the rank-nullity theorem to conclude the proof of Lemma 6. $\square$

[3]For any real (possibly non-symmetric) matrix $\mathbf{A}$ and appropriately sized vector $\mathbf{x}$, the following relationships hold: $\mathbf{x}^T\mathbf{A}\mathbf{x} = (\mathbf{x}^T\mathbf{A}\mathbf{x})^T = \mathbf{x}^T\mathbf{A}^T\mathbf{x} = \mathbf{x}^T(\mathbf{A}/2 + \mathbf{A}^T/2)\mathbf{x} = \mathbf{x}^T\operatorname{Sym}(\mathbf{A})\,\mathbf{x}$.

Lemma 7. *Let* $\mathbf{A} \succ \mathbf{0}$ *and* $\mathbf{B} \succeq \mathbf{0}$ *be square matrices in* $\mathbb{R}^{n\times n}$. *Then* $\mathbf{A}+\mathbf{B} \succ \mathbf{0}$.

Proof. Lemma 7 follows from Lemma 6 and Lemma 5 with $\mathbf{Y} = \mathbf{A}+\mathbf{B}$. □

Lemma 8. *Let* $\mathbf{M} \succeq \mathbf{0}$, $\mathbf{M} \in \mathbb{C}^{n\times n}$ *be a Hermitian matrix, then* $\mathrm{Re}(\mathbf{M}) \succeq \mathbf{0}$, $\mathrm{Null}(\mathrm{Re}(\mathbf{M})) = \mathrm{Null}(\mathbf{M})$, *and* $\mathrm{rank}(\mathrm{Re}(\mathbf{M})) = \mathrm{rank}(\mathbf{M})$.

Proof. As $\mathbf{M}$ is Hermitian and positive-semidefinite, it can be factored as $\mathbf{M} = \mathbf{A}^H\mathbf{A}$. Now we expand $\mathrm{Re}(\mathbf{M})$:

$$\mathrm{Re}(\mathbf{M}) = \mathrm{Re}(\mathbf{A})^T\,\mathrm{Re}(\mathbf{A}) + \mathrm{Im}(\mathbf{A})^T\,\mathrm{Im}(\mathbf{A}) \succeq \mathbf{0}. \quad (30)$$

Note that $\mathrm{Re}(\mathbf{M})$ is symmetric. Applying Lemmas 6 and 3:

$$\mathrm{Null}(\mathrm{Re}(\mathbf{M})) = \mathrm{Null}(\mathrm{Re}(\mathbf{A})) \cap \mathrm{Null}(\mathrm{Im}(\mathbf{A})), \quad (31a)$$
$$\mathrm{Null}(\mathrm{Re}(\mathbf{M})) \subseteq \mathrm{Null}(\mathbf{A}) = \mathrm{Null}(\mathbf{M}). \quad (31b)$$

Recall that $\mathrm{Re}(\mathbf{M}) \succeq \mathbf{0}$. Applying Lemma 5 yields:

$$\mathrm{Null}(\mathbf{M}) \subseteq \mathrm{Null}(\mathrm{Re}(\mathbf{M})). \quad (32)$$

Since $\mathrm{Null}(\mathrm{Re}(\mathbf{M})) \subseteq \mathrm{Null}(\mathbf{M})$ and $\mathrm{Null}(\mathbf{M}) \subseteq \mathrm{Null}(\mathrm{Re}(\mathbf{M}))$, we have that $\mathrm{Null}(\mathrm{Re}(\mathbf{M})) = \mathrm{Null}(\mathbf{M})$. The claim follows after applying the rank-nullity theorem. □

Lemma 9. *Let* $\mathbf{D} \in \mathbb{R}^{m\times m}$ *be a diagonal matrix and let* $\mathbf{A} \in \mathbb{C}^{m\times n}$ *be an arbitrary matrix. Then* $\mathrm{Null}(\mathbf{A}) \subseteq \mathrm{Null}\left(\mathrm{Re}\left(\mathbf{A}^H\mathbf{D}\mathbf{A}\right)\right)$ *and* $\mathrm{rank}\left(\mathrm{Re}\left(\mathbf{A}^H\mathbf{D}\mathbf{A}\right)\right) \leq \mathrm{rank}(\mathbf{A})$.

Proof. We separate the positive and negative entries of $\mathbf{D}$:

$$\mathbf{D} = \mathbf{D}^+ - \mathbf{D}^-, \qquad \mathbf{D}^+, \mathbf{D}^- \succeq \mathbf{0}. \quad (33)$$

We can therefore write $\mathbf{D}^+ = \mathbf{B}^H\mathbf{B}$ and $\mathbf{D}^- = \mathbf{C}^H\mathbf{C}$. Expanding $\mathbf{A}^H\mathbf{D}\mathbf{A}$ in terms of $\mathbf{D}^+$ and $\mathbf{D}^-$ yields:

$$\mathrm{Re}\left(\mathbf{A}^H\mathbf{D}\mathbf{A}\right) = \mathrm{Re}\left(\mathbf{A}^H\mathbf{D}^+\mathbf{A}\right) - \mathrm{Re}\left(\mathbf{A}^H\mathbf{D}^-\mathbf{A}\right). \quad (34)$$

Hence, $\mathrm{Null}\left(\mathrm{Re}\left(\mathbf{A}^H\mathbf{D}\mathbf{A}\right)\right) \supseteq \mathrm{Null}\left(\mathrm{Re}\left(\mathbf{A}^H\mathbf{D}^+\mathbf{A}\right)\right) \cap \mathrm{Null}\left(\mathrm{Re}\left(\mathbf{A}^H\mathbf{D}^-\mathbf{A}\right)\right)$. As both $\mathrm{Re}\left(\mathbf{A}^H\mathbf{D}^+\mathbf{A}\right) \succeq \mathbf{0}$ and $\mathrm{Re}\left(\mathbf{A}^H\mathbf{D}^-\mathbf{A}\right) \succeq \mathbf{0}$, we can apply Lemma 8 to obtain:

$$\mathrm{Null}\left(\mathrm{Re}\left(\mathbf{A}^H\mathbf{D}\mathbf{A}\right)\right) \supseteq \mathrm{Null}\left(\mathbf{A}^H\mathbf{D}^+\mathbf{A}\right) \cap \mathrm{Null}\left(\mathbf{A}^H\mathbf{D}^-\mathbf{A}\right). \quad (35)$$

