Modeling Complex Power Injections in Radial Electrical Networks Using Voltage Magnitudes

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***Abstract*— This paper addresses the problem of modeling complex (active and reactive) power injection perturbations using only perturbations of voltage magnitude measurements in radial electrical networks. We first analyze the sensitivity matrices relating active and reactive power injections to voltage magnitudes, which are two of the four block submatrices of the inverse of the AC power flow Jacobian. We find that, under the assumption that the network is radial, the rates of change—or sensitivities—of the voltage magnitude at a given node with respect to changes in active and reactive power injections at that node will take distinct values. We then develop a sufficient condition for estimating complex power injections from voltage magnitudes with these matrices. We also develop estimation techniques to recover these sensitivity matrices from commonly available measurements with varying levels of sensor availability. Several simulations verify the results and demonstrate engineering use cases.**

I. Introduction and Preliminaries

Sensitivity coefficients describe small changes in an independent variable Δx due to small changes in a dependent variable Δy [1]. The growth in sensor deployment in electric distribution systems has spurred research to develop methods to recover these sensitivity coefficients from these data. This allows for the network behavior to be approximated even when the model is inaccurate, out of date, or unavailable [2]–[4]. These coefficients can be used to form a linear approximation of the non-linear AC power flow equations, which is useful for modeling networks where knowledge of the topology and circuit parameters is limited.

Recent works have shown the effectiveness of matrix completion [5] in power system estimation problems, such as estimating low-observability voltage phasors [6] and evaluating voltage stability [7]. In this work, we extend this to discovering the behavior of a radial network in terms of the voltage *magnitude* sensitivities to active and reactive power.

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We rely on the results of [8] that establish unique solutions for the voltage *phasor* sensitivities in radial networks. Our work also builds upon previous work on adaptive power flow linearizations [9] and measurement-based estimation methods for sensitivity coefficients [3], [4]. In contrast with our work, these methods assume access to voltage angles.

At nodes $i = 1, \ldots, n$ of an electric power system, the net power injections are $\bar{s}_i \triangleq p_i + jq_i \in \mathbb{C}$, where p_i and q_i are the net active and reactive power injections, respectively. These are related with the nodal voltages $\bar{v}_i = v_i \underline{/\theta_i} \in \mathbb{C}$ and the net current injections $\bar{\ell}_i = \ell_i \underline{/\delta_i} \in \mathbb{C}$ as

$$\bar{s}_i = \bar{v}_i \bar{\ell}_i^* = s_i \underline{/\theta_i - \delta_i} \tag{1}$$

where $\bar{\ell}_i^*$ is the complex conjugate of the net current injection and $\theta_i - \delta_i$ is the difference between the phase angles of the voltage and current.

Measurements of θ_i are often unavailable. For instance, the most common data available in electric distribution systems are only the voltage magnitudes $\boldsymbol{v} \in \mathbb{R}^n$ and the active and reactive power injections $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^n$ [10], making it difficult to model (1) with these measurement data. In particular, it is not immediately clear how $\boldsymbol{v}$ relates to $\boldsymbol{p}$ and $\boldsymbol{q}$.

The nodal power factors $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_n]^T$ are

$$\alpha_i \triangleq \cos(\theta_i - \delta_i) = \frac{p_i}{\sqrt{p_i^2 + q_i^2}}, \quad i = 1, \ldots, n. \tag{2}$$

This quantity describes the ratio of active power to the magnitude of the complex power (1), where $s_i = \sqrt{p_i^2 + q_i^2}$. In this paper we relate (2) to when it is possible to model the changes in $\boldsymbol{v}$ from only observations of $\boldsymbol{p}, \boldsymbol{q}$.

The contributions of this paper are:

1) Conditions for modeling complex power injections from voltage magnitudes in radial electrical networks, by showing that the voltage magnitude sensitivities to active and reactive power injections are unique and take distinct values, and establishing a relationship with the nodal power factors $\boldsymbol{\alpha}$.
2) Methodologies to recover or update the matrices describing the sensitivities of voltage magnitudes to active and reactive power injections—which are submatrices of the inverse of the power flow Jacobian—via regression and matrix completion.

The methods presented in this paper assume a three-phase, unbalanced radial electrical network, where the set $\mathcal{N}$ comprises the PQ nodes of the network. We assume that there is a single slack bus, and thus, three slack nodes.

A. Data Input Assumptions

Advanced metering infrastructure (AMI) datasets $\mathcal{D}_i$ for PQ nodes $i = 1, \dots, n \triangleq |\mathcal{N}|$ are defined as:

$$\mathcal{D}_i \triangleq \left\{ \left(v_i^{(t)}, p_i^{(t)}, q_i^{(t)}\right) \right\}_{t=1}^{m}, \tag{3}$$

where $v_i^{(t)}, p_i^{(t)}$, and $q_i^{(t)}$ are the nodal voltage magnitude, net active, and net reactive power injection measurements at node i at time steps $t = 1, \dots, m$, respectively. We will assume the errors of these sensors to be normally distributed with variance that is on the order of 0.5%. AMI sensors typically have errors between 0.07% and 4% depending on the power quality of the load [11]. Throughout the paper we operate under the assumption that voltage regulating devices are held fixed throughout the system. In the next section, we will drop the superscript t for brevity.

B. The Newton-Raphson Power Flow

Consider the power balance equations for a node $i \in \mathcal{N}$:

$$p_i = v_i \sum_{k=1}^{n} v_k \big(G_{ik} \cos(\theta_i - \theta_k) + B_{ik} \sin(\theta_i - \theta_k)\big), \tag{4}$$

$$q_i = v_i \sum_{k=1}^{n} v_k \big(G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k)\big), \tag{5}$$

where v_i, v_k are the voltage magnitudes at nodes i and k and G_{ik}, B_{ik} are the real and imaginary parts of the ik-th entry of the nodal admittance matrix $Y_{ik} \triangleq G_{ik} + jB_{ik}$, respectively. In order to solve the systems (4) and (5), a classical approach is the Newton-Raphson (NR) algorithm, which iteratively solves the system of equations (6):

$$\underbrace{\begin{bmatrix} \Delta \boldsymbol{p} \\ \Delta \boldsymbol{q} \end{bmatrix}}_{(2n\times1)} = \underbrace{\left[\begin{array}{c|c} \frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} & \frac{\partial \mathbf{P}}{\partial \mathbf{V}} \\ \hline \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\theta}} & \frac{\partial \mathbf{Q}}{\partial \mathbf{V}} \end{array}\right]}_{(2n\times2n)} \underbrace{\begin{bmatrix} \Delta \boldsymbol{\theta} \\ \Delta \boldsymbol{v} \end{bmatrix}}_{(2n\times1)} = \mathbf{J} \begin{bmatrix} \Delta \boldsymbol{\theta} \\ \Delta \boldsymbol{v} \end{bmatrix}, \tag{6}$$

where $\Delta \boldsymbol{p}, \Delta \boldsymbol{q} \in \mathbb{R}^n$ are vectors of small deviations in active and reactive power, respectively. The power flow Jacobian $\mathbf{J}$ is known to be relatively constant with respect to small changes in power injections [3], [4]. Let us consider the block submatrices of the inverse power flow Jacobian. We refer to blocks of the inverse Jacobian as sensitivity matrices and their elements as sensitivity coefficients. Denote these blocks as $\left(\mathbf{S}_y^x\right)$. The inverse problem of (6) can be written as:

$$\underbrace{\begin{bmatrix} \Delta \boldsymbol{\theta} \\ \Delta \boldsymbol{v} \end{bmatrix}}_{(2n\times1)} = \underbrace{\left[\begin{array}{c|c} (\mathbf{S}_p^{\theta}) & (\mathbf{S}_q^{\theta}) \\ \hline (\mathbf{S}_p^{v}) & (\mathbf{S}_q^{v}) \end{array}\right]}_{(2n\times2n)} \underbrace{\begin{bmatrix} \Delta \boldsymbol{p} \\ \Delta \boldsymbol{q} \end{bmatrix}}_{(2n\times1)} = \mathbf{J}^{-1} \begin{bmatrix} \Delta \boldsymbol{p} \\ \Delta \boldsymbol{q} \end{bmatrix}. \tag{7}$$

C. Phaseless Approximation of the Power Flow Equations

Particularly in distribution systems, access to phase angle information $\Delta\boldsymbol{\theta}$ may be unavailable due to low penetrations of phasor measurement units (PMUs), making it difficult to analyze (6) and (7) in real time. However, the sensitivity matrices are relatively constant inter-temporally, allowing for model behavior to be linearly approximated [3], [4]. The voltage magnitude of node i, v_i, can be written as a first-order linear approximation around a given operating condition:

$$v_i \approx v_i^0 + \frac{\partial V_i}{\partial \mathbf{P}} \Delta \boldsymbol{p} + \frac{\partial V_i}{\partial \mathbf{Q}} \Delta \boldsymbol{q}, \tag{8}$$

where v_i^0 is the voltage magnitude of node i in the given operating condition and $\frac{\partial V_i}{\partial \mathbf{P}}, \frac{\partial V_i}{\partial \mathbf{Q}}$ are the i-th rows of the matrices describing voltage magnitude sensitivities with respect to the power injections. From (7), we can write a rectangular linearized system which relates voltage magnitude variations to active and reactive power variations:

$$\underbrace{\Delta \boldsymbol{v}}_{(n\times1)} \approx \underbrace{\left[(\mathbf{S}_p^{v}) \ (\mathbf{S}_q^{v})\right]}_{(n\times2n)} \underbrace{\begin{bmatrix} \Delta \boldsymbol{p} \\ \Delta \boldsymbol{q} \end{bmatrix}}_{(2n\times1)} = \tilde{\mathbf{S}} \boldsymbol{x}. \tag{9}$$

The wide sensitivity matrix, $\tilde{\mathbf{S}}$, describes the sensitivity of voltage magnitudes to active and reactive power injections, and is the main quantity of interest in this paper.