We next apply Lemma 3:

$$\mathrm{Null}\left(\mathrm{Re}\left(\mathbf{A}^H\mathbf{D}\mathbf{A}\right)\right) \supseteq \mathrm{Null}(\mathbf{B}\mathbf{A}) \cap \mathrm{Null}(\mathbf{C}\mathbf{A}). \quad (36)$$

Notice that $\mathrm{Null}(\mathbf{A}) \subseteq \mathrm{Null}(\mathbf{B}\mathbf{A})$ and $\mathrm{Null}(\mathbf{A}) \subseteq \mathrm{Null}(\mathbf{C}\mathbf{A})$. Therefore, we have $\mathrm{Null}(\mathbf{B}\mathbf{A}) \cap \mathrm{Null}(\mathbf{C}\mathbf{A}) \supseteq \mathrm{Null}(\mathbf{A})$. Substituting this result in (36) yields:

$$\mathrm{Null}\left(\mathrm{Re}\left(\mathbf{A}^H\mathbf{D}\mathbf{A}\right)\right) \supseteq \mathrm{Null}(\mathbf{A}). \quad (37)$$

Finally, the claim follows from the rank-nullity theorem. □

B. Admittance Matrix Invertibility Theorem

We now have the tools to present the amended version of Theorem 1 and prove its validity under several conditions.

Theorem 1 (Generalized) *If the graph* $(\mathcal{N}, \mathcal{L})$ *defines a connected network and Hypotheses 1 to 3 hold, then:*

$$\mathrm{rank}(\mathbf{Y}_\mathcal{N}) = \begin{cases} \mathrm{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) & \textit{if } \ \mathbf{y}_\mathcal{T} = \mathbf{0}, \\ |\mathcal{N}| & \textit{otherwise}, \end{cases} \quad (38)$$

as long as at least one of the following conditions hold.

Condition 0: $\mathrm{Re}(y_l) \geq 0$ *for all* y_l *of* $\mathbf{y}_\mathcal{L}$, $\mathrm{Re}(y_t) \geq 0$ *for all* y_t *of* $\mathbf{y}_\mathcal{T}$ *and* $\mathrm{rank}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathrm{Re}(\mathbf{Y}_\mathcal{L})\,\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) = \mathrm{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right)$.

Proof. While somewhat convoluted, this condition is useful in proving the other conditions. We write $\mathbf{y}_\mathcal{L}$ in rectangular form as $\mathbf{y}_\mathcal{L} = \mathbf{g}_\mathcal{L} + j\mathbf{b}_\mathcal{L}$ and $\mathbf{Y}_\mathcal{L} = \mathbf{G}_\mathcal{L} + j\mathbf{B}_\mathcal{L}$ where $\mathbf{G}_\mathcal{L} = \mathrm{diag}(\mathbf{g}_\mathcal{L})$ and $\mathbf{B}_\mathcal{L} = \mathrm{diag}(\mathbf{b}_\mathcal{L})$. Let $\mathbf{A}_R$ and $\mathbf{A}_I$ denote the real and imaginary parts of $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$. First consider the case where $\mathbf{y}_\mathcal{T} = \mathbf{0}$. We can write the following relationships:

$$\mathbf{Y}_\mathcal{N} = \mathbf{G}_\mathcal{N} + j\mathbf{B}_\mathcal{N}, \quad (39a)$$
$$\mathbf{G}_\mathcal{N} = \mathrm{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{Y}_\mathcal{L}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right), \quad (39b)$$
$$\mathbf{G}_\mathcal{N} = \mathbf{A}_R^T\mathbf{G}_\mathcal{L}\mathbf{A}_R + \mathbf{A}_I^T\mathbf{G}_\mathcal{L}\mathbf{A}_I + \mathbf{A}_I^T\mathbf{B}_\mathcal{L}\mathbf{A}_R - \mathbf{A}_R^T\mathbf{B}_\mathcal{L}\mathbf{A}_I, \quad (39c)$$
$$\mathrm{Sym}(\mathbf{G}_\mathcal{N}) = \mathbf{A}_R^T\mathbf{G}_\mathcal{L}\mathbf{A}_R + \mathbf{A}_I^T\mathbf{G}_\mathcal{L}\mathbf{A}_I \succeq \mathbf{0}. \quad (39d)$$

Hence $\mathbf{G}_\mathcal{N}$ is also positive semi-definite. Notice that:

$$\mathbf{A}_R^T\mathbf{G}_\mathcal{L}\mathbf{A}_R + \mathbf{A}_I^T\mathbf{G}_\mathcal{L}\mathbf{A}_I = \mathrm{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_\mathcal{L}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right). \quad (40)$$

Since $\mathbf{G}_\mathcal{L} \succeq \mathbf{0}$ as stated in Condition 0, $\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_\mathcal{L}\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ must be Hermitian. Applying Lemma 8 and Condition 0:

$$\mathrm{rank}(\mathrm{Sym}(\mathbf{G}_\mathcal{N})) = \mathrm{rank}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_\mathcal{L}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right), \quad (41a)$$
$$\mathrm{rank}(\mathrm{Sym}(\mathbf{G}_\mathcal{N})) = \mathrm{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right). \quad (41b)$$

Applying Lemma 5 we get that $\mathrm{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) \leq \mathrm{rank}(\mathbf{Y}_\mathcal{N})$, and since $\mathbf{Y}_\mathcal{N}$ involves a product by $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$, we also have that $\mathrm{rank}(\mathbf{Y}_\mathcal{N}) \leq \mathrm{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right)$ (see [24, Theorem 3.7]). The only integral solution of the inequality is, of course:

$$\mathrm{rank}(\mathbf{Y}_\mathcal{N}) = \mathrm{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right). \quad (42)$$