II. Analytical Results

Prior numerical results have empirically indicated that the voltage magnitude sensitivities for active and reactive power injections are distinct [12]–[14]. In this section, we show that the voltage magnitude sensitivities to active and reactive power injections take distinct non-zero values in radial networks. The complex power injection $\bar{s}_i$ at node i can be related to the network's phasor voltages as:

$$\bar{s}_i = \bar{v}_i \left(\sum_{j \in \mathcal{N} \cup \mathcal{S}} Y_{ij} \bar{v}_j \right)^* \ \forall i \in \mathcal{N}, \tag{10}$$

where $\bar{v}_i$ is the voltage phasor of node i and $(\cdot)^*$ denotes the complex conjugate operator. Following [8], differentiating (10) with respect to active and reactive power individually yields the linear differential equations (11) and (12) respectively, whose solutions are the sensitivities of *phasor* voltages to active and reactive power injections:

$$\delta_{il} = \frac{\partial \bar{V}_i^*}{\partial P_l} \sum_{j \in \mathcal{S} \cup \mathcal{N}} Y_{ij} \bar{v}_j + \bar{v}_i^* \sum_{j \in \mathcal{N}} Y_{ij} \frac{\partial \bar{V}_j}{\partial P_l}, \tag{11}$$

$$-j\delta_{il} = \frac{\partial \bar{V}_i^*}{\partial Q_l} \sum_{j \in \mathcal{S} \cup \mathcal{N}} Y_{ij} \bar{v}_j + \bar{v}_i^* \sum_{j \in \mathcal{N}} Y_{ij} \frac{\partial \bar{V}_j}{\partial Q_l}, \tag{12}$$

where δ_{il} is the Kronecker delta: $\delta_{il} = \begin{cases} 1 & \text{if } i = l, \\ 0 & \text{otherwise.} \end{cases}$ The voltage *phasor* sensitivities to active and reactive power injections, $\frac{\partial \bar{V}_i}{\partial P_l}, \frac{\partial \bar{V}_i}{\partial Q_l}$ are of particular interest in distribution systems, because they are known to have a unique solution [8] for radial networks.

Remark 1: For the nontrivial solutions of the equations in the systems (11) and (12), i.e., where $\delta_{il} \neq 0$, the unknowns $\frac{\partial \bar{V}_i}{\partial Q_l}$ and $\frac{\partial \bar{V}_i}{\partial P_l}$ achieve distinct complex values.

For the next propositions we denote by $\Re\{\cdot\}$ and $\Im\{\cdot\}$ the real and imaginary part operators, respectively.

Lemma 1: The voltage magnitude sensitivity coefficients of a network can be written as (13) and (14).

$$\frac{\partial V_i}{\partial Q_l} = \frac{1}{v_i}\Re\left\{\bar{v}_i{}^{*}\frac{\partial \bar{V}_i}{\partial Q_l}\right\}, \tag{13}$$

$$\frac{\partial V_i}{\partial P_l} = \frac{1}{v_i}\Re\left\{\bar{v}_i{}^{*}\frac{\partial \bar{V}_i}{\partial P_l}\right\}. \tag{14}$$

Proof: Let v_i and θ_i be the real-valued magnitude and angle of the voltage phasor. Write the voltage phasor sensitivity at node i to a quantity X as:

$$\frac{\partial \bar{V}_i}{\partial X} = \frac{\partial}{\partial X}\{v_i e^{j\theta_i}\} = \frac{\partial V_i}{\partial X}e^{j\theta_i} + j v_i \frac{\partial \theta_i}{\partial X}e^{j\theta_i}, \tag{15}$$

so we have that:

$$e^{-j\theta_i}\frac{\partial \bar{V}_i}{\partial X} = \frac{\partial V_i}{\partial X} + j v_i\frac{\partial \theta_i}{\partial X} \implies \Re\left\{e^{-j\theta_i}\frac{\partial \bar{V}_i}{\partial X}\right\} = \frac{\partial V_i}{\partial X}. \tag{16}$$

Next, observe that $\bar{v}_i^{*}/v_i = e^{-j\theta}$. Therefore,

$$\Re\left\{e^{-j\theta_i}\frac{\partial \bar{V}_i}{\partial X}\right\} = \frac{1}{v_i}\Re\left\{\bar{v}_i^{*}\frac{\partial \bar{V}_i}{\partial X}\right\}, \tag{17}$$

$$\frac{\partial V_i}{\partial X} = \frac{1}{v_i}\Re\left\{\bar{v}_i^{*}\frac{\partial \bar{V}_i}{\partial X}\right\}. \tag{18}$$

Make $X = P_l$ or $X = Q_l$ to get the desired result. ■

A. Unique Voltage Magnitude Sensitivities

Next, we will show that if $\frac{\partial \bar{V}_i}{\partial Q_l}$ and $\frac{\partial \bar{V}_i}{\partial P_l}$ have unique solutions then we can say the same for the voltage magnitudes.

Lemma 2: Let the rectangular form of the complex sensitivities be $\frac{\partial \bar{V}_i}{\partial Q_l} = \alpha + j\beta$ and $\frac{\partial \bar{V}_i}{\partial P_l} = \gamma + j\delta$ respectively. If

$$(\alpha, \beta) \notin \{(\alpha,\beta) : \Re\{\bar{v}_i\}\alpha + \Im\{\bar{v}_i\}\beta = 0\}, \tag{19}$$

$$(\gamma, \delta) \notin \{(\gamma,\delta) : \Re\{\bar{v}_i\}\gamma + \Im\{\bar{v}_i\}\delta = 0\}, \tag{20}$$

then $\frac{\partial V_i}{\partial Q_l} \neq \frac{\partial V_i}{\partial P_l}$ $\forall i, l$.