We now consider the case where $\mathbf{y}_\mathcal{T} \neq \mathbf{0}$. We can write $\mathbf{y}_\mathcal{T}$ in rectangular form as $\mathbf{y}_\mathcal{T} = \mathbf{g}_\mathcal{T} + j\mathbf{b}_\mathcal{T}$ and $\mathbf{Y}_\mathcal{T} = \mathbf{G}_\mathcal{T} + j\mathbf{B}_\mathcal{T}$ where $\mathbf{G}_\mathcal{T} = \mathrm{diag}(\mathbf{g}_\mathcal{T})$ and $\mathbf{B}_\mathcal{T} = \mathrm{diag}(\mathbf{b}_\mathcal{T})$. We then have:

$$\mathbf{G}_\mathcal{N} = \mathrm{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{Y}_\mathcal{L}\mathbf{A}_{(\mathcal{N},\mathcal{L})} + \mathbf{G}_\mathcal{T}\right), \quad (43a)$$
$$\mathrm{Sym}(\mathbf{G}_\mathcal{N}) = \mathrm{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_\mathcal{L}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) + \mathbf{G}_\mathcal{T} \succeq \mathbf{0}. \quad (43b)$$

From the statement of Condition 0, we have $\mathbf{G}_\mathcal{T} \succeq \mathbf{0}$ and we also know that $\mathrm{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_\mathcal{L}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) \succeq \mathbf{0}$ as discussed above. Applying Lemma 6, we conclude that $\mathrm{Sym}(\mathbf{G}_\mathcal{N}) \succeq \mathbf{0}$ and its null space is the intersection of the null spaces of $\mathrm{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_\mathcal{L}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right)$ and $\mathbf{G}_\mathcal{T}$. We next apply Lemma 5:

$$\mathrm{Null}(\mathbf{Y}_\mathcal{N}) \subseteq \mathrm{Null}(\mathbf{G}_\mathcal{N}) \subseteq \mathrm{Null}\left(\mathrm{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_\mathcal{L}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right)\right). \quad (44)$$

Applying Lemma 8 yields:

$$\mathrm{Null}(\mathbf{Y}_\mathcal{N}) \subseteq \mathrm{Null}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_\mathcal{L}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right). \quad (45)$$

Notice that $\mathrm{Null}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) \subseteq \mathrm{Null}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_\mathcal{L}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right)$. From the statement of Condition 0, we know that $\mathrm{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) = \mathrm{rank}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_\mathcal{L}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right)$, so their null spaces have the same dimension. Therefore, we have:

$$\mathrm{Null}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) = \mathrm{Null}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_\mathcal{L}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right). \quad (46)$$

In conclusion, $\mathrm{Null}(\mathbf{Y}_\mathcal{N}) \subseteq \mathrm{Null}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right)$. If $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ is full-rank, then $\mathbf{Y}_\mathcal{N}$ is also full-rank. Otherwise, let $\mathbf{x}$ be a vector in the null space of $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$. From Lemma 1, $\mathbf{x} = \alpha\mathbf{v}$ with $\mathbf{v}$ being a vector with no null entries. We now calculate $\mathbf{Y}_{\mathcal{N}}\mathbf{x}$:

$$\mathbf{Y}_{\mathcal{N}}\mathbf{x} = \alpha\mathbf{Y}_{\mathcal{N}}\mathbf{v}, \tag{47a}$$

$$\mathbf{Y}_{\mathcal{N}}\mathbf{x} = \alpha\left(\mathbf{A}^T_{(\mathcal{N},\mathcal{L})}\mathbf{Y}_{\mathcal{L}}\mathbf{A}_{(\mathcal{N},\mathcal{L})} + \mathbf{Y}_{\mathcal{T}}\right)\mathbf{v}, \tag{47b}$$

$$\mathbf{Y}_{\mathcal{N}}\mathbf{x} = \alpha\left(\mathbf{A}^T_{(\mathcal{N},\mathcal{L})}\mathbf{Y}_{\mathcal{L}}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\mathbf{v} + \mathbf{Y}_{\mathcal{T}}\mathbf{v}\right), \tag{47c}$$

$$\mathbf{Y}_{\mathcal{N}}\mathbf{x} = \alpha\mathbf{Y}_{\mathcal{T}}\mathbf{v}. \tag{47d}$$

Since $\mathbf{Y}_{\mathcal{T}} = \operatorname{diag}(\mathbf{y}_{\mathcal{T}})$, $\mathbf{y}_{\mathcal{T}} \neq \mathbf{0}$, and $\mathbf{v}$ has no null entries, we observe that $\mathbf{Y}_{\mathcal{N}}\mathbf{x}$ cannot be $\mathbf{0}$ unless $\alpha = 0$. This means the only vector in the null space of $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ that is also in the null space $\mathbf{Y}_{\mathcal{N}}$ is $\mathbf{0}$. This implies that $\mathbf{Y}_{\mathcal{N}}$ is full-rank. □

Condition 1: $\operatorname{Re}(\mathbf{Y}_{\mathcal{N}}) \succ \mathbf{0}$.

Proof. If $\mathbf{y}_{\mathcal{T}} \neq \mathbf{0}$, we aim to show that $\mathbf{Y}_{\mathcal{N}}$ must be full-rank as stated in Theorem 1. If $\mathbf{y}_{\mathcal{T}} = \mathbf{0}$, we can write $\operatorname{Sym}(\operatorname{Re}(\mathbf{Y}_{\mathcal{N}})) = \operatorname{Sym}(\mathbf{G}_{\mathcal{N}})$ as:

$$\operatorname{Sym}(\mathbf{G}_{\mathcal{N}}) = \operatorname{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_{\mathcal{L}}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right). \tag{48}$$

Applying Lemma 9, we obtain:

$$\operatorname{rank}(\operatorname{Sym}(\mathbf{G}_{\mathcal{N}})) \leq \operatorname{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right). \tag{49}$$

Since $\mathbf{G}_{\mathcal{N}} \succ \mathbf{0}$, we know that $\operatorname{Sym}(\mathbf{G}_{\mathcal{N}}) \succ \mathbf{0}$ and $\operatorname{Sym}(\mathbf{G}_{\mathcal{N}})$ must also be full-rank, so $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ is full column rank. Theorem 1 thus claims that $\mathbf{Y}_{\mathcal{N}}$ must be full-rank. For both cases $\mathbf{y}_{\mathcal{T}} \neq \mathbf{0}$ and $\mathbf{y}_{\mathcal{T}} = \mathbf{0}$, the claim follows from Lemma 5 and the fact that $\operatorname{Sym}(\mathbf{G}_{\mathcal{N}})$ is full-rank. □

Condition 2: $\operatorname{Re}(y_l) > 0$ *for all* y_l *of* $\mathbf{y}_{\mathcal{L}}$ *and* $\operatorname{Re}(y_t) \geq 0$ *for all* y_t *of* $\mathbf{y}_{\mathcal{T}}$.