Proof: Using (13), the voltage magnitude sensitivity coefficients for node i to reactive power at node l is:

$$\frac{\partial V_i}{\partial Q_l} = \frac{1}{v_i}\Re\left\{(\Re\{\bar{v}_i\} - j\Im\{\bar{v}_i\})\frac{\partial \bar{V}_i}{\partial Q_l}\right\}, \tag{21}$$

and in the same way, for active power, we have:

$$\frac{\partial V_i}{\partial P_l} = \frac{1}{v_i}\Re\left\{(\Re\{\bar{v}_i\} - j\Im\{\bar{v}_i\})\frac{\partial \bar{V}_i}{\partial P_l}\right\}. \tag{22}$$

Simplifying the above results in:

$$\frac{\partial V_i}{\partial Q_l} = \frac{1}{v_i}(\Re\{\bar{v}_i\}\alpha + \Im\{\bar{v}_i\}\beta), \tag{23}$$

$$\frac{\partial V_i}{\partial P_l} = \frac{1}{v_i}(\Re\{\bar{v}_i\}\gamma + \Im\{\bar{v}_i\}\delta). \tag{24}$$

From Remark 1, if $\frac{\partial \bar{V}_i}{\partial Q_l} \neq \frac{\partial \bar{V}_i}{\partial P_l}$ for nonzero solutions this implies that either $\alpha \neq \gamma$ or $\beta \neq \delta$. So, provided that the sensitivities are not zero, i.e., (19) and (20) are satisfied, then it must also be true that $\frac{\partial V_i}{\partial Q_l} \neq \frac{\partial V_i}{\partial P_l}$ $\forall i, l$. ■

B. Encoding Reactive Power Perturbations

Propositions 1 and 2 imply that the matrices $\mathbf{S}_p^v$, $\mathbf{S}_q^v$ in (7) are full rank for a radial network, and the matrix $\tilde{\mathbf{S}}$ has full column rank for any subset of the columns whose cardinality is less than $\frac{n}{2}$. Essentially, the voltage magnitude sensitivities to active and reactive power injections in a radial network take distinct values. Practically, what this means is that it is possible to quantify both active and reactive power impacts on distribution systems using only voltage magnitudes.

Remark 2: Consider a node $l \in \mathcal{N}$ in a radial distribution network with unknown complex power injections. Given a vector of voltage magnitude perturbations $\Delta \boldsymbol{v} \in \mathbb{R}^n$ and a voltage magnitude sensitivity matrix

$$\tilde{\mathbf{S}} = \left[\frac{\partial \mathbf{V}}{\partial P_l}, \frac{\partial \mathbf{V}}{\partial Q_l}\right] \in \mathbb{R}^{n\times 2}, \tag{25}$$

then there is a unique least squares solution for the complex power perturbation $\Delta \boldsymbol{x} = [\Delta p_l, \Delta q_l]^T$ such that

$$\Delta \boldsymbol{x} = \left(\tilde{\mathbf{S}}^T\tilde{\mathbf{S}}\right)^{-1}\tilde{\mathbf{S}}^T\Delta \boldsymbol{v}. \tag{26}$$

By Lemma 2, the system

$$\tilde{\mathbf{S}}\Delta \boldsymbol{x} = \frac{\partial \mathbf{V}}{\partial P_l}\Delta p_l + \frac{\partial \mathbf{V}}{\partial Q_l}\Delta q_l = 0 \tag{27}$$

has a solution if and only if $\Delta \boldsymbol{x} = \mathbf{0}$. Therefore, $\tilde{\mathbf{S}}$ has full rank and (26) will always exist.

Lemma 3: Let α_i, $i = 1, \ldots, n$ denote the nodal power factors and assume that they have fixed values. Then the voltage deviation vector $\Delta \boldsymbol{v}$ can be written as:

$$\Delta \boldsymbol{v} = \tilde{\mathbf{S}}_\dagger \Delta \boldsymbol{p}, \tag{28}$$

where we define $\tilde{\mathbf{S}}_\dagger \triangleq \left(\mathbf{S}_p^v + \mathbf{S}_q^v\mathbf{K}\right)$, $\mathbf{K} \triangleq \mathrm{dg}\left(\frac{\sqrt{1-\alpha_i^2}}{\alpha_i}\right) \in \mathbb{R}^{n\times n}$, and $\mathrm{dg}(\cdot)$ returns a diagonal matrix over $i = 1\ldots, n$.

Proof: For any fixed power factor α_i at node $i \in \mathcal{N}$, via the implicit function theorem we can express the reactive power injection as a locally linear function of the active power injection:

$$q_i = \frac{\sqrt{1-\alpha_i^2}}{\alpha_i}p_i \tag{29}$$

where p_i is the active power measurement at bus i. Therefore, we can express the voltage deviation vector as

$$\Delta \boldsymbol{v} = \mathbf{S}_p^v\Delta \boldsymbol{p} + \mathbf{S}_q^v\mathbf{K}\Delta \boldsymbol{p}, \tag{30}$$

$$= \left(\mathbf{S}_p^v + \mathbf{S}_q^v\mathbf{K}\right)\Delta \boldsymbol{p} = \tilde{\mathbf{S}}_\dagger\Delta \boldsymbol{p}, \tag{31}$$

which is what we wanted to show. ■

We now state assumptions that allow us to discuss further:

Assumption 1: The power flow Jacobian is nonsingular and the symmetric part of the unknown angle sensitivity submatrix $\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}$ (that is, $\frac{1}{2}(\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} + \frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}^T)$) is positive definite.

The non-singularty of the Jacobian is a reasonable assumption as power systems operate far from the point of collapse. While counterexamples do exist (see [15], [16]), the full power flow Jacobian can often be expected to be nonsingular in normal network operating conditions. The assumption of $\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}$ being positive definite is not restrictive, as it holds in

TABLE I
VALIDITY OF ASSUMPTION 1 ON PQ NODES OR PQ AND PV NODES OF TEST FEEDERS

	Test Case:	case3	case4_dist	case5	case5_x	case6	case9	case14	case18	case24	case30	case118zh
Assumption	$\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} \succ 0$	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
True?	$\mathbf{J}$ Invertible	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes

most practical cases. To see this, we first consider the vector of active powers excluding slack nodes

$$\boldsymbol{p} = \Re\left\{\mathrm{dg}(\boldsymbol{v})(\mathbf{G}' - j\mathbf{B}')\mathrm{dg}(\boldsymbol{v}^*)\right\}, \tag{32}$$

where $\boldsymbol{v}$ is the vector of node voltages excluding slack nodes, $\mathrm{dg}(\boldsymbol{x})$ gives a diagonal matrix whose entries correspond to the entries of $\boldsymbol{x}$, and $\mathbf{Y}' = \mathbf{G}' + j\mathbf{B}'$ is the system admittance matrix with the rows and columns of the slack nodes removed. From the rules of differentiation for complex matrices (see [17]), we get that

$$\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} = -\Im\left\{\mathrm{dg}(\boldsymbol{v})(\mathbf{G}' - j\mathbf{B}')\mathrm{dg}(\boldsymbol{v}^*)\right\} - \mathrm{dg}(\boldsymbol{q}), \tag{33}$$

where $\boldsymbol{q}$ is the vector of reactive powers excluding slack nodes. Writing the voltages in rectangular form as $\boldsymbol{v} = \boldsymbol{v}_R + j\boldsymbol{v}_I$, we obtain

$$\begin{aligned}\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} = &-\mathrm{dg}(\boldsymbol{q}) - \mathrm{dg}(\boldsymbol{v}_I)\,\mathbf{G}'\,\mathrm{dg}(\boldsymbol{v}_R) + \mathrm{dg}(\boldsymbol{v}_R)\,\mathbf{G}'\,\mathrm{dg}(\boldsymbol{v}_I) \\ &+ \mathrm{dg}(\boldsymbol{v}_R)\,\mathbf{B}'\,\mathrm{dg}(\boldsymbol{v}_R) + \mathrm{dg}(\boldsymbol{v}_I)\,\mathbf{B}'\,\mathrm{dg}(\boldsymbol{v}_I). \end{aligned} \tag{34}$$

Define the matrix $\mathbf{B}'_s = \frac{1}{2}(\mathbf{B}' + \mathbf{B}'^T)$. Any matrix is positive definite if and only if its symmetric part is also positive semidefinite,[1] so now we consider the symmetric part of $\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}$:

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} + \frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}^T\right) = &\ \mathrm{dg}(\boldsymbol{v}_R)\,\mathbf{B}'_s\,\mathrm{dg}(\boldsymbol{v}_R) \\ &+ \mathrm{dg}(\boldsymbol{v}_I)\,\mathbf{B}'_s\,\mathrm{dg}(\boldsymbol{v}_I) - \mathrm{dg}(\boldsymbol{q}). \end{aligned} \tag{35}$$

$\mathbf{B}'_s$ can be viewed as the DC admittance matrix of the system, excluding slack nodes and making the branches connecting to slack nodes into shunts. In practice, all these branches are inductive, making $\mathbf{B}'_s$ the admittance matrix of the only-inductor network. According to [18], $\mathbf{B}'_s$ is invertible and positive semidefinite, and thus positive definite. Moreover, the reference angle can be chosen such that $\boldsymbol{v}_R, \boldsymbol{v}_I$ have no zero entries, therefore $\mathrm{dg}(\boldsymbol{v}_R)\,\mathbf{B}'_s\,\mathrm{dg}(\boldsymbol{v}_R) \succ 0$ and $\mathrm{dg}(\boldsymbol{v}_I)\,\mathbf{B}'_s\,\mathrm{dg}(\boldsymbol{v}_I) \succ 0$. Lastly, $\boldsymbol{q}$ typically has non-positive entries, as loads are mostly inductive in practice. For all these reasons most real-life systems have $\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} \succ 0$.

We test Assumption 1 in `PowerModels.jl` [19] by using and extending the `calc_basic_jacobian_matrix` function, the results of which are shown in Table 1.[2]

Assumption 2: The difference between the maximum and minimum elements of $\mathbf{K}$,

$$\Delta k \triangleq k_{\max} - k_{\min} = \frac{\sqrt{1-\alpha_{\min}^2}}{\alpha_{\min}} - \frac{\sqrt{1-\alpha_{\max}^2}}{\alpha_{\max}}, \tag{36}$$

is sufficiently small relative to an expression that depends on the power to voltage phase angle sensitivity matrices $\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}$ and $\frac{\partial \mathbf{Q}}{\partial \boldsymbol{\theta}}$, which will be defined explicitly in (47).

Theorem 1: Let $\Delta \boldsymbol{v} \in \mathbb{R}^n$ be a vector of voltage magnitude perturbations. If Assumptions 1 and 2 hold, there exists unique power perturbations $\Delta \boldsymbol{x} \triangleq [\Delta \boldsymbol{p}^T, \Delta \boldsymbol{q}^T]^T \in \mathbb{R}^{2n}$ such that $\Delta \boldsymbol{v} = \tilde{\mathbf{S}}\Delta \boldsymbol{x}$.