Proof. From Condition 2, we know that $\mathbf{G}_{\mathcal{L}} = \operatorname{Re}(\mathbf{Y}_{\mathcal{L}})$ is positive definite and can thus be written as $\mathbf{G}_{\mathcal{L}} = \mathbf{C}^H\mathbf{C}$ where $\mathbf{C}$ is full-rank. Applying Lemma 3 and Lemma 4 yields:

$$\operatorname{rank}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_{\mathcal{L}}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) = \operatorname{rank}\left(\mathbf{C}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) = \operatorname{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right). \tag{50}$$

Hence, Condition 2 implies Condition 0. □

Condition 3: $\operatorname{Re}(y_l) \geq 0$ *for all* y_l *of* $\mathbf{y}_{\mathcal{L}}$ *and* $\operatorname{Re}(y_t) > 0$ *for all* y_t *of* $\mathbf{y}_{\mathcal{T}}$.

Proof. We can write $\operatorname{Sym}(\mathbf{G}_{\mathcal{N}})$ as:

$$\operatorname{Sym}(\mathbf{G}_{\mathcal{N}}) = \operatorname{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_{\mathcal{L}}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) + \mathbf{G}_{\mathcal{T}}. \tag{51}$$

Write $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ in rectangular form as $\mathbf{A}_{(\mathcal{N},\mathcal{L})} = \mathbf{A}_R + j\mathbf{A}_I$. Then, we can write (51) as:

$$\operatorname{Sym}(\mathbf{G}_{\mathcal{N}}) = \mathbf{A}_R^T\mathbf{G}_{\mathcal{L}}\mathbf{A}_R + \mathbf{A}_I^T\mathbf{G}_{\mathcal{L}}\mathbf{A}_I + \mathbf{G}_{\mathcal{T}}, \tag{52}$$

where (because of Condition 3) $\mathbf{G}_{\mathcal{L}} \succeq \mathbf{0}$ and $\mathbf{G}_{\mathcal{T}} \succ \mathbf{0}$, which also means that both $\mathbf{A}_R^T\mathbf{G}_{\mathcal{L}}\mathbf{A}_R \succeq \mathbf{0}$ and $\mathbf{A}_I^T\mathbf{G}_{\mathcal{L}}\mathbf{A}_I \succeq \mathbf{0}$. We apply Lemma 7 and conclude that $\operatorname{Sym}(\mathbf{G}_{\mathcal{N}}) \succ \mathbf{0}$, so $\mathbf{G}_{\mathcal{N}} \succ \mathbf{0}$ and hence Condition 3 implies Condition 1. □

Condition 4: $\operatorname{Re}(y_l) \geq 0$ *for all* y_l *of* $\mathbf{y}_{\mathcal{L}}$, $\operatorname{Re}(y_t) \geq 0$ *for all* y_t *of* $\mathbf{y}_{\mathcal{T}}$*, there exists a spanning tree with edges* $\mathcal{S} \subseteq \mathcal{L}$ *such that* $\operatorname{Re}(y_s) > 0$ *for all* $s \in \mathcal{S}$*, and there is at least one* y_t *of* $\mathbf{y}_{\mathcal{T}}$ *such that* $\operatorname{Re}(y_t) > 0$ *(there is spanning tree of positive conductances including the ground node).*

Proof. We first write the following block form of $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$:

$$\mathbf{A}_{(\mathcal{N},\mathcal{L})} = \begin{bmatrix} \mathbf{A}_{(\mathcal{N},\mathcal{S})} \\ \mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{S})} \end{bmatrix}, \tag{53}$$

where $\mathbf{A}_{(\mathcal{N},\mathcal{S})}$ is the generalized incidence matrix of the edges in $\mathcal{S}$ and $\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{S})}$ is the generalized incidence matrix of the remaining edges. Notice that, according to Lemma 1 (extended) and the fact that $\mathbf{A}_{(\mathcal{N},\mathcal{S})}$ has exactly $|\mathcal{N}|-1$ rows (branches in the spanning tree $\mathcal{S}$), $\operatorname{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{S})}\right) = |\mathcal{N}|-1$. Similarly, we can write $\mathbf{G}_{\mathcal{L}}$ in blocks as follows:

$$\mathbf{G}_{\mathcal{L}} = \begin{bmatrix} \mathbf{G}_{\mathcal{S}} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{\mathcal{L}/\mathcal{S}} \end{bmatrix}, \tag{54}$$

where $\mathbf{G}_{\mathcal{S}} \succ \mathbf{0}$ is the square diagonal matrix with the conductances of edges in $\mathcal{S}$ and $\mathbf{G}_{\mathcal{L}/\mathcal{S}}$ is the square diagonal matrix grouping the conductances of all other edges. Solving the matrix product, we write $\operatorname{Sym}(\mathbf{G}_{\mathcal{N}})$ as follows:

$$\operatorname{Sym}(\mathbf{G}_{\mathcal{N}}) = \operatorname{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L})}\mathbf{G}_{\mathcal{L}}\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) + \mathbf{G}_{\mathcal{T}}, \tag{55a}$$

$$\operatorname{Sym}(\mathbf{G}_{\mathcal{N}}) = \operatorname{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{S})}\mathbf{G}_{\mathcal{S}}\mathbf{A}_{(\mathcal{N},\mathcal{S})}\right) + \mathbf{G}_{\mathcal{T}} + \operatorname{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L}/\mathcal{S})}\mathbf{G}_{\mathcal{L}/\mathcal{S}}\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{S})}\right). \tag{55b}$$