Proof: Using Lemma 3 it now suffices to show that

$$\tilde{\mathbf{S}}_\dagger \triangleq \left(\mathbf{S}_p^v + \mathbf{S}_q^v \mathbf{K}\right) \tag{37}$$

is invertible to complete the proof.

If Assumption 1 holds, then both $\mathbf{J}$ and $\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}$ are invertible. Thus we can apply the Schur Complement to write the reactive power voltage sensitivity matrix in terms of the blocks of the Jacobian in (7) as:

$$\mathbf{S}_q^v = \left(\frac{\partial \mathbf{Q}}{\partial \mathbf{V}} - \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\theta}}\left(\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}\right)^{-1}\frac{\partial \mathbf{P}}{\partial \mathbf{V}}\right)^{-1}, \tag{38}$$

and the active power voltage sensitivity matrix is then:

$$\begin{aligned}\mathbf{S}_p^v &= -\left(\frac{\partial \mathbf{Q}}{\partial \mathbf{V}} - \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\theta}}\left(\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}\right)^{-1}\frac{\partial \mathbf{P}}{\partial \mathbf{V}}\right)^{-1}\frac{\partial \mathbf{Q}}{\partial \boldsymbol{\theta}}\left(\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}\right)^{-1}, && (39)\\ &= -\mathbf{S}_q^v\frac{\partial \mathbf{Q}}{\partial \boldsymbol{\theta}}\left(\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}\right)^{-1}. && (40)\end{aligned}$$

Combining (37), (38), and (40), we can express $\tilde{\mathbf{S}}_\dagger$ as:

$$\tilde{\mathbf{S}}_\dagger = \mathbf{S}_q^v\left(\mathbf{K} - \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\theta}}\left(\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}\right)^{-1}\right). \tag{41}$$

Thus we have that

$$(\mathbf{S}_q^v)^{-1}\tilde{\mathbf{S}}_\dagger\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} = \mathbf{K}\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} - \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\theta}}. \tag{42}$$

Let $k_{\max}$ and $k_{\min}$ denote the maximum and minimum entries of $\mathbf{K}$, respectively. We also define $\Delta k = k_{\max} - k_{\min}$ and $\Delta \mathbf{K} = k_{\max}\mathbf{I} - \mathbf{K}$. Then we can write

$$(\mathbf{S}_q^v)^{-1}\tilde{\mathbf{S}}_\dagger\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} = \underbrace{k_{\max}\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} - \frac{\partial \mathbf{Q}}{\partial \boldsymbol{\theta}}}_{\triangleq \mathbf{M} \succ \mathbf{0}} - \Delta \mathbf{K}\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}, \tag{43}$$

$$(\mathbf{S}_q^v)^{-1}\tilde{\mathbf{S}}_\dagger\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}} = \mathbf{M}\left(\mathbf{I} - \mathbf{M}^{-1}\Delta \mathbf{K}\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}\right). \tag{44}$$

The inverse of the term in parenthesis can be computed using Neumann series. This inverse is guaranteed to exist if:

$$\left\|\mathbf{M}^{-1}\Delta \mathbf{K}\frac{\partial \mathbf{P}}{\partial \boldsymbol{\theta}}\right\|_2 < 1, \tag{45}$$

where $\|\cdot\|_2$ denotes the the largest singular value, or the spectral norm of the argument. The sub-multiplicative property of

[1]This stems from the fact that for any real (possibly non-symmetric) matrix $\mathbf{A}$ and appropriately sized vector $\mathbf{x}$, the following relationships hold: $\mathbf{x}^T\mathbf{A}\mathbf{x} = (\mathbf{x}^T\mathbf{A}\mathbf{x})^T = \mathbf{x}^T\mathbf{A}^T\mathbf{x} = \mathbf{x}^T(\mathbf{A}/2 + \mathbf{A}^T/2)\mathbf{x}$.

[2]A Julia script to reproduce the results shown is publicly available at `https://github.com/samtalki/PowerSensitivities.jl`

this norm allows us to use the following stronger inequality:

$$\left\|\mathbf{M}^{-1}\right\|_2 \|\Delta\mathbf{K}\|_2 \left\|\frac{\partial\mathbf{P}}{\partial\boldsymbol{\theta}}\right\|_2 < 1. \tag{46}$$

Notice that $\|\Delta\mathbf{K}\|_2 = k_{\max} - k_{\min} = \Delta k$, hence the inverse exists if

$$\Delta k < \left\|\mathbf{M}^{-1}\right\|_2^{-1} \left\|\frac{\partial\mathbf{P}}{\partial\boldsymbol{\theta}}\right\|_2^{-1}, \tag{47}$$

which holds for close enough power factors. In conclusion, the right hand side of (44) is invertible, so the left hand side is invertible too. Under the condition (47), as both $(\mathbf{S}_q^v)^{-1}$ and $\frac{\partial\mathbf{P}}{\partial\boldsymbol{\theta}}$ are invertible, then $\tilde{\mathbf{S}}_\dagger$ is invertible too. This means that for any $\Delta\boldsymbol{v}$ we have a unique $\Delta\boldsymbol{p}$ and a unique $\Delta\boldsymbol{x}$. ■

If we assume that the wide $\tilde{\mathbf{S}}$ has rapidly decreasing singular values, then $\tilde{\mathbf{S}}$ can be well approximated via a truncated singular value decomposition (SVD) as:

$$\tilde{\mathbf{S}} \approx \sum_{k=1}^{R} \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^T, \tag{48}$$

where $\sigma_k, \boldsymbol{u}_k$, and $\boldsymbol{v}_k$, $k = 1, \ldots, R$ are the R largest singular values and corresponding singular vectors. The low-rank structure inherent in this matrix can be verified empirically in Fig. 1, which visualizes a spectral analysis of the voltage sensitivities for the IEEE 13-bus test feeder.

Remark 3: The approximate low-rank structure of $\tilde{\mathbf{S}}$ results from the columns belonging to a union of low-rank subspaces. Empirically, we have found these are related to groupings of the injection type (P/Q) and phase (A/B/C).

III. Applications

In this section, we develop applications of the voltage magnitude sensitivity matrices for low-observability distribution system modeling settings. We review known regression techniques and develop matrix completion techniques to compute an estimate $\tilde{\mathbf{S}}^{\#}$ of the sensitivity matrix $\tilde{\mathbf{S}}$.

A. Least Squares

It is well known that, given perturbations of (3) of the form $\Delta\mathbf{V}, \Delta\mathbf{P}, \Delta\mathbf{Q} \in \mathbb{R}^{m' \times n}$, $m' \geq 2n$, and defining $\Delta\mathbf{X} \triangleq [\Delta\mathbf{P}^T, \Delta\mathbf{Q}^T]^T \in \mathbb{R}^{m' \times 2n}$, a least-squares solution for $\tilde{\mathbf{S}}$ can be found via the Moore-Penrose Pseudoinverse as $\tilde{\mathbf{S}}^{\#} = (\Delta\mathbf{X}^T\Delta\mathbf{X} + \lambda\mathbf{I})^{-1}\Delta\mathbf{X}^T\Delta\mathbf{V}$, where λ is a Tikhonov regularization parameter.

B. Matrix Completion Update

Suppose that we have an incomplete sensitivity matrix $\tilde{\mathbf{S}}_0 := [\mathbf{S}_{p,0}^v, \mathbf{S}_{q,0}^v]$ where the set $\Omega = \{i, j : [\tilde{\mathbf{S}}_0]_{i,j} = 0\}$ represents $|\Omega|$ entries of $\tilde{\mathbf{S}}_0$ for which we do not have access to voltage sensitivity relationships. The full matrix $\tilde{\mathbf{S}}$ can be recovered as the solution to the following program, subject to the constraint that the estimated matrix has rank R:

$$\tilde{\mathbf{S}}^{\#} = \operatorname*{arg\,min}_{\mathbf{S} \in \mathbb{R}^{n \times 2n}} ||\tilde{\mathbf{S}}_0 - \mathbf{S}||_F^2 \quad \text{s.t.} \quad \operatorname{rank}(\mathbf{S}) = R, \tag{49}$$

where $||\cdot||_F^2$ is the squared Frobenius norm, which is defined for a matrix $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2}$ as $||\mathbf{X}||_F^2 = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} |X_{i,j}|^2$.

The program (49) is non-convex, but a closed form solution can be tractably found by truncating the SVD as in (48). Choosing R is equivalent to tuning a real-valued hyperparameter $\lambda \geq 0$ in the Lagrangian of this program,

$$\operatorname*{arg\,min}_{\mathbf{S}} \left(||\tilde{\mathbf{S}}_0 - \mathbf{S}||_F^2 + \lambda \left(\operatorname{rank}(\mathbf{S})\right) \right). \tag{50}$$

The rank constraint on the optimization variable $\mathbf{S}$ is also non-convex, and the solution requires *hard-thresholding*, i.e., selecting an integer R in (48). Additionally, we cannot solve (50) in this way, as we cannot take the truncated SVD of a matrix with unknown values. Following [5], [6], this leads to the convex relaxation (51), which replaces the rank penalty term with the *nuclear norm* of the decision matrix:

$$\begin{aligned} &\operatorname*{arg\,min}_{\mathbf{S}} \left(||\tilde{\mathbf{S}}_0 - \mathbf{S}||_F^2 + \lambda||\mathbf{S}||_* \right), \\ &\text{s.t.} \quad ||\tilde{\mathbf{S}}_0 - \mathbf{S}_\Omega||_F \leq \delta, \end{aligned} \tag{51}$$

where $[\tilde{\mathbf{S}}_\Omega]_{i,j} = 0 \ \forall (i,j) \in \Omega$. The operator $||\cdot||_*$ is the nuclear norm, which is the sum of the singular values of $\mathbf{S}$. The hyperparameter δ reflects how accurately we wish to match the coefficients that are known beforehand in $\tilde{\mathbf{S}}_0$. The program (50) promotes solutions with skewed singular values, which are "approximately low-rank".