Notice that all the terms composing $\operatorname{Sym}(\mathbf{G}_{\mathcal{N}})$ are positive-semidefinite, so $\operatorname{Sym}(\mathbf{G}_{\mathcal{N}}) \succeq \mathbf{0}$. Applying Lemmas 5 and 6:

$$\operatorname{Null}(\mathbf{Y}_{\mathcal{N}}) \subseteq \operatorname{Null}\left(\operatorname{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{S})}\mathbf{G}_{\mathcal{S}}\mathbf{A}_{(\mathcal{N},\mathcal{S})}\right)\right) \cap \operatorname{Null}(\mathbf{G}_{\mathcal{T}}) \cap \operatorname{Null}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{L}/\mathcal{S})}\mathbf{G}_{\mathcal{L}/\mathcal{S}}\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{S})}\right), \tag{56a}$$

$$\operatorname{Null}(\mathbf{Y}_{\mathcal{N}}) \subseteq \operatorname{Null}\left(\operatorname{Re}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{S})}\mathbf{G}_{\mathcal{S}}\mathbf{A}_{(\mathcal{N},\mathcal{S})}\right)\right) \cap \operatorname{Null}(\mathbf{G}_{\mathcal{T}}). \tag{56b}$$

Applying Lemma 8 yields:

$$\operatorname{Null}(\mathbf{Y}_{\mathcal{N}}) \subseteq \operatorname{Null}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{S})}\mathbf{G}_{\mathcal{S}}\mathbf{A}_{(\mathcal{N},\mathcal{S})}\right) \cap \operatorname{Null}(\mathbf{G}_{\mathcal{T}}). \tag{57}$$

As $\mathbf{G}_{\mathcal{S}} \succ \mathbf{0}$, we can apply Lemma 3 and Lemma 4 to show that $\operatorname{Null}\left(\mathbf{A}^H_{(\mathcal{N},\mathcal{S})}\mathbf{G}_{\mathcal{S}}\mathbf{A}_{(\mathcal{N},\mathcal{S})}\right) = \operatorname{Null}\left(\mathbf{A}_{(\mathcal{N},\mathcal{S})}\right)$. Substituting into (57) yields:

$$\operatorname{Null}(\mathbf{Y}_{\mathcal{N}}) \subseteq \operatorname{Null}\left(\mathbf{A}_{(\mathcal{N},\mathcal{S})}\right) \cap \operatorname{Null}(\mathbf{G}_{\mathcal{T}}). \tag{58}$$

Since $\mathbf{y}_{\mathcal{T}} \neq \mathbf{0}$, Theorem 1 states that $\mathbf{Y}_{\mathcal{N}}$ must be full-rank. From Lemma 1 (extended), recall that the basis vector $\mathbf{v}$ of the null space of $\mathbf{A}_{(\mathcal{N},\mathcal{S})}$ has no null entries. Also recall that $\mathbf{G}_{\mathcal{T}} \neq \mathbf{0}$ from the statement of Condition 4. We thus conclude:

$$\dim\left(\operatorname{Null}\left(\mathbf{A}_{(\mathcal{N},\mathcal{S})}\right) \cap \operatorname{Null}(\mathbf{G}_{\mathcal{T}})\right) = 0. \tag{59}$$

Hence, $\mathbf{Y}_{\mathcal{N}}$ is full-rank and Theorem 1 follows. □

Condition 5: $\operatorname{Re}(y_l) \geq 0$ *for all* y_l *of* $\mathbf{y}_{\mathcal{L}}$, $\operatorname{Re}(y_t) \geq 0$ *for all* y_t *of* $\mathbf{y}_{\mathcal{T}}$, $k \cdot \operatorname{Im}(y_l) \geq 0$ *for all* y_l *of* $\mathbf{y}_{\mathcal{L}}$*, and* $k \cdot \operatorname{Im}(y_t) \geq 0$ *for all* y_t *of* $\mathbf{y}_{\mathcal{T}}$*, where* $k = 1$ *or* $k = -1$ *(there are no imaginary parts with opposite signs).*

Proof. Consider an alternative network with a set of nodes $\mathcal{N}'$ identical to $\mathcal{N}$, a set of branches $\mathcal{L}'$ such that $\mathbf{A}_{(\mathcal{N}',\mathcal{L}')} = \mathbf{A}_{(\mathcal{N},\mathcal{L})}$ and $\mathbf{Y}_{\mathcal{L}'} = (1-jk)\,\mathbf{Y}_{\mathcal{L}}$, and a set of shunts $\mathcal{T}'$ such that $\mathbf{Y}_{\mathcal{T}'} = (1-jk)\,\mathbf{Y}_{\mathcal{T}}$. The admittance matrix of the alternate network is:

$$\mathbf{Y}_{\mathcal{N}'} = \mathbf{A}^H_{(\mathcal{N}',\mathcal{L}')}\mathbf{Y}_{\mathcal{L}'}\mathbf{A}_{(\mathcal{N}',\mathcal{L}')} + \mathbf{Y}_{\mathcal{T}'}, \tag{60a}$$

$$\mathbf{Y}_{\mathcal{N}'} = (1-jk)\,\mathbf{Y}_{\mathcal{N}}. \tag{60b}$$

Since the constant $(1-jk)$ is not zero, we have that $\operatorname{rank}(\mathbf{Y}_{\mathcal{N}'}) = \operatorname{rank}(\mathbf{Y}_{\mathcal{N}})$. Thus, if Theorem 1 holds for the alternate network, the theorem also holds for the original network. (Notice that $\mathbf{Y}_{\mathcal{T}'}=0$ if and only if $\mathbf{Y}_{\mathcal{T}}=0$.) Next, observe that:

$$\mathbf{Y}_{\mathcal{L}'}=(1-jk)\left(\mathbf{G}_{\mathcal{L}}+j\mathbf{B}_{\mathcal{L}}\right)=\mathbf{G}_{\mathcal{L}'}+j\mathbf{B}_{\mathcal{L}'}, \tag{61a}$$
$$\mathbf{Y}_{\mathcal{T}'}=(1-jk)\left(\mathbf{G}_{\mathcal{T}}+j\mathbf{B}_{\mathcal{T}}\right)=\mathbf{G}_{\mathcal{T}'}+j\mathbf{B}_{\mathcal{T}'}. \tag{61b}$$