C. Online Matrix Completion With Time Horizon Penalty

We can use the sequential datastreams in (3) to perform real-time estimation of the sensitivity matrix at time t, $\tilde{\mathbf{S}}_t^{\#}$, by solving the online convex optimization problem (52):

$$\begin{aligned} \tilde{\mathbf{S}}_t^{\#} &= \operatorname*{arg\,min}_{\mathbf{S}} ||\Delta\boldsymbol{v}_t - \mathbf{S}\Delta\boldsymbol{x}_t||_2^2 + \lambda||\mathbf{S}||_* + c\sum_{s=1}^{t-1}\gamma^s||\tilde{\mathbf{S}}_{t-s}^{\#} - \mathbf{S}||_F^2, \\ &\text{s.t.} \quad ||\tilde{\mathbf{S}}_0 - \mathbf{S}_\Omega||_F \leq \delta. \end{aligned} \tag{52}$$

The summation term in the optimization is a penalty term: if we consider $\tilde{\mathbf{S}}_t^{\#}$ for all t as a time series, then the summation is equivalent to an exponential smoother. The time constant of the smoother is $\gamma \in (0,1)$ and the strength of this penalty term is given by hyperparameter c. The purpose of this term is to smooth out any sharp difference between the various $\tilde{\mathbf{S}}_t^{\#}$ at contiguous time steps. The voltage and power perturbations at time t are the vectors $\Delta\boldsymbol{v}_t \in \mathbb{R}^n$, $\boldsymbol{x}_t \in \mathbb{R}^{2n}$.

IV. Case Study

We compute the voltage sensitivities to active and reactive power injections for the IEEE 13-bus test case using OpenDSS as a baseline for comparison. The default loadshape is used for all loads. CVXPY [20] is used to implement the matrix completion algorithms. To represent a varying degree of sensor penetration, we vary the number of observed sensitivity coefficients $|\Omega|$, from 20% to 90% of the total number of entries. We vary the nuclear norm penalty λ between 1×10^{-6} and 8×10^{-6}. We fix $\delta = 6 \times 10^{-3}$. We reconstruct the active and reactive power sensitivities with a mean absolute error below 1.25×10^{-6} for all sensor levels.

At the sensor observability level of 20%, we used the online model update (52) to estimate $\tilde{\mathbf{S}}_t^{\#}$ in real time. With a smoothing factor of $\gamma = 0.9$, a nuclear norm penalty of $\lambda = 1.25 \times 10^{-4}$, and a gain of $c = 1 \times 10^{-8}$ for the smoothing term, we run the online optimization problem for 15-minute time steps at 10 different noise levels for the IEEE 13-bus test case, as shown in Fig. 2. The errors

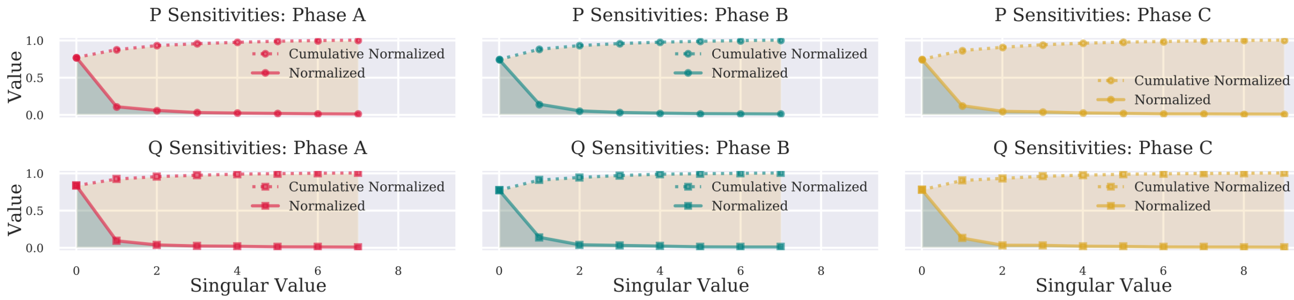

Fig. 1. Spectral analysis of the matrix in (9) by phase and injection type for the IEEE 13-bus test feeder showing approximate low-rank structure.

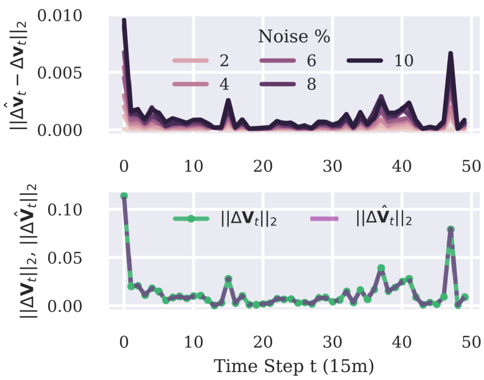

Fig. 2. Online estimation results for the IEEE 13-bus test feeder. The top figure shows the Euclidean distance vs. time between the observed and predicted voltage deviations at all nodes for varying noise levels. The bottom figure shows the Euclidean norm vs. time of the predicted (purple) and the true (green) voltage deviations for all nodes given the lowest noise level.

are approximately an order of magnitude smaller than the values of $\|\Delta \boldsymbol{v}_t\|$ and $\|\Delta \hat{\boldsymbol{v}}_t\|$ themselves at all noise levels, which indicates the predictive performance of the method. This shows that $\tilde{\mathbf{S}}$ can accurately and precisely model the voltage impact of complex power injections.

V. Conclusion

This paper analyzed the voltage magnitude sensitivity matrices for radial electrical networks. We showed that these matrices achieve distinct values, proposed methods for recovering and updating them in low-observability modeling scenarios, and developed conditions for solving the underdetermined linear system formed by them based on the nodal power factors. The results indicate that many networks may be able to be modeled with significantly reduced data input requirements. Problems that require phase angle measurements may be able to be solved using voltage magnitude measurements by exploiting the voltage sensitivities created by active and reactive power injections. The presented results are useful for updating a linearization of the node voltages, and identifying the behavior of grids in settings where measurements are limited.

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