Using the statement of Condition 5, we can write $\mathbf{B}_{\mathcal{L}}=k\mathbf{B}_L$ and $\mathbf{B}_{\mathcal{T}}=k\mathbf{B}_T$, where $\mathbf{B}_L$ and $\mathbf{B}_T$ are the matrices of element-wise absolute values of $\mathbf{B}_{\mathcal{L}}$ and $\mathbf{B}_{\mathcal{T}}$, respectively. Observe that $k^2=1$. Substituting these relationships into (61):

$$\mathbf{G}_{\mathcal{L}'}=\mathbf{G}_{\mathcal{L}}+\mathbf{B}_L, \qquad \mathbf{G}_{\mathcal{T}'}=\mathbf{G}_{\mathcal{T}}+\mathbf{B}_T. \tag{62}$$

Hence, $\mathrm{Re}\left(y_{l'}\right)\geq 0$ for all $l'\in\mathcal{L}'$ and $\mathrm{Re}\left(y_{t'}\right)\geq 0$ for all $t'\in\mathcal{T}'$. Thus, the following relationship holds:

$$\mathbf{A}_{(\mathcal{N}',\mathcal{L}')}^H\mathbf{G}_{\mathcal{L}'}\mathbf{A}_{(\mathcal{N}',\mathcal{L}')}=\mathbf{A}_{(\mathcal{N},\mathcal{L})}^H\left(\mathbf{G}_{\mathcal{L}}+\mathbf{B}_L\right)\mathbf{A}_{(\mathcal{N},\mathcal{L})}. \tag{63}$$

Per Hypothesis 1, we know that $\mathbf{Y}_{\mathcal{L}}$ is full-rank, so $y_l\neq 0$ for all $l\in\mathcal{L}$. Taking the Manhattan norm (1-norm) yields:

$$\|y_l\|_1=|g_l|+|b_l|>0, \qquad \forall l\in\mathcal{L}. \tag{64}$$

Per Condition 5, we have that $|g_l|=g_l$, and hence $g_l+|b_l|>0$. Notice that $\mathbf{G}_{\mathcal{L}'}=\mathbf{G}_{\mathcal{L}}+\mathbf{B}_L$ is a diagonal matrix with diagonal entries equal to $g_l+|b_l|$ for all $l\in\mathcal{L}$, so for the admittance $y_{l'}$ of each branch $l'\in\mathcal{L}'$ we have $\mathrm{Re}\left(y_{l'}\right)=g_l+|b_l|>0$ for some $l\in\mathcal{L}$. Hence, the alternative network satisfies Condition 2, and thus Theorem 1 follows. □

Condition 6: $\mathrm{Re}\left(y_l\right)\geq 0$ *for all* y_l *of* $\mathbf{y}_{\mathcal{L}}$*,* $\mathrm{Re}\left(y_t\right)\geq 0$ *for all* y_t *of* $\mathbf{y}_{\mathcal{T}}$*,* $\mathrm{Re}\left(y_l\right)>0$ *for all* y_l *of* $\mathbf{y}_{\mathcal{L}}$ *such that* $k\cdot\mathrm{Im}\left(y_l\right)<0$*, and* $\mathrm{Re}\left(y_t\right)>0$ *for all* y_t *of* $\mathbf{y}_{\mathcal{T}}$ *such that* $k\cdot\mathrm{Im}\left(y_t\right)<0$*, where* $k=1$ *or* $k=-1$ *(the elements with susceptances of opposite sign have positive conductance).*

Proof. Let $\mathcal{L}'\subseteq\mathcal{L}$ be the largest set of branches satisfying $k\cdot\mathrm{Im}\left(y_l\right)<0$ for $l\in\mathcal{L}'$. We write $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ and $\mathbf{Y}_{\mathcal{L}}$ in blocks:

$$\mathbf{A}_{(\mathcal{N},\mathcal{L})}=\begin{bmatrix}\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{L}')}\\ \mathbf{A}_{(\mathcal{N},\mathcal{L}')}\end{bmatrix}, \quad \mathbf{Y}_{\mathcal{L}}=\begin{bmatrix}\mathbf{Y}_{\mathcal{L}/\mathcal{L}'} & \mathbf{0}\\ \mathbf{0} & \mathbf{Y}_{\mathcal{L}'}\end{bmatrix},$$

where $\mathbf{Y}_{\mathcal{L}'}$ is the square diagonal matrix with diagonal entries consisting of the admittances of the edges in $\mathcal{L}'$ and $\mathbf{Y}_{\mathcal{L}/\mathcal{L}'}$ is the square diagonal matrix grouping the admittances of all other edges. Note that $\mathbf{Y}_{\mathcal{L}'}$ can be written as $\mathbf{Y}_{\mathcal{L}'}=\mathbf{G}_{\mathcal{L}'}-jk\mathbf{B}_{L'}$ with both $\mathbf{G}_{\mathcal{L}'}\succ\mathbf{0}$ and $-k\mathbf{B}_{L'}\succ\mathbf{0}$ being diagonal matrices. Similarly, let $\mathcal{T}'\subseteq\mathcal{T}$ be the set of branches satisfying $k\cdot\mathrm{Im}\left(y_t\right)<0$ for $t\in\mathcal{T}'$. We then write $\mathbf{Y}_{\mathcal{T}}$ as:

$$\mathbf{Y}_{\mathcal{T}}=\mathbf{Y}_{\mathcal{T}/\mathcal{T}'}+\mathbf{Y}_{\mathcal{T}'}. \tag{65}$$

Note that $\mathbf{Y}_{\mathcal{T}'}$ can be written as $\mathbf{Y}_{\mathcal{T}'}=\mathbf{G}_{\mathcal{T}'}-jk\mathbf{B}_{T'}$ with both $\mathbf{G}_{\mathcal{T}'}\succ\mathbf{0}$ and $-k\mathbf{B}_{T'}\succ\mathbf{0}$ being diagonal matrices with non-negative entries. From the statement of Condition 6, the admittance of every shunt in $\mathcal{T}'$ has non-null real and imaginary parts. Hence, the positions of the null columns of $\mathbf{G}_{\mathcal{T}'}$ and $-k\mathbf{B}_{T'}$ are the same. This along with the fact that both matrices are diagonal imply they have same null space:

$$\mathrm{Null}\left(\mathbf{G}_{\mathcal{T}'}\right)=\mathrm{Null}\left(-k\mathbf{B}_{T'}\right). \tag{66}$$

Expanding $\mathbf{Y}_{\mathcal{N}}$ in terms of the block matrices yields:

$$\begin{aligned}\mathbf{Y}_{\mathcal{N}}=&\,\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{L}')}^H\mathbf{Y}_{\mathcal{L}/\mathcal{L}'}\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{L}')}\\&+\mathbf{A}_{(\mathcal{N},\mathcal{L}')}^H\mathbf{Y}_{\mathcal{L}'}\mathbf{A}_{(\mathcal{N},\mathcal{L}')}+\mathbf{Y}_{\mathcal{T}/\mathcal{T}'}+\mathbf{Y}_{\mathcal{T}'}.\end{aligned} \tag{67}$$

Let $\mathbf{Y}_{\mathcal{N}'}$ denote the admittance matrix of an alternative network similar to the original one, with the only difference being that the admittances of the branches in $\mathcal{L}'$ and shunts in $\mathcal{T}'$ are purely resistive with the same conductance values of the original network (this alternative network does not have susceptances with opposite signs, and thus satisfies Condition 5):

$$\begin{aligned}\mathbf{Y}_{\mathcal{N}'}=&\,\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{L}')}^H\left(\mathbf{G}_{\mathcal{L}/\mathcal{L}'}+j\mathbf{B}_{\mathcal{L}/\mathcal{L}'}\right)\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{L}')}\\&+\mathbf{A}_{(\mathcal{N},\mathcal{L}')}^H\mathbf{G}_{\mathcal{L}'}\mathbf{A}_{(\mathcal{N},\mathcal{L}')}+\mathbf{G}_{\mathcal{T}/\mathcal{T}'}+j\mathbf{B}_{\mathcal{T}/\mathcal{T}'}+\mathbf{G}_{\mathcal{T}'}.\end{aligned} \tag{68}$$

Substituting $\mathbf{Y}_{\mathcal{N}'}$ into (67) yields:

$$\mathbf{Y}_{\mathcal{N}}=\mathbf{Y}_{\mathcal{N}'}+j\left(\mathbf{A}_{(\mathcal{N},\mathcal{L}')}^H\left(-k\mathbf{B}_{L'}\right)\mathbf{A}_{(\mathcal{N},\mathcal{L}')}-k\mathbf{B}_{T'}\right). \tag{69}$$

Let $\mathbf{G}_{\mathcal{N}'}$ denote the real part of $\mathbf{Y}_{\mathcal{N}'}$. From our definition of $\mathbf{Y}_{\mathcal{N}'}$, we have $\mathbf{G}_{\mathcal{N}}=\mathbf{G}_{\mathcal{N}'}$, and hence:

$$\begin{aligned}\mathrm{Sym}\left(\mathbf{G}_{\mathcal{N}}\right)=&\,\mathrm{Sym}\left(\mathbf{G}_{\mathcal{N}'}\right)=\mathrm{Re}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{L}')}^H\mathbf{G}_{\mathcal{L}/\mathcal{L}'}\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{L}')}\right)\\&+\mathrm{Re}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L}')}^H\mathbf{G}_{\mathcal{L}'}\mathbf{A}_{(\mathcal{N},\mathcal{L}')}\right)+\mathbf{G}_{\mathcal{T}/\mathcal{T}'}+\mathbf{G}_{\mathcal{T}'}.\end{aligned} \tag{70}$$

Since all of the terms composing $\mathrm{Sym}\left(\mathbf{G}_{\mathcal{N}}\right)$ are positive-semidefinite, $\mathrm{Sym}\left(\mathbf{G}_{\mathcal{N}}\right)\succeq\mathbf{0}$. Applying Lemmas 5 and 6:

$$\begin{aligned}\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right)\subseteq&\,\mathrm{Null}\left(\mathrm{Re}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{L}')}^H\mathbf{G}_{\mathcal{L}/\mathcal{L}'}\mathbf{A}_{(\mathcal{N},\mathcal{L}/\mathcal{L}')}\right)\right)\\&\cap\mathrm{Null}\left(\mathrm{Re}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L}')}^H\mathbf{G}_{\mathcal{L}'}\mathbf{A}_{(\mathcal{N},\mathcal{L}')}\right)\right)\\&\cap\mathrm{Null}\left(\mathbf{G}_{\mathcal{T}/\mathcal{T}'}\right)\cap\mathrm{Null}\left(\mathbf{G}_{\mathcal{T}'}\right),\end{aligned} \tag{71a}$$
$$\begin{aligned}\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right)\subseteq&\,\mathrm{Null}\left(\mathrm{Re}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L}')}^H\mathbf{G}_{\mathcal{L}'}\mathbf{A}_{(\mathcal{N},\mathcal{L}')}\right)\right)\\&\cap\mathrm{Null}\left(\mathbf{G}_{\mathcal{T}'}\right).\end{aligned} \tag{71b}$$

Applying Lemma 8 yields:

$$\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right)\subseteq\mathrm{Null}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L}')}^H\mathbf{G}_{\mathcal{L}'}\mathbf{A}_{(\mathcal{N},\mathcal{L}')}\right)\cap\mathrm{Null}\left(\mathbf{G}_{\mathcal{T}'}\right) \tag{72}$$

Since $\mathbf{G}_{\mathcal{L}'}\succ\mathbf{0}$, we apply Lemma 3 and Lemma 4 to obtain:

$$\mathrm{Null}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L}')}^H\mathbf{G}_{\mathcal{L}'}\mathbf{A}_{(\mathcal{N},\mathcal{L}')}\right)=\mathrm{Null}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L}')}\right). \tag{73}$$

Substituting (73) into (72) yields:

$$\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right)\subseteq\mathrm{Null}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L}')}\right)\cap\mathrm{Null}\left(\mathbf{G}_{\mathcal{T}'}\right). \tag{74}$$

Since the null spaces of $\mathbf{G}_{\mathcal{T}'}$ and $-k\mathbf{B}_{T'}$ are equal, we have:

$$\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right)\subseteq\mathrm{Null}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L}')}\right)\cap\mathrm{Null}\left(-k\mathbf{B}_{T'}\right). \tag{75}$$

We next calculate the rank of $\mathbf{Y}_{\mathcal{N}}$. Consider an arbitrary vector $\mathbf{x}$ in the null space of $\mathbf{Y}_{\mathcal{N}}$. Then, from (69), we obtain:

$$\mathbf{0}=\mathbf{Y}_{\mathcal{N}'}\mathbf{x}+j\left(\mathbf{A}_{(\mathcal{N},\mathcal{L}')}^H\left(-k\mathbf{B}_{L'}\right)\mathbf{A}_{(\mathcal{N},\mathcal{L}')}-k\mathbf{B}_{T'}\right)\mathbf{x}. \tag{76}$$

Notice that, according to (75), $\mathbf{x}$ is also in the null space of $\mathbf{A}_{(\mathcal{N},\mathcal{L}')}$ and $-k\mathbf{B}_{T'}$. Hence, we have:

$$\mathbf{0}=\mathbf{Y}_{\mathcal{N}'}\,\mathbf{x}. \tag{77}$$

In conclusion, if $\mathbf{Y}_{\mathcal{N}}\,\mathbf{x}=\mathbf{0}$ then $\mathbf{Y}_{\mathcal{N}'}\,\mathbf{x}=\mathbf{0}$, and thus:

$$\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}}\right)\subseteq\mathrm{Null}\left(\mathbf{Y}_{\mathcal{N}'}\right). \tag{78}$$

Applying the rank-nullity theorem, we obtain:

$$\mathrm{rank}\left(\mathbf{Y}_{\mathcal{N}'}\right)\leq\mathrm{rank}\left(\mathbf{Y}_{\mathcal{N}}\right). \tag{79}$$

Recall that $\mathbf{Y}_{\mathcal{N}'}$ is the admittance matrix of an alternative network as shown in (68). This alternative network satisfies Condition 5. Note that $\mathbf{Y}_{\mathcal{N}'}$ includes all shunts in $\mathcal{T}/\mathcal{T}'$ and the real part of all shunts in $\mathcal{T}'$ (which are never null). Thus, if $\mathbf{y}_{\mathcal{T}} \neq \mathbf{0}$, then $\mathbf{Y}_{\mathcal{N}'}$ will have at least one shunt. For this case, Theorem 1 with Condition 5 indicates that $\mathbf{Y}_{\mathcal{N}'}$ will be full-rank, and hence $\mathbf{Y}_{\mathcal{N}}$ will also be full rank due to (79). When $\mathbf{y}_{\mathcal{T}} = \mathbf{0}$, notice that the generalized incidence matrix of $\mathbf{Y}_{\mathcal{N}'}$ is also $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$. Thus, according to Theorem 1, we have:

$$\operatorname{rank}\left(\mathbf{Y}_{\mathcal{N}'}\right) = \operatorname{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right). \tag{80}$$

Substituting this into (79) yields:

$$\operatorname{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right) \leq \operatorname{rank}\left(\mathbf{Y}_{\mathcal{N}}\right). \tag{81}$$

Moreover, for the case where $\mathbf{y}_{\mathcal{T}} = \mathbf{0}$, we can write:

$$\mathbf{Y}_{\mathcal{N}} = \mathbf{A}^{H}_{(\mathcal{N},\mathcal{L})} \mathbf{Y}_{\mathcal{L}} \mathbf{A}_{(\mathcal{N},\mathcal{L})}. \tag{82}$$

Observe that $\mathbf{Y}_{\mathcal{N}}$ involves a product by $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$, and thus $\operatorname{rank}\left(\mathbf{Y}_{\mathcal{N}}\right) \leq \operatorname{rank}\left(\mathbf{A}_{(\mathcal{N},\mathcal{L})}\right)$ (see [24, Theorem 3.7]). Clearly, this implies that the rank of $\mathbf{Y}_{\mathcal{N}}$ and the rank of $\mathbf{A}_{(\mathcal{N},\mathcal{L})}$ are equal, so Theorem 1 holds for $\mathbf{Y}_{\mathcal{N}}$. □

C. Practical Implications

Conditions 2, 3, 4, and 6 suggest possible regularization procedures to enforce invertibility of the admittance matrix. To enforce Condition 2, we can add a small series resistance to purely inductive lines or transformers. To enforce Condition 3, we can apply a small Tikhonov regularization to the matrix (equivalent to adding small shunt conductances to each node). Condition 4 only considers the conductances associated with a spanning tree of the network, so unlike Condition 2 it can ensure invertibility without adding series resistances to all purely inductive elements, since the spanning tree may have many fewer edges than the entire network. However, Condition 4 requires at least one shunt with positive resistance, so it does not generalize Condition 2. Condition 5 guarantees the validity of the theorem for purely inductive (or resistive-inductive) networks, so we only have to enforce the presence of a single shunt to ensure the invertibility of the admittance matrix. Condition 6 is particularly relevant for networks which are predominantly inductive but also have a small number of shunt or series capacitors. Condition 6 indicates that we would only need to add resistances to account for these few capacitors in order to ensure invertibility of the admittance matrix.

IV. Conclusions

This paper studied the invertibility of the admittance matrix for balanced networks. First, we analyzed a theorem from the literature regarding conditions guaranteeing invertibility of the admittance matrix, and we found a technical issue in the proof of that theorem. Next, we developed a framework of lemmas and hypotheses that allowed us to amend the proof of previous claims, developing relaxed conditions that guarantee the invertibility of the admittance matrix and generalizing the results to systems with branches modeled as purely reactive elements and transformers with off-nominal tap ratios. Finally, we discussed these conditions' implications, their wide practical applicability, and their usefulness in enforcing invertibility.

The theory developed in this paper has solely considered admittance matrices for balanced single-phase equivalent network representations. With rapidly increasing penetrations of distributed energy resources in unbalanced distribution systems, extending the theory developed here to address the admittance matrices associated with polyphase networks is an important direction for future work. The authors of [17] considered this topic in [18], where they generalize Theorem 1 to polyphase networks. However, the theory in [18] also relies on the incorrectly stated Lemma 3 and hence may also benefit from amendments and extensions similar to those in this paper.

Acknowledgements

The authors greatly appreciate technical discussions with Mario Paolone and Andreas Kettner.

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